## 7. Lecture #6: Global well-posedness for the $H^1(\mathbb{R}^n)$ critical NLS - Part I

We recall that the $H^1$ critical exponent for (1) is $p = 1 + \frac{4}{n-2}$. We also recall the following theorem that can be basically completely proved using either directly or indirectly theorems and arguments already presented in Lecture #4 and Lecture #5:

**Theorem 7.1.** *[Local or global small data well-posedness for the $H^1$ critical NLS] We have the following two results:*

1. *For any $u_0 \in H^1$ there exist $T = T(u_0)$ and a unique solution $u \in S^1_{[T,T]}$ to (1) with $p = 1 + \frac{4}{n-2}$ and $\mu = \pm 1$. Moreover there is continuity with respect to the initial data.*
2. *There exists $\epsilon$ small enough such that for any $u_0$, $\|u_0\|_{H^1} \leq \epsilon$ there exists a unique global solution $u \in S^1$ to (1) with $p = 1 + \frac{4}{n-2}$ and $\mu = \pm 1$. Moreover there is continuity with respect to the initial data and scattering in the sense that there exists $u_\pm \in H^1$ such that*
$$\|u(t) - S(t)u_\pm\|_{H^1} \longrightarrow 0 \quad as \quad t \to \pm\infty.$$

*Proof.* It is clear that the part about well-posedness is a summary of what has been proved in Lecture #4. The part about scattering instead can be proved as in Lecture #5 and by simply observing that Proposition 6.16 follows directly from the well-posedness proof thanks to the small data assumption. □

*Remark* 7.2. We first remark that this theorem doesn't *see* the focusing or defocusing nature of the equation. This clearly means that in Theorem 7.1 the NLS is treat as a "small" perturbation of the linear problem. Due to the criticality of the problem and hence the fact that $T$ depends also on the profile of the initial data an iteration argument based on the conservation of mass and energy is not possible. It is also clear that even increasing the regularity of the data the large data problem doesn't become any easier.

The first break through on this problem is due to Bourgain [13]. He considers the defocusing case with $n = 3, 4$ and assumes radial symmetry for the problem. He proves the second part of Theorem 7.1 for arbitrarily large radially symmetric data. Here we summarize the main steps of Bourgain's proof for $n = 3$, which doesn't really do justice to the novelty and depth of the proof itself. The background argument is done by induction on the size of the energy $E$, the only quantity, besides the mass that here doesn't play much of a role, that remains controlled over time. From Theorem 7.1 the first step of the induction (small $E$) is in place. Let's now assume the second induction assumption that if $E < E_0$ then the theorem is true. We take $E = E_0$ and we want to prove that also in this case the theorem is true. One first shows that the theorem follows if and only if a priori one knows that the norm $L^{10}_t L^{10}_x$ of the solution remains bounded (see Theorem 4.8). Then the proof proceeds by contradiction. One supposes that there is a solution $u$ such that $\|u\|_{L^{10}_t L^{10}_x}$ is arbitrarily large and $E = E_0$. The heart of the proof is on showing that at some time $t_0$ there is concentration of the $H^1$ norm: There exists a small ball centered at the origin $B_0$ such that $\|u(t_0)\|_{H^1(B_0)} > \delta$, and this ball is "sufficiently isolated" from the rest of the solution . It is here that the radial assumption is used. At this point one restarts the evolution at time $t_0$ by splitting the data as
$$\psi_0 = u(t_0)\chi_{B_0} \quad \text{and} \quad \psi_1 = u(t_0)(1 - \chi_{B_0}),$$
where $\chi_{B_0}$ is the indicator function for the ball $B_0$, and evolving $\psi_0$ with NLS and $\psi_1$ with a difference equation so that the sum of the two evolutions give the solution to NLS. Since now $\psi_0 \in H^1$ and $x\psi \in L^2$ it follows[25] that the evolution $v$ of $\psi_0$ is global in time. Moreover

[25] This result is for example proved in [19] as a consequence of the pseudo-conformal transformation and a monotonicity formula linked to it.

since $E(\psi_0) \sim \delta^2$ it follows that $E(\psi_1) < E_0 - \delta^2$. Hence for the difference equation we are in the induction assumption. This is not quite like to have the equation under the induction assumption, but with some relatively straightforward perturbation theory[26] one also gets that the evolution $w$ of $\psi_1$ is global. Hence we have a global evolution for the solution $u = v + w$ to NLS and as a consequence a uniform bound for $\|u\|_{L^{10}_t L^{10}_x}$ which is a contradiction.

Almost at the same time, with the same radial symmetry assumption above, Grillakis [41] proved a slighter weaker result than Bourgain's, namely existence and uniqueness for smooth global solution. It took few more years to remove the radial assumption and obtain the following theorem and its corollary [29]:

**Theorem 7.3.** *For any $u_0$ with finite energy, $E(u_0) < \infty$, there exists a unique*[27] *global solution $u \in C^0_t(\dot{H}^1_x) \cap L^{10}_{t,x}$ to (1) with $p = 5, n = 3, \mu = 1$ such that*

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} |u(t,x)|^{10} \, dxdt \leq C(E(u_0)). \tag{129}$$

*for some constant $C(E(u_0))$ that depends only on the energy.*

As one can see from Theorem 4.8 and from the arguments in Lecture #5, the $L^{10}_{t,x}$ bound above also gives scattering and and persistence of regularity:

**Corollary 7.4.** *Let $u_0$ have finite energy. Then there exist finite energy solutions $u_\pm(t,x)$ to the free Schrödinger equation $(i\partial_t + \Delta)u_\pm = 0$ such that*

$$\|u_\pm(t) - u(t)\|_{\dot{H}^1} \to 0 \text{ as } t \to \pm\infty.$$

*Furthermore, the maps $u_0 \mapsto u_\pm(0)$ are homeomorphisms from $\dot{H}^1(\mathbb{R}^3)$ to $\dot{H}^1(\mathbb{R}^3)$. Finally, if $u_0 \in H^s$ for some $s > 1$, then $u(t) \in H^s$ for all time $t$, and one has the uniform bounds*

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s} \leq C(E(u_0), s)\|u_0\|_{H^s}.$$

Most of the rest of this lecture and Lecture # 7 will be devoted to give an idea of the proof for Theorem 7.3. Still for the defocusing case and for $n > 3$ we recall first the result of Tao [70], where an equivalent of Theorem 7.3 is proved still under the radial assumption, the result of Ryckman and Visan [67] for $n = 4$, where the radial assumption is removed, and finally the full generalization for any $n \geq 5$ by Visan [74].

The situation in the focusing case was first considered successfully by Kenig and Merle. They prove the following theorem [50]:

**Theorem 7.5.** *Assume that $E(u_0) < E(W), \|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, where $n = 3, 4, 5$ and $u_0$ is radial and $W$ is the stationary solution (soliton). Then the solution $u$ to the critical $H^1$ focusing IVP (1) with data $u_0$ at $t = 0$ is defined for all time and there exists $u_\pm \in \dot{H}^1$ such that*

$$\|S(t)u_\pm - u(t)\|_{\dot{H}^1} \to 0 \text{ as } t \to \pm\infty.$$

*Moreover for $u_0$ radial, $E(u_0) < E(W)$, but $\|u_0\|_{\dot{H}^1} > \|W\|_{\dot{H}^1}$, the solution must break down in finite time.*

This result has been extend in every dimension and for general data in [54]. Moreover a similar result has been proved by Kenig and Merle for the critical wave equation without the radial assumption [51], see also [52]. The proof of Theorem 7.5 introduces a new point of view for these problems. Using a compensated compactness argument the authors reduce matters to a

[26] That works tanks to the fact that the ball is "sufficiently" isolatd from the rest of the solution.

[27] In fact, uniqueness actually holds in the larger space $C^0_t(\dot{H}^1_x)$ (thus eliminating the constraint that $u \in L^{10}_{t,x}$) [29].

rigidity theorem, which is proved with the aid of a localized virial identity (in the spirit of Merle [59, 60]). The radiality enters only in the proof of the rigidity theorem. In the case of the critical wave equation other consideration of *elliptic nature* are used to remove the radial assumption. The authors also use in their approach a profile decomposition proved in the context of the Schrödinger equation by Keraani [53].

7.6. **Idea of the proof of Theorem 7.3.** To give a complete proof of this theorem in less than two lectures is impossible, so we will first outline the idea of the proof and then we only show rigourously few parts of it.

First the naive approach: We follow the strategy of induction/contraddiction introduced by Bourgain. We define $E_{crit}$ the critical energy below which the $L^{10}_t L^{10}_x$ norm of a solutions stays bounded by some constant depending on the energy. We then identify a smooth *minimal energy blow up solution* $u$ of energy $E_{crit}$ such that

$$\|u\|_{L^{10}_t L^{10}_x} > M, \tag{130}$$

where $M$ is as large as we please. For this solution we then show a series of properties that at the end will actually give

$$\|u\|_{L^{10}_t L^{10}_x} \leq C(E_{crit}), \tag{131}$$

contradicting (130).

This is in order the summary of the properties we prove for the *minimal energy blow up solution* on a fixed (compact) interval of time $I$:

(1) **Frequency and space localization:** For each $t \in I$ there exists $N(t) > 0$ and $x(t) \in \mathbb{R}^3$ such that $\hat{u}(t)$ is *mostly* supported at frequency of size proportional to $N(t)$ and $u(t)$ is mostly supported on a ball centered at $x(t)$ and radius proportional to $\frac{1}{N(t)}$. To prove the frequency localization part one uses the intuition that the *minimal energy blow up solution* $u$, at a given time $t_0$, cannot have two components $u_-$ and $u_+$ which Fourier transforms are supported respectively in $|\xi| \leq N$ and $|\xi| \geq KN$, $K >> 1$, and such that both pieces carrie a large amount of energy. The reason for this is that the energy relative to $u_-$ will make the energy relative to $u_+$ smaller than $E_{crit}$ and viceversa. Hence both $u_-$ and $u_+$ can flow globally. On the other hand if $K$ is large enough their nonlinear interaction is basically negligible, hence perturbation says that basically $u \sim u_- + u_+$, hence $u$ exists globally and its $L^{10}_t L^{10}_x$ norm is uniformly bounded, a contradiction. A similar, but just a bit more complicated, argument gives also space localization.

(2) **Frequency localized interaction Morawetz inequality:** As we mentioned several times whenever a problem is not a perturbation of the linear one, like the critical ones for example, in order to obtain a global statement we need to have a global space-time bound. We learned that the Morawetz estimates for the defocusing problem and the Viriel identity for the focusing one are the types of estimates that we want to have. Bourgain in fact used the classical Morawetz estimate that appears in (30) with $p = 5$. Here the presence of the denominator forced the radial symmetry. Here instead we would like to use the Interaction Morawetz estimate (116). This is weaker in the sense that we only have the fourth power, but it is also stronger since we do not have a denominator. We keep in mind that our final goal is to show boundedness of the $L^{10}_I L^{10}_x$ norm of the *minimal energy blow up solution* $u$ so we need to upgrade the $L^4_I L^4_x$ norm. We believe that for the low frequencies, where the energy is very small thanks to localization, Strichartz estimates will be enough to give us the bound in the $L^{10}_I L^{10}_x$ norm. For the high frequencies we also have small energy, but we expect that the Strichartz estimates

are too weak here. So the idea is to first prove (116) for the high frequency part of the solution. We have for all $N_* < N_{min}$

$$\int_I \int |P_{\geq N_*} u(t,x)|^4 \, dxdt \lesssim \eta_1 N_*^{-3}, \tag{132}$$

where $N_{min} = \inf_{t \in I} N(t)$ for which one can prove $N_{min} > 0$ and $\eta_1$ is a small quantity. Note that the quantities appearing in the right hand side of (132) are independent of $I$.

(3) **Uniform boudedness of time interval $I$:** Assuming that $N(t)$ doesn't run to infinity, use the $L^4_I L^4_x$ bound, which is uniform in $I$, to get a uniform bound on the length of time interval $I$ itself. With this information now, since most of the solution remains on a uniformly bounded frequency window, perturbation will provide the final uniform bound for the $L^{10}_I L^{10}_x$ norm.

(4) **Uniform Boudedness of $N(t)$:** We mentioned above that there exists $N_{min}$ such that $0 < N_{min} \leq N(t)$, and this in not hard to prove. In fact by rescaling[28] one can assume that

$$N_{min} = 1.$$

The difficult part is to show that there exists $N_{max} < \infty$ such that

$$N(t) \leq N_{max}.$$

Again by contradiction one assumes that given $R >> 1$ there exists $t_R$ such that $N(t_R) > R$ and by definition most of the energy is located on frequencies $R < N(t_R) \lesssim |\xi|$. But then one can prove by a simple application of the "I-method" that although the energy has migrated on very large frequencies, some *littering* of mass has been left on medium frequencies. But mass on medium frequencies is equivalent to energy, hence there is some significant energy left over on medium frequencies. If them $R$ is large enough these two pieces of the solution $u$, the one at very high frequencies and the one at medium frequencies, are very separated and each has a significant amount of energy. But this cannot happen for a *energy critical blow up solution*. Hence $N_{max}$ must be bounded.

In order to proceed with the outline given above we use heavily Strichartz estimates (12) and (13), improved bilinear estimate (14) and multilinear estimates of different kinds. A very important tool that was mentioned often above is the theory of perturbation that in practice is made of a serious of perturbation lemmas. These lemmas are particularly useful when we have to claim that if $u$ is a solution to NLS and $v$ is a solution of an equation which is a small perturbation of NLS, then $u$ and $v$ are close to each other and if one exists the other does too. Here we report two examples of such lemmas.

**Lemma 7.7** (Short-time perturbations)**.** *Let $I$ be a compact interval, and let $\tilde{u}$ be a function on $I \times \mathbb{R}^3$ which is a near-solution to* (1) *with $p = 5$ and $\mu = 1$ in the sense that*

$$(i\partial_t + \frac{1}{2}\Delta)\tilde{u} = |\tilde{u}|^4 \tilde{u} + e \tag{133}$$

*for some function $e$. Suppose that we also have the energy bound*

$$\|\tilde{u}\|_{L^\infty_t \dot{H}^1_x(I \times \mathbb{R}^3)} \leq E$$

*for some $E > 0$. Let $t_0 \in I$, and let $u(t_0)$ be close to $\tilde{u}(t_0)$ in the sense that*

$$\|u(t_0) - \tilde{u}(t_0)\|_{\dot{H}^1_x} \leq E' \tag{134}$$

[28] Since the problem is $H^1$ critical and we only use the energy, nothing will change by rescaling!

*for some $E' > 0$. Assume also that we have the smallness conditions*

$$\|\nabla \tilde{u}\|_{L_t^{10} L_x^{30/13}(I \times \mathbb{R}^3)} \leq \epsilon_0 \tag{135}$$

$$\|\nabla e^{i(t-t_0)\Delta}(u(t_0) - \tilde{u}(t_0))\|_{L_t^{10} L_x^{30/13}(I \times \mathbb{R}^3)} \leq \epsilon \tag{136}$$

$$\|\nabla e\|_{L_t^2 L_x^{6/5}} \leq \epsilon \tag{137}$$

*for some $0 < \epsilon < \epsilon_0$, where $\epsilon_0$ is some constant $\epsilon_0 = \epsilon_0(E, E') > 0$.*

*We conclude that there exists a solution $u$ to (1) with $p = 5$ and $\mu = 1$ on $I \times \mathbb{R}^3$ with the specified initial data $u(t_0)$ at $t_0$, and furthermore*

$$\|u - \tilde{u}\|_{\dot{S}^1(I \times \mathbb{R}^3)} \lesssim E' \tag{138}$$

$$\|u\|_{\dot{S}^1(I \times \mathbb{R}^3)} \lesssim E' + E \tag{139}$$

$$\|u - \tilde{u}\|_{L_{t,x}^{10}(I \times \mathbb{R}^3)} \lesssim \|\nabla(u - \tilde{u})\|_{L_t^{10} L_x^{30/13}(I \times \mathbb{R}^3)} \lesssim \epsilon \tag{140}$$

$$\|\nabla(i\partial_t + \frac{1}{2}\Delta)(u - \tilde{u})\|_{L_t^2 L_x^{6/5}(I \times \mathbb{R}^3)} \lesssim \epsilon. \tag{141}$$

Note that $u(t_0) - \tilde{u}(t_0)$ is allowed to have large energy, albeit at the cost of forcing $\epsilon$ to be smaller, and worsening the bounds in (138). From the Strichartz estimate (12), we see that the hypothesis (136) is redundant if one is willing to take $E' = O(\varepsilon)$.

*Proof.* By the well-posedness theory presented in Lecture #4, it suffice to prove (138) - (141) as *a priori* estimates[29]. We establish these bounds for $t \geq t_0$, since the corresponding bounds for the $t \leq t_0$ portion of $I$ are proved similarly.

First note that the Strichartz estimate (12) and (13) give,

$$\|\tilde{u}\|_{\dot{S}^1(I \times \mathbb{R}^3)} \lesssim E + \|\tilde{u}\|_{L_{t,x}^{10}(I \times \mathbb{R}^3)} \cdot \|\tilde{u}\|^4_{\dot{S}^1(I \times \mathbb{R}^3)} + \varepsilon.$$

By (135) and Sobolev embedding we have $\|\tilde{u}\|_{L_{t,x}^{10}(I \times \mathbb{R}^3)} \lesssim \varepsilon_0$. A standard continuity argument in $I$ then gives (if $\varepsilon_0$ is sufficiently small depending on $E$)

$$\|\tilde{u}\|_{\dot{S}^1(I \times \mathbb{R}^3)} \lesssim E. \tag{142}$$

Define $v := u - \tilde{u}$. For each $t \in I$ define the quantity

$$S(t) := \|\nabla(i\partial_t + \frac{1}{2}\Delta)v\|_{L_t^2 L_x^{6/5}([t_0,t] \times \mathbb{R}^3)}.$$

From using again Strichartz estimates and the definition of $S^1$, (136), we have

$$\begin{aligned} \|\nabla v\|_{L_t^{10} L_x^{30/13}([t_0,t] \times \mathbb{R}^3)} &\lesssim \|\nabla(v - e^{i(t-t_0)\frac{1}{2}\Delta}v(t_0))\|_{L_t^{10} L_x^{30/13}([t_0,t] \times \mathbb{R}^3)} \\ &\quad + \|\nabla e^{i(t-t_0)\frac{1}{2}\Delta}v(t_0)\|_{L_t^{10} L_x^{30/13}([t_0,t] \times \mathbb{R}^3)} \\ &\lesssim \|v - e^{i(t-t_0)\frac{1}{2}\Delta}v(t_0)\|_{\dot{S}^1([t_0,t] \times \mathbb{R}^3)} + \varepsilon \end{aligned} \tag{143}$$

$$\lesssim S(t) + \varepsilon. \tag{144}$$

On the other hand, since $v$ obeys the equation

$$(i\partial_t + \frac{1}{2}\Delta)v = |\tilde{u} + v|^4(\tilde{u} + v) - |\tilde{u}|^4\tilde{u} - e = \sum_{j=1}^{5} \emptyset(v^j \tilde{u}^{5-j}) - e$$

[29] That is, we may assume the solution $u$ already exists and is smooth on the entire interval $I$.

where $\O(v_1, v_2, v_3, v_4, v_5)$ denotes any combination of $v_i$ and $\bar{v_j}$. By some standard multilinear estimates, (135), (137), (144) then

$$S(t) \lesssim \varepsilon + \sum_{j=1}^{5} (S(t)+\varepsilon)^j \varepsilon_0^{5-j}.$$

If $\varepsilon_0$ is sufficiently small, a standard continuity argument then yields the bound $S(t) \lesssim \varepsilon$ for all $t \in I$. This gives (141), and (140) follows from (144). Applying Strichartz inequalities again, (134) we then conclude (138) (if $\varepsilon$ is sufficiently small), and then from (142) and the triangle inequality we conclude (139). □

We will actually be more interested in iterating the above Lemma[30] to deal with the more general situation of near-solutions with finite but arbitrarily large $L^{10}_{t,x}$ norms.

**Lemma 7.8** (Long-time perturbations)**.** *Let $I$ be a compact interval, and let $\tilde{u}$ be a function on $I \times \mathbb{R}^3$ which obeys the bounds*

$$\|\tilde{u}\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \leq M \tag{145}$$

*and*

$$\|\tilde{u}\|_{L^\infty_t \dot{H}^1_x(I\times\mathbb{R}^3)} \leq E \tag{146}$$

*for some $M, E > 0$. Suppose also that $\tilde{u}$ is a near-solution to* (1) *with $p = 5$ and $\mu = 1$ in the sense that it solves* (133) *for some $e$. Let $t_0 \in I$, and let $u(t_0)$ be close to $\tilde{u}(t_0)$ in the sense that*

$$\|u(t_0) - \tilde{u}(t_0)\|_{\dot{H}^1_x} \leq E'$$

*for some $E' > 0$. Assume also that we have the smallness conditions,*

$$\begin{aligned} \|\nabla e^{i(t-t_0)\frac{1}{2}\Delta}(u(t_0) - \tilde{u}(t_0))\|_{L^{10}_t L^{30/13}_x(I\times\mathbb{R}^3)} &\leq \varepsilon \\ \|\nabla e\|_{L^2_t L^{6/5}_x(I\times\mathbb{R}^3)} &\leq \varepsilon \end{aligned} \tag{147}$$

*for some $0 < \varepsilon < \varepsilon_1$, where $\varepsilon_1$ is some constant $\varepsilon_1 = \varepsilon_1(E, E', M) > 0$. We conclude there exists a solution $u$ to* (1) *with $p = 5$ and $\mu = 1$ on $I \times \mathbb{R}^3$ with the specified initial data $u(t_0)$ at $t_0$, and furthermore*

$$\begin{aligned} \|u - \tilde{u}\|_{\dot{S}^1(I\times\mathbb{R}^3)} &\leq C(M, E, E') \\ \|u\|_{\dot{S}^1(I\times\mathbb{R}^3)} &\leq C(M, E, E') \\ \|u - \tilde{u}\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \leq \|\nabla(u - \tilde{u})\|_{L^{10}_t L^{30/13}_x(I\times\mathbb{R}^3)} &\leq C(M, E, E')\varepsilon. \end{aligned}$$

Once again, the hypothesis (147) is redundant by the Strichartz estimate if one is willing to take $E' = O(\varepsilon)$; however it will be useful in our applications to know that this Lemma can tolerate a perturbation which is large in the energy norm but whose free evolution is small in the $L^{10}_t \dot{W}^{1,30/13}_x$ norm.

This lemma is already useful in the $e = 0$ case, as it says that one has local well-posedness in the energy space whenever the $L^{10}_{t,x}$ norm is bounded; in fact one has locally Lipschitz dependence on the initial data. For similar perturbative results see [13], [12].

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[30]We are grateful to Monica Visan for pointing out an incorrect version of Lemma 7.8 in a previous version of this paper, and also in simplifying the proof of Lemma 7.7.

*Proof.* As in the previous proof, we may assume that $t_0$ is the lower bound of the interval $I$. Let $\varepsilon_0 = \varepsilon_0(E, 2E')$ be as in Lemma 7.7. (We need to replace $E'$ by the slightly larger $2E'$ as the $\dot{H}^1$ norm of $u - \tilde{u}$ is going to grow slightly in time.)

The first step is to establish a $\dot{S}^1$ bound on $\tilde{u}$. Using (145) we may subdivide $I$ into $C(M, \varepsilon_0)$ time intervals such that the $L^{10}_{t,x}$ norm of $\tilde{u}$ is at most $\varepsilon_0$ on each such interval. By using (146) and Strichartz estimates, as in the proof of (142), we see that the $\dot{S}^1$ norm of $\tilde{u}$ is $O(E)$ on each of these intervals. Summing up over all the intervals we conclude

$$\|\tilde{u}\|_{\dot{S}^1(I\times\mathbb{R}^3)} \leq C(M, E, \varepsilon_0)$$

and in particular

$$\|\nabla\tilde{u}\|_{L^{10}_t L^{30/13}_x(I\times\mathbb{R}^3)} \leq C(M, E, \varepsilon_0).$$

We can then subdivide the interval $I$ into $N \leq C(M, E, \varepsilon_0)$ subintervals $I_j \equiv [T_j, T_{j+1}]$ so that on each $I_j$ we have,

$$\|\nabla\tilde{u}\|_{L^{10}_t L^{30/13}_x(I_j\times\mathbb{R}^3)} \leq \varepsilon_0.$$

We can then verify inductively using Lemma 7.7 for each $j$ that if $\varepsilon_1$ is sufficiently small depending on $\varepsilon_0$, $N$, $E$, $E'$, then we have

$$\begin{aligned}
\|u - \tilde{u}\|_{\dot{S}^1(I_j\times\mathbb{R}^3)} &\leq C(j)E' \\
\|u\|_{\dot{S}^1(I_j\times\mathbb{R}^3)} &\leq C(j)(E' + E) \\
\|\nabla(u - \tilde{u})\|_{L^{10}_t L^{30/13}_x(I_j\times\mathbb{R}^3)} &\leq C(j)\varepsilon \\
\|\nabla(i\partial_t + \frac{1}{2}\Delta)(u - \tilde{u})\|_{L^2_t L^{6/5}_x(I_j\times\mathbb{R}^3)} &\leq C(j)\varepsilon
\end{aligned}$$

and hence by Strichartz we have

$$\begin{aligned}
&\|\nabla e^{i(t-T_{j+1})\frac{1}{2}\Delta}(u(T_{j+1}) - \tilde{u}(T_{j+1}))\|_{L^{10}_t L^{30/13}_x(I\times\mathbb{R}^3)} \\
&\leq \|\nabla e^{i(t-T_j)\frac{1}{2}\Delta}(u(T_j) - \tilde{u}(T_j))\|_{L^{10}_t L^{30/13}_x(I\times\mathbb{R}^3)} + C(j)\varepsilon
\end{aligned}$$

and

$$\|u(T_{j+1}) - \tilde{u}(T_{j+1})\|_{\dot{H}^1} \leq \|u(T_j) - \tilde{u}(T_j)\|_{\dot{H}^1} + C(j)\varepsilon$$

allowing one to continue the induction (if $\varepsilon_1$ is sufficiently small depending on $E$, $N$, $E'$, $\varepsilon_0$, then the quantity in (134) will not exceed $2E'$). The claim follows. $\square$

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