8. Lecture #7:Global well-posedness for the $H^1(\mathbb{R}^n)$ critical NLS -Part II

We start by recalling the critical H^1 defocusing IVP for which we want to prove global well-posedness and scattering for large data:

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = |u|^4 u \\ u(0,x) = u_0(x). \end{cases} \tag{148}$$

where $u(t,x)$ is a complex-valued field in spacetime $\mathbb{R}_t \times \mathbb{R}^3_x$. This equation has as Hamiltonian,

$$E(u(t)) := \int \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{6}|u(t,x)|^6 \, dx. \tag{149}$$

We now outline the proof of Theorem 7.3, breaking it down into a number of smaller propositions.

8.1. **Zeroth stage: Induction on energy.**

The first observation is that in order to prove Theorem 7.3, it suffices to do so for Schwartz solutions. Indeed, once one obtains a uniform $L^{10}_{t,x}(I \times \mathbb{R}^3)$ bound for all Schwartz solutions and all compact I, one can then approximate arbitrary finite energy initial data by Schwartz initial data and use Lemma 7.8 to show that the corresponding sequence of solutions to (148) converge in the homogeneous Strichartz space $\dot{S}^1(I \times \mathbb{R}^3)$ to a finite energy solution to (148). We omit the standard details.

For every energy $E \geq 0$ we define the quantity $0 \leq M(E) \leq +\infty$ by

$$M(E) := \sup\{\|u\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)}\}$$

where $I_* \subset \mathbb{R}$ ranges over all compact time intervals, and u ranges over all Schwartz solutions to (148) on $I_* \times \mathbb{R}^3$ with $E(u) \leq E$. We shall adopt the convention that $M(E) = 0$ for $E < 0$. By the above discussion, it suffices to show that $M(E)$ is finite for all E.

In the argument of Bourgain [13] (see also [12]), the finiteness of $M(E)$ in the spherically symmetric case is obtained by an induction on the energy E; indeed a bound of the form

$$M(E) \leq C(E, \eta, M(E - \eta^4))$$

is obtained for some explicit $0 < \eta = \eta(E) \ll 1$ which does not collapse to 0 for any finite E, and this easily implies via induction that $M(E)$ is finite for all E. Our argument will follow a similar induction on energy strategy, however it will be convenient to run this induction in the contrapositive, assuming for contradiction that $M(E)$ can be infinite, studying the minimal energy E_{crit} for which this is true, and then obtaining a contradiction using the "induction hypothesis" that $M(E)$ is finite for all $E < E_{crit}$. This will be more convenient for us, especially as we will require more than one small parameter η.

We turn to the details. We assume for contradiction that $M(E)$ is not always finite. From Lemma 7.8 we see that the set $\{E : M(E) < \infty\}$ is open; clearly it is also connected and contains 0. By our contradiction hypothesis, there must therefore exist a *critical energy* $0 < E_{crit} < \infty$ such that $M(E_{crit}) = +\infty$, but $M(E) < \infty$ for all $E < E_{crit}$. One can think of E_{crit} as the minimal energy required to create a blowup solution. For instance, we have

Lemma 8.2 (Induction on energy hypothesis)**.** *Let $t_0 \in \mathbb{R}$, and let $v(t_0)$ be a Schwartz function such that $E(v(t_0)) \leq E_{crit} - \eta$ for some $\eta > 0$. Then there exists a Schwartz global solution $v : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{C}$ to (148) with initial data $v(t_0)$ at time $t = t_0$ such that $\|v\|_{L^{10}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \leq M(E_{crit} - \eta) = C(\eta)$. Furthermore we have $\|v\|_{\dot{S}^1(\mathbb{R}\times\mathbb{R}^3)} \leq C(\eta)$.*

Indeed, this Lemma follows immediately from the definition of E_{crit} and Theorem 4.8. As in the argument in [13], we will need a small parameter $0 < \eta = \eta(E_{crit}) \ll 1$ depending on E_{crit}.

In fact, our argument is somewhat lengthy and we will actually use *seven* such parameters

$$1 \gg \eta_0 \gg \eta_1 \gg \eta_2 \gg \eta_3 \gg \eta_4 \gg \eta_5 \gg \eta_6 > 0.$$

Specifically, we will need a small quantity $0 < \eta_0 = \eta_0(E_{crit}) \ll 1$ assumed to be sufficiently small depending on E_{crit}. Then we need a smaller quantity $0 < \eta_1 = \eta_1(\eta_0, E_{crit}) \ll 1$ assumed sufficiently small depending on E_{crit}, η_0 (in particular, it may be chosen smaller than positive quantities such as $M(E_{crit} - \eta_0^{100})^{-1}$). We continue in this fashion, choosing each $0 < \eta_j \ll 1$ to be sufficiently small depending on all previous quantities $\eta_0, \ldots, \eta_{j-1}$ and the energy E_{crit}, all the way down to η_6 which is extremely small, much smaller than any quantity depending on $E_{crit}, \eta_0, \ldots, \eta_5$ that will appear in our argument. We will always assume implicitly that each η_j has been chosen to be sufficiently small depending on the previous parameters. We will often display the dependence of constants on a parameter, e.g. $C(\eta)$ denotes a large constant depending on η, and $c(\eta)$ will denote a small constant depending upon η. When $\eta_1 \gg \eta_2$, we will understand $c(\eta_1) \gg c(\eta_2)$ and $C(\eta_1) \ll C(\eta_2)$.

Since $M(E_{crit})$ is infinite, it is in particular larger than $1/\eta_6$. By definition of M, this means that we may find a compact interval $I_* \subset \mathbb{R}$ and a smooth solution $u : I_* \times \mathbb{R}^3 \to \mathbb{C}$ to (148) with $E_{crit}/2 \leq E(u) \leq E_{crit}$ so that u is ridiculously large in the sense that

$$\|u\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)} > 1/\eta_6. \tag{150}$$

We will show that this leads to a contradiction[31]. Although u does not actually blow up (it is assumed smooth on all of the compact interval I_*), it is still convenient to think of u as almost[32] blowing up in $L^{10}_{t,x}$ in the sense of (150). We summarize the above discussion with the following,

Definition 8.3. A *minimal energy blowup solution* of (148) is a Schwartz solution on a time interval I_* with energy[33],

$$\frac{1}{2}E_{crit} \leq E(u)(t) = \int \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{6}|u(t,x)|^6 \, dx \leq E_{crit} \tag{151}$$

and $L^{10}_{x,t}$ norm enormous in the sense of (150).

We remark that both conditions (150), (151) are invariant under the scaling (66) (though of course the interval I_* will be dilated by μ^2 under this scaling). Thus applying the scaling (66) to a minimal energy blowup solution produces another minimal energy blowup solution. Some of the proofs of the sub-propositions below will revolve around a specific frequency N; using this scale invariance, we can then normalize that frequency to equal 1 for the duration of that proof. (Different parts of the argument involve different key frequencies, but we will not run into problems because we will only normalize one frequency at a time).

Henceforth we will not mention the E_{crit} dependence of our constants explicitly, as all our constants will depend on E_{crit}. We shall need however to keep careful track of the dependence of our argument on $\eta_0, \ldots, \eta_6$. Broadly speaking, we will start with the largest η, namely η_0, and slowly "retreat" to increasingly smaller values of η as the argument progresses (such a retreat will for instance usually be required whenever the induction hypothesis Lemma 8.2 is

---

[31] Assuming, of course, that the parameters $\eta_0, \ldots, \eta_6$ are each chosen to be sufficiently small depending on previous parameters. It is important to note however that the η_j cannot be chosen to be small depending on the interval I_* or the solution u; our estimates must be uniform with respect to these parameters.

[32] For instance, u might genuinely blow up at some time $T_* > 0$, but I_* is of the form $I_* = [0, T_* - \varepsilon]$ for some very small $0 < \varepsilon \ll 1$, and thus u remains Schwartz on $I_* \times \mathbb{R}^3$.

[33] We could modify our arguments below to allow the assumption here $E(u) = E_{crit}$. For example, the arguments in the proof of Proposition 8.5 below also show that the function $\tilde{M}(s) := \sup_{E(u)=s}\{\|u\|_{L^{10}_{x,t}}\}$ is a nondecreasing function of s. On first reading, the reader may imagine $E(u) = E_{crit}$ in Definition 8.3.

invoked). However we will only retreat as far as η_5, not η_6, so that (150) will eventually lead to a contradiction when we show that

$$\|u\|_{L^{10}_{t,x}(I^* \times \mathbb{R}^3)} \leq C(\eta_0, \dots, \eta_5).$$

Together with our assumption that we are considering a minimal energy blowup solution u as in Definition 8.3, Sobolev embedding implies the bounds on kinetic energy

$$\|u\|_{L^\infty_t \dot{H}^1_x(I_* \times \mathbb{R}^3)} \sim 1 \tag{152}$$

and potential energy

$$\|u\|_{L^\infty_t L^6_x(I_* \times \mathbb{R}^3)} \lesssim 1 \tag{153}$$

(since our implicit constants are allowed to depend on E_{crit}). Note that we do not presently have any *lower* bounds on the potential energy, but see below.

Having displayed our preliminary bounds on the kinetic and potential energy, we briefly discuss the mass $\int_{\mathbb{R}^3} |u(t,x)|^2 \, dx$, which is another conserved quantity. Because of our *a priori* assumption that u is Schwartz, we know that this mass is finite. However, we cannot obtain uniform control on this mass using our bounded energy assumption, because the very low frequencies of u may simultaneously have very small energy and very large mass. Furthermore it is dangerous to rely too much on this conserved mass for this energy-critical problem as the mass is not invariant under the natural scaling (66) of the equation (indeed, it is super-critical with respect to that scaling). On the other hand, from (152) we know that the *high frequencies* of u have small mass:

$$\|P_{>M} u\|_{L^2(\mathbb{R}^3)} \lesssim \frac{1}{M} \text{ for all } M \in 2^{\mathbb{Z}}. \tag{154}$$

Thus we will still be able to use the concept of mass in our estimates as long as we restrict our attention to sufficiently high frequencies.

8.4. **First stage: Localization control on u.** We aim to show that a*minimal energy blowup solution* as in Definition 8.3 does not exist. Intuitively, as we already mentioned in Lecture #6, it seems reasonable to expect that a *minimal-energy blowup solution* should be "irreducible" in the sense that it cannot be decoupled into two or more components of strictly smaller energy that essentially do not interact with each other (i.e. each component also evolves via (148) modulo small errors), since one of the components must then also blow up, contradicting the minimal-energy hypothesis. In particular, we expect at every time that such a solution should be localized in both frequency and space.

The first main step in the proof of Theorem 7.3 is to make the above heuristics rigorous for our solution u. Roughly speaking, we would like to assert that at each time t, the solution $u(t)$ is localized in both space and frequency to the maximum extent allowable under the uncertainty principle (i.e. if the frequency is localized to $N(t)$, we would like to localize $u(t)$ spatially to the scale $1/N(t)$).

These sorts of localizations already appear for instance in the argument of Bourgain [13], [12], where the induction on energy argument is introduced. Informally[34], the reason that we can expect such localization is as follows. Suppose for contradiction that at some time t_0 the solution $u(t_0)$ can be split into two parts $u(t_0) = v(t_0) + w(t_0)$ which are widely separated in either space or frequency, and which each carry a nontrivial amount $O(\eta^C)$ of energy for some

[34]The heuristic that *minimal energy blowup solutions* should be strongly localized in both space and frequency has been employed in previous literature for a wide variety of nonlinear equations, including many of elliptic or parabolic type. Our formalizations of this heuristic, however, rely on the induction on energy methods of Bourgain and perturbation theory, as opposed to variational or compactness arguments. This last one indeed is the method used in [50, 51].

$\eta_5 \leq \eta \leq \eta_0$. Then by orthogonality we expect v and w to each have strictly smaller energy than u, e.g. $E(v(t_0)), E(w(t_0)) \leq E_{crit} - O(\eta^C)$. Thus by Lemma 8.2 we can extend $v(t)$ and $w(t)$ to all of $I_* \times \mathbb{R}^3$ by evolving the nonlinear Schrödinger equation (148) for v and w separately, and furthermore we have the bounds

$$\|v\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)}, \|w\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)} \leq M(E_{crit} - O(\eta^C)) \leq C(\eta).$$

Since v and w both solve (148) separately, and v and w were assumed to be widely separated, we thus expect $v + w$ to solve (148) *approximately.* The idea is then to use the perturbation theory from Lecture #6 to obtain a bound of the form $\|u\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)} \leq C(\eta)$, which contradicts (150) if η_6 is sufficiently small.

As recalled in Lecture #6, model example of this type of strategy occurs in Bourgain's argument [12], where substantial effort is invested in locating a "bubble" - a small localized pocket of energy - which is sufficiently isolated in physical space from the rest of the solution. One then removes this bubble, evolves the remainder of the solution, and then uses perturbation theory, augmented with the additional information about the isolation of the bubble, to place the bubble back in. We will use arguments similar to these in the sequel, but first we need instead to show that a solution of (148) which is sufficiently delocalized in frequency space is globally spacetime bounded. More precisely, we have:

Proposition 8.5 (Frequency delocalization implies spacetime bound)**.** *Let $\eta > 0$, and suppose there exists a dyadic frequency $N_{lo} > 0$ and a time $t_0 \in I_*$ such that we have the energy separation conditions*

(155) $$\|P_{\leq N_{lo}} u(t_0)\|_{\dot{H}^1(\mathbb{R}^3)} \geq \eta$$

and

(156) $$\|P_{\geq K(\eta) N_{lo}} u(t_0)\|_{\dot{H}^1(\mathbb{R}^3)} \geq \eta.$$

If $K(\eta)$ is sufficiently large depending on η, i.e.

$$K(\eta) \geq C(\eta)$$

then we have

(157) $$\|u\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)} \leq C(\eta).$$

Clearly the conclusion of Proposition 8.5 is in conflict with the hypothesis (150), and so we should now expect the solution to be localized in frequency for every time t. This is indeed the case:

Corollary 8.6 (Frequency localization of energy at each time)**.** *A minimal energy blowup solution of* (148) *(see Definition 8.3) satisfies: For every time $t \in I_*$ there exists a dyadic frequency $N(t) \in 2^{\mathbb{Z}}$ such that for every $\eta_5 \leq \eta \leq \eta_0$ we have small energy at frequencies $\ll N(t)$,*

(158) $$\|P_{\leq c(\eta) N(t)} u(t)\|_{\dot{H}^1} \leq \eta,$$

small energy at frequencies $\gg N(t)$,

(159) $$\|P_{\geq C(\eta) N(t)} u(t)\|_{\dot{H}^1} \leq \eta,$$

and large energy at frequencies $\sim N(t)$,

(160) $$\|P_{c(\eta) N(t) < \cdot < C(\eta) N(t)} u(t)\|_{\dot{H}^1} \sim 1.$$

Here $0 < c(\eta) \ll 1 \ll C(\eta) < \infty$ are quantities depending on η.

Informally, this Corollary asserts that at every given time t the solution u is essentially concentrated at a single frequency $N(t)$. Note however that we do not presently have any information as to how $N(t)$ evolves in time; obtaining long-term control on $N(t)$ will be a key objective of later stages of the proof.

Proof. For each time $t \in I_*$, we define $N(t)$ as

$$N(t) := \sup\{N \in 2^{\mathbb{Z}} : \|P_{\leq N} u(t)\|_{\dot{H}^1} \leq \eta_0\}.$$

Since $u(t)$ is Schwartz, we see that $N(t)$ is strictly larger than zero; from the lower bound in (152) we see that $N(t)$ is finite. By definition of $N(t)$, we have

$$\|P_{\leq 2N(t)} u(t)\|_{\dot{H}^1} > \eta_0.$$

Now let $\eta_5 \leq \eta \leq \eta_0$. Observe that we now have (159) if $C(\eta)$ is chosen sufficiently large, because if (159) failed then Proposition 8.5 would imply that $\|u\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)} \leq C(\eta)$, contradicting (150) if η_6 is sufficiently small. In particular we have (159) for $\eta = \eta_0$. Since we also have (158) for $\eta = \eta_0$ by construction of $N(t)$, we thus see from (152) that we have (160) for $\eta = \eta_0$, which of course then implies (again by (152)) the same bound for all $\eta_5 \leq \eta \leq \eta_0$. Finally, we obtain (158) for all $\eta_5 \leq \eta \leq \eta_0$ if $c(\eta)$ is chosen sufficiently small, since if (158) failed then by combining it with (160) and Proposition 8.5 we would once again imply that $\|u\|_{L^{10}_{t,x}(I_* \times \mathbb{R}^3)} \leq C(\eta)$, contradicting (150). □

Having shown that any *minimal energy blowup solution* u must be localized in frequency at each time, we now turn to showing that such a u is also localized in physical space. This turns out to be somewhat more involved, although it still follows the same general strategy. We first borrow a useful trick from [13]; since u is Schwartz, we may divide the interval I_* into three consecutive pieces $I_* := I_- \cup I_0 \cup I_+$ where each of the three intervals contains a third of the $L^{10}_{t,x}$ density:

$$\int_I \int_{\mathbb{R}^3} |u(t,x)|^{10}\ dxdt = \frac{1}{3} \int_{I_*} \int_{\mathbb{R}^3} |u(t,x)|^{10}\ dxdt \text{ for } I = I_-, I_0, I_+.$$

In particular from (150) we have

$$\|u\|_{L^{10}_{t,x}(I \times \mathbb{R}^3)} \gtrsim 1/\eta_6 \text{ for } I = I_-, I_0, I_+. \tag{161}$$

Thus to contradict (150) it suffices to obtain $L^{10}_{t,x}$ bounds on just one of the three intervals I_-, I_0, I_+.

It is in the middle interval I_0 that we can obtain physical space localization; this shall be done in several stages. The first step is to ensure that the potential energy $\int_{\mathbb{R}^3} |u(t,x)|^6\ dx$ is bounded from below.

Proposition 8.7 (Potential energy bounded from below)**.** *For any minimal energy blowup solution of* (148) *(see Definition 8.3) we have for all* $t \in I_0$*,*

$$\|u(t)\|_{L^6_x} \geq \eta_1. \tag{162}$$

The proof of this proposition is inspired by a similar argument of Bourgain [13]. Using (162) and some simple Fourier analysis, we can thus establish the following concentration result:

Proposition 8.8 (Physical space concentration of energy at each time)**.** *Any minimal energy blowup solution of* (148) *satisfies: For every* $t \in I_0$*, there exists an* $x(t) \in \mathbb{R}^3$ *such that*

$$\int_{|x - x(t)| \leq C(\eta_1)/N(t)} |\nabla u(t,x)|^2\ dx \gtrsim c(\eta_1) \tag{163}$$

and

$$\int_{|x-x(t)|\leq C(\eta_1)/N(t)} |u(t,x)|^p \, dx \gtrsim c(\eta_1)N(t)^{\frac{p}{2}-3} \tag{164}$$

for all $1 < p < \infty$*, where the implicit constant can depend on* p*. In particular we have*

$$\int_{|x-x(t)|\leq C(\eta_1)/N(t)} |u(t,x)|^6 \, dx \gtrsim c(\eta_1), \tag{165}$$

Similar results were obtained in [13], [41] in the radial case; see also [9]. Informally, the above estimates assert that $u(t,x)$ is roughly of size $N(t)^{1/2}$ on the average when $|x - x(t)| \lesssim 1/N(t)$; observe that this is consistent with bounded energy (152) as well as with Corollary 8.6 and the uncertainty principle.

It turns out that in our argument, it is not enough to know that the energy concentrates at one location $x(t)$ at each time; we must also show that the energy is small at all other locations, where $|x - x(t)| \gg 1/N(t)$. The main tool for achieving this is

Proposition 8.9 (Physical space localization of energy at each time)**.** *For any minimal energy blowup solution of* (148) *we have for every* $t \in I_0$

$$\int_{|x-x(t)|>1/(\eta_2 N(t))} |\nabla u(t,x)|^2 \, dx \lesssim \eta_1. \tag{166}$$

The proof follows a similar strategy to that used to prove Corollary 8.5; the main difference is that we now consider spatially separated components of u rather than frequency separated components, and instead of using multilinear Strichartz estimates to establish the decoupling of these components, we shall rely instead on approximate finite speed of propagation and on the pseudoconformal identity.

To summarize, at each time t we have a location $x(t)$, around which the kinetic and potential energy are large, and away from which the kinetic energy is small (and one can also show the potential energy is small, although we will not need this). From this and a little Fourier analysis we obtain an important conclusion:

Proposition 8.10 (Reverse Sobolev inequality)**.** *Assuming* u *is a minimal energy blowup solution (and hence* (151)*,* (158)*-*(166) *hold), we have that for every* $t_0 \in I_0$*, any* $x_0 \in \mathbb{R}^3$*, and any* $R \geq 0$*,*

$$\int_{B(x_0,R)} |\nabla u(t_0,x)|^2 \, dx \lesssim \eta_1 + C(\eta_1,\eta_2) \int_{B(x_0,C(\eta_1,\eta_2)R)} |u(t_0,x)|^6 \, dx \tag{167}$$

Thus, up to an error of η_1, we are able to control the kinetic energy locally by the potential energy[35]. This fact will be crucial in the interaction Morawetz portion of our argument when we have an error term involving the kinetic energy, and control of a positive term which involves the potential energy; the reverse Sobolev inequality is then used to control the former by the latter.

To summarize, the statements above tell us that any minimal energy blowup solution (Definition 8.3) to the equation (148) must be localized in both frequency and physical space at every time. We are still far from done: we have not yet precluded blowup in finite time (which would happen if $N(t) \to \infty$ as $t \to T_*$ for some finite time T_*), nor have we eliminated soliton or soliton-like solutions (which would correspond, roughly speaking, to $N(t)$ staying close to

[35]Note that this is a special property of the *minimal energy blowup solution*, reflecting the very strong physical space localization properties of such a solution; it is false in general, even for solutions to the free Schrödinger equation. Of course, Proposition 8.7 is similarly false in general, for instance for solutions of the free Schrödinger equation, the L^6_x norm goes to zero as $t \to \pm\infty$.

constant for all time t). To achieve this we need spacetime integrability bounds on u. Our main tool for this is a frequency-localized version of the interaction Morawetz estimate (116), to which we now turn.

8.11. **Second stage: Localized Morawetz estimate.** In order to localize the interaction Morawetz inequality, it turns out to be convenient to work at the "minimum" frequency attained by u.

We observe that

$$\|P_{c(\eta_0)N(t)<\cdot<C(\eta_0)N(t)}u(t)\|_{\dot H^1} \leq C(\eta_0)N(t)\|u\|_{L_t^\infty L_x^2}$$

Comparing this with (160) we obtain the lower bound

$$N(t) \geq c(\eta_0)\|u\|^{-1}_{L_t^\infty L_x^2}$$

for $t \in I_0$. Since u is Schwartz, the right-hand side is nonzero, and thus the quantity

$$N_{min} := \inf_{t\in I_0} N(t)$$

is strictly positive.

From (158) we see that the low frequency portion of the solution - where $|\xi| \leq c(\eta_0)N_{min}$ - has small energy; one might then hope to use Strichartz estimates to obtain some spacetime control on these low frequencies. However, we do not yet have much control on the high frequencies $|\xi| \geq c(\eta_0)N_{min}$, apart from the energy bounds (152) and (153) of course.

Our initial spacetime bound in the high frequencies is provided by the following interaction Morawetz estimate.

Proposition 8.12 (Frequency-localized interaction Morawetz estimate)**.** *Assuming u is a minimal energy blowup solution of* (148) *(and hence* (151), (158)-(167) *all hold), we have for all* $N_* < c(\eta_3)N_{min}$

$$\int_{I_0}\int |P_{\geq N_*}u(t,x)|^4 \, dxdt \lesssim \eta_1 N_*^{-3}. \tag{168}$$

Remark 8.13. The factor N_*^{-3} on the right-hand side of (168) is mandated by scale-invariance considerations (cf. (66)). The η_1 factor on the right side reflects our smallness assumption on N_*: if we think of N_* as being very small and then scale the solution so that $N_* = 1$, we are pushing the energy to very high frequencies so heuristically it's not unreasonable to expect the supercritical $L^4_{x,t}$ norm on the left hand side to be small.

Regarding the size of N_*: write for the moment $\tilde{c}(\eta_3)$ as the constant appearing in Corollary 8.6 with $\eta = \eta_3$. The constant $c(\eta_3)$ appearing in Proposition 8.12 is chosen so $c(\eta_3) \lesssim \tilde{c}(\eta_3)\cdot\eta_3$, hence at all times we know there is very little energy at frequencies below $\frac{N_*}{\eta_3}$, and (ignoring factors of N_* which can be scaled to 1) above frequency N_* there is very little (at most η_3/N_*) L^2 mass.

This small η_1 factor will be used to close a bootstrap argument in the proof of the important estimate on the movement of energy to very low frequencies.

The key thing about this estimate is that the right-hand side does not depend on I_0; thus for instance it is already useful in eliminating soliton or pseudosoliton solutions, at least for frequencies close to N_{min}. (Frequencies much larger than N_{min} still cause difficulty, and will be dealt with later in the argument). Proposition 8.12 roughly corresponds to the localized Morawetz inequality used by Bourgain [13], [12] and Grillakis [41]. The main advantage of (168) is that it is not localized to near the spatial origin, in contrast with the standard (30) Morawetz inequalities.

Although this proposition is based on the interaction Morawetz inequality developed in the references given above, there are significant technical difficulties in truncating that inequality to the high frequencies. As a consequence the proof of this proposition is somewhat involved. Also, we caution the reader that the above proposition is not proved as an *a priori* estimate; indeed the proof relies crucially on the assumption that u is a *minimal energy blowup solution* in the sense of 8.3, and in particular verifies the reverse Sobolev inequality (167).

Combining Proposition 8.12 with Proposition 8.8 gives us the following integral bound on $N(t)$.

Corollary 8.14. *For any minimal energy blowup solution of* (148)*, we have*

$$\int_{I_0} N(t)^{-1}\ dt \lesssim C(\eta_1, \eta_3) N_{min}^{-3}. \tag{169}$$

Proof. Let $N_* := c(\eta_3) N_{min}$ for some sufficiently small $c(\eta_3)$. Then from Proposition 8.12 we have

$$\int_{I_0}\int_{\mathbb{R}^3} |P_{\geq N_*} u(t,x)|^4\ dxdt \lesssim \eta_1 N_*^{-3} \lesssim C(\eta_1,\eta_3) N_{min}^{-3}.$$

On the other hand, from Bernstein inequality and (152) we have for each $t \in I_0$ that

$$\int_{|x-x(t)|\leq C(\eta_1)/N(t)} |P_{<N_*} u(t,x)|^4\ dx \lesssim N(t)^{-3} \|P_{<N_*} u(t)\|_{L^\infty_x}^4 \lesssim C(\eta_1) N(t)^{-3} N_*^2,$$

so by (164) and the triangle inequality we have (noting that $N_* \leq c(\eta_3) N(t)$)

$$\int_{\mathbb{R}^3} |P_{\geq N_*} u(t,x)|^4\ dx \gtrsim c(\eta_1) N(t)^{-1}.$$

Comparing this with the previous estimate, the claim follows. □

Remark 8.15. The estimate (169) is scale-invariant under the natural scaling (32) (N has the units of $length^{-1}$, and t has the units of $length^2$). In the radial case, a somewhat similar estimate was obtained by Bourgain [13] and implicitly also by Grillakis [41]; in our notation, this bound would be the assertion that

$$\int_I N(t)\ dt \lesssim |I|^{1/2} \tag{170}$$

for all $I \subseteq I_0$; indeed in the radial case (when $x(t) = 0$) this bound easily follows from Proposition 8.8 and a local version of (30). Both estimates are equally good at estimating the amount of time for which $N(t)$ is comparable to N_{min}, but Corollary 8.14 is much weaker than (170) when it comes to controlling the times for which $N(t) \gg N_{min}$. Indeed if we could extend (170) to the nonradial case one could obtain a significantly shorter proof of Theorem 7.3, however we were unable to prove this bound directly, although it can be deduced from Corollary 8.14 and Proposition 8.19 below).

This Corollary allows us to obtain some useful $L^{10}_{t,x}$ bounds in the case when $N(t)$ is bounded from above.

Corollary 8.16 (Nonconcentration implies spacetime bound)**.** *Let $I \subseteq I_0$, and suppose there exists a $N_{max} > 0$ such that $N(t) \leq N_{max}$ for all $t \in I$. Then for any localized minimal energy blowup solution of* (148) *we have*

$$\|u\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \lesssim C(\eta_1, \eta_3, N_{max}/N_{min})$$

and furthermore

$$\|u\|_{\dot{S}^1(I\times\mathbb{R}^3)} \lesssim C(\eta_1, \eta_3, N_{max}/N_{min}).$$

Proof. We may use scale invariance (32) to rescale $N_{min} = 1$. From Corollary 8.14 we obtain the useful bound

$$|I| \lesssim C(\eta_1, \eta_3, N_{max}).$$

Let $\delta = \delta(\eta_0, N_{max}) > 0$ be a small number to be chosen later. We may partition I into $O(|I|/\delta)$ intervals $I_1, \ldots, I_J$ of length at most δ. Let I_j be any of these intervals, and let t_j be any time in I_j. Observe from Corollary 8.6 and the hypothesis $N(t_j) \le N_{max}$ that

$$\|P_{\ge C(\eta_0)N_{max}} u(t_j)\|_{\dot H^1} \le \eta_0$$

(for instance). Now let $\tilde u(t) := e^{i(t-t_j)\Delta} P_{<C(\eta_0)N_{max}} u(t_j)$ be the free evolution of the low and medium frequencies of u. The above estimate then becomes

$$\|u(t_j) - \tilde u(t_j)\|_{\dot H^1} \le \eta_0.$$

On the other hand, from Bernstein inequality and (152) we have

$$\|\tilde u(t)\|_{L^{10}_x} \lesssim C(\eta_0, N_{max}) \|\tilde u(t_j)\|_{\dot H^1} \lesssim C(\eta_0, N_{max})$$

for all $t \in I_j$, and hence

$$\|\tilde u\|_{L^{10}_{t,x}(I_j \times \mathbb{R}^3)} \lesssim C(\eta_0, N_{max}) \delta^{1/10}.$$

Similarly we have

$$\|\nabla(|\tilde u(t)|^4 \tilde u(t))\|_{L^{6/5}_x} \lesssim \|\nabla \tilde u(t)\|_{L^6_x} \|\tilde u(t)\|^4_{L^6_x} \lesssim C(\eta_0, N_{max}) \|\tilde u(t_j)\|^5_{\dot H^1} \lesssim C(\eta_0, N_{max})$$

and hence

$$\|\nabla(|\tilde u(t)|^4 \tilde u(t))\|_{L^2_t L^{6/5}_x(I_j \times \mathbb{R}^3)} \lesssim C(\eta_0, N_{max}) \delta^{1/2}.$$

From these two estimates, the energy bound (152), and Lemma 7.7 with $e = -|\tilde u|^4 \tilde u$, we see (if δ is chosen sufficiently small) that

$$\|u\|_{L^{10}_{t,x}(I_j \times \mathbb{R}^3)} \lesssim 1$$

Summing this over each of the $O(|I|/\delta)$ intervals I_j we obtain the desired $L^{10}_{t,x}$ bound. The $\dot S^1$ bound then follows as in Theorem 4.8. □

This above corollary gives the desired contradiction to (161) when N_{max}/N_{min} is bounded, i.e. $N(t)$ stays in a bounded range.

8.17. **Third stage: Nonconcentration of energy.** Of course, any global well-posedness argument for (148) must eventually exclude a blowup scenario (self-similar or otherwise) where $N(t)$ goes to infinity in finite time, and indeed by Corollary 8.16 this is the only remaining possibility for a *minimal energy blowup solution.* Corollary 8.6 implies that in such a scenario the energy must almost entirely evacuate the frequencies near N_{min}, and instead concentrate at frequencies much larger than N_{min}. While this scenario is consistent with conservation of energy, it turns out to not be consistent with the time and frequency distribution of mass.

More specifically, we know there is a $t_{min} \in I_0$ so that for all $t \in I_0, N(t) \ge N(t_{min}) := N_{min} > 0$. By Corollary 8.6, at time t_{min} the solution has the bulk of its energy near the frequency N_{min}, and hence the medium frequencies at that time have mass bounded below by,

$$\|P_{c(\eta_0)N_{min} \le \cdot \le C(\eta_0)N_{min}} u(t_{min})\|_{L^2} \gtrsim c(\eta_0) N^{-1}_{min}. \tag{171}$$

The idea is to prove the following approximate mass conservation law for these high frequencies[36], which states that while some mass might slip to very low frequencies, it can't all do so.

[36]It is necessary to truncate to the high frequencies in order to exploit mass conservation because the low frequencies contain an unbounded amount of mass. This strategy of mollifying the solution in frequency space in order to exploit a conservation law that would otherwise be unbounded or useless is inspired by the "I-method" for sub-critical dispersive equations discussed in Lecture # 4.

Lemma 8.18 (Some mass freezes away from low frequencies)**.** *Suppose u is a minimal energy blowup solution of* (148)*. Then for all $t \in I_0$,*

$$\|P_{\geq \eta_4^{100} N_{min}} u(t)\|_{L^2} \gtrsim \eta_1. \tag{172}$$

Lemma 8.18 will quickly show that the evacuation scenario - wherein the solution cleanly concentrates energy to very high frequencies - cannot occur. Instead the solution always leaves a nontrivial amount of mass and energy behind at medium frequencies. This "littering" of the solution will serve (via Corollary 8.6) to keep $N(t)$ from escaping to infinity[37] and gives us,

Proposition 8.19 (Energy cannot evacuate from low frequencies)**.** *For any minimal energy blowup solution of* (148) *we have*

$$N(t) \lesssim C(\eta_5) N_{min} \tag{173}$$

for all $t \in I_0$.

By combining Proposition 8.19 with Corollary 8.16, we encounter a contradiction to (161) which completes the proof of Theorem 7.3.

We conclude this lecture by summarizing the role of the parameters $\eta_i, i = 0, \dots 5$ which have now all been introduced. The number η_1 represents the amount of potential energy that must be present at every time in a minimal energy blowup solution. (Proposition 8.7); it also represents the extent of concentration of energy (on the scale of $1/N(t)$) that must occur in physical space at every time in a minimal energy blowup solutions (Proposition 8.8). The number η_2 is introduced in Proposition 8.9, where $1/\eta_2$ represents the extent that there is localization (on the scale of $1/N(t)$) of energy in a minimal energy blowup solution. The number η_3 measures, on the scale of the quantity N_{min}, what we mean by "high frequency" when we say Proposition 8.12 is an interaction Morawetz estimate localized to high frequencies. The number η_4 measures the frequency (on the scale of N_{min}) below which the evolution can't move a certain portion (namely, η_1) of the L^2 mass. Finally, the number η_0 enters in Corollary 8.6 and various other points in the paper where we simply use its value as a small, universal constant.

[37] It is interesting to note that one must exploit conservation of energy, conservation of mass, *and* conservation of momentum (via the Morawetz inequality) in order to prevent blowup for the equation (148); the same phenomenon occurs in the previous arguments [13], [41] in the radial case, even though the details of those arguments are in many ways quite different to those here.