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Cartier isomorphism and Hodge Theory in the non-commutative case

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Abstract. These lectures attempt to give an elementary introduction to my recent paper "Non-commutative Hodge-to-de Rham degeneration via the method of Deligne-Illusie".

Contents

1. Introduction

One of the standard ways to compute the cohomology groups of a smooth complex manifold X is by means of the de Rham theory: the de Rham cohomology groups

$$H^\bullet_{DR}(X) = \mathbb{H}^\bullet(X\ \Omega^\bullet_{DR}) \tag{1.1}$$

are by definition the hypercohomology groups of X with coefficients in the (holomorphic) de Rham complex $\Omega^\bullet_{DR}$, and since, by the Poincaré Lemma, $\Omega^\bullet_{DR}$ is a resolution of the constant sheaf $\mathbb{C}$, we have $H^\bullet_{DR}(X) \cong H^\bullet(X\ \mathbb{C})$. If X is in fact algebraic, then $\Omega^\bullet_{DR}$ can also be defined algebraically, so that the right-hand side in (1.1) can be understood in two ways: either as the hypercohomology of an analytic space, or as the hypercohomology of a scheme equipped with the Zariski topology. One can show that the resulting groups $H^\bullet_{DR}(X)$ are the same (for compact X, this is just the GAGA principle; in the non-compact case this is a difficult but true fact established by Grothendieck [**Gro66**]).

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Of course, an algebraic version of the Poincaré Lemma is false, since the Zariski topology is not fine enough—no matter how small a Zariski neighborhood of a point one takes, it usually has non-trivial higher de Rham cohomology. However, the Lemma survives on the formal level: the completion $\widehat{\Omega^{\bullet}_{DR}}$ of the de Rham complex near a closed point $x \in X$ is quasi-isomorphic to $\mathbb{C}$ placed in degree 0.

Assume now that our X is a smooth algebraic variety over a perfect field k of characteristic $p > 0$. Does the de Rham cohomology still make sense?

The de Rham complex $\Omega^{\bullet}_{DR}$ itself is well-defined: Ω^1 is just the sheaf of Kähler differentials, which makes sense in any characteristic and comes equipped with the universal derivation $d : \mathcal{O}_X \to \Omega^1$, and $\Omega^{\bullet}_{DR}$ is its exterior algebra, which is also well-defined in characteristic p. However, the Poincaré Lemma breaks down completely—the homology of the de Rham complex remains large even after taking completion at a closed point.

In degree 0, this is actually very easy to see: for any local function f on X, we have $df^p = pf^{p-1}df = 0$, so that all the p-th powers of functions are closed with respect to the de Rham differential. Since we are in characteristic p, these powers form a subsheaf of algebras in $\mathcal{O}_X$ which we denote by $\mathcal{O}_X^p \subset \mathcal{O}_X$. This is a large subsheaf. In fact, if we denote by $X^{(1)}$ the scheme X with $\mathcal{O}_X^p$ as the structure sheaf, then $X \cong X^{(1)}$ as abstract schemes, with the isomorphism given by the Frobenius map $f \mapsto f^p$. Fifty years ago P. Cartier proved that in fact *all* the functions in $\mathcal{O}_X$ closed with respect to the de Rham differential are contained in $\mathcal{O}_X^p$, and moreover, one has a similar description in higher degrees: there exist natural isomorphisms

$$C : \mathcal{H}^{\bullet}_{DR} \cong \Omega^{\bullet}_{X^{(1)}} \tag{1.2}$$

where on the left we have the homology sheaves of the de Rham complex, and on the right we have the sheaves of differential forms on the scheme $X^{(1)}$. These isomorphisms are known as *Cartier isomorphisms*.

The Cartier isomorphism has many applications, but one of the most unexpected was discovered in 1987 by P. Deligne and L. Illusie: one can use the Cartier isomorphism to give a purely algebraic proof of the following purely algebraic statement, which is normally proved by the highly transcendental Hodge Theory.

THEOREM 1.1 ([**DI87**]). *Assume given a smooth proper variety X over a field K of characteristic 0. Then the Hodge-to-de Rham spectral sequence*

$$H^{\bullet}(X\ \Omega^{\bullet}) \Rightarrow H^{\bullet}_{DR}(X)$$

associated to the stupid filtration on the de Rham complex $\Omega^{\bullet}$ degenerates at the first term.

The proof of Deligne and Illusie was very strange, because it worked by reduction to positive characteristic, where the statement is not true for a general X. What they proved is that if one imposes two additional conditions on X, then the Cartier isomorphisms can be combined together into a quasi-isomorphism

$$\Omega^{\bullet}_{DR} \cong \bigoplus_i \mathcal{H}^i_{DR}[-i] \cong \bigoplus_i \Omega^i_{X^{(1)}}[-i] \tag{1.3}$$

in the derived category of coherent sheaves on $X^{(1)}$. The degeneration follows from this immediately for dimension reasons. The additional conditions are:

(i) X can be lifted to a smooth scheme over $W_2(k)$, the ring of second Witt vectors of the perfect field k (e.g. if $k = \mathbb{Z}\ p\mathbb{Z}$, X has to be liftable to $\mathbb{Z}\ p^2\mathbb{Z}$), and

(ii) we have $p > \dim X$.

To deduce Theorem 1.1, one finds by the standard argument a proper smooth model X_R of X defined over a finitely generated subring $R \subset K$, one localizes R so that it is unramified over $\mathbb{Z}$ and all its residue fields have characteristic greater than $\dim X$, and one deduces that all the special fibers of $X_R\ \ R$ satisfy the assumptions above; hence the differentials in the Hodge-to-de Rham spectral sequence vanish at all closed points of $\operatorname{Spec} R$, which means they are identically 0 by Nakayama.

The goal of these lectures is to present in a down-to-earth way the results of two recent papers [**Kal05**], [**Kal08**], where the story summarized above has been largely transferred to the setting of *non-commutative geometry*.

To explain what I mean by this, let us first recall that a non-commutative version of differential forms has been known for quite some time now. Namely, assume given an associative unital algebra A over a field k, and an A-bimodule M. Then its *Hochschild homology* $HH_\bullet(A\ \ M)$ of A with coefficients in M is defined as

$$HH_\bullet(A) = \operatorname{Tor}^\bullet_{A^{opp}\otimes A}(A\ \ M) \tag{1.4}$$

where $A^{opp} \otimes A$ is the tensor product of A and the opposite algebra A^{opp}, and the A-bimodule M is treated as a left module over $A^{opp} \otimes A$. Hochschild homology $HH_\bullet(A)$ is the Hochschild homology of A with coefficients in itself.

Assume for a moment that A is in fact commutative, and $\operatorname{Spec} A$ is a smooth algebraic variety over K. Then it has been proved back in 1962 in the paper [**HKR62**] that we have canonical isomorphisms $HH_i(A) \cong \Omega^i(A\ \ k)$ for any $i \geq 0$. Thus for a general A, one can treat Hochschild homology classes as a replacement for differential forms.

Moreover, in the early 1980s it was discovered by A. Connes [**Con83**], J.-L. Loday and D. Quillen [**LQ83**], and B. Feigin and B. Tsygan [**FT83**], that the de Rham differential also makes sense in the general non-commutative setting. Namely, these authors introduced a new invariant of an associative algebra A called *cyclic homology*; cyclic homology, denoted $HC_\bullet(A)$, is related to the Hochschild homology $HH_\bullet(A)$ by a spectral sequence

$$HH_\bullet(A)[u^{-1}] \Rightarrow HC_\bullet(A) \tag{1.5}$$

which in the smooth commutative case reduces to the Hodge-to-de Rham spectral sequence (here u is a formal parameter of cohomological degree 2, and $HH_\bullet(A)[u^{-1}]$ is shorthand for "polynomials in u^{-1} with coefficients in $HH_\bullet(A)$").

It has been conjectured for some time now that the spectral sequence (1.5), or a version of it, degenerates under appropriate assumptions on A (which imitate the assumptions of Theorem 1.1). Following [**Kal08**], we will attack this conjecture by the method of Deligne and Illusie. To do this, we will introduce a certain non-commutative version of the Cartier isomorphism, or rather, of the "globalized" isomorphism (1.3) (in the process of doing it, we will need to introduce some conditions on A which precisely generalize the conditions (i), (ii) above). Then we prove a version of the degeneration conjecture as stated by M. Kontsevich and Ya.