Large random planar maps and their scaling limits
Lectures 4 and 5: Planar maps, bijective methods and first convergence results

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Abstract

These are the notes for Lectures 4 and 5 in a series of 8 lectures given jointly with Jean-François Le Gall at the Clay Mathematical Institute Summer School in Buzios, July 11 – August 7, 2010. In this lecture, we introduce the notion of planar maps, and describe the first properties of the Cori-Vauquelin-Schaeffer correspondence which will be of crucial importance later in this course. Then, we draw the first consequences of the CVS bijection in the context of scaling limits of random quadrangulations

5 Planar maps

5.1 Definitions

A map is a combinatorial object, that can be best visualized as a class of graphs embedded in a surface. In these lectures, we will exclusively focus on the case of *plane* (or *planar*) maps, in which case the surface is the 2-dimensional sphere $\mathbb{S}^2$

Let us first formalize the notion of map. We will not enter into details, referring the reader to the book by Mohar and Thomassen [18] for a very complete exposition. Another useful reference, discussing in depth the different equivalent ways to define maps (in particular through purely algebraic notions) is the book by Lando and Zvonkin [8, Chapter 1].

An *oriented edge* in $\mathbb{S}^2$ is a mapping $e : [0,1] \to \mathbb{S}^2$ that is continuous, and such that either e is injective, or the restriction of e to $[0,1)$ is injective and $e(0) = e(1)$. In the latter case, e is also called a loop. An oriented edge will always be considered up to reparametrization by a continuous increasing function from $[0,1]$ to $[0,1]$, and we will always be interested in properties of edges that do not depend on a particular parameterization. The origin and target of e are the points $e^- = e(0)$ and $e^+ = e(1)$. The reversal of e is the oriented edge $\overline{e} = e(1 - \cdot)$. An *edge* is a pair $\mathbf{e} = \{e, \overline{e}\}$, where e is an oriented edge. The *interior* of $\mathbf{e}$ is defined as $e((0,1))$.

An *embedded graph* in $\mathbb{S}^2$ is a graph[1] $G = (V, E)$ such that

- V is a (finite) subset of $\mathbb{S}^2$
- E is a (finite) set of edges in $\mathbb{S}^2$
- the vertices incident to $\mathbf{e} = \{e, \overline{e}\} \in E$ are $e^-, e^+ \in V$
- the interior of an edge $\mathbf{e} \in E$ does not intersect V nor the edges of E distinct from $\mathbf{e}$

[1] all the graphs considered here are finite, and are multigraphs in which double edges and loops are allowed

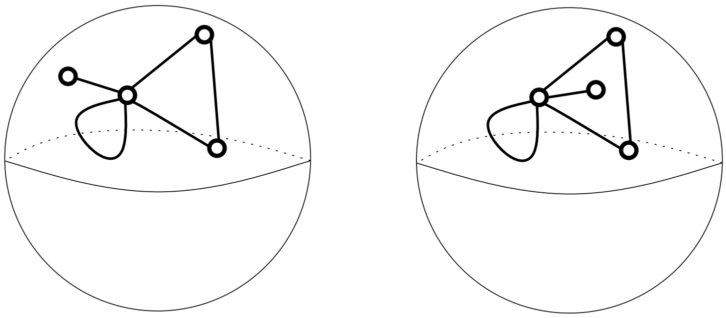

Figure 1: Two planar maps, with 4 vertices and 3 faces of degrees 1,3,6 and 1,4,5 respectively

The support of an embedded graph $G = (V, E)$ is

$$\mathrm{supp}\,(G) = V \cup \bigcup_{\mathbf{e}=\{e,\overline{e}\}\in E} e([0,1])\,.$$

A *face* of the embedding is a connected component of the set $\mathbb{S}^2 \setminus \mathrm{supp}\,(G)$.

Definition 5.1. A (planar) map is a connected embedded graph. Equivalently, a map is an embedded graph whose faces are all homeomorphic to the Euclidean unit disk in $\mathbb{R}^2$.

Topologically, one would say that a map is the 1-skeleton of a CW-complex decomposition of $\mathbb{S}^2$. We will denote maps using bold characters $\mathbf{m}, \mathbf{q}, \ldots$

Let $\mathbf{m} = (V, E)$ be a map, and let $\overrightarrow{E} = \{e \in \mathbf{e} : \mathbf{e} \in E\}$ be the set of oriented edges. Since $\mathbb{S}^2$ is oriented, it is possible to define, for every oriented edge $e \in \overrightarrow{E}$, a unique face f_e of $\mathbf{m}$, located to the left of the edge e. We call f_e the face *incident* to e. Note that the edges incident to a given face form a closed curve in $\mathbb{S}^2$, but not necessarily a Jordan curve (it can happen that $f_e = f_{\overline{e}}$ for some e). The degree of a face f is defined as

$$\deg(f) = \#\{e \in \overrightarrow{E} : f_e = f\}\,.$$

The oriented edges incident to a given face f, are arranged cyclically in counterclockwise order around the face in what we call the *facial ordering.* With every oriented edge e, we can associate a *corner* incident to e, which is a small simply connected neighborhood of e^- intersected with f_e, in such a way that the corners of two different oriented edges do not intersect.

Of course, the degree of a vertex $u \in V$ is the usual graph-theoretical notion

$$\deg(u) = \#\{e \in \overrightarrow{E} : e^- = u\}\,.$$

Similarly as for faces, the outgoing edges from u are organized cyclically in counterclockwise order around u.

A *rooted* map is a pair $(\mathbf{m}, e)$ where $\mathbf{m} = (V, E)$ is a map and $e \in \overrightarrow{E}$ is a distinguished oriented edge, called the root. We often omit the mention of e in the notation.

5.2 Euler's formula

An important property of maps is the so-called *Euler formula.* For $\mathbf{m}$ a map, we let $V(\mathbf{m}), E(\mathbf{m}), F(\mathbf{m})$ denote the sets of vertices, edges and faces of $\mathbf{m}$. Then (using the notation $|A|$ or $\#A$ for the cardinality of $|A|$),

$$|V(\mathbf{m})| - |E(\mathbf{m})| + |F(\mathbf{m})| = 2\,. \tag{1}$$

This is a relatively easy result in the case of interest (the planar case): one can remove the edges of the graph one by one until a spanning tree $\mathbf{t}$ of the graph is obtained, for which the result is trivial (it has one face, and $|V(\mathbf{t})| = |E(\mathbf{t})| + 1$).

5.3 Isomorphism, automorphism and rooting

In the sequel, we will always consider maps "up to deformation" in the following sense.

Definition 5.2. The maps $\mathbf{m}, \mathbf{m}'$ on $\mathbb{S}^2$ are isomorphic if there exists an orientation-preserving homeomorphism h of $\mathbb{S}^2$ onto itself, such that h induces a graph isomorphism of $\mathbf{m}$ with $\mathbf{m}'$.

The rooted maps $(\mathbf{m}, e)$ and $(\mathbf{m}', e')$ are isomorphic if $\mathbf{m}$ and $\mathbf{m}'$ are isomorphic through a homeomorphism h that maps e to e'.

In the sequel, we will almost always identify two isomorphic maps $\mathbf{m}, \mathbf{m}'$. This of course implies that the (non-embedded, combinatorial) graphs associated with $\mathbf{m}, \mathbf{m}'$ are isomorphic, but this is stronger: for instance the two maps of Figure 1 are not isomorphic, since a map isomorphism easily preserves the degrees of faces.

An *automorphism* of a map $\mathbf{m}$ is an isomorphism of $\mathbf{m}$ with itself. It should be interpreted as a *symmetry* of the map. An important fact is the following.

Proposition 5.1. *An automorphism of* $\mathbf{m}$ *that fixes an oriented edge, fixes all the oriented edges.*

Loosely speaking, the only automorphism of a rooted map is the identity. This explains why rooting is an important tool in the combinatorial study of maps, as it "kills the symmetries". The idea of the proof of the previous statement is to see that if e is fixed by the automorphism, then all the edges incident to e^- should also be fixed (since an automorphism preserves the orientation). One can thus progress is the graph (by connectedness) and show that all the edges are fixed.

In a rooted map, the face f_e incident to the root edge e, is often called the *external face*, or root face. The other faces are called *internal*. The vertex e^- is called the root vertex.

From now on, unless otherwise specified, all maps will be rooted.

We end the presentation of map by introducing the notion of *graph distance* in a map $\mathbf{m}$. A *chain* of length $k \geq 1$ is a sequence $e_{(1)}, \ldots, e_{(k)}$ of oriented edges in $\overrightarrow{E}(\mathbf{m})$, such that $e_{(i)}^+ = e_{(i+1)}^-$ for $1 \leq i \leq k-1$, and we say that the chain links the vertices $e_{(1)}^-$ and $e_{(k)}^+$. We also allow, for every vertex $u \in V(\mathbf{m})$, a chain with length 0, starting and ending at u. The *graph distance* $d_{\mathbf{m}}(u, v)$ between two vertices $u, v \in V(\mathbf{m})$, is the minimal k such that there exists a chain with length k linking u and v. A chain with minimal length between two vertices is called a *geodesic chain*.

5.4 The Cori-Vauquelin-Schaeffer bijection

With the identification of maps up to isomorphisms, the set of maps becomes a countable set. For instance, the set $\mathbf{M}_n$ of rooted maps with n edges is a finite set: the $2n$ oriented edges should be organized around a finite family of polygons (the faces of the map), and the number of ways to associate the boundary edges of these polygons is finite. A natural question to ask is "what is the cardinality of $\mathbf{M}_n$?".

Tutte answered this question (and many other counting problems for maps), motivated in part by the 4-color problem. He developed a powerful method, the "quadratic method", to solve the apparently ill-defined equations for the generating functions of maps. For recent developments in this direction, see the article by Bousquet-Mélou and Jehanne [3]. The method, however, is a kind of "black box" which solves such counting problems without giving much extra information about the structure of maps. One obtains

$$\#\mathbf{M}_n = \frac{2}{n+2} 3^n \mathrm{Cat}_n \,,$$

where $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. We also mention the huge literature on the enumeration of maps using matrix integrals, initiating in [20, 5], which is particularly popular in the physics literature. See [8, Chapter 4] for an introduction to this approach.

Motivated by the very simple form of the formula enumerating $\mathbf{M}_n$, Cori and Vauquelin [7] gave in 1981 a bijective approach for this formula. These approaches reached their full potential with the work of Schaeffer starting in his 1998 thesis [19]. We now describe this approach.

5.4.1 Quadrangulations

A map $\mathbf{q}$ is a quadrangulation if all its faces are of degree 4. We let $\mathbf{Q}_n$ be the set of (rooted) quadrangulations with n faces. Quadrangulations are a very natural family of maps to consider, in virtue of the fact that there exists a very natural bijection between $\mathbf{M}_n$ and $\mathbf{Q}_n$, which can be described as follows.

Let $\mathbf{m}$ be a map with n edges, and imagine that the vertices of $\mathbf{m}$ are colored in black. We then create a new map by adding inside each face of $\mathbf{m}$ a white vertex, and joining this white vertex to every corner of the face f it belongs to, by non-intersecting edges inside the face f. In doing so, notice that some black vertices may be joined to the same white vertex with several edges. Lastly, we erase the interiors of the edges of the map $\mathbf{m}$. We end up with a map $\mathbf{q}$, which is a plane quadrangulation with n faces, each face containing exactly one edge of the initial map. We adopt a rooting convention, for instance, we root $\mathbf{q}$ at the first edge coming after e in counterclockwise order around e^-, where e is the root of $\mathbf{m}$.

Notice that $\mathbf{q}$ also comes with a bicoloration of its vertices in black and white, in which two adjacent vertices have different colors. This says that $\mathbf{q}$ is *bipartite*, and as a matter of fact, every (planar!) quadrangulation is bipartite. So this coloring is superfluous: one can recover it by declaring that the black vertices are those at even distance of the root vertex of $\mathbf{q}$, and the white vertices are those at odd graph distance of the root vertex.

Conversely, starting from a rooted quadrangulation $\mathbf{q}$, we can recover a bipartite coloration as above, by declaring that the vertices at even distance of the root edge are black. Then, we draw the diagonal linking the two black corners incident to every face of $\mathbf{q}$. Finally, we remove the interior of the edges of $\mathbf{q}$ and root the resulting map $\mathbf{m}$ at the first outgoing diagonal from e^- in clockwise order from the root edge e of $\mathbf{q}$. One checks that this is indeed a left- and right-inverse of the previous mapping from $\mathbf{M}_n$ to $\mathbf{Q}_n$.

For the record, we state the following useful fact.

Proposition 5.2. *A (planar) map is bipartite if and only if its faces all have even degrees.*

5.4.2 The CVS bijection

Recall that $\mathbf{Q}_n$ is the set of rooted quadrangulation with n faces. A simple application of Euler's formula shows that any element of $\mathbf{Q}_n$ has $2n$ edges ($4n$ oriented edges, 4 for each face) and $n+2$ vertices.

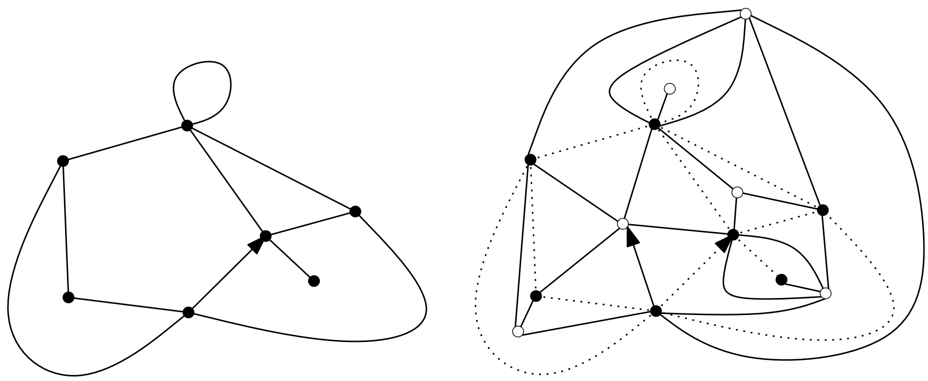

Figure 2: The so-called "trivial" bijection

Let $\mathbf{T}_n$ be the set of labeled trees, (τ, ℓ), as defined in Lecture 3. Recall that this means that τ is a rooted plane tree with n edges, and $\ell : \tau \to \mathbb{Z}$ is a label function on τ, such that $\ell(\varnothing) = 0$ and

$$|\ell(u) - \ell(\pi(u))| \leq 1\,, \qquad \text{for every } u \in \tau \setminus \{\varnothing\}\,.$$

In order to avoid trivialities, we now assume that $n \geq 1$. It will be convenient here to view a plane tree τ as a planar map, by embedding it in $\mathbb{S}^2$, and rooting it at the edge going from $\varnothing$ to (1). Let $\varnothing = v_0, v_1, \ldots, v_{2n} = \varnothing$ be the contour exploration of the vertices of the tree τ. For $i \in \{0, 1, \ldots, 2n-1\}$, we let e_i be the oriented edge from v_i to v_{i+1}, and extend the sequences (u_i) and (e_i) to infinite sequences by $2n$-periodicity. With each edge e_i, recall that we can associate a corner around e_i^-. We will often identify e_i with this corner, and adopt the notation $\ell(e_i) = \ell(e_i^-)$. In particular, note that $\ell(e_i) = V_i, 0 \leq i \leq 2n$ is the label contour sequence as defined in Lecture 3.

For every $i \geq 0$, we define the *successor* of i by

$$s(i) = \inf\{j > i : \ell(e_j) = \ell(e_i) - 1\}\,,$$

with the convention that $\inf \emptyset = \infty$. Note that $s(i) = \infty$ if and only if $\ell(e_i)$ equals $\min\{\ell(v) : v \in \tau\}$. This is a simple consequence of the fact that the integer-valued sequence $(\ell(e_i), i \geq 0)$ can decrease only by taking unit steps.

Consider a point v_* in $\mathbb{S}^2$ that does not belong to the support of τ, and denote by e_∞ a corner around v_*, i.e. a small neighborhood of v_* with v_* excluded, not intersecting the corners $e_i, i \geq 0$. By convention, we let $\ell(v_*) = \ell(e_\infty) = \min\{\ell(u) : u \in \tau\} - 1$. The successor of the corner e_i is then defined by

$$s(e_i) = e_{s(i)}\,.$$

The CVS construction consists in drawing, for every $i \in \{0, 1, \ldots, 2n-1\}$, an *arc*, which is an edge from the corner e_i to the corner $s(e_i)$ inside $\mathbb{S}^2 \setminus (\{v_*\} \cup \operatorname{supp}(\tau))$. See Figure 3 for illustration.

Lemma 5.1. *It is possible to draw the arcs in such a way that the graph with vertex-set $\tau \cup \{v_*\}$ and edge-set the edges of τ and the arcs, is an embedded graph.*

Proof. Since τ is a tree, we can see it as a map with a unique face $\mathbb{S}^2 \setminus \operatorname{supp}(\tau)$. The latter can in turn be seen as an open polygon, bounded by the edges $e_0, e_1, \ldots, e_{2n-1}$ in counterclockwise order. Hence, the result will follow if we can show that the arcs do not cross, i.e. that it is not

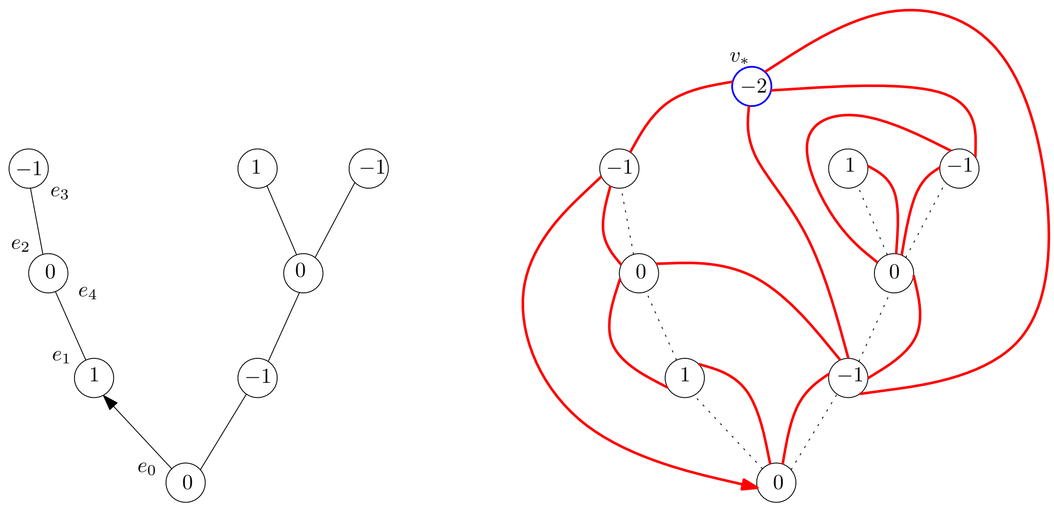

Figure 3: Illustration of the Cori-Vauquelin-Schaeffer bijection, in the case $\epsilon = 1$. For instance, e_3 is the successor of e_0, e_2 the successor of e_1, and so on.

possible to find pairwise distinct corners $e^{(1)}, e^{(2)}, e^{(3)}, e^{(4)}$ that arise in this order in the cyclic order induced by the contour exploration, and such that $e^{(3)} = s(e^{(1)})$ and $e^{(4)} = s(e^{(2)})$.

If it were the case, then $\ell(e^{(2)}) \geq \ell(e^{(1)})$, as otherwise the successor of $e^{(1)}$ would be between $e^{(1)}$ and $e^{(2)}$. Similarly, $\ell(e^{(3)}) \geq \ell(e^{(2)})$. But by definition, $\ell(e^{(3)}) = \ell(e^{(1)}) - 1$, giving $\ell(e^{(2)}) \geq \ell(e^{(3)}) + 1 \geq \ell(e^{(2)}) + 1$, a contradiction. □

We call $\mathbf{q}$ the graph with vertex-set $V(\tau) \cup \{v_*\}$ and edge-set formed by the arcs, now excluding the (interiors of the) edges of τ.

Lemma 5.2. *The embedded graph* $\mathbf{q}$ *is a quadrangulation with* n *faces.*

Proof. First we check that $\mathbf{q}$ is connected, and hence is a map. But this is obvious since the consecutive successors of any given corner e, given by $e, s(e), s(s(e)), \ldots$, form a finite sequence ending at e_∞. Hence, every vertex in $\mathbf{q}$ can be joined by a chain to v_*, and the graph is connected.

To check that $\mathbf{q}$ is a quadrangulation, let us consider an oriented edge in τ, which separates in two oriented edges $e, \overline{e}$. Let us assume that $\ell(e^+) = \ell(e^-) - 1$. Then obviously, the successor of e is incident to e^+, so the arc from e ends at e^+. Now, let e' be the corner following $\overline{e}$ in the contour exploration around τ. Then $\ell(e') = \ell(e^-) = \ell(\overline{e}) + 1$, giving that $s(\overline{e}) = s(s(e'))$. Indeed, $s(e')$ is the first corner coming after e' in contour order and with label $\ell(e') - 1 = \ell(e) - 1$, while $s(s(e'))$ is the first corner coming after e' with label $\ell(e) - 2$. Therefore, it has to be the first corner coming after $\overline{e}$, with label $\ell(e) - 2 = \ell(\overline{e}) - 1$.

We deduce that the arcs joining the corners $e, s(e)$, $\overline{e}, s(\overline{e})$, $e', s(e')$ et $s(e'), s(s(e')) = s(\overline{e})$, form a quadrangle, that contains the edge $\{e, \overline{e}\}$, and no other edge of τ.

If $\ell(e^+) = \ell(e^-) + 1$, the situation is the same by exchanging the roles of e and $\overline{e}$.

The only case that remains is when $\ell(e^+) = \ell(e^-)$. In this case, if e' and e'' are the corners following e and $\overline{e}$ respectively in the contour exploration of τ, then $\ell(e) = \ell(e') = \ell(\overline{e}) = \ell(e'')$, so that $s(e) = s(e')$ on the one hand and $s(\overline{e}) = s(e'')$ on the other hand. We deduce that the edge $\{e, \overline{e}\}$ is the diagonal of a quadrangle formed by the arcs linking $e, s(e)$ $e', s(e') = s(e)$, $\overline{e}, s(\overline{e})$ and $e'', s(e'') = s(\overline{e})$. These situations are summed up in figure 4.

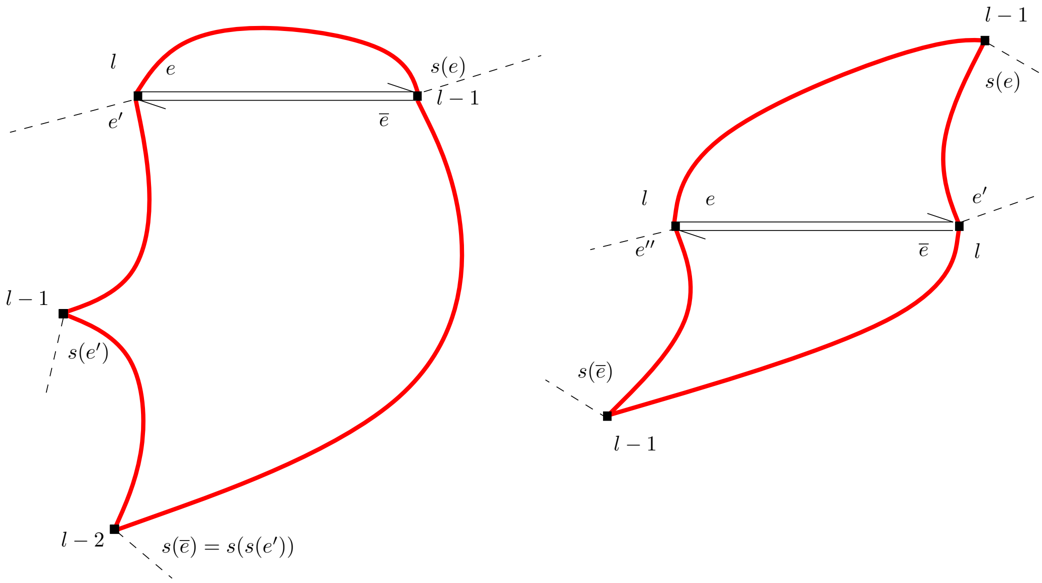

Figure 4: Illustration of the proof of Lemma 5.2 where we let $l = \ell(e)$

Now, notice that $\mathbf{q}$ has $2n$ edges (one per corner of τ) and $n+2$ vertices, so it must have n faces by Euler's formula. So all the faces must be of the form described above. This ends the proof. □

Note that the quadrangulation $\mathbf{q}$ has a distinguished vertex v_*, but for now it is not a rooted quadrangulation. To fix this root, we will need an extra parameter $\epsilon \in \{-1, 1\}$. If $\epsilon = 1$ we let the root edge of $\mathbf{q}$ be the arc linking e_0 with $s(e_0)$, and oriented from $s(e_0)$ from e_0. If $\epsilon = -1$, the root edge is this same arc, but oriented from e_0 to $s(e_0)$.

In this way, we have defined a mapping Φ, from $\mathbf{T}_n \times \{-1, 1\}$ to the set $\mathbf{Q}_n^\bullet$ of pairs $(\mathbf{q}, v_*)$, where $\mathbf{q} \in \mathbf{Q}_n$ and $v_* \in V(\mathbf{q})$. We call such pairs *pointed quadrangulations*. The following result is due to Chassaing and Schaeffer, see [6, Theorem 4]

Theorem 5.1. *For every $n \geq 1$, the mapping Φ is a bijection from $\mathbf{T}_n \times \{-1, 1\}$ onto $\mathbf{Q}_n^\bullet$.*

We will not give a proof of this result. See Tutorial #5 for the inverse construction, going from $\mathbf{Q}_n^\bullet$ to $\mathbf{T}_n \times \{-1, 1\}$.

Corollary 5.1. *We have the following formula for every $n \geq 1$:*

$$\#\mathbf{M}_n = \#\mathbf{Q}_n = \frac{2}{n+2} 3^n \mathrm{Cat}_n$$

Proof. We first notice that $\#\mathbf{Q}_n^\bullet = (n+2)\#\mathbf{Q}_n$, since every quadrangulation $\mathbf{q} \in \mathbf{Q}_n$ has $n+2$ vertices, each of which induces a distinct element of $\mathbf{Q}_n^\bullet$. On the other hand, it is obvious that

$$\#\mathbf{T}_n \times \{-1, 1\} = 2 \cdot 3^n \#\mathbf{A}_n = 2 \cdot 3^n \mathrm{Cat}_n \, .$$

The result follows from Theorem 5.1. □

The probabilistic counterpart of this can be stated as follows.

Corollary 5.2. *Let Q_n be a uniform random element in $\mathbf{Q}_n$, and conditionally given Q_n, let v_* be a uniform random element in $V(Q_n)$.*

On the other hand, let θ_n be a random uniform element in $\mathbf{T}_n$, and let ϵ be independent of θ_n, uniform in $\{-1,1\}$. Then $\Phi(\theta_n,\epsilon)$ has same distribution as (Q_n, v_*).

The proof is obvious, since the probability that (Q_n, v_*) equals some particular $(\mathbf{q}, v) \in \mathbf{Q}_n^\bullet$ equals $((n+2)\#\mathbf{Q}_n)^{-1} = (\#\mathbf{Q}_n^\bullet)^{-1}$.

5.4.3 Interpretation of the labels

The CVS bijection will be of crucial importance to us when we will deal with metric properties of random elements of $\mathbf{Q}_n$, because the labels on $\mathbf{q}$ that are inherited from the CVS construction from a labeled tree turn out to measure certain distances in $\mathbf{q}$. Recall that the set τ is identified with $V(\mathbf{q}) \setminus \{v_*\}$, if (τ,ℓ) and $\mathbf{q}$ are associated through the CVS bijection (the choice of ϵ is irrelevant here). Hence, the function ℓ is also a function on $V(\mathbf{q}) \setminus \{v_*\}$, and we extend it by letting, as previously, $\ell(v_*) = \min\{\ell(u) : \mathcal{U} \in \tau\} - 1$. For simplicity, we simply denote

$$\min \ell = \min\{\ell(u) : u \in \tau\}\,,$$

which should not be misinterpreted for the minimum of ℓ on the vertex set of $\mathbf{q}$.

Proposition 5.3. *For every $v \in V(\mathbf{q})$, it holds that*

$$d_{\mathbf{q}}(v, v_*) = \ell(v) - \min \ell + 1\,, \tag{2}$$

where $d_{\mathbf{q}}$ is the graph distance on $\mathbf{q}$.

Proof. Let $v \in V(\mathbf{q}) \setminus \{v_*\} = \tau$, and let e be a corner (in τ) incident to v. The the chain of arcs

$$e \to s(e) \to s^2(e) \to \ldots \to e_\infty$$

is a chain of length $\ell(e) - \ell(e_\infty) = \ell(v) - \ell(v_*)$ between v and v_*. Therefore, $d_{\mathbf{q}}(v, v_*) \le \ell(v) - \ell(v_*)$. On the other hand, if $v = v_0, v_1, \ldots, v_d = v_*$ are the consecutive vertices of any chain linking v to v_*, then since $|\ell(e) - \ell(s(e))| = 1$ by definition for any corner e and since the edges of $\mathbf{q}$ all joint a corner to its successor, we get

$$d = \sum_{i=1}^{d} |\ell(v_i) - \ell(v_{i-1})| \ge |\ell(v_0) - \ell(v_d)| = \ell(v) - \ell(v_*)\,,$$

as wanted. □

Note that the second part of the argument is valid for any chain linking any two vertices u, v of $\mathbf{q}$, and we obtain the useful (but very rough) bound

$$d_{\mathbf{q}}(u, v) \ge |\ell(u) - \ell(v)|\,. \tag{3}$$

This bound will be improved in the next section.

As a consequence, we obtain for instance that the "volume of spheres" around v_* can be simply interpreted in terms of ℓ: for every $k \ge 0$,

$$|\{v \in V(\mathbf{q}) : d_{\mathbf{q}}(v, v_*) = k\}| = |\{u \in \tau : \ell(u) - \min \ell + 1 = k\}|\,.$$

5.4.4 Two useful bounds

The general philosophy in the forthcoming study of random planar maps is then the following: information about the labels in a random labeled tree, which are well-understood if θ is a uniform element in $\mathbf{T}_n$, allows to obtain information about distances in the associated quadrangulation by the CVS bijection. One major problem with this approach is that exact information on distances is only available for distances to a distinguished vertex v_*. It is uneasy to deduce the distances between two vertices distinct from v_* using the labels in the tree. However, more advanced properties of the CVS bijection allow to get useful bounds on these distances. Recall that $e_0, e_1, e_2, \ldots$ is the contour sequence of corners (or oriented edges) around a tree $\tau \in \mathbf{A}_n$, starting from the root. We view $(e_i, i \geq 0)$ as cyclically ordered, and for two corners e, e' of τ, we let $[e, e']$ be the set of corners encountered when starting from e, following the cyclic contour order, and stopping at the first encounter of e'.

Proposition 5.4. *Let $((\tau, \ell), \epsilon)$ be an element in $\mathbf{T}_n \times \{-1, 1\}$, and $(\mathbf{q}, v_*) = \Phi(((\tau, \ell), \epsilon))$. Let u, v be two vertices in $V(\mathbf{q}) \setminus \{v_*\}$, and let e, e' be two corners of τ such that $e^- = u, (e')^- = v$.*
(i) *It holds that*

$$d_{\mathbf{q}}(u, v) \leq \ell(u) + \ell(v) - 2 \min_{e'' \in [e, e']} \ell(e'') + 2\,,$$

(ii) *It holds that*

$$d_{\mathbf{q}}(u, v) \geq \ell(u) + \ell(v) - 2 \min_{w \in [[u, v]]} \ell(w)\,,$$

where $[[u, v]]$ is the set of vertices lying on the unique chain from u to v in τ.

Proof. For simplicity let $m = \min_{e'' \in [e, e']} \ell(e'')$. Let e'' be the first corner between $[e, e']$ such that $\ell(e'') = m$ The corner $s^k(e)$, whenever it is well defined (i.e. whenever $d_{\mathbf{q}}(e^-, v_*) \geq k$), is called the k-th successor of e. Then e'' is the $(\ell(e) - m)$-th successor of e. Moreover, by definition, $s(e'')$ does not belong to $[e, e']$ since it has lesser label than e'', and necessarily, $s(e'')$ is the $(\ell(e') - m + 1)$-th successor of e'. Hence, the "successors" geodesic path from $u = e^-$ to $s(e'')$, concatenated with the similar geodesic path from v to $s(e'')$ is a path of length

$$\ell(u) + \ell(v) - 2m + 2\,,$$

and the distance $d_{\mathbf{q}}(u, v)$ is less than or equal to this quantity. This proves (i).

For (ii), let $w \in [[u, v]]$ be such that $\ell(w) = \min\{\ell(w') : w' \in [[u, v]]$. If $w \in \{u, v\}$ then the statement follows trivially from (3). So we assume otherwise. We can then write τ as the union of two connected subgraphs $\tau = \tau_1 \cup \tau_2$, such that $\tau_1 \cap \tau_2 = \{w\}$, such that τ_1 contains u but not v, while τ_2 contains v but not u. There can be several such decompositions, so we just choose one. Now, let $e_{(1)}, \ldots, e_{(d)}$ be a chain from u to v, and let k be the last index i such that $e_{(i)}^-$ belongs to τ_1. Then $e_{(i+1)}^- \in \tau_2$, and there is an arc between $e_{(i)}$ and $e_{(i+1)}$, meaning that either $e_{(i+1)} = s(e_{(i)})$ or vice-versa. We only argue in the first case, the other being similar. In this case, by definition of the successor, it must holds that $\ell(e) \geq \ell(e_{(i)})$ for every $e \in [e_{(i)}, e_{(i+1)}] \setminus \{e_{(i+1)}\}$. But necessarily, one of the corners in this set must be incident to w, since any path from τ_1 to τ_2 passes through w. Consequently, it holds that $\ell(w) \geq \ell(e_{(i)})$. By decomposing the path from u to v into the part from u to $e_{(i)}^-$ and the part from $e_{(i)}^-$ to v, we obtain using (3) that

$$d \geq (\ell(u) - \ell(e_{(i)})) + (\ell(v) - \ell(e_{(i)})) \geq \ell(u) + \ell(v) - 2\ell(w)\,,$$

as wanted. □

6 Basic convergence results for uniform quadrangulations

For the remaining part of this course, our main goal will be to study the scaling limits of random planar quadrangulation chosen according to the uniform probability measure on $\mathbf{Q}_n$. According to Corollary 5.2, the CVS bijection and the study of scaling limits of random labeled trees that was done in the first three lectures is particularly adapted to this kind of problems. Ultimately, the question we would like to address is to study the convergence in distribution of an appropriately rescaled version of the random metric space $(V(Q_n), d_{Q_n})$, in the sense of the Gromov-Hausdorff topology.

One of the motivations for this problem comes from Physics, and we refer the interested reader to [1] for an extensive discussion. In the past 15 years or so, physicists have been starting to view random maps as possible discrete models for a continuum model of random surfaces (called the Euclidean 2-dimensional quantum gravity model), which is still ill-defined from a mathematical point of view. We thus want to investigate whether the scaling limit of Q_n exists in the above sense, and does define a certain random surface. One can also ask the natural question of whether this limiting random surface is *universal*, in the sense that it also arises as the scaling limit of many different models of random maps, for instance, maps chosen uniformly at random in the set of p-angulations with n faces:

$$\mathbf{M}_n^p = \{\mathbf{m} : \deg(f) = p \text{ for every } f \in F(\mathbf{m}), \#F(\mathbf{m}) = n\}\,, \qquad p \geq 3\,.$$

Indeed, most of the results that we will describe in the sequel do have analogs in this more general setting [14, 16, 17, 10], thanks to nice generalizations of the CVS bijection that are due to Bouttier, Di Francesco and Guitter [4].

This is of course analogous to the celebrated Donsker Theorem, according to which Brownian motion is the scaling limit of discrete random walks, as well as to the fact that the Brownian CRT is the scaling limit of many different models of random trees.

6.1 Radius and profile

We will first address a simpler question than the one raised above, which is to determine by what factor we should rescale the distance d_{Q_n} in order to get an interesting scaling limit as $n \to \infty$.

Let $\mathbf{q} \in \mathbf{Q}_n$ be a rooted planar quadrangulation, and v be a vertex of $\mathbf{q}$. As before, let $d_{\mathbf{q}}$ denote the graph distance on the set of vertices of $\mathbf{q}$. We define the *radius* of $\mathbf{q}$ seen from v as

$$\mathcal{R}(\mathbf{q}, v) = \max_{u \in V(\mathbf{q})} d_{\mathbf{q}}(u, v)\,,$$

and the *profile* of $\mathbf{q}$ seen from v as the sequence

$$I_{\mathbf{q},v}(k) = \mathrm{Card}\,\{u \in V(\mathbf{q}) : d_{\mathbf{q}}(u, v) = k\}\,, \qquad k \geq 0$$

which measures the 'volumes' of the spheres centered at v in the graph metric. The latter can be seen as a measure on $\mathbb{Z}_+$ with total volume $n+2$. Our first limit theorem is the following.

Theorem 6.1. *Let Q_n be a random variable with uniform distribution in $\mathbf{Q}_n$, and conditionally on Q_n, let v_* be uniformly chosen among the $n+2$ vertices of Q_n. Let also $(\mathbb{e}, Z)$ denote the head of the Brownian snake, as introduced in Lecture #3.*

(i) We have

$$\left(\frac{9}{8n}\right)^{1/4} \mathcal{R}(Q_n, v_*) \xrightarrow[n\to\infty]{(d)} \sup Z - \inf Z\,.$$

(ii) If v_{**} is another uniform vertex of Q_n chosen independently of v_*,

$$\left(\frac{9}{8n}\right)^{1/4} d_{Q_n}(v_*, v_{**}) \xrightarrow[n\to\infty]{(d)} \sup Z\,.$$

(iii) Finally, the following convergence in distribution holds for the weak topology on probability measures on $\mathbb{R}_+$:

$$\frac{I_{Q_n,v_*}((8n/9)^{1/4}\cdot)}{n+2} \xrightarrow[n\to\infty]{(d)} \mathcal{I}\,,$$

where $\mathcal{I}$ is the occupation measure of Z above its infimum, defined as follows: for every non-negative, measurable $g : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\langle \mathcal{I}, g\rangle = \int_0^1 \mathrm{d}t\, g(Z_t - \inf Z)\,.$$

The points (i) and (iii) are due to Chassaing and Schaeffer [6], and (ii) is due to Le Gall [9], although these references state these properties in a slightly different context, namely, in the case where v_* is the root vertex rather than a uniformly chosen vertex. This indicates that as $n \to \infty$, the root vertex plays no particular role.

Proof. Let $((T_n, L_n), \epsilon)$ be a uniform random element in $\mathbf{T}_n \times \{-1, 1\}$. Then by Corollary 5.2 we can assume that (Q_n, v_*) equals $\Phi(((T_n, L_n), \epsilon))$ where Φ is the CVS bijection.

By Proposition 5.3, the radius of Q_n viewed from v_* then equals

$$\mathcal{R}(Q_n, v_*) = \max L_n - \min L_n + 1 = \max V_n - \min V_n\,,$$

where $V_n(i) = L_n(e_i), 0 \le i \le 2n$ is the label process associated with (T_n, L_n), as in Lecture #3. Recall that $e_0, e_1, \ldots, e_{2n-1}$ are the oriented edges observed in the contour exploration of T_n. Then (i) follows immediately from this and Theorem 3.2.

As for (ii), it is clear that we may in fact assume that v_{**} is uniform among the n vertices of Q_n that are distinct from v_* and the vertex $\varnothing$ of T_n (recall that $V(Q_n) \setminus \{v_*\} = V(T_n)$).

Now, for $s \in [0, 2n)$, we let $\langle s\rangle = \lceil s\rceil$ if C_n has slope $+1$ at s_+, and $\langle s\rangle = \lfloor s\rfloor$ otherwise. Let $u_i = e_i^-, 0 \le i \le 2n-1$ be the vertices of T_n in contour order. Then, note that $u_{\langle s\rangle} = u \in T_n$ if and only if s is a time when the contour exploration around T_n explores either of the two oriented edges between u and its parent $\pi(u)$. Therefore, for every $u \in T_n \setminus \{\varnothing\}$, the Lebesgue measure of $\{s \in [0, 2n) : u_{\langle s\rangle} = u\}$ equals 2. Consequently, if U is a uniform random variable in $[0, 1)$, independent of (T_n, L_n), then $u_{\langle 2nU\rangle}$ is uniform in $T_n \setminus \{\varnothing\}$. Hence, it suffices to prove the result with $u_{\langle 2nU\rangle}$ instead of v_{**}.

Obviously, it also holds that $|s - \langle s\rangle| \le 1$. This entails that

$$\begin{aligned}\left(\frac{8n}{9}\right)^{-1/4} d_{Q_n}(v_*, u_{\langle 2nU\rangle}) &= \left(\frac{8n}{9}\right)^{-1/4}(L_n(u_{\langle 2nU\rangle}) - \min L_n + 1)\\ &= \left(\frac{8n}{9}\right)^{-1/4}(V_n(\langle 2nU\rangle) - \min V_n + 1)\,,\end{aligned}$$

which converges in distribution to $Z_U - \inf Z$, by Theorem 3.2, where U is again supposed to be independent of $(\mathbb{e}, Z)$.

The fact that $Z_U - \inf Z$ has same distribution as $\sup Z$ can be obtained as follows. First, observe that for every $k \in \{0, 1, \ldots, 2n-1\}$, the law of (T_n, L_n) is invariant under the operation (a bijection of $\mathbf{T}_n$ onto itself) that consists in re-rooting T_n at e_k, and replacing L_n by $L_n - L_n(e_k)$.

From this, it follows that $(V_n(k+i) - V_n(k), 0 \leq i \leq 2n)$ has same law as V_n, where $V_n(i)$ is extended to any $i \geq 0$ by $2n$-periodicity. So $\min V_n - V_n(k)$ has same distribution as $\min V_n$, and a limiting argument using Theorem 3.2 entails that $\inf Z - Z_t$ has same distribution as $\inf Z$ for any $t \in [0,1]$. This remains valid if t is replaced by the uniform independent variable U.

Therefore, $Z_U - \inf Z$ has same law as $-\inf Z$, or as $\sup Z$, by an obvious symmetry property this ends the proof of (ii).

Finally, for (iii) we just note that if $I_{(n)} = I_{Q_n,v_*}((8n/9)^{1/4}\cdot)$, then for every continuous bounded $g : \mathbb{R}_+ \to \mathbb{R}$,

$$\begin{aligned} I_{(n)}(g) &= \frac{1}{n+2} \sum_{v \in Q_n} g((8n/9)^{-1/4} d_{Q_n}(v_*, v)) \\ &= E_{**}[g((8n/9)^{-1/4} d_{Q_n}(v_*, v_{**}))] \\ &\underset{n\to\infty}{\longrightarrow} E_U[g(Z_U - \inf Z)] \\ &= \int_0^1 \mathrm{d}t\, g(Z_t - \inf Z)\,, \end{aligned}$$

where E_{**} and E_U means that we take the expectation only with respect to v_{**} and U in the corresponding expressions (these are conditional expectations given (Q_n, v_*) and $(\mathbb{e}, Z)$). In the penultimate step, we used the convergence established in the course of the proof of (ii). This entails the result. □

6.2 Convergence as a metric space

We would like to be able to understand the full scaling limit picture for random maps, in a similar fashion as it was done for trees, where we showed, relying on the basic result (3), that the distances in discrete trees, once rescaled by $\sqrt{2n}$, converge to the distances in the continuum random tree $(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}})$. We thus ask if there is an analog of the CRT, that arises as the limit of the properly rescaled metric spaces (Q_n, d_{Q_n}). In view of Theorem 6.1, the correct normalization for the distance should be $n^{1/4}$.

Assume that (T_n, L_n) is uniform in $\mathbb{T}_n$, let ϵ be uniform in $\{-1, 1\}$, independent of (T_n, L_n), and let Q_n be the random uniform quadrangulation with n faces and with a uniformly chosen vertex v_*, obtained from (T_n, L_n, ϵ) by Schaeffer's bijection. Here we follow Le Gall [10][2]. By the usual identification, the set $\{u_i, i \geq 0\}$ of vertices of T_n explored in contour order, is understood as the set $V(Q_n) \setminus \{v_*\}$. Define a pseudo-metric on $\{0, \ldots, 2n\}$ by letting $d_n(i,j) = d_{Q_n}(u_i, u_j)$. The quotient of this metric space, obtained by identifying i, j whenever $d_n(i,j) = 0$, is isometric to $(V(Q_n) \setminus \{v_*\}, d_{Q_n})$. A major problem is that $d_n(i,j)$ is not a simple functional of (C_n, V_n). Indeed, the distances that we are able to handle in an easy way are distances to v_*, through the following rewriting of (2):

$$d_{Q_n}(v_*, u_i) = V_n(i) - \min V_n + 1\,. \tag{4}$$

We also define

$$d_n^0(i,j) = V_n(i) + V_n(j) - 2 \min_{i \wedge j \leq k \leq i \vee j} V_n(k) + 2\,, \qquad i, j \in \{0, \ldots, 2n\}\,,$$

[2] At this point, it should be noted that [10, 12, 11] consider another version of Schaeffer's bijection, where no distinguished vertex v_* has to be considered. This results in considering pairs (T_n, L_n) in which L_n is conditioned to be positive. The scaling limits of such random variables are still tractable, and in fact, are simple functionals of $(\mathbb{e}, Z)$, as shown in [13, 9]. So there will be some differences with our exposition, but these turn out to be non-important.

so as a consequence of (i) in Proposition 5.4, we have the bound $d_n \le d_n^0$.

We now extend the functions d_n, d_n^0 to $[0,2n]^2$ by letting

$$\begin{aligned} d_n(s,t) &= (\lceil s\rceil - s)(\lceil t\rceil - t)d_n(\lfloor s\rfloor, \lfloor t\rfloor) + (\lceil s\rceil - s)(t-\lfloor t\rfloor)d_n(\lfloor s\rfloor, \lceil t\rceil) \\ &\quad +(s-\lfloor s\rfloor)(\lceil t\rceil - t)d_n(\lceil s\rceil, \lfloor t\rfloor) + (s-\lfloor s\rfloor)(t-\lfloor t\rfloor)d_n(\lceil s\rceil, \lceil t\rceil)\,, \end{aligned} \tag{5}$$

where $\lfloor s\rfloor = \sup\{k \in \mathbb{Z}_+ : k \le s\}$ and $\lceil s\rceil = \lfloor s\rfloor + 1$. The function d_n^0 is extended by the obvious similar formula.

It is easy to check that d_n thus extended is continuous on $[0,2n]^2$ and satisfies the triangular inequality (although is not the case for d_n^0), and that it still holds that $d_n \le d_n^0$. We define a rescaled version of these functions by letting

$$D_n(s,t) = \left(\frac{9}{8n}\right)^{1/4} d_n(2ns, 2nt)\,, \qquad 0 \le s,t \le 1\,,$$

so that the subspace $(\{i/2n, 0 \le i \le 2n\}, D_n)$, quotiented by points at zero D_n-distance, is isometric to $(V(Q_n) \setminus \{v_*\}, (9/8n)^{1/4} d_{Q_n})$. We define similarly the functions D_n^0 on $[0,1]^2$. Then, as a consequence of Theorem 3.2, it holds that

$$(D_n^0(s,t), 0 \le s,t \le 1) \xrightarrow[n\to\infty]{(d)} (D^0(s,t), 0 \le s,t \le 1)\,, \tag{6}$$

for the uniform topology on $\mathcal{C}([0,1]^2)$, where by definition

$$D^0(s,t) = Z_s + Z_t - 2 \inf_{[s\wedge t, s\vee t]} Z\,.$$

We can now state

Proposition 6.1. *The family of laws of $(D_n(s,t), 0 \le s,t \le 1)$, as n varies, is relatively compact for the weak topology on the probability measures on $\mathcal{C}([0,1]^2)$.*

Proof. Let $s,t,s',t' \in [0,1]$. Then by a simple use of the triangular inequality, and the fact that $D_n \le D_n^0$,

$$|D_n(s,t) - D_n(s',t')| \le D_n(s,s') + D_n(t,t') \le D_n^0(s,s') + D_n^0(t,t')\,,$$

which allows to estimate the modulus of continuity at a fixed $\delta > 0$:

$$\sup_{\substack{|s-s'|\le\delta \\ |t-t'|\le\delta}} |D_n(s,t) - D_n(s',t')| \le 2 \sup_{|s-s'|\le\delta} D_n^0(s,s')\,. \tag{7}$$

However, the convergence in distribution (6) entails that for every $\varepsilon > 0$,

$$\limsup_{n\to\infty} P\left(\sup_{|s-s'|\le\delta} D_n^0(s,s') \ge \varepsilon\right) \le P\left(\sup_{|s-s'|\le\delta} D^0(s,s') \ge \varepsilon\right)\,,$$

and the latter goes to 0 when $\delta \to 0$, with a fixed ε, by continuity of D^0 and the fact that $D^0(s,s) = 0$. Hence, taking $\eta > 0$ and letting $\varepsilon = \varepsilon_k = 2^{-k}$, we can choose $\delta = \delta_k$ (tacitly depending also on η) such that

$$\sup_{n\ge1} P\left(\sup_{|s-s'|\le\delta_k} D_n^0(s,s') \ge 2^{-k}\right) \le \eta 2^{-k}\,, \qquad k \ge 1,$$

entailing

$$P\left(\bigcap_{k\geq 1}\left\{\sup_{|s-s'|\leq\delta_k} D_n^0(s,s')\leq 2^{-k}\right\}\right)\geq 1-\eta\,,$$

for all $n\geq 1$. Together with (7), this shows that with probability at least $1-\eta$, the function D_n is in the set of functions $f:[0,1]^2\to\mathbb{R}$ such that $f(0,0)=0$ and for every $k\geq 1$,

$$\sup_{\substack{|s-s'|\leq\delta_k\\ |t-t'|\leq\delta_k}} |f(s,t)-f(s',t')|\leq 2^{-k}\,,$$

the latter set being compact by the Arzela-Ascoli Theorem. The conclusion follows from Prokhorov's tightness Theorem [2]. □

At this point, we are allowed to say that the random distance functions D_n admit a limit in distribution, up to taking $n\to\infty$ along a subsequence:

$$(D_n(s,t),0\leq s,t\leq 1)\xrightarrow{(d)}(D(s,t),0\leq s,t\leq 1) \tag{8}$$

for the uniform topology on $\mathcal{C}([0,1]^2)$. In fact, we are going to need a little more than the convergence of D_n. From the relative compactness of its components, we see that the family of laws of $((2n)^{-1}C_n(2n\cdot),(9/8n)^{1/4}V_n(2n\cdot),D_n),n\geq 1$ is relatively compact in the set of probability measures on $\mathcal{C}([0,1])^2\times\mathcal{C}([0,1]^2)$. Therefore, it is possible to choose an extraction $(n_k,k\geq 1)$ so that this triple converges in distribution to a limit, which we call $(\mathbb{e},Z,D)$ with a slight abuse of notation. The joint convergence to the triple $(\mathbb{e},Z,D)$ gives a coupling of D,D^0 such that $D\leq D^0$, since $D_n\leq D_n^0$ for every n.

Define a random equivalence relation on $[0,1]$ by letting $s\approx t$ if $D(s,t)=0$. We let $M=[0,1]/\approx$ be the quotient space, endowed with the quotient distance, which we still denote by D. The canonical projection $[0,1]\to M$ is denoted by $\mathbf{p}$.

Finally, let $s_*\in[0,1]$ be such that $Z_{s_*}=\inf Z$ (such a s_* turns out to be unique a.s., see [13]), and let $\rho=\mathbf{p}(s_*)$. We can now state the main result of this Lecture.

Proposition 6.2. *The random space (M,D) is the limit in distribution of the isometry class of $(V(Q_n),(9/8n)^{-1/4}d_{Q_n})$, for the Gromov-Hausdorff topology, along the subsequence $(n_k,k\geq 1)$. Moreover, it holds that a.s. for every $x\in M$ and $s\in[0,1]$ such that $\mathbf{p}(s)=x$,*

$$D(\rho,x)=D(s_*,s)=Z_s-\inf Z\,.$$

Note that, in the discrete model, a point at which the minimal label in T_n is attained lies at distance 1 from v_*. Therefore, the point ρ should be seen as the continuous analog of the distinguished vertex v_*. The last equation in the statement is then of course the continuous analog of (2) and (4).

Proof. For the purposes of this proof, it is useful to assume, using the Skorokhod representation theorem, that the convergence of $((2n)^{-1}C_n(2n\cdot),(9/8n)^{1/4}V_n(2n\cdot),D_n)$ to $(\mathbb{e},Z,D)$ holds a.s. along the subsequence (n_k).

First of all, note that it is enough to show the same result where $V(Q_n)$ is replaced by $V(Q_n)\setminus\{v_*\}$, since obviously $(V(Q_n),d_{Q_n})$ and $(V(Q_n)\setminus\{v_*\},d_{Q_n})$ lie at Gromov-Hausdorff distance less than or equal to 1. We then use the fact established before that $(V(Q_n)\setminus\{v_*\},d_{Q_n})$ is isometric to the quotient space of $\{0,1,\ldots,2n\}$ by the relation $\{d_n=0\}$.

Now, let us construct a correspondence $\mathcal{R}_n$ between $\{0,\ldots,2n\}$ and $[0,1]$, by letting $(i,s)\in\mathcal{R}_n$ if and only if $i=\lfloor 2ns\rfloor$. We estimate the distortion of $\mathcal{R}_n$ with respect to the pseudo-metrics $(8n/9)^{-1/4}d_n$ and D. Let (i,s) and (j,t) be in $\mathcal{R}_n$, then

$$\left|\left(\frac{8n}{9}\right)^{-1/4} d_n(i,j)-D(s,t)\right| = \left|D_n(\lfloor 2ns\rfloor/2n,\lfloor 2nt\rfloor/2n)-D(s,t)\right| .$$

By the a.s. uniform convergence of D_n to D, we obtain that the supremum of these quantities in $(i,s),(j,t)\in\mathcal{R}_n$ converges to 0 as $n\to\infty$.

Obviously, D_n induces a correspondence with same vanishing distortion between the two quotient spaces of $\{0,\ldots,2n\}$ and $[0,1]$ by the relations $\{D_n=0\}$ and $\{D=0\}$, and endowed with the distances $(8n/9)^{-1/4}d_n$ and D. We thus obtain that

$$\mathrm{d}_{\mathrm{GH}}((V(Q_n)\setminus\{v_*\},(8n/9)^{-1/4}d_{Q_n}),(M,D))\to 0\,,$$

as wanted.

We now turn to the last statement. For every n, let $i_*^{(n)}$ be any index in $\{0,1,\ldots,2n\}$ such that $V_n(i_*^{(n)})=\min V_n$. Then for every $v\in V(Q_n)$, it holds that

$$|d_{Q_n}(v_*,v)-d_{Q_n}(u_{i_*^{(n)}},v)|\leq 1$$

because $d_{Q_n}(v_*,u_{i_*^{(n)}})=1$ (the two vertices are linked by an arc in the CVS bijection). Moreover, since $(8n/9)^{-1/4}V_n(2n\cdot)$ converges to Z uniformly on $[0,1]$, and since we know[3] that Z attains its overall infimum at a unique point s_*, it is easy to obtain that $i_*^{(n)}/2n$ converges as $n\to\infty$ towards s_*. By combining this with the uniform convergence of D_n to D along (n_k), we obtain that for every $s\in[0,1]$,

$$\begin{aligned} D(s,s_*) &= \lim_{n\to\infty} D_n(\lfloor 2ns\rfloor/2n, i_*^{(n)}/2n)\\ &= \lim_{n\to\infty}\left(\frac{8n}{9}\right)^{-1/4} d_{Q_n}(v_*,u_{\lfloor 2ns\rfloor})\\ &= \lim_{n\to\infty}\left(\frac{8n}{9}\right)^{-1/4} (V_n(\lfloor 2ns\rfloor)-\min V_n+1)\\ &= Z_s-\inf Z\,, \end{aligned}$$

as wanted. □

It is tempting to call (M,D) the "Brownian map", or the "Brownian continuum map", by analogy with the fact that the "Brownian continuum random tree" is the scaling limit of random uniform elements of $\mathbf{A}_n$. However, the choice of the subsequence in Proposition 6.2 poses a problem of uniqueness of the limit. As we see in the previous statement, only the distances to ρ are *a priori* defined as simple functionals of the process Z. Distances between other points in M seem to be harder to handle, and it is not known whether they are indeed uniquely defined. Of course, it is natural make the following

Conjecture 6.1. *The spaces $(V(Q_n),n^{-1/4}d_{Q_n})$ converge in distribution, for the Gromov-Hausdorff topology.*

Marckert and Mokkadem [15] and Le Gall [10] give a natural candidate for the limit (called the Brownian map in [15]) but at present, it has not been identified as the correct limit. The rest of this course will be devoted to some properties that are nevertheless satisfied by *any* limit of the form (M,D) as appearing in Proposition 6.2, along some subsequence.

[3] We could also perform the proof without using this fact, but it makes things a little easier

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