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# Birational geometry for number theorists

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ABSTRACT. We introduce some of the ideas and tools of birational geometry which play a role in conjectures by Bombieri, Lang, Vojta and Campana on the relationship between arithmetic and geometry. After a brief discussion of geometry and arithmetic on curves in Section 0, we discuss Kodaira dimension of a variety and its conjectural relationship with arithmetic properties in Section 1. In Section 2 we outline Campana's approach aiming for a more solid conjectural relationship with arithmetic through the core map. Section 3 outlines the minimal model program and discusses its current status. In Section 4 we review Vojta's conjectures and their relationship to Campana's conjectures and to the *abc* conjecture of Masser-Oesterlé.

## CONTENTS



## Introduction

When thinking about the course "birational geometry for number theorists" I so naïvely agreed to give at the Göttingen summer school, I could not avoid imagining the spirit of the late Serge Lang, not so quietly beseeching one to do things right, keeping the theorems functorial with respect to ideas, and definitions natural. But most important is the fundamental tenet of Diophantine geometry, for which Lang was one of the strongest and loudest advocates, which was so aptly summarized in the introduction of Hindry-Silverman [**HS00**]:

**GEOMETRY DETERMINES ARITHMETIC.**

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To make sense of this, largely conjectural, epithet, it is good to have some loose background in birational geometry, which I will try to provide. For the arithmetic motivation I will explain conjectures of Bombieri, Lang and Vojta, and new and exciting versions of those due to Campana. In fact, I imagine Lang would insist (strongly, as only he could) that Campana's conjectures most urgently need further investigation, and indeed in some sense they form the centerpiece of these notes.

Birational geometry is undergoing revolutionary developments these very days: large portions of the minimal model program were solved soon after the Göttingen lectures [**BCHM06**], and it seems likely that more is to come. Also, a number of people seem to have made new inroads into the long standing resolution of singularities problem. I am not able to report on the latter, but I will give a brief account of the minimal model program as it seems to stand at this point in time.

Our convention: a variety over $k$ is an *absolutely* reduced and irreducible scheme of finite type over $k$.

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## 0. Geometry and arithmetic of curves

The arithmetic of algebraic curves is one area where basic relationships between geometry and arithmetic are known, rather than conjectured. Much of the material here is covered in Darmon's lectures of this summer school.

**0.1. Closed curves.** Consider a smooth projective algebraic curve $C$ defined over a number field $k$. We are interested in a qualitative relationship between its arithmetic and geometric properties.

We have three basic facts:

0.1.1. A curve of genus 0 becomes rational after at most a quadratic extension $k'$ of $k$, in which case its set of rational points $C(k')$ is infinite (and therefore dense in the Zariski topology).

0.1.2. A curve of genus 1 has a rational point after a finite extension $k'$ of $k$ (though the degree is not a priori bounded), and has positive Mordell–Weil rank after a further quadratic extension $k''\ k'$, in which case again its set of rational points $C(k'')$ is infinite (and therefore dense in the Zariski topology).

We can immediately introduce the following definition:

Definition 0.1.3. Let $X$ be an algebraic variety defined over $k$. We say that rational points on $X$ are potentially dense if there is a finite extension $k'\ k$ such that the set $X(k')$ is dense in $X_{k'}$ in the Zariski topology.

Thus rational points on a curve of genus 0 or 1 are potentially dense.

Finally we have

THEOREM 0.1.4 (Faltings, 1983). *Let $C$ be an algebraic curve of genus $> 1$ over a number field $k$. Then $C(k)$ is finite.*

See, e.g. [**Fal83, HS00**].

In other words, rational points on a curve $C$ of genus $g$ are potentially dense if and only if $g \leq 1$.

0.1.5. So far there isn't much birational geometry involved, because we have the old theorem:

THEOREM 0.1.6. *A smooth algebraic curve is uniquely determined by its function field.*

But this is an opportunity to introduce a tool: on the curve $C$ we have a canonical divisor class $K_C$, such that $\mathcal{O}_C(K_C) = \Omega^1_C$, the sheaf of differentials, also known by the notation $\omega_C$—the dualizing sheaf. We have:

(1) $\deg K_C = 2g - 2 = -\chi^{\mathrm{top}}(C_{\mathbb{C}})$, where $\chi^{\mathrm{top}}(C_{\mathbb{C}})$ is the topological Euler characteristic of the complex Riemann surface $C_{\mathbb{C}}$.
(2) $\dim H^0(C, \mathcal{O}_C(K_C)) = g$.

For future discussion, the first property will be useful. We can now summarize, following [**HS00**]:

0.1.7.

| Degree of $K_C$ | rational points |
|---|---|
| $2g - 2 \leq 0$ | potentially dense |
| $2g - 2 > 0$ | finite |

**0.2. Open curves.**

0.2.1. Consider a smooth quasi-projective algebraic curve $C$ defined over a number field $k$. It has a unique smooth projective completion $C \subset \overline{C}$, and the complement is a finite set $\Sigma = \overline{C} \smallsetminus C$. Thinking of $\Sigma$ as a reduced divisor of some degree $n$, a natural line bundle to consider is $\mathcal{O}_{\overline{C}}(K_{\overline{C}} + \Sigma) \simeq \omega_{\mathcal{O}}(\Sigma)$, the sheaf of differentials with logarithmic poles on $\Sigma$, whose degree is again $-\chi^{\mathrm{top}}(C) = 2g - 2 + n$. The sign of $2g - 2 + n$ again serves as the geometric invariant to consider.

0.2.2. Consider for example the affine line. Rational points on the affine line are not much more interesting than those on $\mathbb{P}^1$. But we can also consider the behavior of *integral* points, where interesting results do arise. However, what does one mean by integral points on $\mathbb{A}^1$? The key is that integral points are an invariant of an "integral model" of $\mathbb{A}^1$ over $\mathbb{Z}$.

0.2.3. Consider the ring of integers $\mathcal{O}_k$ and a finite set $S \subset \operatorname{Spec} \mathcal{O}_k$ of finite primes. One can associate to it the ring $\mathcal{O}_{k,S}$ of $S$-integers, of elements in $K$ which are in $\mathcal{O}_\wp$ for any prime $\wp \notin S$.

Now consider a *model* of $C$ over $\mathcal{O}_{k,S}$, namely a scheme $\mathcal{C}$ of finite type over $\mathcal{O}_{k,S}$ with an isomorphism of the generic fiber $\mathcal{C}_k \simeq C$. It is often useful to start with a model $\overline{\mathcal{C}}$ of $\overline{C}$, and take $\mathcal{C} = \overline{\mathcal{C}} \smallsetminus \overline{\Sigma}$, where $\overline{\Sigma}$ is the closure of $\Sigma$ in $\overline{\mathcal{C}}$.