

6. LECTURE # 5: INTERACTION MORAWETZ ESTIMATES AND SCATTERING

In the last lecture we discussed the question of global well-posedness. Once one can prove that given an initial data a unique solution evolving from that data exists for all times it becomes natural to ask how this solution looks like as $t \rightarrow \pm\infty$. The theory that addresses these questions is called *scattering theory*. In order to put scattering in a more general context we need few definitions. We will give them by assuming that the solution for (1) is defined globally in time with respect to the energy space H^1 , but it will be easy to generalize them when more general Sobolev spaces are considered.

Definition 6.1 (Scattering). Given a global solution $u \in H^1$ to (1) we say that u *scatters* to $u_+ \in H^1$ if

$$(89) \quad \|u(t) - S(t)u_+\|_{H^1} \longrightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

Clearly a similar definition is given if $t \rightarrow -\infty$.

Remark 6.2. Using the properties of the group $S(t)$ it is easy to see that (89) is equivalent to

$$(90) \quad \|S(-t)u(t) - u_+\|_{H^1} \longrightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

Since by the Duhamel formula (6)

$$S(-t)u(t) - u_+ = u_0 - u_+ - i \int_0^t S(-t')|u(t')|^{p-1}u(t') dt'$$

it is clear that scattering is **equivalent** to showing that the improper time integral

$$\int_0^\infty S(-t')|u(t')|^{p-1}u(t') dt'$$

converges in H^1 and in particular this will give the formula for u_+ , i.e.

$$(91) \quad u_+ = u_0 - i \int_0^\infty S(-t')|u(t')|^{p-1}u(t') dt'.$$

One can also consider an inverse problem: assume $u_+ \in H^1$, can we find an initial data $u_0 \in H^1$ such that the global solution u for (1) scatters to u_+ ?

Definition 6.3 (Wave Operator). Assume that for any $u_+ \in H^1$ there exists $u_0 \in H^1$ such that the solution u to (1) scatters to u_+ in the sense of (91). Then we define the wave operator

$$\Omega^+ : H^1 \longrightarrow H^1 \quad \text{such that } \Omega^+(u_+) = u_0$$

.

In order to prove the existence of Ω_+ it is useful to write the solution u in terms of u_+ . In fact using the Duhamel representation (6) and (91) above we can write

$$(92) \quad u(t) = S(t)u_+ + i \int_t^\infty S(t-t')(|u(t')|^{p-1}u(t')) dt',$$

and being able to define Ω_+ is equivalent to being able to define (92) for $t = 0$.

Remark 6.4. From the two definitions given above it is clear that proving scattering is equivalent to proving that the wave operator Ω^+ is invertible. In this case we also say that we have *Asymptotic Completeness*.

At first, from the definitions, it is not clear what is harder to prove, if existence of the wave operator or asymptotic completeness. But in practice the former is easier. One of the reasons is that the existence of the wave operator usually follows from the strong²² dispersive estimates (10) and from iteration of local well-posedness. On the other hand to prove scattering one needs global space time bounds that are very difficult to get. Here we do not address the question of existence of the wave operator (see [19]), but we will concentrate on the scattering issue. The bibliography on scattering is quite large, but certainly the work of Ginibre and Velo (see for example [40]) takes a special stand in it. But in this lecture we will take a different and more recent approach that is based on the so called *Interaction Morawetz Estimates* [28, 75].

6.5. Interaction Morawetz Estimates. At this point there are several ways one can present these estimates: as weighted overages of the classical Morawetz estimates presented in Lecture #2 [28, 75], as classical Morawetz estimates applied to tensors of solutions to (1) [20, 42, 43], or as more general and refined calculations dealing with vector fields [32, 65]. Here we describe the first one, which was also the original one given in 3 dimensions²³.

In the following we introduce an interaction potential generalization of the classical Morawetz action and associated inequalities. We first recall the standard Morawetz action centered at a point and the proof that this action is monotonically increasing with time when the nonlinearity is defocusing. The interaction generalization is introduced in the second subsection. The key consequence of the analysis in this section is the $L_{x,t}^4$ estimate (116).

The discussion in this section will be carried out in the context of the following generalization of (1):

$$(93) \quad i\partial_t u + \alpha\Delta u = \mu f(|u|^2)u, \quad u : \mathbb{R} \times \mathbb{R}^3 \mapsto \mathbb{C},$$

$$(94) \quad u(0) = u_0.$$

Here f is a smooth function $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ and α and μ are real constants that permit us to easily distinguish in the analysis below those terms arising from the Laplacian or the nonlinearity. We also define $F(z) = \int_0^z f(s)ds$.

We will use polar coordinates $x = r\omega$, $r > 0, \omega \in S^2$, and write Δ_ω for the Laplace-Beltrami operator on S^2 . For ease of reference below, we record some alternate forms of the equation in (93):

$$(95) \quad u_t = i\alpha\Delta u - i\mu f(|u|^2)u,$$

$$(96) \quad \bar{u}_t = -i\alpha\Delta\bar{u} + i\mu f(|u|^2)\bar{u},$$

$$(97) \quad u_t = i\alpha u_{rr} + i\frac{2\alpha}{r}u_r + i\frac{\alpha}{r^2}\Delta_\omega u - i\mu f(|u|^2)u,$$

$$(98) \quad (ru_t) = i\alpha(ru)_{rr} + i\frac{\alpha}{r}\Delta_\omega u - i\mu r f(|u|^2)u,$$

$$(99) \quad (r\bar{u}_t) = -i\alpha(r\bar{u})_{rr} - i\frac{\alpha}{r}\Delta_\omega\bar{u} + i\mu f(|u|^2)\bar{u}.$$

²²Especially in higher dimensions.

²³The reader will see below that for $n = 1, 2$ the argument breaks down. In fact for $n = 1$ one needs to use tensors of solutions [20] and for $n = 2$ one either is happy with a local in time estimate [37] or needs to introduce a much more refined argument [32]. For $n > 3$ the argument below can be used but the estimates are less “clean” than the $L_t^4 L_x^4$ norm we find below. But some use of standard harmonic analysis leads to a better space time estimates which is as good as the one we prove here [67, 74, 75].

6.6. Standard Morawetz action and inequalities. We will call the following quantity the *Morawetz action centered at 0* for the solution u of (93) and this should be compared with (29),

$$(100) \quad M_0[u](t) = \int_{\mathbb{R}^3} \operatorname{Im}[\bar{u}(t, x) \nabla u(t, x)] \cdot \frac{x}{|x|} dx.$$

We check using the equation that,

$$(101) \quad \partial_t(|u|^2) = -2\alpha \nabla \cdot \operatorname{Im}[\bar{u}(t, x) \nabla u(t, x)],$$

hence we may interpret M_0 as the spatial average of the radial component of the L^2 -mass current. We might expect that M_0 will increase with time if the wave u scatters since such behavior involves a broadening redistribution of the L^2 -mass. The following proposition of Lin and Strauss [58] that is equivalent to (29), indeed gives $\frac{d}{dt} M_0[u](t) \geq 0$ for defocusing equations.

Proposition 6.7. [58] *If u solves (93)-(94) then the Morawetz action at 0 satisfies the identity (102)*

$$\partial_t M_0[u](t) = 4\pi\alpha |u(t, 0)|^2 + \int_{\mathbb{R}^3} \frac{2\alpha}{|x|} |\nabla_0 u(t, x)|^2 dx + \mu \int_{\mathbb{R}^3} \frac{2}{|x|} \{ |u|^2 f(|u|^2)(t) - F(|u|^2) \} dx.$$

where ∇_0 is the angular component of the derivative,

$$(103) \quad \nabla_0 u = \nabla u - \frac{x}{|x|} \left(\frac{x}{|x|} \cdot \nabla u \right).$$

In particular, M_0 is an increasing function of time if the equation (93) satisfies the repulsivity condition,

$$(104) \quad \mu \{ |u|^2 f(|u|^2)(t) - F(|u|^2) \} \geq 0.$$

Note that for pure power potentials $F(x) = \frac{2}{p+1} x^{\frac{p+1}{2}}$, where the nonlinear term in (93) is $|u|^{p-1}u$, the function $|u|^2 f(|u|^2) - F(|u|^2) = \frac{p-1}{2} F(|u|^2)$. Hence condition (104) holds.

We may center the above argument at any other point $y \in R^3$ with corresponding results. Toward this end, define the *Morawetz action centered at y* to be,

$$(105) \quad M_y[u](t) = \int_{\mathbb{R}^3} \operatorname{Im}[\bar{u}(x) \nabla u(x)] \cdot \frac{x-y}{|x-y|} dx.$$

We shall often drop the u from this notation, as we did previously in writing $M_0(t)$.

Corollary 6.8. *If u solves (93) the Morawetz action at y satisfies the identity*

$$(106) \quad \frac{d}{dt} M_y = 4\pi\alpha |u(t, y)|^2 + \int_{\mathbb{R}^3} \frac{2\alpha}{|x-y|} |\nabla_y u(t, x)|^2 dx + \int_{\mathbb{R}^3} \frac{2\mu}{|x-y|} \{ |u|^2 f(|u|^2) - F(|u|^2) \} dx,$$

where $\nabla_y u \equiv \nabla u - \frac{x-y}{|x-y|} \left(\frac{x-y}{|x-y|} \cdot \nabla u \right)$. In particular, M_y is an increasing function of time if the nonlinearity satisfies the repulsivity condition (104).

Corollary 6.8 shows that a solution is, on average, repulsed from any fixed point y in the sense that $M_y[u](t)$ is increasing with time.

For our scattering results, we'll need the following pointwise bound for $M_y[u](t)$.

Lemma 6.9. *Assume u is a solution of (93) and $M_y[u](t)$ as in (105). Then,*

$$(107) \quad |M_y(t)| \lesssim \|u(t)\|_{\dot{H}_x^{\frac{1}{2}}}^2.$$

Proof. Without loss of generality we take $y = 0$. This is a refinement of the easy bound using Cauchy-Schwarz $|M_y(t)| \lesssim \|u(t)\|_{L_x^2} \|\nabla u(t)\|_{L_x^2}$. By duality

$$|\operatorname{Im} \int_{\mathbb{R}^3} \overline{u(x,t)} \partial_r u(x,t) dx| \leq \|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \cdot \|\partial_r u\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)}.$$

It suffices to show $\|\partial_r u\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} \leq \|u\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)}$. By duality and the definition $\partial_r \equiv \frac{x}{|x|} \cdot \nabla$, it remains to prove,

$$(108) \quad \left\| \frac{x}{|x|} f \right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leq \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)},$$

for any f for which the right hand side is finite. Inequality (108) follows from interpolating between the following two bounds,

$$\begin{aligned} \left\| \frac{x}{|x|} f \right\|_{L^2(\mathbb{R}^3)} &\leq \|f\|_{L^2(\mathbb{R}^3)} \\ \left\| \frac{x}{|x|} f \right\|_{\dot{H}^1(\mathbb{R}^3)} &\lesssim \|f\|_{\dot{H}^1(\mathbb{R}^3)} \end{aligned}$$

the first of which is trivial, the second of which follows from Hardy's inequality,

$$\begin{aligned} \left\| \nabla \left(\frac{x}{|x|} f \right) \right\|_{L^2} &\leq \left\| \frac{x}{|x|} \cdot \nabla f \right\|_{L^2} + \left\| \frac{1}{|x|} f \right\|_{L^2} \\ &\lesssim \|\nabla f\|_{L^2}. \end{aligned}$$

□

The well-known Morawetz-type inequalities which have proven useful in proving local decay or scattering for (93) arise by integrating the identity (102) or (106) in time. For nonlinear Schrödinger equations, this argument appears in the work of Lin and Strauss [58], who cite as motivation earlier work on Klein-Gordon equations by Morawetz [61].

Corollary 6.10 (Morawetz estimate centered at y). *Suppose u solves (93)-(94). Then for any $y \in \mathbb{R}^3$,*

$$(109) \quad \begin{aligned} 2 \sup_{t \in [0, T]} \|u(t)\|_{\dot{H}_x^{\frac{1}{2}}}^2 &\gtrsim 4\pi\alpha \int_0^T |u(t, y)|^2 dt + \int_0^T \int_{\mathbb{R}^3} \frac{2\alpha}{|x-y|} |\nabla_y u(t, x)|^2 dx dt \\ &+ \int_0^T \int_{\mathbb{R}^3} \frac{2\mu}{|x-y|} \{ |u|^2 f(|u|^2) - F(|u|^2) \} dx dt. \end{aligned}$$

Assuming (93) has a repulsive nonlinearity as in (104), all terms on the right side of the inequality (109) are positive. The inequality therefore gives in particular a bound uniform in T for the quantity $\int_0^T \int_{\mathbb{R}^3} \frac{|u(t, x)|^4}{|x-y|} dx dt$, for solutions u of the defocusing (1).

In their proof of scattering in the energy space for the cubic defocusing problem (1), Ginibre and Velo [40] combine this relatively localized²⁴ decay estimate with a bound surrogate for finite propagation speed in order to show the solution is in certain global-in-time Lebesgue spaces $L^q([0, \infty), L^r(\mathbb{R}^3))$. Scattering follows rather quickly, as will be shown later.

In the following section, we show how to establish an unweighted, global in time Lebesgue space bound directly. The argument below involves the identity (106), but our estimate arises eventually from the linear part of the equation, more specifically from the first term on the right of (106), rather than the third (nonlinearity) term.

²⁴The bound mentioned here may be considered localized since it implies decay of the solution near the fixed point y , but doesn't preclude the solution staying large at a point which moves rapidly away from y , for example.

6.11. Morawetz interaction potential. Given a solution u of (93), we define the *Morawetz interaction potential* to be

$$(110) \quad M(t) = \int_{\mathbb{R}^3} |u(t, y)|^2 M_y(t) dy.$$

The bound (107) immediately implies

$$(111) \quad |M(t)| \lesssim \|u(t)\|_{L^2}^2 \|u(t)\|_{\dot{H}_x^{\frac{1}{2}}}^2.$$

If u solves (93) then the identity (106) gives us the following identity for $\frac{d}{dt} M(t)$,

$$(112) \quad \begin{aligned} \frac{d}{dt} M(t) &= 4\pi\alpha \int_y |u(y)|^4 dy + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{2\alpha}{|x-y|} |u(y)|^2 |\nabla_y u(x)|^2 dx dy \\ &\quad + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{2\mu}{|x-y|} |u(y)|^2 \{ |u(x)|^2 f(|u(x)|^2) - F(|u(x)|^2) \} dx dy \\ &\quad + \int_{\mathbb{R}^3} \partial_t (|u(t, y)|^2) M_y(t) dy. \end{aligned}$$

We write the right side of (112) as $I + II + III + IV$, and work now to rewrite this as a sum involving nonnegative terms.

Proposition 6.12. *Referring to the terms comprising (112), we have*

$$(113) \quad IV \geq -II.$$

Consequently, solutions of (93) satisfy

$$(114) \quad \frac{d}{dt} M(t) \geq 4\pi\alpha \int_{\mathbb{R}^3} |u(t, y)|^4 dy + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{2\mu}{|x-y|} |u(t, y)|^2 \{ |u|^2 f(|u|^2) - F(|u|^2) \} dx dy.$$

In particular, $M(t)$ is monotone increasing for equations with repulsive nonlinearities.

Assuming Proposition 6.12 for the moment, we combine (111) and (114) to obtain the following estimate which plays the major new role in our scattering analysis below,

Corollary 6.13. *Take u to be a smooth solution to the initial value problem (93)-(94) above, under the repulsivity assumption (104). Then we have the following interaction Morawetz inequalities,*

$$(115) \quad \begin{aligned} 2\|u(0)\|_{L^2}^2 \sup_{t \in [0, T]} \|u(t)\|_{\dot{H}_x^{\frac{1}{2}}}^2 &\gtrsim 4\pi\alpha \int_0^T \int_{\mathbb{R}^3} |u(t, y)|^4 dy dt \\ &\quad + \int_0^T \int_y \int_x \frac{2\mu}{|x-y|} |u(t, y)|^2 \{ |u|^2 f(|u|^2) - F(|u|^2) \} (t, x) dx dy dt. \end{aligned}$$

In particular, we obtain the following spacetime $L^4([0, \infty) \times \mathbb{R}^3)$ estimate,

$$(116) \quad \int_0^T \int_{\mathbb{R}^3} |u(t, y)|^4 dy dt \lesssim \|u_0\|_{L^2(\mathbb{R}^3)}^2 \sup_{t \in [0, T]} \|u(t)\|_{\dot{H}_x^{\frac{1}{2}}}^2.$$

Of course, for solutions of the defocusing IVP (1) starting from finite energy initial data, the right side of (116) is uniformly bounded by energy considerations - leading to a rather direct proof of the result in [40] of scattering in the energy space that we will present below.

Proof. We now turn to the proof of Proposition 6.12. Use (101) to write

$$\begin{aligned} IV &= - \int_{\mathbb{R}_y^3} \nabla \cdot \text{Im}[2\alpha\bar{u}(y)\nabla u(y)]M_y(t)dy \\ &= - \int_y \int_x \partial_{y_l} \text{Im}[2\alpha\bar{u}(y)\partial_{y_l} u(y)] \text{Im}[\bar{u}(x)\frac{x_m - y_m}{|x - y|}\partial_{x_m} u(x)]dxdy, \end{aligned}$$

where repeated indices are implicitly summed. We integrate by parts in y , moving the leading ∂_{y_l} to the unit vector $\frac{x-y}{|x-y|}$. Note that,

$$(117) \quad \partial_{y_l} \left(\frac{x_m - y_m}{|x - y|} \right) = \frac{-\delta_{lm}}{|x - y|} + \frac{(x_l - y_l)(x_m - y_m)}{|x - y|^3}.$$

Write $\mathbf{p}(x) = \text{Im}[\bar{u}(x)\nabla u(x)]$ for the mass current at x and use (117) to obtain

$$(118) \quad IV = -2\alpha \int_y \int_x \left[\mathbf{p}(y) \cdot \mathbf{p}(x) - (\mathbf{p}(y) \cdot \frac{x-y}{|x-y|})(\mathbf{p}(x) \cdot \frac{x-y}{|x-y|}) \right] \frac{dxdy}{|x-y|}.$$

The preceding integrand has a natural geometric interpretation. We are removing the inner product of the components of $\mathbf{p}(y)$ and $\mathbf{p}(x)$ parallel to the vector $\frac{x-y}{|x-y|}$ from the full inner product of $\mathbf{p}(y)$ and $\mathbf{p}(x)$. This amounts to taking the inner product of $\pi_{(x-y)^\perp} \mathbf{p}(y) \cdot \pi_{(x-y)^\perp} \mathbf{p}(x)$ where we have introduced the projections onto the subspace of \mathbb{R}^3 perpendicular to the vector $\frac{x-y}{|x-y|}$. But

$$(119) \quad |\pi_{(x-y)^\perp} \mathbf{p}(y)| = \left| \mathbf{p}(y) - \frac{x-y}{|x-y|} \left(\frac{x-y}{|x-y|} \cdot \mathbf{p}(y) \right) \right| = |\text{Im}[\bar{u}(y)\nabla_x u(y)]| \leq |u(y)| \cdot |\nabla_x u(y)|.$$

A similar identity and inequality holds upon switching the roles of x and y in (119). We have thus shown that

$$(120) \quad IV \geq -2\alpha \int_y \int_x |u(x)| \cdot |\nabla_y u(x)| \cdot |u(y)| \cdot |\nabla_x u(y)| \frac{dxdy}{|x-y|}.$$

The conclusion (113) follows by applying the elementary bound $|ab| \leq \frac{1}{2}(a^2 + b^2)$ with $a = |u(y)| \cdot |\nabla_y u(x)|$ and $b = |u(x)| \cdot |\nabla_x u(y)|$. \square

We now state the following theorem as an example of how to use Morawetz interaction estimates in order to prove scattering

Theorem 6.14. *Consider the cubic, defocusing, NLS (1) in \mathbb{R}^3 . Then the wave operator exists and there is asymptotic completeness.*

Remark 6.15. Theorem 6.14 is not the best known result for this cubic NLS. In fact in [28] this same IVP was considered and the $L_t^4 L_x^4$ Morawetz estimate was used to prove scattering below H^1 . For other H^1 subcritical scattering results one should also consult [75] when $n \geq 3$, [32] when $n = 2$ and [20] when $n = 1$. In these cases if one wants to show scattering with regularity $s < 1$, for example when $n = 3$ in [28], the argument is more complicated than the one described for H^1 since one has to prove that the H^s norm of the solution is bounded by using the ‘‘I-method’’ as in Lecture # 4. The basic idea though is the same.

Proof. Existence of Ω_+ : we go back to the formula (92). The idea is to go first from $t = +\infty$ to $t = T$ for some $T > 0$ using some smallness and then solve the problem in the finite interval of time backward from T to 0.

We know already in what kind of spaces we can argue by contraction method: the space S^1 containing all the admissible Strichartz norm of the function and its derivatives and possibly also those that are embedded into these norms by the Sobolev theorem. But in this case there is

one more request that we want to make. We want a smallness assumption, possibly obtained by shrinking the time interval or better by taking the time interval at infinity where the “tail” of the function lives. For this reason we should avoid any norm that contains a L_t^∞ . So we proceed in two steps first we consider the smaller space \tilde{S}^1 given by the norm

$$\|f\|_{\tilde{S}^1} = \|f\|_{L_t^5 L_x^5} + \|f\|_{L_t^{10/3} W_x^{1,10/3}}.$$

Notice that by Sobolev

$$\|f\|_{L_t^5 L_x^5} \lesssim \|f\|_{L_t^5 W_x^{1,30/11}}$$

and $(5, 30/11)$ is a Strichartz admissible pair. It follows that if $u_+ \in H^1$ then by (12)

$$(121) \quad \|S(t)u_+\|_{\tilde{S}^1_{[T,\infty)}} \leq \epsilon$$

for T large enough. From (92) if we define

$$(122) \quad Lv(t) = S(t)u_+ + i \int_t^\infty S(t-t')(|v(t')|^{p-1}v(t')) dt',$$

and we use (13), where we pick the couple $(\tilde{q}, \tilde{r}) = (10/3, 10/3)$, we have

$$(123) \quad \|Lv\|_{\tilde{S}^1_{[T,\infty)}} \leq \epsilon + C\| |v|^2 \nabla v \|_{L_{[T,\infty)}^{10/7} L_x^{10/7}} \leq \epsilon + C\|v\|_{L_{[T,\infty)}^5 L_x^5} \|v\|_{L_{[T,\infty)}^{10/3} W_x^{1,10/3}} = \epsilon + C\|v\|_{\tilde{S}^1_{[T,\infty)}}^3.$$

and with a similar estimate

$$(124) \quad \|Lv - Lw\|_{\tilde{S}^1_{[T,\infty)}} \leq C(\|v\|_{\tilde{S}^1_{[T,\infty)}} + \|w\|_{\tilde{S}^1_{[T,\infty)}})\|v - w\|_{\tilde{S}^1_{[T,\infty)}}.$$

and thanks to the presence of ϵ one can proceed with the contraction argument. This would give a solution, which in particular has the property that

$$(125) \quad \|u\|_{\tilde{S}^1_{[T,\infty)}} \lesssim \epsilon$$

But we didn't prove that this solution is in $C([T, \infty), H^1)$ for example. To do this we need to go back and estimate the solution u in the Strichartz space $S^1_{[T,\infty)}$. We in fact have by (12) and (13)

$$\|u\|_{S^1} \leq C\|u_+\|_{H^1} + C\| |u|^2 \nabla u \|_{L_{[T,\infty)}^{10/7} L_x^{10/7}}$$

and from (125)

$$\|u\|_{S^1} \leq C\|u_+\|_{H^1} + C\|u\|_{\tilde{S}^1_{[T,\infty)}}^3 \lesssim \|u_+\|_{H^1},$$

and we are done in the interval $[T, \infty)$.

We now need to proceed from $t = T$ back to $t = 0$. Since the problem is subcritical, an iteration of local well-posedness like we presented in Lecture # 4, using the conservation of the energy and mass, will suffice to cover the finite interval $[0, T]$.

Invertibility of Ω_+ : This is the proof of scattering and we need to go back to (91). From here we see that we only need to show that the integral involving the global solution u

$$\int_0^\infty S(t)(|u|^2 u)(t) dt$$

converges in H^1 . By the dual of the homogeneous Strichartz estimate (12) we have that

$$\begin{aligned} \left\| \int_0^\infty S(t)|u|^2 u(t) dt \right\|_{H^1} &\lesssim \| |u|^2 \nabla u \|_{L_t^{10/7} L_x^{10/7}} \\ &\lesssim C\|u\|_{L_t^5 L_x^5}^2 \|u\|_{L_t^{10/3} W_x^{1,10/3}} \lesssim \|u\|_{S^1}^3. \end{aligned}$$

Clearly to conclude it would be enough to show that $\|u\|_{S^1} \leq C$. This is in fact proved in the following proposition. □

Proposition 6.16. *Assume that u is the H^1 global solution to the cubic, defocusing NLS in \mathbb{R}^3 . Then*

$$\|u\|_{S^1} \leq C.$$

Proof. We first observe that (116) provides a bound in $L_t^4 L_x^4$. It is to be noted that in \mathbb{R}^3 this norm is not an admissible Strichartz norm so we need to do a bit more work. We start by picking $\epsilon \ll 1$ to be defined later and intervals of time I_k , $k = 1, \dots, M < \infty$ such that

$$(126) \quad \|u\|_{L_{I_k}^4 L_x^4} \leq \epsilon,$$

for all $k = 1, \dots, M$. We now work on each separate interval and at the end we put everything back together. Since for now I_k is fixed we drop the index k and we set $I = [a, b]$. By the Duhamel principle and (12) and (13) we have as above

$$(127) \quad \|u\|_{S_I^1} \lesssim \|u(a)\|_{H^1} + \|u\|_{L_I^5 L_x^5}^2 \|u\|_{L_I^{10/3} W_x^{1,10/3}}.$$

It is important to notice that $10/3 < 4 < 5 < 10$, where $(10/3, 10/3)$ is an admissible pair in the L^2 sense and $(10, 10)$ is admissible in the H^1 sense since by Sobolev

$$\|u\|_{L_t^{10} L_x^{10}} \leq \|u\|_{L_t^{10} W_x^{1,30/13}},$$

and $(10, 30/13)$ is an admissible pair. It follows by interpolation and (126) that

$$\|u\|_{L_I^5 L_x^5} \lesssim \epsilon^\alpha \|u\|_{S_I^1}^{1-\alpha},$$

for some $\alpha > 0$. As a consequence (127) gives

$$\|u\|_{S_I^1} \lesssim \|u(a)\|_{H^1} + \epsilon^{2\alpha} \|u\|_{S_I^1}^{3-2\alpha},$$

and since the H^1 norm is uniformly bounded by energy and mass we have

$$(128) \quad \|u\|_{S_I^1} \lesssim 1 + \epsilon^{2\alpha} \|u\|_{S_I^1}^{3-2\alpha}.$$

We now use a continuity argument. Set $X(t) = \|u\|_{S_{[a, a+t]}^1}$. One can easily prove that $X(t)$ is continuous. From (128) we have

$$X(t) \lesssim 1 + \epsilon^{2\alpha} X(t)^{3-2\alpha}.$$

Then if ϵ is small enough there exist $X_0 < X_1$ such that either $X(t) \leq X_0$ or $X(t) \geq X_1$. But since $X(0) \lesssim 1$ and $X(t)$ is continuous it follows that $X(t) \leq X_0$ for all $t \in I$. This conclusion can be made for all I_k , $k = 1, \dots, M$ and this concludes the proof. □