9. Lecture #8: The periodic NLS

So far we only talked about the Schrödinger equation on $\mathbb{R}^n$, and one can certainly define this equation in more general manifolds M by replacing the usual Laplacian Δ with the Laplace-Beltrami operator Δ_M. In recent years there has been a flurry of activity concerning well-posedness and blow up of the IVP (1) on different manifolds, see for example in the setting of compact Riemannian manifolds $(M, \mathbf{g})$ [8, 7, 17, 18]. In this case the conclusions are generally weaker than those in Euclidean spaces: there is no scattering to linear solutions, or some other type of asymptotic control of the nonlinear evolution as $t \to \infty$. Moreover, in certain cases such as the spheres $\mathbb{S}^n$, the well-posedness theory requires sufficiently subcritical nonlinearities, due to concentration of certain spherical harmonics, see [16]. The situation is different when we are in the setting of symmetric spaces of noncompact type[38]. The simplest such spaces are the hyperbolic spaces $\mathbb{H}^n$, $n \geq 2$. On hyperbolic spaces one can in fact prove *stronger* theorems than on Euclidean spaces. For the linear flow one can exhibit a larger class of global in time Strichartz estimates [1, 45], (for radial functions these were already proved in [3, 4, 5, 64]). For the nonlinear flow with $N(u) = u|u|^{p-1}$ one can prove noneuclidean Morawetz inequalities, and scattering in H^1 in the full subcritical range $p \in (1, 1+4/(n-2))$, [45], These stronger theorems are possible because of the more robust geometry at infinity of noncompact symmetric spaces compared to Euclidean spaces; for example, the scattering result for the nonlinear Schrödinger equation can be interpreted as the absence of long range effects of the nonlinearity.

Here we cannot clearly address all the work mentioned above, but instead we will consider the spacial case of the periodic NLS (1), or in other words the problem on the torus $\mathbb{T}^n$. The first work on this goes back to Bourgain [8]. Since we already learned from Lecture #1 that the first step to take is to analyze in the best possible way the linear problem, we will do this now. We cannot hope to prove Strichartz estimates starting from a dispersive estimate since there is no dispersion here in the sense introduced in Lecture #1. This is because the periodic condition at the boundary doesn't no allow the solution to decay in time. So one needs to use a different analysis. We start by saying that the torus that will be considered here is the one on which

$$\widehat{\Delta_{\mathbb{T}^n} f}(k) = (\sum_{i=1}^{n} k_i^2)\hat{f}(k).$$

The situation is very different if instead one consider general[39] tori $\tilde{\mathbb{T}}^n$ where

$$\widehat{\Delta_{\tilde{\mathbb{T}}^n} f}(k) = (\sum_{i=1}^{n} a_i^2 k_i^2)\hat{f}(k), \tag{174}$$

where $a_i^2 > 0$ for $i = 1, .., n$. in this case the theorems below are either not proved or the results are much weaker, [15].

Let's go back to $\mathbb{T}^n$. We will show here only one bilinear estimate that is particularly instructive:

Theorem 9.1. *Assume ϕ_i has Fourier transform supported at frequency N_i for $i = 1, 2$ and that $S(t)\phi_i$ is the linear solution for the linear IVP (4) on $\mathbb{T}^2$ with data ϕ_i . Then if $N_1 \geq N_2$, for any $\epsilon > 0$ we have*

$$\|\chi(t)S(t)\phi_1\chi(t)\overline{S(t)\phi_2}\|_{L^2_t L^2_{\mathbb{T}^2}} \lesssim N_2^{\epsilon}\|\phi_1\|_{L^2}\|\phi_2\|_{L^2}, \tag{175}$$

where $\chi(t)$ is a smooth cut off function in time near $t = 0$.

[38]The symmetric spaces of noncompact type are simply connected Riemannian manifolds of nonpositive sectional curvature, without Euclidean factors, and for which every geodesic symmetry defines an isometry.

[39]These are also called *irrational tori.*

This theorem is only part of a more general conjecture of Bourgian [8] (see also [39]) that we now recall. Assume that ϕ is supported at frequency N and assume that

$$\|\chi(t)S(t)\phi\|_{L^r_t L^r_{\mathbb{T}^n}} \leq K(n,r,N)\|\phi\|_{L^2_{\mathbb{T}^n}},$$

then we have the following estimates for $K(n,r,N)$

Conjecture 9.2. *With the above assumptions*

$$K(n,r,N) \quad < \quad C_p \quad \textit{for} \quad r < \frac{2(n+2)}{n} \tag{176}$$

$$K(n,r,N) \quad << \quad N^{\epsilon} \quad \textit{for} \quad r = \frac{2(n+2)}{n} \tag{177}$$

$$K(n,r,N) \quad < \quad C_p N^{\frac{n}{2}-\frac{n+2}{r}} \quad \textit{for} \quad r > \frac{2(n+2)}{n} \tag{178}$$

For a partial proof of this conjecture see [8].

Remark 9.3. It is important to note that (175) can also be read as the $L^4_{[-1,1]}L^4_x$ Strichartz estimate, since in fact $(4,4)$ is an admissible pair in this case. Based on this and on the techniques to prove well-posedness in Lecture #3, we can immediately deduce for example that for the H^1 subcritical IVP (1) in $\mathbb{T}^2$ l.w.p. is available for $0 < s \leq 1$ when the nonlinearity is not algebraic and $0 < s$ when it is. Also it should be stressed that l.w.p. for $s = 0$ cannot be proved using (175) because the loss of regularity represented by N^{ϵ}. It should be said that this loss can be proved to be even smaller, of the order of $\log(N)$, see footnote at the end of the lecture.

Problem 9.4. *Prove that there exists ϕ such that*

$$\|\chi(t)S(t)\phi\|_{L^4_t L^4_{\mathbb{T}^2}} \sim \log(N)\|\phi\|_{L^2_{\mathbb{T}^2}},$$

(see [39]*).*

The proof of (175) is based on some number theoretic facts that we recall in the following three lemmas; see also related estimates in the work of Bourgain [7, 15] and [35].

The following lemma is known as **Pick's Lemma** [66]:

Lemma 9.5. *Let Ar be the area of a simply connected lattice polygon. Let E denote the number of lattice points on the polygon edges and I the number of lattice points in the interior of the polygon. Then*

$$Ar = I + \frac{1}{2}E - 1.$$

Lemma 9.6. *Let $\mathcal{C}$ be a circle of radius R. If γ is an arc on $\mathcal{C}$ of length $|\gamma| < \left(\frac{3}{4}R\right)^{1/3}$, then γ contains at most 2 lattice points.*

Proof. We prove the lemma by contradiction. Assume that there are 3 lattice points P_1, P_2 and P_3 on an arc $\gamma = AB$ of $\mathcal{C}$, and denote by $T(P_1,P_2,P_3)$ the triangle with vertices P_1, P_2 and P_3. Then, by Lemma 9.5 we have

$$\text{Area of } T(P_1,P_2,P_3) = I + \frac{1}{2}E - 1 \geq I + \frac{3}{2} - 1 = I + \frac{1}{2} \geq \frac{1}{2}.$$

We shall prove that under the assumption that $|\gamma| < \left(\frac{3}{4}R\right)^{1/3}$, then

$$\text{Area of } T(P_1,P_2,P_3) < \frac{1}{2}, \tag{179}$$

hence γ must contain at most two lattice points.

We observe that (see Figure 1)

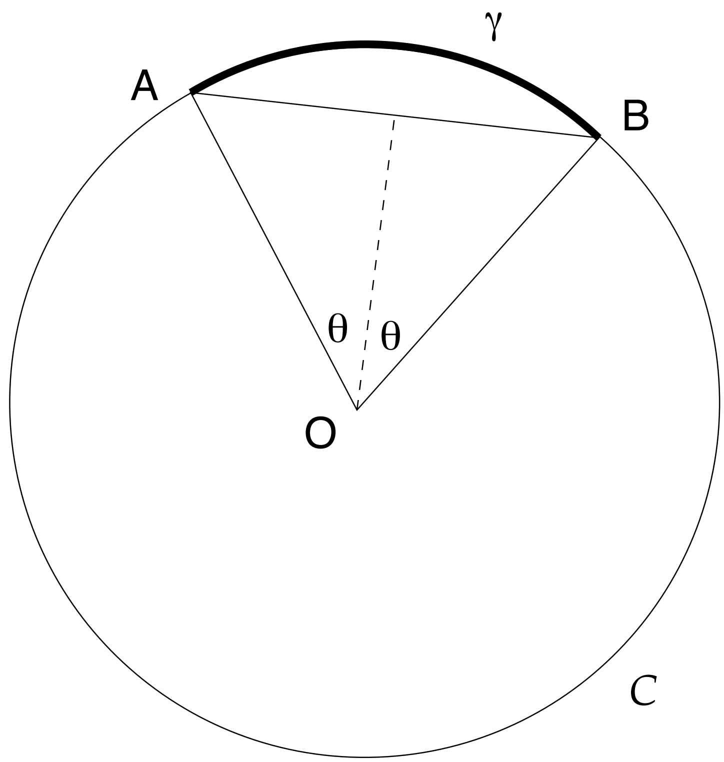

FIGURE 1. Triangle area.

$$\text{Area of the sector } ABO = R^2\theta,$$
$$\text{Area of the triangle } ABO = R^2 \sin\theta\cos\theta.$$

Hence, for any P_1, P_2, P_3 on γ we have

$$\text{Area of } T(P_1, P_2, P_3) \le R^2\theta - R^2\sin\theta\cos\theta = R^2(\theta - \frac{1}{2}\sin(2\theta)). \tag{180}$$

One can easily check that

$$\theta - \frac{1}{2}\sin(2\theta) \le \frac{2}{3}\theta^3. \tag{181}$$

Thus (180), (181) and the fact that $|\gamma| = 2R\theta$ imply that

$$\text{Area of } T(P_1, P_2, P_3) \le \frac{2}{3}R^2\theta^3 = \frac{1}{12}R^2(|\gamma|R^{-1})^3 < \frac{1}{2},$$

where to obtain the last inequality we used the assumption that $|\gamma| < \left(\frac{3}{4}R\right)^{1/3}$. Therefore (179) is proved.

□

Also we recall the following result of Gauss, see, for example [46]

Lemma 9.7. *Let K be a convex domain in $\mathbb{R}^2$. If*

$$N(\lambda) = \#\{\mathbb{Z}^2 \cap \lambda K\},$$

then, for $\lambda >> 1$

$$N(\lambda) = \lambda^2|K| + O(\lambda),$$

where $|K|$ denotes the area of K and $\#A$ denotes the number of points of a set A.

We are now ready for the proof of Theorem 9.1

Proof. Let ψ be a positive even Schwartz function such that $\psi = \hat{\chi}$. Then we have (here we use for simplicity $\int dk = \sum_k$)

$$B = \|\chi(t)(S(t)\phi_1)\ \chi(t)\overline{(S(t)\phi_2)}\|_{L^2_t L^2_x}$$

$$= \left\| \int_{k=k_1+k_2,\, \tau=\tau_1+\tau_2} \widehat{\phi_1}(k_1)\widehat{\phi_2}(k_2)\psi(\tau_1 - k_1^2)\ \psi(\tau_2 - k_2^2)\ dk_1\ dk_2\ d\tau_1\ d\tau_2 \right\|_{L^2_\tau L^2_k}$$

$$\lesssim \left\| \left(\int_{k=k_1+k_2} \widetilde{\psi}(\tau - k_1^2 - k_2^2)\ dk_1\ dk_2 \right)^{1/2} \times \right.$$

$$\left. \times \left(\int_{k=k_1+k_2} \widetilde{\psi}(\tau - k_1^2 - k_2^2)\ |\widehat{\phi}_1(k_1)|^2\ |\widehat{\phi}_2(k_2)|^2\ \ dk_1\ dk_2 \right)^{1/2} \right\|_{L^2_\tau L^2_k}, \tag{182}$$

where to obtain (182) we used Cauchy-Schwartz and the following definition of $\widetilde{\psi} \in \mathcal{S}$

$$\int_{\tau=\tau_1+\tau_2} \psi(\tau_1 - k_1^2)\ \psi(\tau_2 - k_2^2)\ d\tau_1 d\tau_2 = \widetilde{\psi}(\tau - k_1^2 - k_2^2).$$

An application of Hölder gives us the following upper bound on (182)

$$M \left\| \left(\int_{k=k_1+k_2} \widetilde{\psi}(\tau - k_1^2 - k_2^2)\ |\widehat{\phi}_1(k_1)|^2\ |\widehat{\phi}_2(k_2)|^2\ \ dk_1\ dk_2 \right)^{1/2} \right\|_{L^2_\tau L^2_k}, \tag{183}$$

where

$$M = \left\| \int_{k=k_1+k_2} \widetilde{\psi}(\tau - k_1^2 - k_2^2)\ dk_1\ dk_2 \right\|^{1/2}_{L^\infty_\tau L^\infty_k}.$$

Now by integration in τ followed by Fubini in k_1, k_2 and two applications of Plancharel we have

$$\left\| \left(\int_{k=k_1+k_2} \widetilde{\psi}(\tau - k_1^2 - k_2^2)\ |\widehat{\phi}_1(k_1)|^2\ |\widehat{\phi}_2(k_2)|^2\ \ dk_1\ dk_2 \right)^{1/2} \right\|_{L^2_{k,\tau}}$$

$$\lesssim \|\phi_1\|_{L^2_x} \|\phi_2\|_{L^2_x},$$

which combined with (182), (183) gives

$$B \lesssim M \|\phi_1\|_{L^2_x} \|\phi_2\|_{L^2_x}. \tag{184}$$

We find an upper bound on M as follows:

$$M \lesssim \left(\sup_{\tau,k} \#S \right)^{\frac{1}{2}}, \tag{185}$$

where

$$S = \{k_1 \in \mathbb{Z}^2 \mid |k_1| \sim N_1,\ |k - k_1|\ \sim N_2,\ |k|^2 - 2k_1 \cdot (k - k_1) = \tau + O(1)\},$$

and $\#A$ denotes the number of lattice points of a set A.

For notational purposes, let us rename $k_1 = z$, that is

$$S = \{z \in \mathbb{Z}^2 \mid |z| \sim N_1,\ |k - z|\ \sim N_2,\ |k|^2 + 2|z|^2 - 2k \cdot z = \tau + O(1)\}.$$

Let z_0 be an element of S i.e.

$$|z_0| \sim N_1,\ \ |k - z_0| \sim N_2, \tag{186}$$

and

$$|k|^2 + 2|z_0|^2 - 2k \cdot z_0 = \tau + O(1). \tag{187}$$

In order to obtain an upper bound on $\#S$, we shall count the number of l's $\in \mathbb{Z}^2$ such that $z_0 + l \in S$ where z_0 satisfies (186) - (187). Thus such l's must satisfy

$$|z_0 + l| \sim N_1, \;\; |z_0 + l - k| \sim N_2, \tag{188}$$

and

$$|k|^2 + 2\,|z_0 + l|^2 - 2k \cdot (z_0 + l) = \tau + O(1). \tag{189}$$

However by (187) we can rewrite the left hand side of (189) as follows

$$\begin{aligned} &|k|^2 + 2\,|z_0 + l|^2 - 2k \cdot (z_0 + l) \\ &= |k|^2 + 2|z_0|^2 + 2\,|l|^2 + 4z_0 \cdot l - 2k \cdot z_0 - 2k \cdot l \\ &= \tau + O(1) + 2\,|l|^2 + 4z_0 \cdot l - 2k \cdot l. \end{aligned}$$

Therefore (189) holds if

$$|l|^2 + 2l \cdot (z_0 - \frac{k}{2}) = O(1). \tag{190}$$

Moreover, (186) and (188) yield

$$|l| = |l + z_0 - k - z_0 + k| \lesssim N_2 + N_2,$$

that is

$$|l| \lesssim N_2. \tag{191}$$

Finally we observe that (186) together with the assumption that $N_1 >> N_2$ implies that

$$N_1 \sim N_1 - N_2 \sim \left| \left| \frac{z_0}{2} - \frac{k}{2} \right| - \left| \frac{z_0}{2} \right| \right| \leq \left| z_0 - \frac{k}{2} \right| \leq \left| \frac{z_0}{2} - \frac{k}{2} \right| + \left| \frac{z_0}{2} \right| \sim N_2 + N_1 \lesssim N_1,$$

i.e.

$$\left| z_0 - \frac{k}{2} \right| \sim N_1. \tag{192}$$

Hence, it suffices to count the $l's \in \mathbb{Z}^2$ satisfying (190) and (191) where z_0 is such that (192) holds.

Let $w = (a, b)$ denote the vector $z_0 - \frac{k}{2}$. Thus we need to count the number of points in the set A

$$A = \{l \in \mathbb{Z}^2 : \; \big|\, |l|^2 + 2l \cdot w \,\big| = O(1), \; |l| \lesssim N_2, \; |w| \sim N_1\}, \tag{193}$$

or equivalently,

$$\begin{aligned} &A = \\ &\{(x, y) \in \mathbb{Z}^2 : \; \left|x^2 + y^2 + 2(ax + by)\right| \leq c, \; x^2 + y^2 \leq (\sigma_2 N_2)^2, \; a^2 + b^2 \sim N_1^2\}, \end{aligned} \tag{194}$$

for some $c, \sigma_2 > 0$. Let $\mathcal{C}_-$, $\mathcal{C}_+$ be the following circles,

$$\begin{aligned} \mathcal{C}_- &: \; (x + a)^2 + (y + b)^2 = -c + (a^2 + b^2) \\ \mathcal{C}_+ &: \; (x + a)^2 + (y + b)^2 = c + (a^2 + b^2) \end{aligned}$$

and for any integer n, let $\mathcal{C}_n$ be the circle

$$\mathcal{C}_n : \; (x + a)^2 + (y + b)^2 = n + (a^2 + b^2).$$

Finally, let $\mathcal{D}$ denote the disk

$$\mathcal{D}: \; x^2 + y^2 \le (\sigma_2 N_2)^2.$$

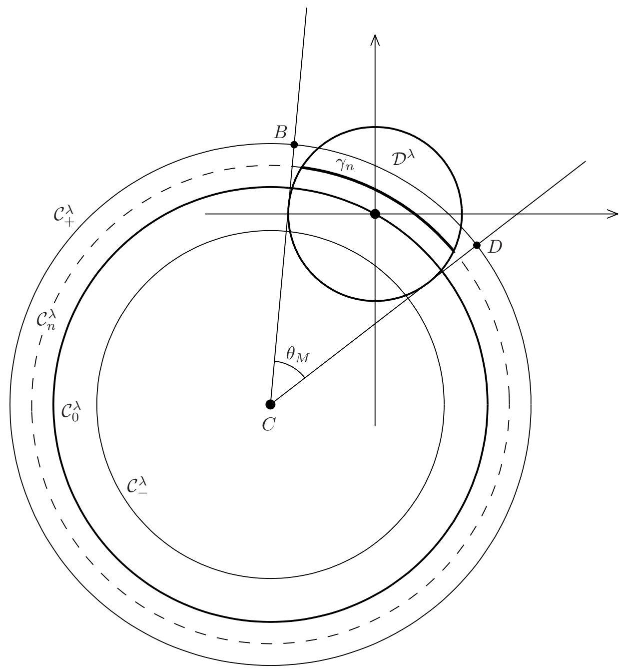

FIGURE 2. Circular sector (here ignore λ).

We need to count the number of lattice points inside $\mathcal{D}$ that are on arcs of circles $\mathcal{C}_n$, with

$$-c \le n \le c.$$

Precisely, the total number of lattice point in A can be bounded from above by

$$2c \times \#(\mathcal{C}_n \cap \mathcal{D}). \tag{195}$$

Denote by γ_n the arc of circle $\mathcal{C}_n$ which is contained in $\mathcal{D}$. Notice that (see Figure 2)

$$|\gamma_n| \le R_M \theta_M \tag{196}$$

where $R_M = \sqrt{c + \sigma_1 N_1^2}$ for some constant $\sigma_1 > 0$, and θ_M is the angle between the line segment CB and CD, which lie along the tangent lines from $C = (-a, -b)$ to the circle $x^2+y^2 = (\sigma_2 N_2)^2$. Hence,

$$\sin\theta_M \le \sigma \frac{N_2}{N_1},$$

for some constant $\sigma > 0$. Since $N_1 >> N_2$, we can assume that $\sin\theta_M > \frac{1}{2}\theta_M$. Hence,

$$\theta_M < 2\sigma \frac{N_2}{N_1}. \tag{197}$$

In order to count efficiently the number of lattice points on each γ_n, we distinguish two cases based on the application of Lemma 9.6.

Case 1: $2\sigma \frac{N_2}{N_1} < \left(\frac{3}{4}\right)^{\frac{1}{3}} R_M^{-\frac{2}{3}}$**.**
In this case (196)-(197) guarantee that the hypothesis of Lemma 9.6 is satisfied by each arc of circle γ_n. Hence, on each γ_n there are at most two lattice points.

Case 2: $2\sigma\frac{N_2}{N_1} \geq \left(\frac{3}{4}\right)^{\frac{1}{3}} R_M^{-\frac{2}{3}}$**.**
In this case we approximate the number of lattice points on γ_n by the number[40] of lattice points on $\mathcal{C}_n$ (see for example [6, 8]):

$$\#\mathcal{C}_n \lesssim R_M^{\epsilon} \sim (N_1)^{\epsilon} \lesssim (N_2)^{3\epsilon} \tag{198}$$

for any $\epsilon > 0$.

Combining the estimate in (195), **Case 1** and **Case 2** we conclude that

$$\#S \lesssim 1 + N_2^{\epsilon},$$

for any $\epsilon > 0$. Since $N_2 \geq 1$, together with (185), this implies that

$$M \lesssim N_2^{\epsilon},$$

for all positive ϵ's. Hence (175) follows.

□

[40]Actually by Gauss Theorem one can get a even better logarithmic estimate in terms of the radius.