

4. LECTURE # 3: LOCAL AND GLOBAL WELL-POSEDNESS FOR THE  $H^1(\mathbb{R}^n)$  SUBCRITICAL NLS

Our intuition suggests that if one assumes enough regularity then l.w.p. should be true basically for any  $p > 1$ . We do not prove this here but one can check this in [19, 69], or use the argument that we will present below and the fact that for  $s > n/2$  the space  $H^s$  is an algebra to obtain this result directly. Here we consider instead the IVP (1) with a nonlinearity that is  $H^1$  subcritical, that is  $1 < p < 1 + \frac{4}{n-2}$  for  $n \geq 3$  and  $1 < p < \infty$  for  $n = 1, 2$ . To prove l.w.p for  $H^s(\mathbb{R}^n)$ , the general strategy that we will follow is based on the contraction method. This method is based on these four steps:

(1) Definition of the operator

$$L(v) = \chi(t/T)S(t)u_0 + c\chi(t/T) \int_0^t S(t-t')|v|^{p-1}v(t') dt'$$

where  $\chi(r)$  denotes a smooth nonnegative bump even function, supported on  $-2 \leq r \leq 2$  and satisfying  $\chi(r) = 1$  for  $-1 \leq r \leq 1$ .

(2) Definition of a Banach space  $X$  such that  $X \subset L^\infty([-T, T], H^s(\mathbb{R}^n))$ .

(3) Proof of the fact that for any ball  $B \subset H^s(\mathbb{R}^n)$ , there exist  $T$  and a ball  $B_X \subset X$  such that the operator  $L$  sends  $B_X$  into itself and it is a contraction there.

(4) Extension of the uniqueness result in  $B_X$  to a unique result in the whole space  $X$ .

We observe that the continuity with respect to the initial data will be a consequence of the fact that the solution is found through a contraction argument. In fact in this case we obtain way more than just continuity.

**Problem 4.1.** *Discuss the regularity of the map  $u_0 \rightarrow u$  from  $H^s$  into  $C([-T, T], H^s(\mathbb{R}^n))$  when l.w.p. is proved by contraction method.*

We state the main theorem:

**Theorem 4.2.** *Assume that  $1 < p < 1 + \frac{4}{n-2}$  for  $n \geq 3$  and  $1 < p < \infty$  for  $n = 1, 2$ . Then the IVP (1) is l.w.p in  $H^s(\mathbb{R}^n)$  for all  $s_c < s \leq 1$ , where  $s_c = \frac{n}{2} - \frac{2}{p-1}$ . Moreover if the nonlinearity is algebraic, that is  $n = 2, 3$  and  $p = 3$ , then there is persistence of regularity, that is if  $u_0 \in H^m$ ,  $m \geq 1$  then the solution  $u(t) \in H^m(\mathbb{R}^n)$ , for all  $t$  in its time of existence. If in (1) we assume that  $\lambda = 1$  (defocusing) then the IVP is uniformly global well-posed for  $s = 1$ .*

Here we prove a less general version of this theorem, namely that in the conditions given above on  $p$  there is uniform g.w.p in  $H^1$ . We do not prove l.w.p. for  $s_c < s \leq 1$  since we would need to introduce a product rule for fractional derivatives and it would become too technical.

Our starting point is the definition of a Banach space  $X$  based on the norms we introduced with the Strichartz estimates.

**Definition 4.3.** Assume  $I = [-T, T]$  is fixed. The space  $S^0(I \times \mathbb{R}^n)$  is the closure of the Schwartz functions under the norm

$$\|f\|_{S^0(I \times \mathbb{R}^n)} = \sup_{(q,r) \text{ } n\text{-admissible}} \|f\|_{L_t^q L_x^r}.$$

We then define the space  $S^1(I \times \mathbb{R}^n)$  where the closure is taken with respect to the norm

$$\|f\|_{S^1(I \times \mathbb{R}^n)} = \|f\|_{S^0(I \times \mathbb{R}^n)} + \|\nabla f\|_{S^0(I \times \mathbb{R}^n)}.$$

*Proof.* We consider the operator  $Lv$  and using (12) and (13) we obtain

$$(38) \quad \|Lv\|_{S^1(I \times \mathbb{R}^n)} \leq C_1 \|u_0\|_{H^1} + C_2 \| |v|^{p-1} |\nabla v| \|_{L_t^{q'} L_x^{r'}},$$

where  $(q, r)$  is a Strichartz admissible pair. The best couple to use in this context is the one that solves the system

$$(39) \quad \frac{2}{q} + \frac{n}{r} = \frac{n}{2} \quad \text{Strichartz Condition}$$

$$(40) \quad (p-1) \left( \frac{1}{r} - \frac{s}{n} \right) = \frac{1}{r'} - \frac{1}{r},$$

and the meaning of the second equation will become clear below. The solutions to the system is

$$\frac{1}{r} = \frac{1}{(p+1)} + \frac{(p-1)s}{(p+1)n} \quad \text{and} \quad \frac{1}{q} = \frac{(p-1)(n-2s)}{4(p+1)}.$$

From here it follows that<sup>12</sup>

$$\frac{1}{q'} > \frac{p}{q} \implies s > s_c = \frac{n}{2} - \frac{2}{p-1}.$$

Then by Hölder inequality repeated

$$\| |v|^{p-1} |\nabla v| \|_{L_t^{q'} L_x^{r'}} \leq T^\alpha \| |v|^{p-1} |\nabla v| \|_{L_t^{q/p} L_x^{r'}} \leq \| \nabla v \|_{L_t^q L_x^r} \| v \|_{L_t^q L_x^{\tilde{r}}}^{p-1}$$

where  $\frac{1}{\tilde{r}} = \frac{1}{r} - \frac{s}{n}$ . By Sobolev embedding

$$(41) \quad \| v \|_{L_t^q L_x^{\tilde{r}}} \lesssim \| (1 + \Delta^{\frac{s}{2}}) v \|_{L_t^q L_x^r},$$

and since we are assuming that we are in the  $H^1$  subcritical regime  $1 < p < 1 + \frac{4}{n-2}$  it also follows that  $s \leq 1$  and as a consequence

$$\| |v|^{p-1} |\nabla v| \|_{L_t^{q'} L_x^{r'}} \leq T^\alpha \| v \|_{S^1}^p.$$

We can now conclude that

$$(42) \quad \| Lv \|_{S^1(I \times \mathbb{R}^n)} \leq C_1 \| u_0 \|_{H^1} + C_2 T^\alpha \| v \|_{S^1}^p.$$

With similar arguments one also obtains

$$(43) \quad \| Lv - Lw \|_{S^1(I \times \mathbb{R}^n)} \leq C_2 T^\alpha (\| v \|_{S^1}^{p-1} + \| w \|_{S^1}^{p-1}) \| v - w \|_{S^1}.$$

We are now ready to set up the contraction: pick  $R = 2C_1 \| u_0 \|_{H^1}$  and  $T$  such that

$$(44) \quad C_2 T^\alpha R^{p-1} < \frac{1}{2} \iff T \lesssim \| u_0 \|_{H^1}^{\frac{1-p}{\alpha}},$$

then clearly from (42), (43) and (44) it follows that  $L : B_R \rightarrow B_R$ , where  $B_R$  is the ball centered at zero and radius  $R$  in  $S^1$ , and  $L$  is a contraction. There is a unique fixed point  $u \in B_R$  that is in fact a solution to our integral equation. The next two properties for  $u$  that we need to show are continuity with respect to time, that is  $u \in C([-T, T], H^1)$  and uniqueness in the whole space  $S^1$ . The first is left to the reader since it is a simple consequence of the representation of  $u$  through Duhamel formula (6). For the second we assume that there exists another solution  $\tilde{u} \in S^1$  for the IVP (1). Using again the Duhamel formula for both  $u$  and  $\tilde{u}$  and the estimates presented above for  $Lv$  we obtain that on an interval of time  $\delta$

$$\| u - \tilde{u} \|_{S_\delta^1} \leq C_2 \delta^\alpha (\| \tilde{u} \|_{S_T^1}^{p-1} + \| u \|_{S_T^1}^{p-1}) \| u - \tilde{u} \|_{S_\delta^1}$$

<sup>12</sup>As mentioned above here we only address l.w.p. in  $H^1$ , but it is clear that if one uses fractional derivatives and (41) l.w.p in  $H^s$ ,  $s > s_c$  can also be obtained based on the fact that  $r$  and  $q$  are given in terms of  $s$  and  $s > s_c$ .

where here we use the lower index  $\delta$  or  $T$  to stress that in the first case the space  $S^1$  is relative to the interval  $[-\delta, \delta]$  and in the second to  $[-T, T]$ . Since  $u$  and  $\tilde{u}$  are fixed we can introduce

$$M = \max(\|\tilde{u}\|_{S_T^1}^{p-1} + \|u\|_{S_T^1}^{p-1})$$

and if  $\delta$  is small enough in terms of  $C_2, \alpha$  and  $M$  we obtain

$$\|u - \tilde{u}\|_{S_\delta^1} \leq \frac{1}{2} \|u - \tilde{u}\|_{S_\delta^1}$$

which forces  $u = \tilde{u}$  in  $[-\delta, \delta]$ . To cover the whole interval  $[-T, T]$  then one iterates this argument  $\frac{T}{\delta}$  times and the conclusion follows.

Before going to the proof of g.w.p we would like to consider the question of **propagation of regularity**. As mentioned above with this we mean the answer to the following question: assume that in (1), with the restrictions on  $p$  above, we start with  $u_0 \in H^m$ ,  $m \geq 1$ . Is it true that the unique solution  $u \in S^1$  has the same property at any later time  $t \in [0, T]$ ? The answer to this depends on the regularity of the non-linear term, more precisely the regularity of the function  $f(z) = |z|^{p-1}z$ . This function is not  $C^\infty$  for all  $p$ , hence one cannot expect propagation of regularity for all  $p$  in the considered range. On the other hand if  $f$  is algebraic, namely when  $p - 1 = 2k$  for some  $k \in \mathbb{N}$ , then propagation of regularity follows from the estimates we presented above. Briefly we can go back to (42) and if we repeat the same argument we obtain that for the solution  $u$  that we already found using only  $H^1$  regularity we also have

$$\|D^m u\|_{S^0} \leq C_1 \|u_0\|_{H^m} + C_2 T^\alpha \|u\|_{S^1}^{2k} \|D^m u\|_{S^0}$$

because when we apply the operator  $D^m$  the term with  $D^m u$  appears linearly<sup>13</sup>. Since we already know that  $C_2 T^\alpha \|u\|_{S^1}^{2k} \leq \frac{1}{2}$  we then obtain<sup>14</sup> that

$$\|D^m u\|_{S^0} \lesssim \|u_0\|_{H^m}.$$

We are now ready for the iteration of the local in time solution  $u$  to a uniformly global one<sup>15</sup>. The first step is to go back to (44) and notice that  $T$  depends on the  $H^1$  norm of the initial data. From the previous lecture we learned that for a *smooth*<sup>16</sup> solution  $u$  to (1) the conservation of the energy and mass gives an a priori uniform bound

$$\|u(t)\|_{H^1} \leq C^*(\|u_0\|_H^1),$$

so if we take now  $T^* \sim (C^*)^{\frac{1-p}{\alpha}}$  we can repeat the argument above with no changes. In particular when we get to time  $T^*$  we can apply the argument again with the new initial data  $u(T^*)$  and the same  $T^*$  will work. In this way we can cover the whole time real line and the well-posedness becomes global. But in the argument we just outlined there is a *caviat* in the sense that if  $u_0 \in H^1$  we do not have a *smooth* solution  $u$ . This obstacle can be overcome by introducing various smoothing tools. The precise argument can be found in [19].  $\square$

<sup>13</sup>Here we are cheating a little since we are ignoring the mixed lower order derivatives. For this reason the constant  $C_2$  is the same as the one in (42). If one does this calculation correctly then that constant  $C_2$  will need to be replaced by a larger one, which will shrink the time  $T$ . To cover the whole interval  $[-T, T]$  then one uses the iteration we introduced while proving uniqueness in  $S^1$ .

<sup>14</sup>Here we are cheating again in the sense that in principle we cannot even talk about  $D^m u$  since we don't know yet that this expression makes sense. The rigorous procedure tells us to start with a smooth approximation of the initial data, the associated solution exists and is unique. Only at this point one can use the argument proposed here to get the uniform bound independent of the approximation.

<sup>15</sup>This argument only works when a uniform  $H^1$  bound in time for the solution is available, for example in the defocusing case or when the  $L^2$  norm of the initial data is small enough.

<sup>16</sup>Here with smooth we also mean zero at infinity.

*Remark 4.4.* We are not addressing in this first part of the course g.w.p. for the focusing NLS (1) even in the subcritical case. In order to address this issue we need to introduce stationary solutions (or solitons) and this will be done later. But assuming that the readers know about solitons and the Gagliardo-Nirenberg inequality, then it is easy to see that l.w.p can be extended to g.w.p. as long as the mass of the initial data is strictly below the mass of the stationary solution. This condition in fact can be used with the Gagliardo-Nirenberg inequality to show also in this case that the  $H^1$  norm of the solutions remains uniformly bounded in time.

*Remark 4.5.* By carefully keeping track of the various exponents that have been introduced in order to get to (42) one can see that for the **critical**  $H^1$  problem, that is  $p = 1 + \frac{4}{n-2}$ , the estimates are border line. In fact one gets

$$(45) \quad \|Lv\|_{S^1(I \times \mathbb{R}^n)} \leq C_1 \|u_0\|_{H^1} + C_2 \|v\|_{S^1}^p.$$

The main difference between this and (42) is that there is no time factor appearing in the right hand side. This of course makes the contraction more difficult to attain by shrinking the time. On the other hand if one starts with small data  $\|u_0\|_{H^1} \leq \epsilon$  and calls now  $R = 2C_1\epsilon$ , then a sufficient condition on  $\epsilon$  to have a contraction would be

$$C_2 R^{p-1} = C_2 (2C_1\epsilon)^{p-1} \leq \frac{1}{2}.$$

This would also guarantee a uniform global solution in  $H^1$ .

One could ask if at least l.w.p could be still achieved for large data. The following theorem gives a positive answer.

**Theorem 4.6.** [*l.w.p. for  $H^1$  critical NLS*] Assume that  $p = 1 + \frac{4}{n-2}$  and  $u_0 \in H^1$ . Assume also that

$$(46) \quad \|S(t)u_0\|_{L_{[-T,T]}^{\frac{2(n+2)}{n-2}} W_x^{1, \frac{2n(n+2)}{n^2+4}}} \leq \epsilon$$

for  $\epsilon$  small enough. Then (1) is  $H^1$  well posed in  $[-T, T]$ .

*Proof.* We first notice that the pair  $(\frac{2(n+2)}{n-2}, \frac{2n(n+2)}{n^2+4})$  is Strichartz admissible. We define the new space  $\tilde{S}^1$  using the following norm

$$\|f\|_{\tilde{S}^1} := T \|f\|_{S^1} + \|f\|_{L_{[-T,T]}^{\frac{2(n+2)}{n-2}} W_x^{1, \frac{2n(n+2)}{n^2+4}}},$$

The idea is to use a contraction method in this space based on the smallness assumption (46). As we did in the proof of Theorem 4.2 we estimate  $Lv$  in the space  $\tilde{S}^1$ :

$$\begin{aligned} \|Lv\|_{\tilde{S}^1} &\lesssim T \|u_0\|_{H^1} + \|S(t)u_0\|_{L_{[-T,T]}^{\frac{2(n+2)}{n-2}} W_x^{1, \frac{2n(n+2)}{n^2+4}}} \\ &+ \| |v|^{\frac{4}{n-2}} |\nabla v| \|_{L_{[-T,T]}^{\tilde{q}'} L_x^{\tilde{r}'}} \end{aligned}$$

Now we pick the Strichartz pair  $(\tilde{q}, \tilde{r}) = (2, \frac{2n}{n-2})$  and we obtain by Hölder

$$\| |v|^{\frac{4}{n-2}} |\nabla v| \|_{L_{[-T,T]}^{\tilde{q}'} L_x^{\tilde{r}'}} \lesssim \|v\|_{L_{[-T,T]}^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}} \|v\|_{L_{[-T,T]}^{\frac{2(n+2)}{n-2}} W_x^{1, \frac{2n(n+2)}{n^2+4}}}.$$

By the Sobolev embedding theorem we then have

$$\|v\|_{L_{[-T,T]}^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}} \lesssim \|v\|_{L_{[-T,T]}^{\frac{2(n+2)}{n-2}} W_x^{1, \frac{2n(n+2)}{n^2+4}}},$$

hence the final bound

$$(47) \quad \|Lv\|_{\tilde{S}^1} \lesssim T \|u_0\|_{H^1} + \|S(t)u_0\|_{L_{[-T,T]}^{\frac{2(n+2)}{n-2}} W_x^{1, \frac{2n(n+2)}{n^2+4}}} + \|v\|_{L_{[-T,T]}^{\frac{2(n+2)}{n-2}} W_x^{1, \frac{2n(n+2)}{n^2+4}}}^{1 + \frac{4}{n-2}}.$$

Now if  $T$  is small enough, in particular  $T \sim \epsilon \|u_0\|_{H^1}^{-1}$ , using (46), we deduce from (47) that

$$\|Lv\|_{\tilde{S}^1} \leq 2C_0\epsilon + C_1 \|v\|_{L_{[-T,T]}^{\frac{2(n+2)}{n-2}} W_x^{1, \frac{2n(n+2)}{n^2+4}}}^{1 + \frac{4}{n-2}}.$$

We then take a ball  $B$  of radius  $R = 4C_0\epsilon$  and if  $\epsilon$  is small enough then  $L$  sends  $B$  into itself and it is a contraction. The rest is now routine. This argument proved the theorem in the interval of time of length approximately  $\epsilon \|u_0\|_{H^1}^{-1}$ . In order to cover an arbitrary interval  $[-T, T]$ , then one has to use again the conservation of energy and mass that gives a uniform bound on  $\|u\|_{H^1}$ .  $\square$

*Remark 4.7.* We have the following two facts:

- (1) By the homogeneous Strichartz estimate (12) it follows that

$$\|S(t)u_0\|_{L_{[-T,T]}^{\frac{2(n+2)}{n-2}} W_x^{1, \frac{2n(n+2)}{n^2+4}}} \lesssim \|u_0\|_{H^1}$$

hence we recover above the small data g.w.p we discussed in Remark 4.5.

- (2) Given any data  $u_0 \in H^1$ , again by (12) we have

$$\|S(t)u_0\|_{L_{[-T,T]}^{\frac{2(n+2)}{n-2}} W_x^{1, \frac{2n(n+2)}{n^2+4}}} \leq C,$$

so we can use the time integral to claim that for  $T$  small enough (46) is satisfied. This gives l.w.p. but it is important to notice that in this case  $T = T(u_0)$  depends also on the profile of the initial data, not only on its  $H^1$  norm.

**Theorem 4.8.** [*G.w.p. for  $H^1$  critical NLS with  $L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}$  bound*] Assume that  $p = 1 + \frac{4}{n-2}$  and  $u_0 \in H^1$ . Assume also the a priori estimate

$$(48) \quad \|u\|_{L_{[-T,T]}^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}} \leq C$$

for any solution  $u$  to (1) with  $p = 1 + \frac{4}{n-2}$ . Then this IVP is  $H^1$  globally well posed.

*Proof.* Fix  $\epsilon$  to be determined later. Using (48) we can find finitely many intervals of time  $I_1, \dots, I_M$  such that

$$(49) \quad \|u\|_{L_{I_j}^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}} \leq \epsilon$$

for all  $j = 1, \dots, M$ . The goal here is to prove that as a consequence of (49) one actually has the stronger bound

$$(50) \quad \|u\|_{S_{I_j}^1} \leq C$$

for all  $j = 1, \dots, M$  and putting all the intervals together

$$(51) \quad \|u\|_{S^1} \leq C$$

How do we use now this bound? We use it to continue the solution. More precisely: since  $u_0 \in H^1$  we already proved that there exists  $T$  and a unique solution  $u \in S_{[-T,T]}^1$ . Let now

$T_{max}$  be the maximum time for well-posedness. Clearly if  $T_{max} = +\infty$  there is nothing to prove. Suppose by contradiction that  $T_{max} < +\infty$ . We can use then (51) to claim in particular that

$$\|u(T_{max})\|_{H^1} \leq C.$$

Then we can continue our solution and obtain a contradiction.

It is now time to prove (50). Using estimates like the ones in the proof of Theorem 4.6 this time applied to the Duhamel representation of a solution  $u$  we have

$$\|u\|_{S^1} \leq C_1 \|u_0\|_{H^1} + C_2 \|u\|_{L_{I_j}^{\frac{4}{n-2}}}^{\frac{4}{2(n+2)}} \|u\|_{L_x^{\frac{2(n+2)}{n-2}}} \|u\|_{S_{I_j}^1} \leq C_1 \|u_0\|_{H^1} + C_2 \epsilon^{\frac{4}{n-2}} \|u\|_{S_{I_j}^1}$$

and if  $C_2 \epsilon^{\frac{4}{n-2}} < 1/2$  then (50) follows. □

We end this section by announcing that similar theorems, replacing  $H^1$  with  $L^2$  are available for the  $L^2$  subcritical NLS, that is when  $1 < p < 1 + \frac{4}{n}$ . We do not list them here, but they can be found in [69].