3. Lecture # 2: The Nonlinear Linear Schrödinger Equation (NLS) in $\mathbb{R}^n$-conservation Laws, classical Morawetz and Virial Identity, inveriances for the equation

In this section we consider the (NLS) IVP (1) and we formally talk about the solution $u(x,t)$ as an object that exists, is smooth ecc. Of course to be able to use whatever we say here later we will need to work on making this formal assumption true!

Given an equation it is always a good idea to read as much as possible out of it. So one should always ask what are the rigid constraints that an equation imposes on its solutions a-priori. Here we will look at conservation laws (in this case integrals involving the solution that are independent of time), some inequalities (or monotonicity formulas) that a solution has to satisfy, symmetries and invariances that a solution to (1) can be subject too. All three of these elements are somehow related (see for example Noether's theorem [69]) and here we will not exploit ALL the possible connections. It is true though that in describing these important features of the equation one often has to recall some basic principles/quantities coming from physics like conservation of mass and energy, the notion of density, interaction of particles, resonance ecc.

3.1. **Conservation laws.** A simple way to interpret physically the function $u(x,t)$ solving a Schrödinger equation is to think about $|u(x,t)|^2$ as the particle density at place x and at time t. Then it shouldn't come as a surprise that the density, momentum and energy are conserved in time. More precisely if we introduce the *pseudo-stress-energy tensor* $T_{\alpha,\beta}$ for $\alpha,\beta = 0,1,...,n$ then

$$T_{00} = |u|^2 \quad \text{(mass density)} \tag{17}$$

$$T_{0j} = T_{j0} = Im(\bar{u}\partial_{x_j}u) \quad \text{(momentum density)} \tag{18}$$

$$T_{jk} = Re(\partial_{x_j}u\overline{\partial_{x_k}u}) - \frac{1}{4}\delta_{j,k}\Delta(|u|^2) + \lambda\frac{p-1}{p+1}\delta_{jk}|u|^{p+1} \quad \text{(stress tensor)} \tag{19}$$

then by using the equation one can show that

$$\partial_t T_{00} + \partial_{x_j}T_{0j} = 0 \ \text{ and } \ \partial_t T_{j0} + \partial_{x_k}T_{jk} = 0 \tag{20}$$

for all $j,k = 1,...,n$.

Problem 3.2. *Prove* (20) *using the equation.*

The conservation laws summarized in (20) are said to be *local* in the sense that they hold pointwise in the physical space. Clearly by integrating in space and assuming that u vanishes at infinity one also has the conserved integrals

$$m(t) = \int T_{00}(x,t)\,dx = \int |u|^2(x,t)\,dx \quad \text{(mass)} \tag{21}$$

$$p_j(t) = -\int T_{0j}(x,t) = -\int Im(\bar{u}\partial_{x_j}u)\,dx \quad \text{(momentum).} \tag{22}$$

We observe here that the stress tensor in (19) is not conserved, but it plays an important role in some "sophisticated" monotonicity formulas involving the solution u. To obtain the conservation of energy $E(t)$ we need to remember that the total energy of a system at time t is

$$E(t) = K(t) + P(t)$$

the sum of kinetic and potential energy. In our case

$$K(t) = \frac{1}{2}\int |\nabla u|^2(x,t)\,dx \quad \text{and} \quad P(t) = \frac{2\lambda}{p+1}\int |u(t,x)|^{p+1}\,dx$$

and hence

$$E(t) = \frac{1}{2}\int |\nabla u|^2(x,t)\,dx + \frac{2\lambda}{p+1}\int |u(t,x)|^{p+1}\,dx = E(0). \tag{23}$$

We immediately observe that now the sign of λ plays a very important role since by picking $\lambda = -1$ one can produce a negative energy. We will discus this later in greater details.

Problem 3.3. *Prove the conservation of energy* (23) *by using the equation.*

As we will see, to have an a-priori control in time of an energy like in (23) when $\lambda = 1$ is an essential tool in order to prove that a solution exists for all times. But it is also true that often this is not sufficient. This is indeed the case when the problem is *critical* [9]. We need then other a-priori controls on norms for the solution u. This is the content of next subsection.

3.4. **Viriel and Classical Morawetz Identities.** The Viriel identity was first introduced by Glassey [38] to show blow up for certain focusing ($\lambda = -1$) NLS problems. The classical[10] Morawetz identity was introduced instead by Moraweyz in the context of the wave equations [61]. In the NLS case it was were introduced by Lin and Strauss [58]. Morawetz type identities are useful in the defocusing setting ($\lambda = 1$).

In general these identities are used in order to show that a positive quantity (often a norm) involving the solution u has a monotonic behavior in time. Monotonic quantities are used systematically in the context of elliptic equations and although both the Viriel and Morawetz estimates go back to the 70' only recently they have been used, together with their variations, in a surprisingly powerful way.

Suppose that a function $a(x)$ is measuring a particular quantity for our system[11] and we want to look at its overage value and in particular at its change in time. To do so we integrate $a(x)$ against the mass density tensor in (17) and we compute using (20) and integration by parts

$$\partial_t \int a(x)|u|^2(x,t)\,dx = \int \partial_{x_j} a(x) Im(\bar{u}\partial_{x_j} u)(t,x)\,dx. \tag{24}$$

At this stage there is no obvious sign for the right hand side of the equality. The integrals appearing above have special names. In fact we can introduce the following definition:

Definition 3.5. Given the IVP (1), we define the associated virial potential

$$V_a(t) = \int a(x)|u(t,x)|^2 dx \tag{25}$$

and the associated Morawetz action

$$M_a(t) = \int \partial_{x_j} a(x) \mathrm{Im}(\overline{u}\partial_{x_j} u) dx. \tag{26}$$

By taking the second derivative in time and by using again (20), we obtain

$$\begin{aligned} \partial_t^2 V_a(t) &= \partial_t^2 \int a(x)|u|^2(x,t)\,dx = \partial_t M_a(t) = \int (\partial_{x_j}\partial_{x_k} a(x)) Re(\partial_{x_j} u \overline{\partial_{x_k} u}\,dx \\ &+ \frac{\lambda(p-1)}{p+1}\int |u(t,x)|^{p+1}\Delta a(x)\,dx - \frac{1}{4}|u|^2(x,t)\Delta^2 a(x)\,dx. \end{aligned}$$

Now let's make a particular choice for $a(x)$.

[9] The notion of criticality will be introduced below.

[10] Here we talk about the *classical* Morawetz type identities in order to distinguish them from the Interaction Morawetz ones.

[11] For example $a(x)$ could represent the distance to a particular point, or the characteristic function of a particular domain.

- If $a(x) = |x|^2$, then $\Delta^2 a(x) = 0$ and $\Delta a(x) = 2n$ so

$$\partial_t^2 \int |x|^2 |u|^2(x,t)\, dx = 4E + \frac{2\lambda}{p+1}[n(p-1)-4] \int |u|^{p+1}\, dx. \tag{27}$$

Remark 3.6. For example in the focusing case $\lambda = -1$, when $n = 3$ and $p > \frac{7}{3}$, if one starts with $E < 0$, then the function $f(t) = \int |x|^2|u|^2(x,t)\, dx$ is concave down and positive ($f'(t)$ is monotone decreasing). Hence there exists $T^* < \infty$ such that there the function cannot longer exists. This was in fact the original argument of Glassey to show the existence of blow up time for certain defocusing NLS equations.

- If $a(x) = |x|$, then (24) becomes

$$\partial_t \int |x||u|^2(x,t)\, dx = \int Im(\bar{u}\frac{x}{|x|} \cdot \nabla u)(t,x)\, dx, \tag{28}$$

and from here

$$\begin{aligned} \partial_t M_{|x|} &= \partial_t \int Im(\bar{u}\frac{x}{|x|} \cdot \nabla u)(t,x)\, dx = \int \frac{|\not\nabla u(t,x)|^2}{|x|}\, dx \\ &+ \frac{2(n-1)(p-1)\lambda}{p+1} \int \frac{|u(t,x)|^{p+1}}{|x|}\, dx - \frac{1}{4}\int |u(x,t)|^2 (\Delta^2 |x|)\, dx, \end{aligned} \tag{29}$$

where $\not\nabla u := \nabla - \frac{x}{|x|}(\frac{x}{|x|} \cdot \nabla)$ denotes the angular gradient of u.

Problem 3.7. *Above we used $a(x) = |x|$ which is clearly non smooth at zero. Check that if we take $n \geq 3$ and we replace $|x|$ with $\sqrt{x^2+\epsilon^2}$ and let $\epsilon \to 0$, then the identity* (29) *is correct.*

One can then compute that for $n \geq 3$, $(\Delta^2|x|) \leq 0$ in the sense of distributions. As a consequence, in the defocusing case $\lambda = 1$, after integrating in time over an interval $[t_0, t_1]$ one has

$$\int_{t_0}^{t_1} \frac{|\not\nabla u(t,x)|^2}{|x|}\, dx,\ \int_{t_0}^{t_1}\int \frac{|u(t,x)|^{p+1}}{|x|}\, dx \lesssim \sup_{[t_0,t_1]} \left| \int Im(\bar{u}\frac{x}{|x|} \cdot \nabla u)(t,x)\, dx \right|. \tag{30}$$

One can easily estimate the right hand side as

$$\sup_{[t_0,t_1]} \left| \int Im(\bar{u}\frac{x}{|x|} \cdot \nabla u)(t,x)\, dx \right| \lesssim \|u_0\|_{L^2} E^{1/2}$$

by using both conservation of mass and energy. But if less regularity is preferable then one can use the Hardy inequality (see Lemma A.10in [69]) as in Lemma 6.9 that will be introduced later in Lecture #5, to obtain

$$\int_{t_0}^{t_1} \frac{|\not\nabla u(t,x)|^2}{|x|}\, dx,\ \int_{t_0}^{t_1}\int \frac{|u(t,x)|^{p+1}}{|x|}\, dx \lesssim \sup_{[t_0,t_1]} \|u(t)\|^2_{H^{1/2}}, \tag{31}$$

where now the disadvantage is the fact that the $H^{1/2}$ norm of u is not uniformly bounded in time.

3.8. **Invariances and symmetries.** In this section we only list invariances and symmetries but we do not attempt to describe their usefulness and applications except for one of them that we will start using in today's lecture.

(1) **Scaling Symmetry:** If u solve the IVP (1) then

$$u_{\mu}(x,t) = \mu^{-\frac{2}{p-1}} u\left(\frac{t}{\mu^2}, \frac{x}{\mu},\right) \quad \text{and} \quad u_{\mu,0}(x) = \mu^{-\frac{2}{p-1}} u\left(\frac{x}{\mu},\right) \tag{32}$$

solves the IVP for any $\mu \in \mathbb{R}$.

(2) **Galilean Invariance:** If u is again a solution to (1) then

$$e^{ix\cdot v} e^{it|v|^2/2} u(t, x - vt) \quad \text{with initial data} \quad e^{ix\cdot v} u_0(x)$$

for every $v \in \mathbb{R}^n$ also solves the same IVP.

(3) **Obvious Symmetries:** Time and space translation invariance, spatial rotation, phase rotation symmetry $e^{i\theta}u$, time reversal.

(4) **Pseudo-conformal Symmetry:** In the case $p = 1 + \frac{4}{n}$, if u is solution for (1) then also

$$\frac{1}{|t|^{n/2}} u\left(\frac{1}{t}, \frac{x}{t}\right) e^{i|x|^2/2t} \tag{33}$$

for $t \neq 0$ is solution to the same equation.

We now concentrate on the **scaling symmetry** and we show how this can be used to understand for which nonlinerity (or for which $p > 1$) the problem of well-posedness is most difficult to address.

If we compute $\|u_{\mu,0}\|_{\dot{H}^s}$ we see that

$$\|u_{\mu,0}\|_{\dot{H}^s} \sim \mu^{-s+s_c} \|u_0\|_{\dot{H}^s}, \tag{34}$$

where

$$s_c = \frac{n}{2} - \frac{2}{p-1}.$$

From (34) it is clear that if we take $\mu \to +\infty$ then

(1) if $s > s_c$ (**sub-critical case**) the norm of the initial data can be made small while at the same time the interval of time is made longer: our intuition says that this is the best possible setting for well-posedness,

(2) if $s = s_c$ (**critical case**) the norm is invariant while the interval of time is made longer. This looks like a problematic situation.

(3) if $s < s_c$ (**super-critical case**) the norms grow as the time interval gets longer. Scaling is obviously against us.

In order to have a better intuition for scaling that also relates the dispersive part of the solution Δu with the nonlinear part of it $|u|^{p-1}u$, we use an informal argument in [69]. Let's consider a special type of initial wave u_0. We want u_0 such that its support in Fourier space is localized at a large frequency $N >> 1$, its support in space is inside a Ball of radius $1/N$ and its amplitude is A. Here we are making the assumption that scaling is the only symmetry that could interfere with a behavior that goes from linear to nonlinear, but in general this is not the only one. We have

$$\|u_0\|_{L^2} \sim AN^{-n/2}, \quad \|u_0\|_{\dot{H}^s} \sim AN^{s-n/2}.$$

If we want $\|u_0\|_{\dot{H}^s}$ small then we need to ask that $A << N^{n/2-s}$. Now under this restriction we want to compare the liner term Δu with the nonlinear part $|u|^{p-1}u$:

$$|\Delta u| \sim AN^2 \quad \text{while} \quad |u|^p \sim A^p.$$

From here if $AN^2 >> A^p$ we believe that the linear behavior would win otherwise the nonlinear one. Putting everything together we have that

$$A^{p-1} << N^2 \quad \text{and} \quad A << N^{n/2-s} \Longrightarrow s > s_c \quad \text{(more linear)} \tag{35}$$

$$A^{p-1} >> N^2 \quad \text{and} \quad A << N^{n/2-s} \Longrightarrow s < s_c \quad \text{(more nonlinear).} \tag{36}$$

As announced at the beginning the so called "scaling argument" presented here should only be used as a guide line since in delivering it we make a purely formal calculation. On the other hand in some cases ill-posedness results below critical exponent have been obtained (see for example [22, 23]).

Problem 3.9. *Prove the conservation of mass using Forier transform for the IVP* (1) *when* $n = 1$ *and* $p = 3$.

3.10. **Definition of well-posedness.** We conclude this lecture by giving the precise definition of local and global well-posedness for an initial value problem, which in this case we will specify to be of type (1).

Definition 3.11 (Well-posedness)**.** We say that the IVP (1) is *locally well-posed (l.w.p)* in $H^s(\mathbb{R}^n)$ if for any for any ball B in the space $H^s(\mathbb{R}^n)$ there exists a time T and a Banach space of functions $X \subset L^\infty([-T,T], H^s(\mathbb{R}^n))$ such that for each initial data $u_0 \in B$ there exists a unique solution $u \in X \cup C([-T,T], H^s(\mathbb{R}^n))$ for the integral equation

$$u(x,t) = S(t)u_0 + c\int_0^t S(t-t')|u|^{p-1}u(t'))\,dt'. \tag{37}$$

Furthermore the map $u_0 \to u$ is continuous as a map from H^s into $C([-T,T], H^s(\mathbb{R}^n))$. If uniqueness is obtained in $C([-T,T], H^s(\mathbb{R}^n))$, then we say that local well -posedness is *unconditional.*

If this hold for all $T \in \mathbb{R}$ then we say that the IVP is *globally well-posed (g.w.p).*

Remark 3.12. Our notion of global well-posedness does not require that $\|u(t)\|_{H^s(\mathbb{R}^n)}$ remains uniformly bounded in time. In fact, unless $s = 0, 1$ and one can use the conservation of mass or energy, it is not a triviality to show such an uniform bound. This can be obtained as a consequence of scattering, when scattering is available. In general this is a question related to the *weak turbulence theory.*