5. Lecture # 4: Global well-posedness for the $H^1(R^n)$ subcritical NLS and the "I-method"

We learned during last lecture that for the H^1 subcritical NLS, i.e. $1 < p < 1+\frac{4}{n-2}$ and hence $s_c < 1$, l.w.p for (1), either focusing or defocusing, is available in $H^s(\mathbb{R}^n)$ for any $s, s_c \leq s \leq 1$. We also learned that if $s = 1$, in the defocusing case, uniform g.w.p is a consequence of the conservation of mass and energy. We then ask: if $0 \leq s_c < s < 1$ is the defocusing NLS problem globally well posed in H^s? This problem is particularly interesting when we consider the L^2 critical NLS, i.e. $s_c = 0$ and $p = 1 + \frac{4}{n}$. In this case the L^2 norm cannot be used to iterate the l.w.p. since the time interval of existence also depends on the profile of the initial data. It is clear then that this is a difficult question since we are in a regime when the conservation of the L^2 norm is too little of an information and the conservation of the Hamiltonian cannot be used since the data has not enough regularity. It was exactly to answer these kinds of questions that the "I-method" [24, 25, 26, 27, 48, 49] was invented. Unfortunately the method is quite technical to be applied in higher dimensions in its full strength. The results that we will report below are not optimal and in general they concern the L^2 critical case $p = 1 + \frac{4}{n}$ since that one is the most interesting, but similar results are available for the general H^1 subcritical case when $s_c < 1$ (see [20, 75]). We will list below the state of the art at this point for this problem for the L^2 critical case. We will give references but we will not prove these theorems in full generality. At the end of this lecture we will prove a weaker result than the one stated here when $n = 2$, see Theorem 5.2. We should also say here that if one assumes radial symmetry, then the L^2 critical NLS for $n \geq 2$ has been proved to be globally well-posed both in the defocusing case and in focusing case with the assumption that the mass of the initial data is strictly less than the mass of the stationary solution. These results are contained in a series of very recent and deep papers [55, 56, 71, 72]. The point here is instead to address the question of global well-posedness without assuming radial symmetry and to present the "I-method".

Theorem 5.1 (G.w.p for (1) with $\lambda = 1$, $p = 1 + \frac{4}{n}$ and $n \geq 3$)**.** *The initial value problem* (1) *with* $\lambda = 1$, $p = 1 + \frac{4}{n}$ *is globally well-posed in* $H^s(\mathbb{R}^n)$, *for any* $1 \geq s > \frac{\sqrt{7}-1}{3}$ *when* $n = 3$, *and for any* $1 \geq s > \frac{-(n-2)+\sqrt{(n-2)^2+8(n-2)}}{4}$ *for* $n \geq 4$.

Here we have to assume that $s \leq 1$ since in general the non smoothness of the nonlinearity doesn't allow us to prove persistence of regularity. The proof of this theorem can be found in [34].

Theorem 5.2 (G.w.p for (1) with $\lambda = 1$, $p = 1 + \frac{4}{n}$ and $n = 2$)**.** *The initial value problem* (1) *with* $\lambda = 1, n = 2$ *and* $p = 3$ *is globally well-posed in* $H^s(\mathbb{R}^2)$, *for any* $1 > s > \frac{2}{5}$. *Moreover the solution satisfies*

$$\sup_{[0,T]} \|u(t)\|_{H^s} \leq C(1 + T)^{\frac{3s(1-s)}{2(5s-2)}}, \tag{52}$$

where the constant C *depends only on the index* s *and* $\|u_0\|_{L^2}$.

Here the theorem is stated only for $s < 1$ since we already know that global well-posedness for $s \geq 1$ follows from conservation of mass and energy as explained in the previous lecture [17].

[17] It is an open problem to obtain a polynomial bound like in (52) for this problem when $s > 1$ and the data are not radial. In fact if if $p > 3$ a uniform bound follows from scattering. But scattering is still an open problem for general data for the L^2 critical NLS. We should also stress that these kinds of polynomial bounds for higher Sobolev norms are particularly interesting since they are related to the *weak turbulence theory*, a topic that we will not address here.

For the proof of Theorem 5.2 see [32]. The argument is based on a combination of the "I-metod" as in [25, 26, 28] and a refined two dimensional Morawetz interaction inequality. This combination first appeared in [37].

Finally we recall the result for the L^2 critical problem for $n = 1$:

Theorem 5.3 (G.w.p for (1) with $\lambda = 1$, $p = 1 + \frac{4}{n}$ and $n = 1$)**.** *The initial value problem* (1) *with* $\lambda = 1, n = 1$ *and* $p = 5$ *is globally well-posed in* $H^s(\mathbb{R})$*, for any* $1 > s > \frac{1}{3}$*. Moreover the solution satisfies*

$$\sup_{[0,T]} \|u(t)\|_{H^s} \leq C(1+T)^{\frac{s(1-s)}{2(3s-1)}}, \tag{53}$$

where the constant C *depends only on the index* s *and* $\|u_0\|_{L^2}$*.*

For the proof of this theorem see [36].

As promised we sketch now the proof of a weaker result than the one reported in Theorem 5.2, namely g.w.p. for $s > 4/7$. This proof is a summary of the work that appeared in [26]. Since below we will often refer to a particular IVP we write it here once for all

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = |u|^2 u, \\ u(x,0) = u_0(x). \end{cases} \tag{54}$$

To start the argument we need to introduce some notation and state some lemma.

We will use the weighted Sobolev norms,

$$||\psi||_{X_{s,b}} \equiv ||\langle\xi\rangle^s \langle \tau - |\xi|^2\rangle^b \tilde{\psi}(\xi,\tau)||_{L^2(\mathbb{R}^n\times\mathbb{R})}. \tag{55}$$

Here $\tilde{\psi}$ is the space-time Fourier transform of ψ. We will need local-in-time estimates in terms of truncated versions of the norms (55),

$$||f||_{X^\delta_{s,b}} \equiv \inf_{\psi = f \text{on}[0,\delta]} ||\psi||_{X^\delta_{s,b}}. \tag{56}$$

We will often use the notation $\frac{1}{2}+ \equiv \frac{1}{2} + \epsilon$ for some universal $0 < \epsilon \ll 1$. Similarly, we shall write $\frac{1}{2}- \equiv \frac{1}{2} - \epsilon$, and $\frac{1}{2} - - \equiv \frac{1}{2} - 2\epsilon$.

For a Schrödinger admissible pair (q,r) we have what we will call the $L^q_t L^r_x$ Strichartz estimate,

$$||\phi||_{L^q_t L^r_x(\mathbb{R}^{n+1})} \lesssim ||\phi||_{X_{0,\frac{1}{2}+}}, \tag{57}$$

which can be proved to be a consequence of (57).

Finally, we will need a refined version of these estimates due to Bourgain [9].

Lemma 5.4. *Let* $\psi_1, \psi_2 \in X^\delta_{0,\frac{1}{2}+}$ *be supported on spatial frequencies* $|\xi| \sim N_1, N_2$*, respectively. Then for* $N_1 \leq N_2$*, one has*

$$||\psi_1 \cdot \psi_2||_{L^2([0,\delta]\times\mathbb{R}^2)} \lesssim \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} ||\psi_1||_{X^\delta_{0,\frac{1}{2}+}} ||\psi_2||_{X^\delta_{0,\frac{1}{2}+}}. \tag{58}$$

In addition, (58) *holds (with the same proof) if we replace the product* $\psi_1 \cdot \psi_2$ *on the left with either* $\overline{\psi}_1 \cdot \psi_2$ *or* $\psi_1 \cdot \overline{\psi}_2$*.*

This lemma is a consequence of the Theorem 2.4.

Problem 5.5. *Show how to deduce* (57) *and* (58)*.*

Hint: *Consider the space of frequencies both of time and space. Partition it into parabolic strips of approximate unit size. On each of these strips a function* ψ *can be viewed as a solution of the linear problem. Use the appropriate Strichartz or improved Strichartz on each of them and then sum with the appropriate weight.*

For rough initial data, with $s<1$, the energy is infinite, and so the conservation law (23) is meaningless. Instead, here we use the fact that a smoothed version of the solution of the IVP (54) has a finite energy which is *almost* conserved in time. We express this 'smoothed version' as follows.

Given $s<1$ and a parameter $N \gg 1$, define the multiplier operator

$$\widehat{I_N f}(\xi) \equiv m_N(\xi)\hat{f}(\xi), \tag{59}$$

where the multiplier $m_N(\xi)$ is smooth, radially symmetric, nonincreasing in $|\xi|$ and

$$m_N(\xi) = \begin{cases} 1 & |\xi| \leq N \\ \left(\frac{N}{|\xi|}\right)^{1-s} & |\xi| \geq 2N. \end{cases} \tag{60}$$

For simplicity, we will eventually drop the N from the notation, writing I and m for (59) and (60). Note that for solution and initial data u, u_0 of (54), the quantities $||u||_{H^s(\mathbb{R}^n)}(t)$ and $E(I_N u)(t)$ (see (23)) can be compared,

$$E(I_N u)(t) \leq \left(N^{1-s}||u(\cdot,t)||_{\dot{H}^s(\mathbb{R}^2)}\right)^2 + ||u(t,\cdot)||^4_{L^4(\mathbb{R}^2)}, \tag{61}$$

$$||u(\cdot,t)||^2_{H^s(\mathbb{R}^2)} \lesssim E(I_N u)(t) + ||u_0||^2_{L^2(\mathbb{R}^2)}. \tag{62}$$

Indeed, the $\dot{H}^1(\mathbb{R}^2)$ component of the left hand side of (61) is bounded by the right side by using the definition of I_N and by considering separately those frequencies $|\xi| \leq N$ and $|\xi| \geq N$. The L^4 component of the energy in (61) is bounded by the right hand side of (61) by using (for example) the Hörmander-Mikhlin multiplier theorem. The bound (62) follows quickly from (60) and L^2 conservation (21) by considering separately the $\dot{H}^s(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ components of the left hand side of (62).

To prove our result, we may assume that $u_0 \in C_0^\infty(\mathbb{R}^2)$, and show that the resulting global-in-time solution grows at most polynomially in the H^s norm,

$$||u(\cdot,t)||_{H^s(\mathbb{R}^2)} \leq C_1 t^M + C_2, \tag{63}$$

where the constants C_1, C_2, M depend only on $||u_0||_{H^s(\mathbb{R}^2)}$ and not on higher regularity norms of the smooth data. The result then follows immediately from (63), the local-in-time theory discussed in the previous lecture, and a standard density argument.

By (62), it suffices to show

$$E(I_N u)(t) \lesssim (1+t)^{2M}. \tag{64}$$

for some $N = N(t)$. (See (71), (72) below for the definition of N and the growth rate M we eventually establish.) The following proposition, represents an "almost conservation law" and will yield (64).

Proposition 5.6. *Given $s > \frac{4}{7}, N \gg 1$, and initial data $u_0 \in C_0^\infty(\mathbb{R}^2)$ (see preceeding remark) with $E(I_N\phi_0) \leq 1$, then there exists a $\delta = \delta(||u_0||_{L^2(\mathbb{R}^2)}) > 0$ so that the solution*

$$u(x,t) \in C([0,\delta], H^s(\mathbb{R}^2))$$

of (54) satisfies

$$E(I_N u)(t) = E(I_N u)(0) + O(N^{-\frac{3}{2}+}), \tag{65}$$

for all $t \in [0,\delta]$.

We first show that Proposition 5.6 implies (64). Recall that the initial value problem here has a scaling symmetry, and is H^s-subcritical when $1 > s > 0$, and $n = 2$. That is, if u is a solution, so too

$$u_\lambda(x,t) := \frac{1}{\lambda} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right). \tag{66}$$

Using (61), the following energy can be made arbitrarily small by taking λ large,

$$E(I_N u_{\lambda,0}) \leq \left((N^{2-2s})\lambda^{-2s} + \lambda^{-2}\right) \cdot (1 + ||u_0||_{H^s(\mathbb{R}^2)})^4 \tag{67}$$

$$\leq C_0(N^{2-2s}\lambda^{-2s}) \cdot (1 + ||u_0||_{H^s(\mathbb{R}^2)})^4. \tag{68}$$

It is important to remark that since the problem is L^2 critical, $\|u_0\|_{L^2} \sim \|u_{\lambda,0}\|_{L^2}$. Assuming $N \gg 1$ is given[18], we choose our scaling parameter $\lambda = \lambda(N, ||u_0||_{H^s(\mathbb{R}^2)})$

$$\lambda = N^{\frac{1-s}{s}} \left(\frac{1}{2C_0}\right)^{-\frac{1}{2s}} \cdot \left(1 + ||u_0||_{H^s(\mathbb{R}^2)}\right)^{\frac{2}{s}} \tag{69}$$

so that $E(I_N u_{\lambda,0}) \leq \frac{1}{2}$. We may now apply Proposition 5.6 to the scaled initial data $u_{\lambda,0}$, and in fact may reapply this Proposition until the size of $E(I_N u_\lambda)(t)$ reaches 1, that is at least $C_1 \cdot N^{\frac{3}{2}-}$ times. Hence

$$E(I_N u_\lambda)(C_1 N^{\frac{3}{2}-}\delta) \sim 1. \tag{70}$$

We now have to undo the scaling: given any $T_0 \gg 1$, we establish the polynomial growth (64) from (70) by first choosing our parameter $N \gg 1$ so that

$$T_0 \sim \frac{N^{\frac{3}{2}-}}{\lambda^2} C_1 \cdot \delta \sim N^{\frac{7s-4}{2s}-}, \tag{71}$$

where we've kept in mind (69). Note the exponent of N on the right of (71) is positive provided $s > \frac{4}{7}$, hence the definition of N makes sense for arbitrary T_0. In two space dimensions,

$$E(I_N u)(t) = \lambda^2 E(I_N u_\lambda)(\lambda^2 t).$$

We use (69), (70), and (71) to conclude that for $T_0 \gg 1$,

$$E(I_N u)(T_0) \leq C_2 T_0^{\frac{1-s}{\frac{7}{4}s-1}+}, \tag{72}$$

where N is chosen as in (71) and $C_2 = C_2(||u_0||_{H^s(\mathbb{R}^2)}, \delta)$. Together with (62), the bound (72) establishes the desired polynomial bound (63).

It remains then to prove Proposition 5.6. We will need the following modified version of the usual local existence theorem, wherein we control for small times the smoothed solution in the $X^\delta_{1,\frac{1}{2}+}$ norm.

Proposition 5.7. *Assume $\frac{4}{7} < s < 1$ and we are given data for the IVP (54) with $E(Iu_0) \leq 1$. Then there is a constant $\delta = \delta(||u_0||_{L^2(\mathbb{R}^2)})$ so that the solution u obeys the following bound on the time interval $[0, \delta]$,*

$$||Iu||_{X^\delta_{1,\frac{1}{2}+}} \lesssim 1. \tag{73}$$

[18]The parameter N will be chosen shortly.

Proof. We mimic the typical iteration argument showing local existence. We will need the following three estimates involving the $X_{s,\delta}$ spaces (55) and functions $F(x,t), f(x)$. (Throughout this section, the implicit constants in the notation $\lesssim$ are independent of δ.)

$$\|S(t)f\|_{X^{\delta}_{1,\frac{1}{2}+}} \lesssim \|f\|_{H^1(\mathbb{R}^2)}, \tag{74}$$

$$\left\|\int_0^t S(t-\tau)F(x,\tau)d\tau\right\|_{X_{1,\frac{1}{2}+}} \lesssim \|F\|_{X^{\delta}_{1,-\frac{1}{2}+}}, \tag{75}$$

$$\|F\|_{X^{\delta}_{1,-b}} \lesssim \delta^P \|F\|_{X^{\delta}_{1,-\beta}}, \tag{76}$$

where in (76) we have $0 < \beta < b < \frac{1}{2}$, and $P = \frac{1}{2}(1 - \frac{\beta}{b}) > 0$. The bounds (74), (75) are analogous to estimates (3.13), (3.15) in [57]. As for (76), by duality it suffices to show

$$||F||_{X^{\delta}_{-1,\beta}} \lesssim \delta^P ||F||_{X^{\delta}_{-1,b}}.$$

Interpolation gives

$$||F||_{X^{\delta}_{-1,\beta}} \lesssim ||F||^{(1-\frac{\beta}{b})-}_{X^{\delta}_{-1,0}} \cdot ||F||^{\frac{\beta}{b}}_{X^{\delta}_{-1,b}}.$$

As $b \in (0, \frac{1}{2})$, arguing exactly as on page 771 of [33],

$$||F||_{X^{\delta}_{-1,0}} \lesssim \delta^{\frac{1}{2}} ||F||_{X^{\delta}_{-1,b}},$$

and (76) follows.

Duhamel's principle gives us

$$\begin{aligned} ||Iu||_{X^{\delta}_{1,\frac{1}{2}+}} &= \left\|S(t)(Iu_0) + \int_0^t S(t-\tau)I(u\bar{u}u)(\tau)d\tau\right\|_{X^{\delta}_{1,\frac{1}{2}+}} \\ &\lesssim ||Iu_0||_{H^1(\mathbb{R}^2)} + ||I(u\bar{u}u)||_{X^{\delta}_{1,-\frac{1}{2}+}} \\ &\lesssim ||Iu_0||_{H^1(\mathbb{R}^2)} + \delta^{\epsilon}||I(u\bar{u}u)||_{X^{\delta}_{1,-\frac{1}{2}++}}, \end{aligned} \tag{77}$$

where $-\frac{1}{2}++$ is a real number slightly larger than $-\frac{1}{2}+$ and $\epsilon > 0$. By the definition of the restricted norm (56),

$$||Iu||_{X^{\delta}_{1,\frac{1}{2}+}} \lesssim ||Iu_0||_{H^1(\mathbb{R}^2)} + \delta^{\epsilon}||I(u\bar{u}u)||_{X_{1,-\frac{1}{2}++}}, \tag{78}$$

where the function ψ agrees with u for $t \in [0,\delta]$, and

$$||Iu||_{X^{\delta}_{1,\frac{1}{2}+}} \sim \quad ||I\psi||_{X_{1,\frac{1}{2}+}}. \tag{79}$$

We will show shortly that

$$||I(\psi\overline{\psi}\psi)||_{X_{1,-\frac{1}{2}++}} \lesssim \quad ||I\psi||^3_{X_{1,\frac{1}{2}+}}. \tag{80}$$

Setting then $Q(\delta) \equiv ||Iu(t)||_{X^{\delta}_{1,\frac{1}{2}+}}$, the bounds (77), (79) and (80) yield

$$Q(\delta) \lesssim ||Iu_0||_{H^1(\mathbb{R}^2)} + \delta^{\epsilon}(Q(\delta))^3. \tag{81}$$

Note

$$||Iu_0||_{H^1(\mathbb{R}^2)} \lesssim (E(Iu_0))^{\frac{1}{2}} + ||u_0||_{L^2(\mathbb{R}^2)} \lesssim 1 + ||u_0||_{L^2(\mathbb{R}^2)}. \tag{82}$$

As Q is continuous in the variable δ, a bootstrap argument yields (73) from (81), (82).

It remains to show (80). Using the interpolation lemma of [31], it suffices to show

$$||\psi\bar{\psi}\psi||_{X_{s,-\frac{1}{2}++}} \lesssim ||\psi||^3_{X_{s,\frac{1}{2}+}}, \tag{83}$$

for all $\frac{4}{7} < s < 1$. By duality and a "Leibniz" rule[19], (83) follows from

$$\left|\int_{\mathbb{R}}\int_{\mathbb{R}^2} (\langle\nabla\rangle^s u_1)\overline{u_2} u_3 u_4 dxdt\right| \lesssim ||u_1||_{X_{s,\frac{1}{2}+}} \cdot ||u_2||_{X_{s,\frac{1}{2}+}} \cdot ||u_3||_{X_{s,\frac{1}{2}+}} ||u_4||_{X_{0,\frac{1}{2}--}}. \tag{84}$$

Note that since the factors in the integrand on the left here will be taken in absolute value, the relative placement of complex conjugates is irrelevant. Use Hölder's inequality on the left side of (84), taking the factors in, respectively, $L^4_{x,t}, L^4_{x,t}, L^6_{x,t}$ and $L^3_{x,t}$. Using a Strichartz inequality,

$$\begin{aligned} ||\langle\nabla\rangle^s u_1||_{L^4_{x,t}(\mathbb{R}^{2+1})} &\lesssim ||\langle\nabla\rangle^s u_1||_{X_{0,\frac{1}{2}+}} \\ &= ||u_1||_{X_{s,\frac{1}{2}+}}, \end{aligned}$$

and

$$\begin{aligned} ||u_2||_{L^4_{x,t}(\mathbb{R}^{2+1})} &\lesssim ||u_2||_{X_{0,\frac{1}{2}+}} \\ &\lesssim ||u_2||_{X_{s,\frac{1}{2}+}}. \end{aligned}$$

The bound for the third factor uses Sobolev embedding and the $L^6_t L^3_x$ Strichartz estimate,

$$\begin{aligned} ||u_3||_{L^6_t L^6_x(\mathbb{R}^{2+1})} &\lesssim ||\langle\nabla\rangle^{\frac{1}{3}} u_3||_{L^6_t L^3_x(\mathbb{R}^{2+1})} \\ &\lesssim ||\langle\nabla\rangle^{\frac{1}{3}} u_3||_{X_{0,\frac{1}{2}+}} \\ &\leq ||u_3||_{X_{s,\frac{1}{2}+}}. \end{aligned}$$

It remains to bound $||u_4||_{L^3(\mathbb{R}^{2+1})}$. Interpolating between $||u_4||_{L^2_t L^2_x} \leq ||u_4||_{X_{0,0}}$ and the Strichartz estimate $||u_4||_{L^4_t L^4_x} \lesssim ||u_4||_{X_{0,\frac{1}{2}+}}$ yields

$$||u_4||_{L^3_t L^3_x} \lesssim ||u_4||_{X_{0,\frac{1}{2}--}}.$$

This completes the proof of (84), and hence Proposition 5.7. □

Before we proceed to the proof of Proposition 5.6 we would like to present the proof of conservation of mass[20] for (54) using Fourier transform. Understanding this proof is fundamental to understand the types of cancelations that will make $E(Iu)$ almost conserved.

Proposition 5.8. *Assume that u is a solution to* (54) *smooth and decaying at infinity. Then* $\|u(t)\|^2_{L^2} = \|u_0\|^2_{L^2}$.

Proof. We write this L^2 norm using Plancherel formula

$$\|u(t)\|^2_{L^2} = \int \hat{u}(\xi,t)\overline{\hat{u}}(\xi,t)\, d\xi$$

[19] By this, we mean the operator $\langle D\rangle^s$ can be distributed over the product by taking Fourier transform and using $\langle\xi_1 + \dots \xi_4\rangle^s \lesssim \langle\xi_1\rangle^s + \dots \langle\xi_4\rangle^s$.

[20] Actually showing the proof of conservation of energy would be even more appropriate here since in Proposition 5.6 we will be dealing with an energy instead of a mass, but clearly for the mass the calculation is less involved and the ideas are still present in full power!

Using the equation we then have

$$\begin{aligned}
\frac{d}{dt}\|u(t)\|_{L^2}^2 &= 2Re\int (\hat{u}(\xi,t))_t\bar{\hat{u}}(\xi,t)\,d\xi \\
&= -Im\int |\xi|^2\hat{u}(\xi,t)\bar{\hat{u}}(\xi,t)\,d\xi - 2Im\int \widehat{u^2\bar{u}}(\xi)\bar{\hat{u}}(\xi,t)\,d\xi \\
&= -2Im\int_{\xi_1+\xi_2+\xi_3-\xi=0} \hat{u}(\xi_1)\bar{\hat{u}}(-\xi_2)\hat{u}(\xi_3)\bar{\hat{u}}(\xi)\,d\xi d\xi_1 d\xi_2 d\xi_3 \\
&= -2Im\int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \hat{u}(\xi_1)\bar{\hat{u}}(-\xi_2)\hat{u}(\xi_3)\bar{\hat{u}}(-\xi_4)\,d\xi_1 d\xi_2 d\xi_3 d\xi_4
\end{aligned}$$

and by symmetry

$$\begin{aligned}
&2Im\int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \hat{u}(\xi_1)\bar{\hat{u}}(-\xi_2)\hat{u}(\xi_3)\bar{\hat{u}}(-\xi_4)\,d\xi_1 d\xi_2 d\xi_3 d\xi_4 = \\
&Im\int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \hat{u}(\xi_1)\bar{\hat{u}}(-\xi_2)\hat{u}(\xi_3)\bar{\hat{u}}(-\xi_4)\,d\xi_1 d\xi_2 d\xi_3 d\xi_4 \\
+\;&Im\int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \hat{u}(-\xi_2)\bar{\hat{u}}(\xi_1)\hat{u}(-\xi_4)\bar{\hat{u}}(\xi_3)\,d\xi_1 d\xi_2 d\xi_3 d\xi_4 = 0
\end{aligned}$$

□

Problem 5.9. *Prove the conservation of energy* (23) *by using Fourier transform.*

Proof of Proposition 5.6. The usual energy (23) is shown to be conserved by differentiating in time, integrating by parts, and using the equation (54),

$$\begin{aligned}
\partial_t E(u) &= \mathrm{Re}\int_{\mathbb{R}^2} \overline{u_t}(|u|^2u - \Delta u)dx \\
&= \mathrm{Re}\int_{\mathbb{R}^2} \overline{u_t}(|u|^2u - \Delta u - iu_t)dx \\
&= 0.
\end{aligned}$$

We follow the same strategy to estimate the growth of $E(Iu)(t)$,

$$\begin{aligned}
\partial_t E(Iu)(t) &= \mathrm{Re}\int_{\mathbb{R}^2} \overline{I(u)_t}(|Iu|^2Iu - \Delta Iu - iIu_t)dx \\
&= \mathrm{Re}\int_{\mathbb{R}^2} \overline{I(u)_t}(|Iu|^2Iu - I(|u|^2u))dx,
\end{aligned}$$

where in the last step we've applied I to (54). When we integrate in time and apply the Parseval formula[21] it remains for us to bound

$$E(Iu(\delta)) - E(Iu(0)) = \int_0^\delta \int_{\sum_{j=1}^4 \xi_j=0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)\cdot m(\xi_3)\cdot m(\xi_4)}\right) \widehat{\overline{I\partial_t u}}(\xi_1)\widehat{Iu}(\xi_2)\widehat{\overline{Iu}}(\xi_3)\widehat{Iu}(\xi_4). \tag{85}$$

The reader may ignore the appearance of complex conjugates here and in the sequel, as they have no impact on the availability of estimates. (See e.g. Lemma 5.4 above.) We include the complex conjugates for completeness.

[21] That is, $\int_{\mathbb{R}^n} f_1(x)f_2(x)f_3(x)f_4(x)dx = \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)\hat{f}_4(\xi_4)$ where $\int_{\sum_i \xi_i=0}$ here denotes integration with respect to the hyperplane's measure
$\delta_0(\xi_1+\xi_2+\xi_3+\xi_4)d\xi_1 d\xi_2 d\xi_3 d\xi_4$, with δ_0 the one dimensional Dirac mass.

We use the equation to substitute for $\partial_t I(u)$ in (85). Our aim is to show that

$$\mathrm{Term}_1 + \mathrm{Term}_2 \lesssim N^{-\frac{3}{2}+}, \tag{86}$$

where the two terms on the left are

$$\mathrm{Term}_1 \equiv \left| \int_0^\delta \int_{\sum_{i=1}^4 \xi_i = 0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right) \widehat{(\Delta \overline{Iu})}(\xi_1) \cdot \widehat{Iu}(\xi_2) \cdot \widehat{\overline{Iu}}(\xi_3) \cdot \widehat{Iu}(\xi_4) \right| \tag{87}$$

$$\mathrm{Term}_2 \equiv \left| \int_0^\delta \int_{\sum_{i=1}^4 \xi_i = 0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right) \widehat{(\overline{I(|u|^2u)})}(\xi_1) \cdot \widehat{Iu}(\xi_2) \cdot \widehat{\overline{Iu}}(\xi_3) \cdot \widehat{Iu}(\xi_4) \right|. \tag{88}$$

From this point on the proof proceeds with a case by case analysis based on the relative magnitude of various frequencies. The basic cancellation of the type we presented in the proof of Proposition 5.8 are fundamental as is the fact that the multiplier is smooth. We send the reader to the original paper for a complete proof.

□

Remark 5.10. Here we only gave an idea of the "I-method". One can implement it in more effective ways by defining formally families of energies that, if controlled analytically, are proved to be more and more *almost conserved*. This was in fact the case for the one dimensional derivative NLS [24, 25] and the KdV [27] for example. Unfortunately controlling these families of energies becomes more difficult in higher dimensions since orthogonality issues start appearing, see for example [30].