LECTURES ON DEFORMATIONS OF GALOIS REPRESENTATIONS

Mark Kisin

Lecture 3: Representability

(3.1) Quotients by finite (formal) group actions are often representable, and indeed there are general results which guarantee this in certain situations.

In this section we assume that G satisfies the condition Φ_p. We begin with a result from Lecture 1, whose proof had been postponed.

Theorem (3.1.1). *Suppose* $\operatorname{End}_{\mathbb{F}[G]} V_{\mathbb{F}} = \mathbb{F}$. *Then* $D_{V_{\mathbb{F}}}$ *is representable.*

Proof. We already saw that $D^{\square}_{V_{\mathbb{F}}}$ is representable by $\operatorname{Spf} R^{\square}_{V_{\mathbb{F}}} =: X_{V_{\mathbb{F}}}$ where $R^{\square}_{V_{\mathbb{F}}}$ is a complete local $W(\mathbb{F})$-algebra.

Let $\widehat{\mathrm{PGL}}_d$ denote the formal completion of the $W(\mathbb{F})$-group scheme PGL_d along its identity section. Then $\widehat{\mathrm{PGL}}_d$ acts on $X_{V_{\mathbb{F}}}$ and we have

$$X_{V_{\mathbb{F}}} \times \widehat{\mathrm{PGL}}_d \rightrightarrows X_{V_{\mathbb{F}}}; \quad (x, g) \mapsto (x, gx).$$

The action of $\widehat{\mathrm{PGL}}_d$ on $X_{V_{\mathbb{F}}}$ is *free* which means that the induced map

$$X_{V_{\mathbb{F}}} \times \widehat{\mathrm{PGL}}_d \to X_{V_{\mathbb{F}}} \times X_{V_{\mathbb{F}}}.$$

is a closed immersion.

We would like to take the quotient of $X_{V_{\mathbb{F}}}$ by this action. To do this we need a little preparation.

(3.1.2) Let $\widehat{\mathfrak{AR}}_W$ denote the category of complete local Noetherian $W(\mathbb{F})$-algebras, so the opposite category $(\widehat{\mathfrak{AR}}_W)^\circ$ is equivalent to the category of formal spectra of such $W(\mathbb{F})$-algebras.

An *equivalence relation* $R \rightrightarrows X$ in $(\widehat{\mathfrak{AR}}_W)^\circ$ is a pair of morphisms such that

(1) $R \to X \times X$ is a closed embedding.

(2) For all T in $(\widehat{\mathfrak{AR}}_W)^\circ$ $R(T) \subset (X \times X)(T)$ is an equivalence relation.

For example let G be a group object in $(\widehat{\mathfrak{AR}}_W)^\circ$, and $G \times X \to X$ a free action. Then the map

$$G \times X \rightrightarrows X; \quad (g, x) \mapsto (x, gx)$$

is an equivalence relation.

A flat morphism $X \to Y$ in $(\widehat{\mathfrak{AR}}_W)^\circ$ is said to be a quotient of X by R, if the embedding $R \to X \times X$ induces an isomorphism $R \xrightarrow{\sim} X \times_Y X$.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Theorem (3.1.3). *(SGA 3, VIIb, Thm. 1.4): Let* $R \overset{p_0}{\underset{p_1}{\rightrightarrows}} X$ *be an equivalence relation in* $(\widehat{\mathfrak{AR}_W})^\circ$ *such that the first projection* $R \to X$ *is flat. Then the quotient of* X *by* R *exists. If* $X = \operatorname{Spf} B$ *and* $R = \operatorname{Spf} C$, *then* $X/R = \operatorname{Spf} A$, *where*

$$A = \{b \in B : p_0^*(b) = p_1^*(b)\}.$$

We can now complete the proof of Theorem (3.1.1) by applying (3.1.3) to the equivalence relation $X_{V_{\mathbb{F}}} \times \widehat{\mathrm{PGL}}_d \rightrightarrows X_{V_{\mathbb{F}}}$. $\square$

(3.2) Now fix a pseudo-representation $\bar\tau : G \to \bar\tau$. We now want to construct some representable subgroupoids of $\mathrm{Rep}_{\bar\tau} \to D_{\bar\tau}$.

Suppose that for $i = 1, \dots, s$ $\bar\rho_i : G \to \mathrm{GL}_{d_i}(\mathbb{F})$ are pairwise distinct absolutely irreducible representation of G, such that $\bar\tau = \sum_{i=1}^s \bar\rho_i$.

Let $\mathrm{Rep}'_{\bar\tau} \subset \mathrm{Rep}_{\bar\tau}$ be the full subgroupoid of $\mathrm{Rep}_{\bar\tau}$ such that $\mathrm{Rep}'_{\bar\tau}(A, B)$ consists of the objects V_B in $\mathrm{Rep}_{\bar\tau}(A, B)$ such that

$$V_B \otimes_B B/\mathfrak{m}_A B \sim \begin{pmatrix} \bar\rho_1 & c_1 & \cdots & \cdots \\ & \bar\rho_2 & c_2 & \cdots \\ & & \cdots & \cdots \\ & & & \bar\rho_s \end{pmatrix}. \tag{3.2.1}$$

where for $i = 1, \dots, s-1$ c_i is a *non-trivial* extension of $\bar\rho_{i+1}$ by $\bar\rho_i$.

Note that the isomorphism in (3.2.1) is uniquely determined as out conditions imply that the representation on the right has no non-trivial automorphisms.

Theorem (3.2.2). *The groupoid* $\mathrm{Rep}'_{\bar\tau} \to D_{\bar\tau}$ *is representable by a proper formal scheme over* $\operatorname{Spf} R_{\bar\tau}$.

Proof. Let

$$\mathrm{Rep}^{\square,\prime}_{\bar\tau} = \mathrm{Rep}'_{\bar\tau} \times_{\mathrm{Rep}_{\bar\tau}} \mathrm{Rep}^{\square}_{\bar\tau} \subset \mathrm{Rep}^{\square}_{\bar\tau}.$$

This is a locally closed subspace. The group PGL_d acts freely on $\mathrm{Rep}^{\square,\prime}_{\bar\tau}$ as $V_B \in \mathrm{Rep}'_{\bar\tau}(A, B)$ has no non-trivial automorphisms. To take the quotient we need the following

Theorem (3.2.3). *Let* $S = \operatorname{Spec} A$, *with* A *a local Artin ring, and let* X/S *be a finite type* S*-scheme equipped with a free action by a reductive group* G/S.[1] *Suppose that every* $x \in X$ *is contained in an affine,* G*-stable open subset of* X. *Then the quotient* X/G *exists.*

x When S is a field this is explained in Mumford's book [GIT, Ch 1, §4, Prop 1.9]. The general case will be explained by Brian Conrad in another lecture.

We want to apply the theorem to the quotient $\mathrm{Rep}^{\square,\prime}_{\bar\tau}/\mathrm{PGL}_d$.

For simplicity we will consider only the case when $d = 2$ and $\bar\rho_1, \bar\rho_2$ are characters, which we will denote by χ_1 and χ_2 respectively. We have to check the condition that every point of $\mathrm{Rep}^{\square,\prime}_{\bar\tau}$ has an affine PGL_d-stable neighbourhood.

Let $U \subset \mathrm{Ext}^1(\chi_1, \chi_2)\backslash\{0\}$ be affine open and define $\mathrm{Rep}^{\square,U}_{\bar\tau} \subset \mathrm{Rep}^{\square,\prime}_{\bar\tau}$ the subgroupoid consisting of those (V_B, β) in $\mathrm{Rep}^{\square,\prime}_{\bar\tau}$ such that $V_B \xrightarrow{\sim} \left(\begin{smallmatrix} \chi_1 & c \\ 0 & \chi_2 \end{smallmatrix}\right)$ with $c \in U$. Then $\mathrm{Rep}^{\square,U}_{\bar\tau}$ is stable by PGL_2, and its fibre over the closed point of $\operatorname{Spf} R_{\bar\tau}$

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[1] A reductive group over a base S is a smooth group scheme with reductive fibres.

is isomorphic to U. In particular $\mathrm{Rep}^{\square,U}_{\bar{\tau}}$ is affine, and we may apply (3.2.3) with $A = R_{\bar{\tau}}/\mathfrak{m}^n_{R_{\bar{\tau}}}$ for $n = 1, \dots$.

We obtain an (a priori non-separated) scheme $\mathcal{E}_{\bar{\tau},n}$ over $R_{\bar{\tau}}/\mathfrak{m}^n_{R_{\bar{\tau}}}$, and hence a formal scheme $\mathcal{E}_{\bar{\tau}} = \lim_n \mathcal{E}_{\bar{\tau},n}$. Since $\mathcal{E}_{\bar{\tau},1} \xrightarrow{\sim} \mathbb{P}(\mathrm{Ext}^1(\chi_2, \chi_1))$ is proper, we see also that $\mathcal{E}_{\bar{\tau}}$ is proper. $\square$

Remark (3.2.4) I expect that the representing formal scheme is projective and therefore arises from a *scheme* of finite type over $Spec R_{\bar{\tau}}$.

Corollary (3.2.5). *Let* $x \in \mathrm{Rep}^{\square}_{\bar{\tau}}(\mathbb{F})$ *and* $V_{\mathbb{F}}$ *the corresponding representation of* G*. Then the complete local ring at of* $\mathrm{Rep}^{\square}_{\bar{\tau}}$ *at* x *pro-represents* $D^{\square}_{V_{\mathbb{F}}}$.

On the situation of (3.2), if $x \in \mathrm{Rep}^{\square}_{\bar{\tau}}(\mathbb{F})$ *then the complete local ring at of* $\mathrm{Rep}^{\square}_{\bar{\tau}}$ *at* x *is a quotient of* $R^{\square}_{V_{\mathbb{F}}}$.

Proof. This follows immediately from the definitions. $\square$

Corollary (3.2.6). *Let* $E/W(\mathbb{F})[1/p]$ *be a finite extension and* $x : Rep'_{\bar{\tau}} \to E$ *a point such that the corresponding* E*-valued pseudo-representation* τ_x *is absolutely irreducible. Then the map*

$$\mathrm{Rep}'_{\bar{\tau}} \to \mathrm{Spf}\, R_{\bar{\tau}}$$

is a closed embedding over a formal neighbourhood of τ_x.[2]

Proof. First we remark that x is the only point of $\mathrm{Rep}'_{\bar{\tau}}$ lying over τ_x. To see this suppose x' is another such point and let denote by V_x and $V_{x'}$ the corresponding G-representations. Then, by the properness of $\mathrm{Rep}'_{\bar{\tau}}$ x and x' arise from $\mathcal{O}_E$ valued points, which in turn correspond to G-stable lattice $L_x \subset V_x$ and $L'_x \subset V_{x'}$. Since V_x and $V_{x'}$ are absolutely irreducible with the same trace, they are isomorphic. We may choose this isomorphism so that it induces a non-zero map $L_x \to L_{x'}$, whose reduction modulo the radical (π_E) of $\mathcal{O}_E$ is non-zero. Now as $L_x/\pi_E L_x$ and $L_{x'}/\pi_E L_{x'}$ are both non-zero described in (3.2.1), the only non-zero maps between them are isomorphisms. Hence we find that $L_x \xrightarrow{\sim} L_{x'}$.

Next let $\widehat{R}_{\tau_x}$ be the complete local ring at τ_x. By (3.2.4) $\widehat{R}_{\tau_x}$ is the universal deformation ring of τ_x. Similarly the complete local ring at x, $\widehat{R}_x$ is a quotient of the universal deformation ring of V_x, by Lecture 1, Exercise 4 and (3.2.6). Hence the map $\widehat{R}_{\tau_x} \to \widehat{R}_x$ is a surjection by the Theorem of Nyssen-Rouquier (2.3.1). $\square$

Exercises:

Exercise 1: Verify that the quotient produced in the proof of (3.1.1) does represent $R_{V_{\mathbb{F}}}$.

Exercise 2: Complete the proof of (3.2.2) for arbitrary d.

References

[GIT] D. Mumford, *Geometric invariant theory*, Ergeb. der Math. 34, Springer, 1994.

[SGA 3] M. Demazure, A. Grothendieck, *Schemas en Groupes I, II, III*, Lecture notes in math. 151-153, Springer, 1970.

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[2] Note that when we refer to E-valued points of formal schemes over $W(\mathbb{F})$, and complete local rings at such points, what we really mean is the complete local ring at the corresponding point of the p-adic analytic space

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Lecture 4: Presenting global deformation rings over local ones

(4.1) Let F be a number field and S a finite set of primes of F containing the primes dividing p. Fix an algebraic closure $\bar{F}$ of F and denote by $F_S \subset \bar{F}$ the maximal extension of F unramified outside S. Write $G_{F,S} = \mathrm{Gal}(F_S/F)$.

Let $\Sigma \subset S$ and fix an algebraic closure $\bar{F}_v$ of F_v for each $v \in \Sigma$, as well as an embedding $\bar{F} \hookrightarrow \bar{F}_v$. We write $G_{F_v} = \mathrm{Gal}(\bar{F}_v/F_v)$.

Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$ and uniformizer $\pi_{\mathcal{O}}$. We fix a finite dimensional $\mathbb{F}$-vector space $V_{\mathbb{F}}$ equipped with a continuous action of $G_{F,S}$, and a continuous character $\psi : G_{F,S} \rightarrow \mathcal{O}^\times$ such that $\det_{\mathbb{F}} V_{\mathbb{F}} \sim \psi$.

For simplicity we assume in the following that $p \nmid \dim V_{\mathbb{F}}$, although this is not really necessary (see [Ki 2]).

(4.1.1) For each $v \in \Sigma$ fix a basis β_v of $V_{\mathbb{F}}$. For A in $\mathfrak{AR}_W(\mathbb{F})$ denote by $D_v^{\square,\psi}(A)$ the category of framed deformations of $(V_{\mathbb{F}}|_{G_{F_v}}, \beta)$ to A, with determinant ψ. This is pro-representable by a complete local $\mathcal{O}$-algebra $R_v^{\square,\psi}$.

Similarly we denote by $D_{F,S}^{\square,\psi}(A)$ the category whose objects consist of a deformation of $V_{\mathbb{F}}$ to V_A together with a lifting of *each* basis β_v to an A-basis of V_A.

We also have the analogous functors D_v^{ψ} and $D_{F,S}^{\psi}$ for unframed deformations and the universal $\mathcal{O}$-algebras R_v^{ψ} and $R_{F,S}^{\psi}$ when these functors are representable.

We set

$$R_{\Sigma}^{\square,\psi} = \widehat{\otimes}_{\mathcal{O},v\in\Sigma} R_v^{\square,\psi} \text{ and } R_{\Sigma}^{\psi} = \widehat{\otimes}_{\mathcal{O},v\in\Sigma} R_v^{\psi}$$

when the latter ring exists.

Finally we denote by $\mathfrak{m}_{\Sigma}^{\square}$ and $\mathfrak{m}_{F,S}^{\square}$ the radicals of $R_{\Sigma}^{\square,\psi}$ and $R_{F,S}^{\square,\psi}$ respectively, and similarly for $\mathfrak{m}_{\Sigma}$ and $\mathfrak{m}F, S$.

Proposition (4.1.2). *For $i \geq 1$ let h_{Σ}^i and c_{Σ}^i denote respectively the dimension of the kernel and cokernel of*

$$\theta^i : H^i(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) \rightarrow \prod_{v\in\Sigma} H^i(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}})$$

where $\mathrm{ad}^0 V_{\mathbb{F}} \subset \mathrm{ad} V_{\mathbb{F}}$ denotes the space of endomorphisms with trace 0. If R_{Σ}^{ψ} exists then $R_{F,S}^{\psi}$ is a quotient of $R_{\Sigma}^{\psi}[\![x_1, \dots, x_{h_{\Sigma}^1}]\!]$ by $c_{\Sigma}^1 + h_{\Sigma}^2$ relations.

In general let

$$\eta : \mathfrak{m}_{\Sigma}^{\square}/(\mathfrak{m}_{\Sigma}^{\square,2}, \pi_{\mathcal{O}}) \rightarrow \mathfrak{m}_{F,S}^{\square}/(\mathfrak{m}_{F,S}^{\square,2}, \pi_{\mathcal{O}}).$$

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Then $R^{\psi,\square}_{F,S}$ *is a quotient of a power series ring over* $R^{\square}_{\Sigma}$ *in* $\mathrm{dimcoker}\eta$ *variables by* $h^2_{\Sigma} + \mathrm{dimker}\,\eta$ *relations.*

Proof. We prove only the first statement. Choose a surjection

$$\tilde{R} := R^{\psi}_{\Sigma}[\![x_1, \dots, x_{h^1_{\Sigma}}]\!] \to R^{\psi}_{F,S} \tag{4.1.3}$$

which induces a surjection on reduced tangent spaces and denote by J its kernel. We denote by $\tilde{\mathfrak{m}}$ the radical of $\tilde{R}$.

The universal representation $\rho_{R^{\psi}_{F,S}} : G_{F,S} \to \mathrm{GL}_d(R^{\psi}_{F,S})$ has a set theoretic lifting $\tilde{\rho} : G_{F,S} \to G_{F,S} \to \mathrm{GL}_d(\tilde{R}/\tilde{\mathfrak{m}}J)$ such that $\det \tilde{\rho}(\gamma) = \psi(\gamma)$ for all $\gamma \in G_{F,S}$. Such a lifting exists as the fibres of $\det : \mathrm{GL}_d \to \mathrm{GL}_1$are torsors over SL_d, and in particular smooth.

Define

$$c : G^2_{F,S} \to J/\tilde{m}J \otimes_{\mathbb{F}} \mathrm{ad}^0 V_{\mathbb{F}}; \quad c(g_1, g_2) = \tilde{\rho}(g_1 g_2)\tilde{\rho}(g_2)^{-1}\tilde{\rho}(g_1)^{-1}$$

where we regard

$$J/\tilde{m}J \otimes_{\mathbb{F}} \mathrm{ad}^0 V_{\mathbb{F}} \xrightarrow{\sim} \ker\left(\mathrm{GL}_d(\tilde{R}/\tilde{m}J) \to \mathrm{GL}_d(\tilde{R}/J)\right)$$

Then $[c] \in H^2(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) \otimes J/\tilde{\mathfrak{m}}J$ depends only on $\rho_{R^{\psi}_{F,S}}$ and not on $\tilde{\rho}$, and $[c] \sim 0$ if and only if $\tilde{\rho}$ can be chosen to be a homomorphism (see Exercise 2 below).

As $\rho_{R^{\psi}_{F,S}}|_{G_{F_v}}$ is induced by the universal representation over R^{ψ}_v, $\rho_{R^{\psi}_{F,S}}|_{G_{F_v}}$ lifts to $\tilde{R}/\tilde{\mathfrak{m}}J$ and hence $[c|_{G_{F_v}}] \sim 0$. Hence $[c] \in H^2_{\Sigma}(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) \otimes J/\tilde{\mathfrak{m}}J$ where

$$H^2_{\Sigma}(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) := \ker\left(H^2(G_{F,S}, \mathrm{ad}^- V_{\mathbb{F}}) \to \prod_{v \in \Sigma} H^2(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}})\right),$$

and we obtain a map

$$(J/\tilde{\mathfrak{m}}J)^* \to H^2_{\Sigma}(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}); \quad u \mapsto \langle [c], u \rangle. \tag{4.1.4}$$

Let

$$\begin{tikzcd}
I = \ker\left(\mathfrak{m}_{\Sigma}/(\mathfrak{m}^2_{\Sigma}, \pi_{\mathcal{O}}) \right. \arrow[r] \arrow[d, "\sim"] & \left. \mathfrak{m}_{F,S}/(\mathfrak{m}^2_{F,S}, \pi_{\mathcal{O}})\right) \arrow[d, "\sim"] \\
\oplus D_{V_{\mathbb{F}}|_{G_{F_v}}}(F[\epsilon])^* \arrow[r] & D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon])
\end{tikzcd}$$

Note that $I \xrightarrow{\sim} \ker\left(\tilde{\mathfrak{m}}/(\tilde{\mathfrak{m}}^2, \pi_{\mathcal{O}}) \to \mathfrak{m}_{F,S}/(\mathfrak{m}^2_{F,S}, \pi_{\mathcal{O}})\right)$, so reducing mod $\tilde{\mathfrak{m}}$ we get a surjection $J/\tilde{\mathfrak{m}}J \to I$ and an injection $I^* \hookrightarrow (J/\tilde{\mathfrak{m}}J)^*$.

We claim that I^* contains the kernel of (4.1.4). If $0 \neq u \in (J/\tilde{\mathfrak{m}}J)^*$ let $\tilde{R}_u$ be the pushout of $\tilde{R}/\tilde{\mathfrak{m}}J$ by u. Then $R^{\psi}_{F,S} = \tilde{R}_u/I_u$ where $I_u \subset \tilde{R}_u$ is an ideal with $I_u \cdot \tilde{\mathfrak{m}}$ and which is 1-dimensional as an $\mathbb{F}$-vector space. If $\langle [c], u \rangle = 0$ then $\rho^{\psi}_{F,S}$ lifts to a representation into $\mathrm{GL}_d(\tilde{R}_u)$ with determinant ψ. Hence $\tilde{R}_u \to R^{\psi}_{F,S}$ has a section and $\tilde{R}_u = R^{\psi}_{F,S} \oplus I_u$. This implies that $\tilde{R}_u \to R^{\psi}_{F,S}$ does not induce a bijection on reduced tangent spaces. In particular, the composite

$$\ker\left(J/\tilde{\mathfrak{m}}J \to I\right) \to J/\tilde{\mathfrak{m}}J \to I_u$$

is not surjective, and is therefore 0. This means that u factors through I, which proves our claim.

It follows that

$$\dim_{\mathbb{F}}(J/\tilde{\mathfrak{m}}J)^* \leqslant \dim_{\mathbb{F}} I + h^2_{\Sigma} = c^1_{\Sigma} + h^2_{\Sigma}.$$

□

Theorem (4.2). *Suppose $\{v|p\} \subset \Sigma$, $\{v|\infty\} \subset S$, and $S\backslash\Sigma$ contains a finite prime. Then*

$$R^{\psi}_{F,S} \xrightarrow{\sim} R^{\psi}_{\Sigma}[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_{r+s})$$

for some $r \geq 0$, and $s = \sum_{v|\infty, v\notin\Sigma} \dim_{\mathbb{F}}(\mathrm{ad}^0 V_{\mathbb{F}})^{G_{F_v}}$, provided the rings $R^{\psi}_{F,S}$ and R^{ψ}_{Σ} exist.

Moreover,

$$R^{\psi}_{F,S} \xrightarrow{\sim} R^{\psi}_{\Sigma}[\![x_1, \dots, x_{r+\Sigma-1}]\!]/(f_1, \dots, f_{r+s})$$

(4.2.1) To prove the theorem, we need the Poitou-Tate sequence. Late X be a finite abelian p-group equipped with an action of $G_{F,S}$. We denote by $X^{\vee}$ the Pontryagin dual of X, and by $X^* = X^{\vee}(1)$ its Tate dual. Then there is an exact sequence

(PT(X))

$$0 \to H^0(G_{F,S}, X) \to \prod_{v|\infty} \widehat{H}^0(G_{F_v}, X) \times \prod_{v\in S_f} H^0(G_v, X) \to H^0(G_{F,S}, X^*)^{\vee}$$
$$\to H^1(G_{F,S}.X) \to \prod_{v\in S} H^1(G_{F_v}, X) \to H^1(G_{F,S}, X^*)^{\vee}$$
$$\to H^2(G_{F,S}, X) \to \prod_{v\in S} H^2(G_{F_v}, X) \to H^0(G_{F,S}, X^*)^{\vee} \to 0$$

Here $\widehat{H}^0(G_{F_v}, X)$ denotes $H^0(G_{F_v}, X)$ modulo the subgroup of norms in X.

Local Tate duality provides an isomorphism

$$H^i(G_{F_v}, X)^{\vee} \xrightarrow{\sim} H^{2-i}(G_{F_v}, X^*)$$

for v a finite prime and $i = 0, 1, 2$. Using this, one can identify the Pontryagin dual of the sequence $PT(X)$ with $PT(X^*)$.

Proof of (4.2). We will prove only the first statement since the proof of the second statement requires only some extra book keeping.

We apply the above sequence with $X = \mathrm{ad}^0 V_{\mathbb{F}}$. First note that, using the remark on the duality of $PT(X)$ and $PT(X^*)$ one sees that the map

$$\prod_{S\backslash\Sigma} H^2(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}}) \to H^0(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}(1))^{\vee}$$

induced by the final map of $PT(X)$ is surjective, as $S\backslash\Sigma$ contains a finite prime. Hence the map

$$H^2(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) \to \prod_{v\in\Sigma} H^2(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}})$$

is surjective and

$$h^2_{\Sigma} = h^2(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) - \sum_{v\in\Sigma} h^2(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}}).$$

Here we use the convention that $h^i = \dim H^i$.

By (4.1.2) we have $R^{\psi}_{F,S}[\![x_1,\dots,x_r]\!]/(f_1,\dots f_{r+s})$. with

$$(4.2.2) \quad -s = h^1_{\Sigma} - c^1_{\Sigma} - h^2_{\Sigma}$$
$$= h^1(G_{F,S}.\mathrm{ad}^0 V_{\mathbb{F}}) - \sum_{v\in\Sigma} h^1(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}}) - h^2(G_{F,S}.\mathrm{ad}^0 V_{\mathbb{F}}) - \sum_{v\in\Sigma} h^2(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}})$$
$$= -\chi(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) + \sum_{v\in\Sigma} \chi(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}}).$$

Here we have used that fact that the existence of $R^{\psi}_{G_{F,S}}$ and $\mathcal{R}^{\psi}_{\Sigma}$ implies that $\mathrm{ad}^0 V_{\mathbb{F}}$ has no $G_{F,S}$ or G_{F_v} invariants for $v \in \Sigma$.

Now we use Tate's global Euler characteristic formula which says that

$$(4.2.3) \quad \chi(G_{F,S}, \mathrm{ad}^0 V_{\mathbb{F}}) = \sum_{v|\infty} (h^0(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}}) - [F_v : R]\mathrm{dim}_{\mathbb{F}}\mathrm{ad}^0 V_{\mathbb{F}})$$
$$= (\sum_{v|\infty} h^0(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}})) - [F;\mathbb{Q}]\mathrm{dim}_{\mathbb{F}}\mathrm{ad}^0 V_{\mathbb{F}}.$$

The local Euler characteristic $\chi(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}})$ is 0 if $v \nmid p$ is a finite prime. Hence the contributions of the local terms in (4.2.2) is

$$(4.2.4) \quad \sum_{v|\infty, v\in\Sigma} h^0(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}}) - \sum_{v|p} [F_v : \mathbb{Q}_p]\mathrm{dim}_{\mathbb{F}}\mathrm{ad}^0 V_{\mathbb{F}}$$
$$= \sum_{v|\infty, v\in\Sigma} h^0(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}}) - [F : \mathbb{Q}]\mathrm{dim}_{\mathbb{F}}\mathrm{ad}^0 V_{\mathbb{F}}.$$

Subtracting (4.2.4) from (4.2.3) one finds

$$s = \sum_{v|\infty, v\notin S} h^0(G_{F_v}, \mathrm{ad}^0 V_{\mathbb{F}}).$$

□

Exercises:

Exercise 1: Prove the second statement in Proposition (4.1.2).

Exercise 2: Check the statements about the cocycle c in Proposition: That $[c]$ does not depend on $\tilde{\rho}$ and is trivial if and only if $\tilde{\rho}$ can be chosen to be a homomorphism.

Exercise 3: Prove the second statement in Theorem (4.2)

Exercise 4: (This is more difficult.) Formulate and prove Theorem (4.2) without assuming $p \nmid \mathrm{dim} V_{\mathbb{F}}$.

References

[Ki 2] M. Kisin, *Modularity of 2-dimensional Galois representations*, Current developments in mathematics 2005, 2008.

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Lecture 5: Flat deformations

(5.1) Flat deformations: Let $K/\mathbb{Q}_p$ be a finite extension with residue field k. Let $W = W(k)$ and $K_0 = \mathrm{Fr}W$. We consider a finite dimensional $\mathbb{F}$-vector space equipped with a continuous action of G_K. Fix an algebraic closure $\bar{K}$ of K, and let $G_K = \mathrm{Gal}(\bar{K}/K)$.

Recall that a representation of G_K on a finite abelian p-group is called *flat* if it arises from a finite flat group scheme over $\mathcal{O}_K$.

The following result is due to Ramakrishna [Ra]:

Proposition (5.1.1). *Let $A \in \mathfrak{AR}_W$ and V_A in $D_{V_{\mathbb{F}}}(A)$. There exists a quotient A^{fl} of A such that for any morphism $A \to A'$ in $\mathfrak{AR}_W$, $V_{A'} =: V_A \otimes_A A'$ is flat if and only if $A \to A'$ factors through A^{fl}.*

Proof. First note that if V is a finite abelian p-group equipped with a continuous action of G_K and V is flat, then any G_K-stable subgroup $V' \subset V$ is flat. Indeed, suppose $V = \mathcal{G}(\mathcal{O}_{\bar{K}})$ think of V, V' as finite étale group schemes over K. If $\mathcal{G}'$ is the closure of V' in $\mathcal{G}$, then $V' = \mathcal{G}'(\mathcal{O}_{\bar{K}})$.

This remark shows that if $\theta : A \to A'$ is a morphism in $\mathfrak{AR}_W$ then $V_{A'}$ is flat if and only if $V_{\theta(A)}$ is flat. Similarly, if $I, J \subset A$ are ideals and $V_{A/I}$ and $V_{A/J}$ is flat then $V_{A/I\cap J} \subset V_{A/I} \oplus V_{A/J}$ is flat. $\square$

Corollary (5.1.2). *Let $D^{\mathrm{fl}}_{V_{\mathbb{F}}} \subset D_{V_{\mathbb{F}}}$ denote the sub-functor corresponding to flat deformations. Then $D^{\mathrm{fl}}_{V_{\mathbb{F}}} \subset D_{V_{\mathbb{F}}}$ is relatively representable.*

Proof. In the language of groupoids this just means that if ξ in $D_{V_{\mathbb{F}}}$, then $(D^{\mathrm{fl}}_{V_{\mathbb{F}}})_\xi$ is representable, and this follows from (5.1.1). $\square$

(5.2) Weakly admissible modules and Smoothness of the generic fibre:

Proposition (5.2.1). *Suppose that $D_{V_{\mathbb{F}}}$ is pro-represented by $R_{V_{\mathbb{F}}}$ and let $R^{\mathrm{fl}}_{V_{\mathbb{F}}}$ be the quotient of $R_{V_{\mathbb{F}}}$ which pro-represents $D^{\mathrm{fl}}_{V_{\mathbb{F}}}$. Let $E/W(\mathbb{F})[1/p]$ be a finite extension and $x : R^{\mathrm{fl}}_{V_{\mathbb{F}}}[1/p] \to E$ be a point such that* $\ker x$ *has residue field E. Write $\widehat{R}^{\mathrm{fl}}_x$ (resp. $\widehat{R}_x$) for the completion of $R^{\mathrm{fl}}_{V_{\mathbb{F}}}[1/p]$ (resp. $R_{V_{\mathbb{F}}}[1/p]$) at x.*

For any Artinian quotient $\epsilon : \widehat{R}_x \to B$ denote the specialization of the universal deformation by $R_{V_{\mathbb{F}}} \to B$. Then ϵ factors through $R^{\mathrm{fl}}_{V_{\mathbb{F}}}$ if and only if V_B factors through $\widehat{R}^{\mathrm{fl}}_x$ if and only if V_B arises from a p-divisible group. Moreover this condition holds if and only if V_B is crystalline.

Proof. Let B be in $\mathfrak{AR}_E$ and denote by Int_B the set of finite $\mathcal{O}_E$-subalgebras $A \subset B$. Then $R_{V_{\mathbb{F}}} \to B$ factors through some A in Int_B. Denote by V_A the induced G_K representation.

Then V_B arises from a p-divisible group if and only if V_A does. By a result of Raynaud [Ray, 2.3.1], V_A arises from a p-divisible group if and only if V_A/p^nV_A is flat for $n \geq 1$. This is equivalent to asking that $V_A \otimes_A A/\mathfrak{m}_A^n$ be flat for $n \geq 1$, or that $V_A \otimes_A A/\mathfrak{m}_A^n$ is in $D^{\mathrm{fl}}(A/\mathfrak{m}_A^n)$.

For the final statement we use Breuil's result that a crystalline representation with all Hodge-Tate weights equal to $0, 1$ arises from a p-divisible group [Br, Thm. 5.3.2], [Ki 4, 2.2.6]. □

(5.2.2) The deformation theoretic description of $\widehat{R}_x^{\mathrm{fl}}$ allows us to show that this ring is always formally smooth over E, and hence that $R_{V_{\mathbb{F}}}^{\mathrm{fl}}[1/p]$ is formally smooth over $W(\mathbb{F})[1/p]$. To prove this we need a little preparation (see [Ki 3]).

For any weakly admissible filtered φ-module D over K, denote by $C^\bullet(D)$ the complex $C^\bullet(D)$ the complex

$$D \overset{(1-\varphi,\mathrm{id})}{\longrightarrow} D \oplus D_K/\mathrm{Fil}^0 D_K$$

concentrated in degrees $0, 1$.

Lemma (5.2.3). *There is a canonical isomorphism*

$$\mathrm{Ext}^1_{w.adm}(\mathbf{1}, D) \xrightarrow{\sim} H^1(C^\bullet(D)).$$

where $\mathbf{1} = K_0$ *denotes the unit object in the category of weakly admissible module.*

Proof. Let

$$0 \to D \to \tilde{D} \to \mathbf{1} \to 0$$

be an extension. Let $\tilde{d} \in \mathrm{Fil}^0\tilde{D}$ be a lift of $1 \in \mathbf{1}$. Note that $D_K/\mathrm{Fil}^0 D_K \xrightarrow{\sim} \tilde{D}_K/\mathrm{Fil}^0\tilde{D}_K$, so we may regard $\tilde{d} \in D_K/\mathrm{Fil}^0 D_K$. Moreover $(1-\varphi)(\tilde{d}) \in D$. We associate the class

$$((1-\varphi)\tilde{d}, \tilde{d}) \in H^1(C^\bullet(D))$$

to the given extension.

If $(d_0, d_1) \in D \oplus D_K/\mathrm{Fil}^0 D_K$ we construct an extension of $\mathbf{1}$ by D by setting $\tilde{D} = D \oplus 1$ on underlying K_0-vector spaces and defining φ on $\tilde{D}$ by $\varphi(1) = 1 + d_0$ and the filtration by

$$\mathrm{Fil}^i\tilde{D}_K = \mathrm{Fil}^i D_K + K \cdot d_1 \quad i \leqslant 0$$

and $\mathrm{Fil}^i\tilde{D}_K = \mathrm{Fil}^i D_K$ if $i > 0$.

One checks that these two constructions induce the required isomorphism and its inverse. □

(5.2.4) Let $E/\mathbb{Q}_p$ be a finite extension, and D_E a weakly admissible filtered φ-module over K, equipped with an action of E. For B in $\mathfrak{AR}_E$ we denote by $D_{D_E}(B)$ the category of weakly admissible filtered filtered φ-modules D_B, equipped with an action of B, and an isomorphism $D_B \otimes_B E \xrightarrow{\sim} D$, such that D and $\mathrm{gr}^\bullet D_K$ are free B-modules.

For V_E a crystalline representation on a finite dimensional E-vector space, and B in $\mathfrak{AR}_E$ we denote by $D_{\mathrm{cris}}(V_B)$ the category of crystalline deformations of V_E to B.

Lemma (5.2.5). *Let* V_E *be as above and* $D_E = D_{\mathrm{cris}}(V_E)$. *Then* D_{cris} *induces an equivalences of groupoids over* $\mathfrak{AR}_E$ $D_{V_E} \xrightarrow{\sim} D_{D_E}$. *Moreover each of these groupoids is formally smooth.*

Proof. The proof of the first statement is formally similar to (5.3.5) below, and left as an exercise to the reader. The formal smoothness can be proved by a deformation theoretic argument using (5.2.3) and the fact that $H^2(C^\bullet(\mathrm{ad}D_E)) = 0$. □

Proposition (5.2.6). *In the notation of (5.2.1), let* $D_E = D_{\mathrm{cris}}(V_E)$. *The* E-*algebra* $\widehat{R}_x^{\mathrm{fl}}$ *is formally smooth of dimension*

$$\dim_E H^1(\mathcal{C}^\bullet(\mathrm{ad}D_x)) = 1 + \dim_E D_K/\mathrm{Fil}^0 D_K.$$

Proof. By (5.2.1) $\widehat{R}_x^{\mathrm{fl}}$ pro-represents D_{V_E}, which is equivalent to D_{D_E} and hence formally smooth by (5.2.5). The dimension of $\widehat{R}_x^{\mathrm{fl}}$ is equal to

$$\begin{aligned}&\dim_E \mathrm{Ext}^1_{\mathrm{cris}}(V_E, V_E) = \dim_E \mathrm{Ext}^1_{w.adm}(D_E, D_E) = \\ &\qquad \dim_E \mathrm{Ext}^1_{w.adm}(\mathbf{1}, \mathrm{ad}D_x) = \dim_E H^1(C^\bullet(\mathrm{ad}D)) = 1 + \dim_E D_K/\mathrm{Fil}^0 D_K.\end{aligned}$$

Here the first term means crystalline self extensions of V_E, as a representation of $E[G_K]$, the second last equality follows from (5.2.3), and the final one from the fact that the Euler characteristic of a finite complex is equal to that of its cohomology. □

(5.3) The Fontaine-Laffaille functor and Smoothness when $e = 1$: Recall the Fontaine-Laffaille category $\mathrm{MF}^1_{\mathrm{tor}}$ whose objects consist of a finite, torsion W-module M, together with a submodule $M^1 \subset M$, and Frobenius semi-linear maps

$$\varphi : M \to M \text{ and } \varphi^1 : M^1 \to M$$

such that

(1) $\varphi|_{M^1} = p\varphi^1$.
(2) $\varphi(M) + \varphi^1(M^1) = M$.

$\mathrm{MF}^1_{\mathrm{tor}}$ is an abelian subcategory of the category of filtered W-modules of finite length [FL, 1.9.1.10]. In particular, any morphism on $\mathrm{MF}^1_{\mathrm{tor}}$ is strict for filtrations.

Note also that if $p \cdot M = 0$, then $\varphi(M^1) = 0$, and so comparing the lengths of the two sides of (2) above shows that φ^1 is injective and

$$\varphi(M) \oplus \varphi^1(M^1) \xrightarrow{\sim} M.$$

Theorem (5.3.1). *(Fontaine-Laffaille, Raynaud) Suppose that* $K = K_0$ *and* $p > 2$. *Then there exist equivalences of abelian categories*

$$\mathrm{MF}^1_{\mathrm{tor}} \xrightarrow[\mathrm{FL}]{\sim} \{\text{f.flat group schemes}/W\} \xrightarrow{\sim} \{\text{flat repns. of } G_K\}.$$

Proof. The first equivalence is obtained by composing the anti-equivalence [FL, 9,11] with Cartier duality. The second follows from Raynaud's result [Ray, 3.3.6] that when $e(K/K_0) < p - 1$ the functor $\mathcal{G} \mapsto \mathcal{G}(\mathcal{O}_{\bar{K}})$ is fully faithful and the category of finite flat group schemes over $\mathcal{O}_K$ is abelian. □

(5.3.2) We will need a little more information about the functor FL. For a finite flat group scheme $\mathcal{G}$ over W we denote by $t_{\mathcal{G}}$ the tangent space of $\mathcal{G} \otimes_W k$, and by $\mathcal{G}^*$ the Cartier dual of $\mathcal{G}$.

The contravariant version of the functor is constructed via the theory of Honda systems, which is an extension of Dieudonné theory (which classifies finite flat group schemes over k) [FL, 9.7].

In particular, if $\mathcal{G}$ killed by p, and $M = \mathrm{FL}(\mathcal{G})$, there is an exact sequence

$$0 \to (t_{\mathcal{G}^*}{}^\vee \to \sigma^{-1*}M \to t_{\mathcal{G}} \to 0$$

where σ denote the Frobenius on W. Moreover the linear map $1 \otimes \varphi_1 : M^1 \to \sigma^{-1*}M$ identifies M^1 with with $t^\vee_{\mathcal{G}^*}$. In particular

$$\dim_k M^1 = \dim_k \varphi^1(M_1) = \dim_k t_{\mathcal{G}^*}.$$

Now suppose that $\mathcal{G}$ is a *p-divisible group*, $T_p\mathcal{G}$ its Tate module, and $\mathcal{G}^*$ its Cartier dual. Write $V_p\mathcal{G} = T_p\mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then

$$D(\mathcal{G}) := D_{\mathrm{cris}}(T_p\mathcal{G})(1) \xrightarrow{\sim} \mathrm{Hom}_{G_K}(T_p\mathcal{G}^*, B_{\mathrm{cris}})$$

is a weakly admissible module whose associated graded is zero except in degrees $0, 1$. The lattice $T_p\mathcal{G} \subset V_p\mathcal{G}$ corresponds to a *strongly divisible lattice* $M \subset D_{\mathrm{cris}}(T_p\mathcal{G})$ with M/pM canonically isomorphic to $\mathrm{FL}(\mathcal{G}[p])$ as an object of $\mathrm{MF}^1_{\mathrm{tor}}$. In particular

$$\dim_{K_0} \mathrm{Fil}^1 D(\mathcal{G}) = \mathrm{rk}_{\mathcal{O}_{K_0}} \mathrm{Fil}^1 M = \dim_k M^1 = \dim_k t_{\mathcal{G}^*[p]}. \tag{5.3.3}$$

Theorem (5.3.4). *Suppose $K = K_0$ and $p > 2$. Then $D^{\mathrm{fl}}_{V_{\mathbb{F}}}$ is formally smooth. If $\mathcal{G}_{\mathbb{F}}$ denote the unique finite flat model of $V_{\mathbb{F}}$, $\mathcal{G}^*_{\mathbb{F}}$ denotes its Cartier dual, and $t_{\mathcal{G}_{\mathbb{F}}}$ denotes the tangent space of $\mathcal{G}_{\mathbb{F}}$, then*

$$\dim_{\mathbb{F}} D^{\mathrm{fl}}_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) = 1 + \dim_{\mathbb{F}} t_{\mathcal{G}_{\mathbb{F}}} \dim_{\mathbb{F}} t_{\mathcal{G}^*_{\mathbb{F}}}.$$

Proof. Let $M_{\mathbb{F}}$ in $\mathrm{MF}^1_{\mathrm{tor}}$ denote the object corresponding to $V_{\mathbb{F}}$. Then $M_{\mathbb{F}}$ is naturally an $\mathbb{F}$-vector space by the full faithfulness of (5.2.1). Let $D_{M_{\mathbb{F}}}$ denote the groupoid over $\mathfrak{AR}_{W(\mathbb{F})}$ such that $D_{M_{\mathbb{F}}}(A)$ is the category of objects M_A in $\mathrm{MF}^1_{\mathrm{tor}}$ equipped with an action of A, such that such that M_A is a finite free A-module and M^1 is an A-module direct summand, and an isomorphism $M_A \otimes_A \mathbb{F} \xrightarrow{\sim} M_{\mathbb{F}}$ in $\mathrm{MF}^1_{\mathrm{tor}}$.

Lemma (5.3.5). *The Fontaine-Laffaille functor of (5.2.1) induces an equivalence of categories*

$$\mathrm{FL} : D_{M_{\mathbb{F}}} \xrightarrow{\sim} D^{\mathrm{fl}}_{V_{\mathbb{F}}}.$$

Proof. If M_A is in $D_{M_{\mathbb{F}}}(A)$ let V_A be its image under FL. As FL is exact V_A is finite free over A. Indeed, for any finite A-module N we have

$$\mathrm{FL}(M_A) \otimes_A N \xrightarrow{\sim} \mathrm{FL}(M_A \otimes_A N). \tag{5.3.4}$$

This is obvious if N is free over A, and the general case follows by choosing a presentation of N by free modules. As the right hand side is an exact functor in N, so is the left hand side, which shows that V_A is a free A-module. Applying (5.3.4) with $N = \mathbb{F}$, one also sees that V_A is naturally a deformation of $V_{\mathbb{F}}$.

Conversely if V_A is in $D^{\mathrm{fl}}_{V_{\mathbb{F}}}$, and $M_A \in \mathrm{MF}^1_{\mathrm{tor}}$ satisfies $\mathrm{FL}(M_A) \xrightarrow{\sim} V_A$, then M_A is an A-module by the full faithfulness of FL, and since $\mathrm{MF}^1_{\mathrm{tor}}$ is abelian, the same argument as above shows that M_A is free over A, and that $M^1_A \subset M_A$ is an A-module direct summand □

(5.3.6) We return to the proof of (5.3.4). By the lemma to prove the formal smoothness of $D^{\mathrm{fl}}_{V_{\mathbb{F}}}$ it suffices to prove the formal smoothness of $D_{M_{\mathbb{F}}}$. Let A be in $\mathfrak{AR}_{W(\mathbb{F})}$, $I \subset A$ an ideal and $M_{A/I}$ in $D_{M_{\mathbb{F}}}(A/I)$. We have to show that $M_{A/I}$ lifts to an object of $D_{M_{\mathbb{F}}}(A)$.

First choose a lifting of the A/I-module $M_{A/I}$ to an A-module M_A, and a submodule $M^1_A \subset M_A$ which is a direct summand and lifts $M^1_{A/I}$. Next let $L_{A/I} = \varphi^1(M^1_{A/I})$, and choose a lift of $L_{A/I}$ to a direct summand $L_A \subset M_A$ and a lift of the composite

$$\varphi^*(M_A) \to \varphi^*(M_{A/I}) \overset{1\otimes\varphi^1}{\to} M_{A/I}$$

to L_A. Finally one checks that the map $1\otimes p\varphi^1 : \varphi^*(M^1_A) \to M_A$ admits an extension to a map $\varphi^*(M_A) \to M_A$ which induces the given map $\varphi^*(M_{A/I}) \to M_{A/I}$.

(5.3.7) It remains to check that the dimension of $D^{\mathrm{fl}}_{V_{\mathbb{F}}}$ has the claimed dimension. We could do this directly by computing the dimension of $D_{M_{\mathbb{F}}}(\mathbb{F}[\epsilon])$, however it is simpler to use our computation of the dimension of the generic fibre of $R^{\mathrm{fl}}_{V_{\mathbb{F}}}$.

Let $x : R^{\mathrm{fl}}_{V_{\mathbb{F}}}[1/p] \to E$ be a surjective map, where E is a finite extension of $\mathbb{Q}_p$. Write V_x for the corresponding crystalline representation. Since we already know that $R^{\mathrm{fl}}_{V_{\mathbb{F}}}$ is smooth, we need to compute the dimension of the tangent space of $\mathcal{R}^{\mathrm{fl}}_{V_{\mathbb{F}}}[1/p]$ at x. Let $D_x = D_{\mathrm{cris}}(V_x)$.

Using (5.2.6) and (5.3.3)one sees that this dimension is[1]

$$1 + \dim_E(D_x/\mathrm{Fil}^1 D_x)\dim_E \mathrm{Fil}^1 D_x = 1 + \dim_{\mathbb{F}} t_{\mathcal{G}} \dim_{\mathbb{F}} t_{\mathcal{G}^*}.$$

□

Exercises:

Exercise 1: Formulate and prove Proposition (5.2.1) for framed deformations.

Exercise 2: Check that the two constructions used to define the isomorphism

$$\mathrm{Ext}^1_{w.adm}(\mathbf{1}, D) \xrightarrow{\sim} H^1(C^\bullet(D))$$

in (5.2.3) are well defined and inverse.

Exercise 3: Give an explicit description of the isomorphism

$$\mathrm{Ext}^1_{w.adm}(\mathbf{1}, \mathrm{ad}D) \xrightarrow{\sim} \mathrm{Ext}^1_{w.adm}(D, D)$$

used in (5.2.6).

Exercise 4: Show that the functor D_{D_E} in (5.2.5) is formally smooth.

Exercise 5: Show that the category $\mathrm{MF}^1_{\mathrm{tor}}$ is equivalent to the category of *finite Honda systems* defined in Conrad's lectures. This is slightly tricky [FL, 9.4].

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[1] The computation becomes slightly easier if $E = W(\mathbb{F})[1/p]$, which we may assume as $R^{\mathrm{fl}}_{V_{\mathbb{F}}}$ is formally smooth.

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