

Critical site percolation on the triangular lattice

In the whole session, we consider site percolation on the triangular lattice with parameter $p = 1/2$ (the measure is denoted by \mathbb{P}). We use coordinates in the base (e_1, e_2) where $e_1 = (1, 0)$ and $e_2 = (1/2, \sqrt{3}/2)$. A rectangle $[a, b] \times [c, d]$ is given by the sites $ne_1 + me_2$ where $n \in [a, b]$ and $m \in [c, d]$.

When handling percolation, you want to express your artistic qualities: draw some pictures!

Background (FKG inequality) An event A is *increasing* if it is preserved by addition of black sites. The FKG inequality is

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B)$$

for any increasing events A and B .

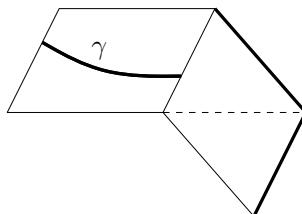
Exercise 1: Russo-Seymour-Welsh theorem for site percolation

A *horizontal crossing* of a rectangle is a sequence of neighboring black sites going from the left side to the right side.

Question 1 What is the probability that there exists an horizontal crossing of $[0, n]^2$?

Question 2 Assume $[0, n]^2$ is crossed from left to right and set Γ to be the lowest horizontal crossing. Let γ be a deterministic path from left to right, prove that $\{\Gamma = \gamma\}$ is measurable with respect to the σ -algebra spanned by the sites **below** γ and the sites of γ . When conditioning on $\{\Gamma = \gamma\}$, what can be said about the law of sites above γ ?

Question 3 Consider the shape in the following figure and assume that the left rectangle $[0, n]^2$ is crossed horizontally. Can you bound from below the probability that the lowest crossing Γ is connected to the bold part by a black path? *Hint: condition on $\{\Gamma = \gamma\}$ and consider the reflected path $\sigma(\gamma)$ with respect to the line $y = n + \frac{1}{2}$.*



Question 4 Deduce that the probability of crossing the rectangle $[0, 2n] \times [0, n]$ horizontally is bounded away from 0 when n goes to infinity.

Question 5 a) Let $\rho > 1$. Deduce that the probability to cross the rectangle $[0, \rho n] \times [0, n]$ horizontally is (uniformly in n) bounded away from 0.

b) Prove that the probability of a black circuit surrounding the origin in the annulus $[-2n, 2n]^2 \setminus [-n, n]^2$ remains bounded away from 0 when n goes to infinity.

c) Show that almost surely there is no infinite cluster at $p = \frac{1}{2}$.

d) What can be said about $\mathbb{P}(0 \leftrightarrow \partial\Lambda_n)$?

e)** Explain a strategy to prove that $p_c = \frac{1}{2}$.

Exercise 2: universal exponents

Let σ be a finite sequence of colors (B for black, W for white). We associate to $n > 0$ and $\sigma = \{\sigma_1, \dots, \sigma_k\}$ the event $A_\sigma(n)$ that there exist paths $\gamma_1, \dots, \gamma_k$ such that:

- γ_i has color σ_i
- γ_i connects the origin to the boundary of $[-n, n]^2$. When $k > 2$, we require only that the paths connect $[-k, k]^2$ to the boundary of $[-n, n]^2$.
- $\gamma_1, \dots, \gamma_k$ can be found in counter-clockwise order.

We define the same event in the upper half-plane (which we denote by $A_\sigma^{\mathbb{H}}(n)$). In this case, the paths must be found in counter-clockwise order, starting from the right.

Question 1 a) Prove that $\mathbb{P}(A_{BW}^{\mathbb{H}}(n)) \geq \frac{c}{n}$ for some universal constant c . *Hint. Use the RSW theorem to construct a point in $\{0\} \times [-n/2, n/2]$ which is connected to the boundary of the box by two arms of distinct colors.*

b) Assume $A_{BW}^{\mathbb{H}}(n)$ holds. We require that the site on the left of 0 is white and that it is the start of the white path, and the site on the right is black and is the start of the black path. Show that one can explore the interface between the black and the white paths without exploring any other site.

c) Let $B(n)$ be the event that there exist a white path connected to the left side of $[-n, n] \times [0, n]$ and a black path connected to the right side. Show that there exists a universal $c_1 > 0$ such that

$$\mathbb{P}(A_{BW}^{\mathbb{H}}(n)) \leq c_1 \mathbb{P}(B(n)).$$

d) Deduce that there exists $c_2 > 0$ such that

$$\mathbb{P}(A_{BW}^{\mathbb{H}}(n)) \leq \frac{c_2}{n}.$$

What was proved?

Question 2* Prove that the exponent for $BWBWW$ in the plane is 2.

Question 3** Prove that the exponent for $BWBW$ in the plane is smaller than 2.

Rudiments of complex analysis

Exercise 1: Riemann mapping theorem

Theorem 0.1 (Riemann mapping theorem) *Let D and D' be two simply connected domains included in \mathbb{C} and different from \mathbb{C} , there exists a conformal map (i.e. a bijection differentiable in the complex variable) between D and D' .*

Question 1 Find a conformal map between the following domains:

- from $\mathbb{R} \times]0, \pi[$ to $\mathbb{H} = \{z, \text{Im}(z) > 0\}$;
- from the disk $\mathbb{D} = \{z, |z| < 1\}$ to \mathbb{H} ;
- from $\mathbb{H} \setminus [0, ir]$ to \mathbb{H} ;
- * from \mathbb{D} to $\mathbb{C} \setminus (-\infty, -\frac{1}{4}]$;
- from $S_\varepsilon = (\mathbb{R} \times]0, 2[) \setminus ((i - \infty, i - \varepsilon] \cup [i + \varepsilon, i + \infty))$ to \mathbb{H}
- ** from \mathbb{H} to an equilateral triangle.

Question 2 a) Show that there is no conformal map from $D(0, 1)$ to \mathbb{C} . *It confirms that the assumption $D \neq \mathbb{C}$ is necessary.*

b) Let D be a simply connected domain and f be a conformal map, why is $f(D)$ simply connected?

Question 3 Which are the conformal maps from $D(0, 1)$ into $D(0, 1)$? *Hint. One can guess what they are and make sure none is left behind using Schwarz's Lemma.* Deduce that there are three (real) degrees of freedom in the choice of a conformal map between two domains in the following sense:

- one can fix the image of one point on the boundary and the image of one point inside the domain;
- one can fix the image of one point inside the domain and the direction of the derivative;
- one can fix the image of three points on the boundary (keeping the order).

Exercise 2: Estimates for conformal maps

Question 1 (Lemme de Schwarz): Let f be a continuous map from $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$ such that $f(0) = 0$ and f is holomorphic inside \mathbb{D} . Show that $|f(z)| \leq |z|$. *Hint. Think about the maximum principle.* Study the case where $|f'(0)| = 1$.

Question 2 * (Koebe 1/4-theorem) Let

$$\mathcal{S} := \{f : \mathbb{D} \rightarrow \mathbb{C}, \text{ analytic, one-to-one with } f(0) = 0 \text{ and } f'(0) = 1\}$$

a) (Area theorem) Let $f \in \mathcal{S}$ and $K = \mathbb{C} \setminus \{1/z, z \in f(\mathbb{D})\}$, prove that

$$\text{area}(K) = \pi \left[1 - \sum_{n=1}^{\infty} n |b_n|^2 \right]$$

where $1/f(1/z) = z + b_0 + \sum_{n \geq 1} \frac{b_n}{z^n}$. Note that it implies $|b_1| \leq 1$.

b)* Prove that if $f = z + a_2 z^2 + \dots$ is in \mathcal{S} , then $|a_2| \leq 2$. *Hint: construct a function $h \in \mathcal{S}$ such that $h(z) = z + \frac{a_2}{2} z^3 + \dots$ and conclude.*

c)* Deduce Koebe 1/4 theorem: if $f \in \mathcal{S}$, then $B(0, \frac{1}{4}) \subset f(\mathbb{D})$.

d) Suppose $f : D \rightarrow D'$ is a conformal transformation with $f(z) = z'$. Then

$$\frac{1}{4} \frac{d(z', \partial D')}{d(z, \partial D)} \leq |f'(z)| \leq 4 \frac{d(z', \partial D')}{d(z, \partial D)}.$$

Exercise 3 **

Prove the Riemann mapping theorem. *Hint. It is sufficient to handle the case of $D' = \mathbb{D}$. Fix ω in D . Consider the set of functions from D into \mathbb{D} such that $f(\omega) = 0$ and $f'(\omega) > 0$. The function f maximizing $f'(\omega)$ will be your conformal map between D and \mathbb{D} .*

Itô's Formula

In this exercise sheet, B is a standard one-dimensional Brownian motion. Let (\mathcal{F}_t) be the filtration associated to the Brownian motion, i.e. $\mathcal{F}_t = \sigma(B_s, s < t)$. The process (M_t) is a *martingale* (with respect to \mathcal{F}_t) if for each $s < t$, $\mathbb{E}[|M_t|] < \infty$ and $\mathbb{E}[M_t | \mathcal{F}_s] = M_s$.

Exercise 1: integration with respect to Brownian motion

Question 1 We call H a *simple process* if it is of the form

$$H_s = \sum_{j=1}^n C_j \mathbb{1}_{[t_{j-1}, t_j)}(s)$$

where (t_j) is increasing and C_j is $\mathcal{F}_{t_{j-1}}$ -measurable.

a) For a (random) process $H = C \mathbb{1}_{[s, t)}$, where C is \mathcal{F}_s -measurable, find a natural candidate for the integral of H against Brownian motion B , in other words, what could $\int_0^\infty H_s dB_s$ be? How could the notion of integral be extended to any simple process?

b) We assume that the integral has been constructed as above. For any simple process H , check that

$$\mathbb{E} \left[\left(\int_0^\infty H_s dB_s \right)^2 \right] = \int_0^\infty \mathbb{E}[H_s^2] ds.$$

Question 2 Let \mathcal{L}^2 the set of square integrable adapted processes (in the sense $\int_0^\infty \mathbb{E}[H_s^2] ds < \infty$). Explain how to extend the definition of integral to \mathcal{L}^2 .

Question 3 For a bounded adapted process H , we define $\int_0^t H_s dB_s$ as $\int_0^\infty H_s \mathbb{1}_{[0, t)} dB_s$. Show that $M_t = \int_0^t H_s dB_s$ is a \mathcal{F}_t -martingale. *Hint. Check it in the case of simple processes first.* ** Show that it is a continuous process.

Remark 0.2 Note that for any bounded adapted process a , $\int_0^t a_s ds$ is straightforward to define. It is also possible to check that $H_t = \int_0^t a_s dB_s + \int_0^t \sigma_s ds$ is a martingale if and only if $\sigma = 0$.

Question 4 a) Let H_s be a bounded continuous adapted process and $t > 0$. Considering subdivisions $0 = t_1^n < \dots < t_n^n = t$ with $\max(t_{i+1}^n - t_i^n) \rightarrow 0$, show that

$$\sum_{i=1}^{n-1} H_{t_i^n} (B_{t_{i+1}^n} - B_{t_i^n}) \xrightarrow{\mathcal{L}^2} \int_0^t H_s dB_s.$$

b) Let H_s be a bounded continuous adapted process and $t > 0$. Considering subdivisions $0 = t_1^n < \dots < t_n^n = t$ with $\max(t_{i+1}^n - t_i^n) \rightarrow 0$, show that

$$\sum_{i=1}^{n-1} H_{t_i^n} (B_{t_{i+1}^n} - B_{t_i^n})^2 \xrightarrow{\mathcal{L}^2} \int_0^t H_s ds.$$

Hint. Recall that $B_t^2 - t$ is a martingale.

c) Prove Itô's formula.

Theorem 0.3 (Itô's formula) For any a, σ bounded adapted processes and $t > 0$, we set $Y_t = \int_0^t a_s dB_s + \int_0^t \sigma_s ds$. Let $\phi: \mathbb{R} \rightarrow \mathbb{R}$ be a function twice continuously derivable, then

$$\phi(Y_t) = \phi(Y_0) + \int_0^t \phi'(Y_s) a_s dB_s + \int_0^t \left[\phi'(Y_s) \sigma_s + \frac{1}{2} \phi''(Y_s) a_s^2 \right] ds.$$

Remark 0.4 In order to write the equality

$$H_t = x + \int_0^t a_s dB_s + \int_0^t \sigma_s ds$$

in a concise way, we often write

$$H_0 = x \quad \text{and} \quad dH_t = a_t dB_t + \sigma_t dt.$$

Exercise 2: A first application, Bessel processes

Let $d > 0$. We **admit** that there exists a unique process, denoted X_t^x , which is solution of the following stochastic differential equation

$$dX_t^x = dB_t + \frac{d-1}{2X_t^x} dt, \quad X_0^x = x$$

up to time $T_x := \inf\{t : X_t^x = 0\}$. This process is called a *d-dimensional Bessel process*. For integer values of d , this process is the norm of a d -dimensional vector with independent Brownian entries. Let $0 < a < x < b < \infty$, τ the first exit time of the set $[a, b]$, and $\phi(x) = \mathbb{P}(X_\tau^x = a)$.

Question 1 Show that $\phi(X_{t \wedge \tau}^x)$ is a martingale with respect to $\mathcal{F}_{t \wedge \tau}$.

Question 2 a) Assume ϕ is twice continuously differentiable. Using Itô's formula, deduce that

$$\frac{1}{2}\phi''(x) + \frac{d-1}{2x}\phi'(x) = 0, \quad a < x < b,$$

and compute ϕ when $d \geq 2$,

b) When $d > 0$, compute $\mathbb{P}(X_\tau^x = a)$. What can you deduce?

Question 3 ** (optional question) Using Itô's formula, show that $\psi(x, t) = \mathbb{P}_x(\tau > t)$ is the solution of a partial differential equation. Deduce an estimate for $\mathbb{P}_x(\tau > t)$ when t goes to infinity. *Hint. I might help you if I am in a good mood.*

Conformal invariance of Brownian motion

Exercise 1: Conformal invariance of Brownian motion

Consider B a Brownian motion in the plane, and for a domain U , $\tau_U := \inf\{t \geq 0 : B_t \notin U\}$ the exit time of U . A conformal map is a bijective bi-holomorphic map. In this exercise, we prove the following theorem:

Theorem 0.5 *Let $z \in U$, and let $f : U \rightarrow V$ be conformal. The law of $\{f(B_t), t \leq \tau_U\}$ is the same as the law of the trace of a Brownian motion in V from $f(z)$ to the boundary.*

Let \tilde{B} be an independent Brownian motion in the plane, introduce the time changes

$$\zeta_s := \int_0^s |f'(B_u)|^2 du \quad \text{and} \quad \sigma_t := \inf\{s \geq 0 : \zeta_s \geq t\}.$$

Define

$$W_t = f(B(\sigma_t \wedge \tau_U)) + \tilde{B}(t) - \tilde{B}(\zeta[\sigma_t \wedge \tau_U]), \quad t \geq 0.$$

It is sufficient to prove that W is a Brownian motion.

Question 0 What does the previous construction boil down to?

Question 1 Show that W is continuous. Let \mathcal{G}_s be the σ -algebra generated by $\{W_u, u \leq s\}$. Show that W is a (\mathcal{G}_s) -Brownian motion if and only if

$$\mathbb{E} \left[e^{\langle \lambda, W_t \rangle} \mid W_s = f(z) \right] = \exp \left(\frac{1}{2} |\lambda|^2 (t-s) + \langle \lambda, f(z) \rangle \right)$$

for every $z \in U$. The quantity $\langle u, v \rangle$ corresponds to the usual scalar product between two complex numbers.

In order to simplify, we prove the statement only for $s = 0$. Consider a conformal map f from U to $f(U)$. Assume first that f and f' are uniformly bounded.

Question 2 a) Show that for every $z \in U$,

$$\mathbb{E} \left[e^{\langle \lambda, W_t \rangle} \mid W(0) = f(z) \right] = \mathbb{E}_z \left[\exp \left(\frac{1}{2} |\lambda|^2 (t - \zeta(\sigma_t \wedge \tau_U)) + \langle \lambda, f(B(\sigma_t \wedge \tau_U)) \rangle \right) \right]$$

where \mathbb{P}_z is the law of a Brownian motion starting at z .

b) Prove that for every $z \in U$,

$$\Delta e^{\langle \lambda, f(z) \rangle} = |\lambda|^2 |f'(z)|^2 e^{\langle \lambda, f(z) \rangle}.$$

c) Using the two-dimensional Itô formula (see below), show that

$$M_s = \exp \left(\frac{1}{2} |\lambda|^2 [t - \zeta(s \wedge \tau_U)] + \langle \lambda, f(B(s \wedge \tau_U)) \rangle \right)$$

is a bounded martingale and conclude.

Question 3 Where did we use the assumption that f and f' are uniformly bounded? How could one get rid of the assumption on f and f' ?

Question 4 What is the probability that a Brownian motion starting at ϵ exits the domain $\mathbb{D} \setminus [-1, 0]$ through $\partial\mathbb{D}$?

Question 5 a) Explain how one could define the Brownian motion in a simply connected domain D between two boundary points a and b . We denote by $\mathbb{P}_{(D,a,b)}^{BM}$ this measure.

b) Sketch a proof of the following conformal invariance property: let (D, a, b) be a simply connected domain and f a conformal map, then

$$f \circ \mathbb{P}_{(D,a,b)}^{BM} = \mathbb{P}_{(f(D),f(a),f(b))}^{BM}.$$

c) Explicit a construction when $D = \mathbb{H}$, $a = 0$ and $b = \infty$.

d) * Let K be a compact set such that $H = \mathbb{H} \setminus K$ is a simply connected domain containing 0 (the set H is called a *hull*). Prove that

$$\mathbb{P}_{(\mathbb{H},0,\infty)}^{BM}(B \text{ stays inside } H) = \phi'_H(0)$$

where ϕ_H is the map from H to \mathbb{H} that maps 0 to 0 and with $\phi_H(z) \sim z$ when z goes to infinity.

Theorem 0.6 (Itô formula in dimension 2) Let $B = B^{(1)} + iB^{(2)}$ be a two-dimensional Brownian motion. Let Σ be an increasing continuous adapted process and

$$X_t = \int_0^t a_s^{(1)} dB_s^{(1)} + i \int_0^t a_s^{(2)} dB_s^{(2)}.$$

Let $f : \mathbb{C} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be a smooth function, then

$$f(X_t, \Sigma_t) = f(X_0, \Sigma_0) + \sum_{i=1}^2 \int_0^t \partial_i f(X_s, \Sigma_s) a_s^{(i)} dB_s^{(i)} + \int_0^t \partial_t f(X_s, \Sigma_s) d\Sigma_s + \frac{1}{2} \sum_{i=1}^2 \int_0^t \partial_{ii}^2 f(X_s, \Sigma_s) (a_s^{(i)})^2 ds.$$

Critical percolation exponents

The goal of this session is to compute explicitly an exponent for radial SLE(6), using diffusion processes. We deduce an exponent for discrete critical percolation on the triangular lattice.

Exercise 1: Disconnection exponent I

Let $x \in (0, 2\pi)$ and let $\mathcal{H}(x, t)$ be the event that one radial SLE(6) starting from 1 does not disconnect e^{ix} from 0 before the time t . The goal of this exercise is to show that there exists $c > 0$ universal such that

$$e^{-\frac{t}{4}} \left(\sin \frac{x}{2} \right)^{\frac{1}{3}} \leq \mathbb{P}[\mathcal{H}(x, t)] \leq ce^{-\frac{t}{4}} \left(\sin \frac{x}{2} \right)^{\frac{1}{3}}.$$

Question 1 Let ζ_t be the driving process of the SLE (it is $\sqrt{6}$ times a standard Brownian motion). Why can one define a real valued process Y_t^x such that $g_t(e^{ix}) = \zeta_t \exp(iY_t^x)$ and $Y_0^x = x$ for every $t < \tau(e^{ix})$ ($\tau(z)$ is the disconnecting time)? Show that

$$dY_t^x = \sqrt{6}dB_t + \cotg(Y_t^x/2)dt.$$

Hint. Recall that the argument is the imaginary part of the logarithm. Moreover, what is $\text{Im}[\partial_t \log g_t(e^{ix})]$?

Question 2 Let $\tau_x := \inf\{t \geq 0 : Y_t^x \in \{0, 2\pi\}\}$, prove that $\mathbb{P}[\mathcal{H}(x, t)] = \mathbb{P}[\tau^x > t]$.

Question 3 Assume that $f(x, t) := \mathbb{P}[\mathcal{H}(x, t)]$ is smooth on $(0, 2\pi) \times [0, \infty)$ (the general theory of diffusion processes guarantees that), show that

$$3f'' + \cotg(x/2)f' = \partial_t f$$

and that $\lim_{x \rightarrow 0^+} f(x, t) = \lim_{x \rightarrow 2\pi^-} f(x, t) = 0$ and $f(x, 0) = 1$.

Question 4 Define $F(x, t) := \mathbb{E}\left[\mathbb{1}_{\mathcal{H}(x, t)} \left(\sin \frac{Y_t^x}{2} \right)^{\frac{1}{3}}\right]$ and show that

$$F(x, t) = e^{-\frac{t}{4}} \left(\sin \frac{x}{2} \right)^{\frac{1}{3}}.$$

Hint. We can admit that the solutions of the PDE are determined by boundary conditions.

Question 5 * Conclude

Exercise 2: One-arm exponent for percolation

Consider critical site percolation on the triangular lattice, we want to prove that

$$\mathbb{P}(0 \leftrightarrow \partial\Lambda_n) = n^{-\frac{5}{48} + o(1)}$$

where $\partial\Lambda_n$ is the boundary of the box of size n . We assume that the result of exercise 3 is known.

Question 1 Let $\pi(r, R)$ be the probability that there exists a path between $\partial\Lambda_r$ and $\partial\Lambda_R$. Show that there exists a constant $c > 0$ uniform in r and R such that

$$c\pi(r_1, r_2)\pi(r_2, r_3) \leq \pi(r_1, r_3) \leq \pi(r_1, r_2)\pi(r_2, r_3).$$

Hint. For the left side of the inequality, use RSW.

Question 2 Consider percolation on a finite subgraph of the triangular lattice with circular shape. Explain how to define a natural exploration process at a discrete level. Towards which process should it converge (we do not ask for a proof!!)?

Question 3 What is the event associated to $\{\partial\Lambda_r \leftrightarrow \partial\Lambda_R\}$ for the exploration process? Show that there exist $0 < c_1, c_2 < \infty$ such that for every R ,

$$c_1 R^{-\frac{5}{48}} \leq \pi(n, nR) \leq c_2 R^{-\frac{5}{48}}$$

when n is large enough.

Question 4 Conclude the proof.

Exercise 3 (optional**): Disconnection exponent II

Let $x \in (0, 2\pi)$ and let $\mathcal{J}(t)$ be the event that one radial SLE(6) starting from 1 does not close any counterclockwise loop before time t . Let ∂_t^1 be the part of $\partial K_t \setminus \partial U$ lying on the left of the end-point γ_t . We set Y_t to be the arc-length of $g_t(\partial_t^1)$.

Question 1 Find a SDE which is satisfied by Y and express $\mathbb{P}[\mathcal{J}(t)]$ in terms of the survival time of Y .

Question 2*** Find a PDE which is satisfied by the function $h(x, t) := \int_0^1 f(x, t+s) ds$ where $f(x, t) := \mathbb{P}(2\pi \notin Y[0, t] | Y_0 = x)$. What are the boundary conditions? *Hint. Try to go back to the discrete model in order to prove that $h(x, t) - h(0, t) = o(x)$.*

Question 3* Explain how one could prove that there exist $0 < c_1, c_2 < \infty$ such that

$$c_1 e^{-\frac{5t}{48}} \leq \mathbb{P}[\mathcal{J}(t)] \leq c_2 e^{-\frac{5t}{48}}.$$

Question 4* Show that the probability of a radial SLE(6) starting from 1 does not close any counterclockwise loop before touching the circle of radius ε is of order $\varepsilon^{-5/48}$.

Some analytic properties of SLE curves

In this session, we prove that the SLE(κ) is space-filling whenever $\kappa \geq 8$. We further compute exponents that will be useful for the derivation of intersecting exponents of Brownian motion and random walks.

Exercise 1: Conformal radius, space-filling SLE and Hausdorff dimension

We consider chordal SLE(κ) in the upper half-plane for $\kappa > 0$. As usual, g_t is the conformal map from H_t to \mathbb{H} with hydrodynamical renormalization. Define

$$\tilde{g}_t(z) := \frac{g_t(z) - g_t(z_0)}{g_t(z) - g_t(z_0)} \quad \text{for all } z \in \mathbb{H}.$$

Question 1 Let $z_0 \in \mathbb{H}$. Prove that there exist constants $0 < c_1, c_2 < \infty$ such that

$$c_1 d[z_0, \gamma[0, t]] \leq |\tilde{g}'_t(z_0)|^{-1} \leq c_2 d[z_0, \gamma[0, t]]$$

for any $t < \tau(z_0)$.

Question 2 a) Assume that $t < \tau(z_0)$. Explain how one could derive the following equality – we do not ask for the (straightforward yet messy) computation.

$$\partial_t \tilde{g}_t(z) = a_t \times \frac{\tilde{\beta}_t \tilde{g}_t(z) (\tilde{g}_t(z) - 1)}{(1 - \tilde{\beta}_t)(\tilde{g}_t(z) - \tilde{\beta}_t)},$$

where

$$\tilde{\beta}_t = \frac{\beta_t - g_t(z_0)}{\beta_t - g_t(z_0)} \quad \text{and} \quad a_t = \frac{2(\tilde{\beta}_t - 1)^4}{(g_t(z_0) - g_t(z_0))^2 \tilde{\beta}_t^2}.$$

Check that $\tilde{\beta}_t \in \partial\mathbb{D}$ and $a_t > 0$ for every $t < \tau(z_0)$.

b) We introduce the time-change $s := \int_0^t a_u du$. Show that $h_s = \tilde{g}_{t(s)}$ satisfies a 'radial Loewner like' equation with 'driving process' α_s where $\exp(i\alpha_s) = \tilde{\beta}_{t(s)}$.

c) By differentiating (with respect to z) the previous equation at z_0 , show that

$$\partial_s h'_s(z_0) = \frac{2h'_s(z_0)}{1 - \tilde{\beta}_s}.$$

Deduce that $|h'_s(z_0)| = |h'_0(z_0)|e^{-s}$ when $s < s(\tau(z_0))$.

Question 3 Explain how one could prove (we do not ask for the computation)

$$\alpha_0^x = x \quad \text{and} \quad d\alpha_t^x = \sqrt{\kappa} dB_t + \frac{\kappa - 4}{2} \cotg(\alpha_t^x/2) dt.$$

Question 4 To which event for the diffusion α does $\{t(s) = \tau(z_0)\}$ correspond? Show that $d(z_0, \gamma[0, \infty)) \asymp e^{-S}$ where S is the survival time of the diffusion α .

Question 5 Prove that the SLE(κ) is dense whenever $\kappa \geq 8$. We assume that it is generated by a transient continuous curve. prove that it is space-filling.

Question 6* What can be said about the Hausdorff dimension of SLE(κ) when $\kappa < 8$?

Exercise 2 (optional): Derivative exponents for SLE

We consider radial SLE(κ) starting from 1. Recall the definition of $\mathcal{H}(x, t)$ from the previous session. We consider $b > 0$ and $\kappa \in (0, 8)$, we want to prove

$$c_1(x) e^{-\nu t} \leq \mathbb{E}[|g'_t(e^{ix})|^b \mathbb{1}_{\mathcal{H}(x, t)}] \leq c_2(x) e^{-\nu t}$$

where $x \in (0, 2\pi)$, $0 < c_1, c_2 < \infty$ and

$$\nu := \frac{8b + \kappa - 4 + \sqrt{(\kappa - 4)^2 + 16b\kappa}}{16}.$$

Question 1 Let Y^x be the diffusion defined by $g_t(e^{ix}) = \zeta_t e^{iY_t^x}$. Explain how one could prove that $dY_t^x = \sqrt{\kappa} dB_t + \cotg(Y_t^x/2) dt$. Set $\tau^x := \inf\{t \geq 0 : Y_t^x \in \{0, 2\pi\}\}$ and show that $\phi_t^x := |g'_t(e^{ix})|$ satisfies

$$\phi_t^x = \partial_x Y_t^x$$

for every $t < \tau^x$.

Question 2 Show that

$$\partial_t \log \phi_t^x = -\frac{1}{2 \sin^2(Y_t^x/2)}.$$

Set $f(x, t) := [|g'_t(e^{ix})|^b \mathbb{1}_{\mathcal{H}(x,t)}]$, deduce that

$$f(x, t) = \left[\exp \left(-\frac{b}{2} \int_0^t \frac{ds}{\sin^2(Y_s/2)} \right) \mathbb{1}_{\mathcal{H}(x,t)} \right]$$

Question 3 Assuming $f(x, t)$ is smooth, show that

$$\frac{\kappa}{2} \partial_{xx} f + \cot g(x/2) \partial_x f - \frac{b}{2 \sin^2(x/2)} f = \partial_t f.$$

What are the boundary conditions?

Question 4 ** Sketch how one could conclude.

Intersection of two Brownian motions

Exercise 1: playing with SLE(6) and other models

Let $V = \{z : 0 < \operatorname{Re}(z), \operatorname{Im}(z) < \sqrt{3}/2 \operatorname{Re}(z)\}$ and $V_0 = \{z \in V, \sqrt{3}/2 \operatorname{Re}(z) + \operatorname{Im}(z) \leq \sqrt{3}/2\}$.

Question 1 Consider a simple random walk in $\{\frac{m}{N} + \frac{n}{N} e^{i\pi/3}, 0 \leq n, m \leq N\}$ on the triangular lattice stopped at the first time it hits $1 + e^{2i\pi/3} \mathbb{R}$. We consider that it is reflected on \mathbb{N} following the transition probabilities

$$\begin{aligned} p(x, x + \frac{1}{N}) &= \frac{1}{3}, & p(x, x - \frac{1}{N}) &= p(x, x + \frac{e^{i\pi/3}}{N}) = p(x, x + \frac{e^{2i\pi/3}}{N}) = p(x, x) = \frac{1}{6}, \\ p(x, x + \frac{e^{i\pi/3}}{N}) &= \frac{1}{3}, & p(x, x + \frac{1}{N}) &= p(x, x + \frac{e^{-i\pi/3}}{N}) = p(x, x + \frac{e^{-2i\pi/3}}{N}) = p(x, x) = \frac{1}{6}. \end{aligned}$$

We assume that the random walk converges (when N goes to ∞) to a continuous process called the Brownian motion which is reflected with an angle of $2\pi/3$ and stopped at the first time it hits $1 + e^{2i\pi/3} \mathbb{R}$.

a) What is the hitting distribution of the reflected Brownian motion on $1 + e^{2i\pi/3} \mathbb{R}$?

b) (optional) We construct the process in any domain by conformal mapping. What does the process look like?

Question 2 Let K be a compact set intersecting \mathbb{R} and not intersecting $(-1, 1)$ such that $\mathbb{H} \setminus K$ is simply connected, show that

$$\mathbb{P}^{RBM}(E \cap K) = 1 - \text{length}(\phi(\partial K))$$

where ϕ is the conformal map from $\mathbb{H} \setminus K$ to V_0 such that $\phi(0) = 0$, $\phi(1) = 1$ and $\phi(-1) = e^{i\pi/3}$. Show that the hull of the reflected Brownian motion up to the first time it hits $\mathbb{R} \setminus (-1, 1)$ is the same as the hull of a chordal SLE(6) up to the first time it hits $\mathbb{R} \setminus (-1, 1)$.

Question 3 ** Let K_t be a chordal SLE(6) from 1 to e^{ix} in \mathbb{D} , τ is the disconnecting time of 0. Let \tilde{K}_t be a radial SLE(6) starting from 1 and going to 0 and $\tilde{\tau}_t$ the disconnecting time of e^{ix} . Explain how one could prove that the two processes $K_{t \wedge \tau}$ and $\tilde{K}_{t \wedge \tilde{\tau}}$ are the same up to time reparametrization.

Exercise 2: Intersection exponent for two Brownian motions

In this exercise, we prove

$$\mathbb{P}(Z^1[\tau_r^1, \tau_1^1] \cap Z^2[\tau_r^2, \tau_1^2] = \emptyset) = R^{-\frac{5}{4}+o(1)}$$

where Z^1 and Z^2 are two Brownian motions started uniformly on the circle of radius 1. The random variables τ_r^1 and τ_r^2 are the hitting time of the ball centered at 0 of size r .

Question 1 (SLE computation) Let D_t be the hull of a chordal SLE(6) starting from 1.

a) let $\ell_t := \int_0^{2\pi} |g'_t(e^{ix})| \mathbb{1}_{\mathcal{H}(x,t)} dx$ be the length of the arc $A_t := \partial D_t \cap \partial \mathbb{U}$ that has not been disconnected by D_t . What is the probability for a Brownian starting at the origin to hit A_t ?

b) Let Z a Brownian motion starting at 0, τ_1 the exit time of the disk. Let Y be a radial SLE(6) starting at 1 and $\tilde{\tau}_r$ the hitting time of the ball of size r . Show that

$$\mathbb{P}(Z[0, \tau_1] \cap Y[0, \tilde{\tau}_r] = \emptyset) \asymp r^{-\frac{5}{4}}$$

when r goes to 0.

c) Let Z a Brownian motion starting at 0, τ_1 the exit time of the disk. Let X be a reflected Brownian motion starting at 1 and $\tilde{\tau}_r$ the hitting time of the ball of size r . Show that

$$\mathbb{P}(Z[0, \tau_1] \cap X[0, \tilde{\tau}_r] = \emptyset) \asymp r^{-\frac{5}{4}}$$

when r goes to 0.

Question 2 ** Sketch a proof of the derivation of ξ .

Question 3 ** Explain (in a very sketchy way) how you would prove

$$\mathbb{P}(X[1, n] \cap Y[1, n] = \emptyset) = n^{-\frac{5}{8}+o(1)}$$

where X and Y are two walks starting at the origin.