INFORMATION ON WUNSCH/MAZZEO LECTURE COURSE

This course will focus on both older and newer tools in microlocal analysis and their application to the study of propagation phenomena for various classes of PDE. We begin with a brief introduction to the classical theory of pseudodifferential operators. Rather than giving a complete development, we treat it as a 'black box', describing the main objects and their properties, but leaving students who have not seen this before to read more about it on their own (ample references will be given).

The first main goal will be the derivation of microlocal energy estimates. These are used to prove Hörmander's celebrated theorem on propagation of singularities for operators of real principal type. Specializing to the wave operator, we show how this propagation theorem has some very strong consequences for the spectral theory of the associated Laplacian, and establishes the amazing link between this spectral theory and dynamical properties of the geodesic flow. Explaining this, and in particular giving a precise statement and (sketch of) proof of the Duistermaat-Guillemin trace theorem will require some discussion of parametrix constructions in the class of Fourier integral operators.

The next set of topics involves the introduction of a semiclassical parameter, which may be Planck's constant itself in problems of a quantum-mechanical nature or which may be interpreted as the inverse of a spectral parameter, which is useful in a wide variety of other applications. The formalization of all this is the subject of semiclassical microlocal analysis. We apply this to give an introduction to quantum chaos, beginning with a proof of Schnirelman and Zelditch's result on quantum ergodicity. (Coverage of these topics may be abbreviated, depending on time constraints.)

The third (and final) pseudofferential calculus we discuss is Melrose's *scattering calculus*. The applications here are to global problems in scattering theory on asymptotically Euclidean (or more generally, asymptotically conic) manifolds. The propagation here occurs along the sphere at infinity, and the analogue of Hörmander's theorem leads to a more geometric formulation (and proof) of the classical Mourre estimate in scattering theory. We describe the Melrose-Zworski construction of the scattering matrix, and sketch the far-reaching generalization of this by Vasy in N-body scattering theory.

Syllabus:

(1) Introduction: energy estimates by commutators; Morawetz's inequality.
(2) What is microlocal analysis? Crash course in pseudodifferential operators.
(3) Elliptic parametrices, wavefront sets, boundedness on Sobolev spaces.
(4) Propagation of singularities, positive commutator arguments.
(5) Trace theorems: Poisson relation, Guillemin-Duistermaat theorem (statement).
(6) Conormal distributions; Lagrangian distributions.
(7) Introduction to Fourier Integral Operators.
(8) Parametrix for the wave operator. Weyl estimate; 'Proof' of trace theorem.

(9) Morawetz estimate revisited; local smoothing for Schrödinger operators.
(10) (Time permitting:) The semiclassical calculus; defect measures, quantum ergodicity and quantum chaos.
(11) The scattering calculus; introduction to scattering theory on asymptotically Euclidean manifolds: resolvent, plane waves, and scattering matrix.
(12) More on scattering theory; resonances.
(13) Connection to time-dependent theory, local energy decay, Strichartz estimates.
(14) Structure of the scattering operator; brief introduction to N-body scattering.
(15) Scattering on other open manifolds.

Reading list:

- L. Hörmander, *Linear differential operators, Vol.1* (for background in distribution theory).
- R. Melrose *Lecture notes on microlocal analysis* (a readable albeit incomplete account of the pseudodifferential calculus and (especially) wavefront sets) available at

  `www-math.mit.edu/~rbm/Lecture_Notes.html`
- A. Grigis, J. Sjöstrand *Microlocal analysis of differential operators* (a terse but rather complete introduction to pseudodifferential and Fourier Integral operators).
- M. Shubin *Pseudodifferential operators and spectral theory* (a more leisurely introduction to pseudodifferential operators, focusing on their applications in spectral theory)
- A. Martinez *Semiclassical analysis* (a very short, elementary book covering both classical and semiclassical calculi of pseudodifferential operators, somewhat geared toward the use of the FBI transform).
- R. Melrose *Geometric scattering theory* (a short and entertaining introduction to the subject, with no proofs).
- L.C. Evans and M. Zworski, *Lectures on semiclassical analysis* (an elementary introduction to semiclassics, with applications primarily in spectral theory) available from

  `www.math.berkeley.edu/~zworski`
- M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical limit* (a speedy introduction, with some sophisticated applications, e.g. to quantum tunnelling).