THE THEORY OF NONLINEAR SCHRÖDINGER EQUATIONS: PART I

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1. Introduction

The title of these lecture notes is certainly too ambitious. In fact here we will mainly consider semilinear Schrödinger initial value problems (IVP)

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = \lambda |u|^{p-1}u, \\ u(x,0) = u_0(x) \end{cases} \tag{1}$$

where $\lambda = \pm 1$, $p > 1$, $u : \mathbb{R} \times M \to \mathbb{C}$, and M is a manifold[1]. Even in this relatively special case we will not be able to mention all the findings and results concerning the initial value problem (1) and for this we apologize from the beginning.

Schrödinger equations are classified as *dispersive* partial differential equations and the justification for this names comes from the fact that if no boundary conditions are imposed their solutions tend to be waves which spread out spatially. But what does this mean mathematically? A simple and complete mathematical characterization of the word *dispersion* is given to us for example by R. Palais in [63]. Although his definition is given for one dimensional wave, the concept is expressed so clearly that it is probably a good idea to follow almost[2] literally his explanation: " *Let us [next] consider* linear *wave equations of the form*

$$u_t + P\left(\frac{\partial}{\partial x}\right) u = 0,$$

where P is polynomial. Recall that a solution $u(x,t)$, which Fourier transform is of the form $e^{i(kx-\omega t)}$, is called a plane-wave solution; k is called the wave number *(waves per unit of length) and ω the (angular)* frequency. *Rewriting this in the form $e^{ik(x-(\omega/k)t)}$, we recognize that this is a traveling wave of velocity $\frac{\omega}{k}$. If we substitute this $u(x,t)$ into our wave equation, we get a formula determining a unique frequency $\omega(k)$ associated to any wave number k, which we can write in the form*

$$\frac{\omega(k)}{k} = \frac{1}{ik}P(ik). \tag{2}$$

This is called the "dispersive relation" for this wave equation. Note that it expresses the velocity for the plane-wave solution with wave number k. For example, $P(\frac{\partial}{\partial x}) = c\frac{\partial}{\partial x}$ gives the linear advection equation $u_t + cu_x = 0$, which has the dispersion relation $\frac{\omega}{k} = c$, showing of course that all plane-wave solutions travel at the same velocity c, and we say that we have trivial dispersion in this case. On the other hand if we take $P\left(\frac{\partial}{\partial x}\right) = -\frac{i}{2}\left(\frac{\partial}{\partial x}\right)^2$, then our wave equation is $iu_t + \frac{1}{2}u_{xx} = 0$, which is the linear Schrödinger equation, and we have the non-trivial dispersion relation $\frac{\omega}{k} = \frac{k}{2}$. In this case, plane waves of large wave-number (and hence high frequency) are

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[1] In most cases M is the euclidian space $\mathbb{R}^n$ and only at the end we will mention some results and references when M is a different kind of manifold.

[2] R. Palais actually uses the Airy equation as an example, while we use the linear Schrödinger equation to be consistent with the topic of the lectures.

traveling much faster than low-frequency waves. The effect of this is to "broaden a wave packet". That is, suppose our initial condition is $u_0(x)$. *We can use the Fourier transform*[3] *to write* u_0 *in the form*

$$u_0(x) = \int \widehat{u_0}(k) e^{ikx}\, dk,$$

and then, by superposition, the solution to our wave equation will be

$$u(x,t) = \int \widehat{u_0}(k) e^{ik(x-(\omega(k)/k)t)}\, dk.$$

Suppose for example that our initial wave form is a highly peaked Gaussian. Then in the case of the linear advection equation all the Fourier modes travel together at the same speed and the Gaussian lump remains highly peaked over time. On the other hand, for the linearized Schrödinger equation the various Fourier modes all travel at different velocities, so after time they start cancelling each other by destructive interference, and the original sharp Gaussian quickly broadens".

As one can imagine dispersive equations are proposed as descriptions of certain phenomena that accour in nature. But it turned out that some of these equations appear also in more abstract mathematical areas like algebraic geometry [44], and certainly We are not in the position to discuss this beautiful part of mathematics.

The questions that we will address here are more phenomenological. Assume that a profile of a wave is given at time $t = 0$, (initial data). Is it possible to prove that there exists a unique wave that "lives" for an interval of time $[0, T]$, that satisfies the equation, and that at time $t = 0$ has the assigned profile? What kind of properties does the wave have at later times? Does it "live" for all times or does it "blow up" in finite time?

Our intuition tells us that, if we start with *nice* and *small* initial data, then all the questions above should be easier to answer. This is indeed often true. In general in this case one can prove that the wave exists for all times, it is unique and its "size", measured taking into account the order of smoothness, can be controlled in a reasonable way. But what happens when we are not in this advantageous setting? These lecture notes are devoted to the understanding of how much of the above is still true when we consider *large* data and *long* interval of times. To be able to give a rigorous setting for the study of the initial value problem in (1) and to avoid any confusion in the future we need a strong mathematical definition for *well-posedness.* We consider the general initial value problem of type

$$\left\{ \begin{array}{ll} \partial_t u + P_m(\partial_{x_1}, \dots, \partial_{x_n})u + N(u, \partial_x^\alpha u) = 0, & \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n \ (\text{or} x \in \mathbb{T}^n, \ t \in \mathbb{R}, \end{array} \right. \tag{3}$$

where $m \in \mathbb{N}$, $P_m(\partial_{x_1}, \dots, \partial_{x_n})$ is a differential operator with constant coefficients of order m and $N(u, \partial_x^\alpha u)$ is the nonlinear part of the equation, that is a nonlinear function that depends on u and derivatives of u up to order $m-1$. The function $u_0(x)$ is the initial condition or initial profile, and most of the time is called initial data. Above we pointed out the fact that finding a solution for an IVP strongly depends on the regularity one asks for the solution itself. So we first have to decide how we "measure" the regularity of a function. The most common way of doing so is to decide where the weak derivatives of the function "live". It is indeed time to recall the definition of Sobolev spaces[4]

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[3] In these lectures we will ignore the absolute constants that may appear in other definitions for the Fourier transform.

[4] In more sophisticated instances one replaces the Sobolev spaces with different ones, like L^p spaces, Hölder spaces, and so on.

Definition 1.1. We say that a function $f \in H^k(\mathbb{R}^n), k \in \mathbb{N}$ if f and all its partial derivatives up to order k are in L^2. We recall that $H^k(\mathbb{R}^n)$ is a Banach space with the norm

$$\|f\|_{H^k} = \sum_{|\alpha|=0}^{k} \|\partial_x^\alpha f\|_{L^2},$$

where $\alpha(\alpha_1, \ldots, \alpha_n)$ and $|\alpha| = \sum_{i=1}^n \alpha_i$ is its length.

We also recall here the definition of the Fourier transform.

Definition 1.2. Assume $f \in L^2(\mathbb{R}^n)$, then the Fourier transform of f is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} f(x)\, dx$$

where $\langle \cdot \rangle$ is the inner product in $\mathbb{R}^n$. We also have an inverse Fourier formula

$$f(x) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \hat{f}(x)\, dx.$$

If the function is defined on the torus $\mathbb{T}^n$ then the Fourier transform is defined as

$$\hat{f}(k) = \int_{\mathbb{T}^n} e^{i\langle x,k\rangle} f(x)\, dx$$

and the inverse Fourier formula is

$$f(x) = \sum_{k\in\mathbb{Z}} e^{-i\langle x,k\rangle} \hat{f}(k).$$

Remark 1.3. Because $\widehat{\partial_x^\alpha f}(\xi) = (i\xi)^\alpha \hat{f}(\xi)$, it is easy to see that $f \in H^k(\mathbb{R}^n)$ if and only if

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1+|\xi|)^{2k}\, d\xi < \infty,$$

and moreover

$$\left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1+|\xi|)^{2k}\, d\xi\right)^{1/2} \sim \|f\|_{H^k}.$$

Then we can generalize our notion of Sobolev space and define $H^s(\mathbb{R}^n), s \in \mathbb{R}$ as the set of functions such that

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1+|\xi|)^{2s}\, d\xi < \infty.$$

Also $H^s(\mathbb{R}^n)$ is a Banach space with norm

$$\left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1+|\xi|)^{2s}\, d\xi\right)^{1/2} \sim \|f\|_{H^s}.$$

Sometimes it is useful to use the *homogeneous* Sobolev space $\dot{H}^s(\mathbb{R}^n)$. This is the space of functions such that

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 |\xi|^{2s}\, d\xi < \infty.$$

Clearly all these observations can be made for Sobolev spaces in $\mathbb{T}^n$, except that in this case $\dot{H}^s(\mathbb{T}^n)$ and $H^s(\mathbb{T}^n)$ coincides.

We use $\|f\|_{L^p}$ to denote the $L^p(\mathbb{R}^n)$ norm. We often need mixed norm spaces, so for example, we say that $f \in L^p_x L^q_t$ if $\|(\|f(x,t)\|_{L^q_t})\|_{L^p_x} < \infty$. Finally, for a fixed interval of time $[0,T]$ and a Banach space of functions Z, we denote with $C([0,T],Z)$ the space of the continuous maps from $[0,T]$ to Z.

We are now ready to give a first definition of well-posedness. We will give a more refined one later in Subsection **??**.

Definition 1.4. We say that the IVP (3) is locally well-posed (l.w.p) in H^s if, given $u_0 \in H^s$, there exist T, a Banach space of functions $X_T \subset C([-T,T];H^s)$ and a unique $u \in X_T$ which solve (3). Moreover we ask that there is continuity with respect to the initial data in the appropriate topology. We say that (3) is globally well-posed (g.w.p) in H^s if the definition above is satisfied in any interval of time $[-T,T]$.

Remark 1.5. The intervals of time are symmetric about the origin because the problems that we study here are all time reversible (i.e. if $u(x,t)$ is a solution, then so is $-u(x,-t)$).

We end this introduction with some notations. Throughout the notes we use C to denote various constants. If C depends on other quantities as well, this will be indicated by explicit subscripting, e.g. $C_{\|u_0\|_2}$ will depend on $\|u_0\|_2$. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$, where C is an absolute constant. We use $a+$ and $a-$ to denote expressions of the form $a+\varepsilon$ and $a-\varepsilon$, for some $0 < \varepsilon \ll 1$.

2. Lecture # 1: The Linear Schrödinger Equation in $\mathbb{R}^n$-Dispersive and Strichartz Estimates

In this lecture we introduce some of the most important estimates relative to the linear Schrödinger IVP

$$\begin{cases} iv_t + \frac{1}{2}\Delta v = 0, \\ v(x,0) = u_0(x). \end{cases} \tag{4}$$

It is important to understand as much as possible the solution v of (4) that we will denote with $v(x,t) = S(t)u_0(x)$, since by the Duhamel principle one can write the solution of the associated forced or nonlinear problem

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = F(u), \\ u(x,0) = u_0(x). \end{cases} \tag{5}$$

as

$$u(x,t) = S(t)u_0 + c\int_0^t S(t-t')F(u(t'))\,dt'. \tag{6}$$

Problem 2.1. *Prove the Duhamel Principle* (6).

The solution of the linear problem (4) is easily computable by taking Fourier transform. In fact by fixing the frequency ξ the problem (4) transforms into the ODE

$$\begin{cases} i\hat{v}_t(t,\xi) - \frac{1}{2}|\xi|^2\hat{v}(t,\xi) = 0, \\ \hat{v}(\xi,0) = \hat{u}_0(\xi) \end{cases} \tag{7}$$

and we can write its solution as

$$\hat{v}(t,\xi) = e^{-i\frac{1}{2}|\xi|^2 t}\hat{u}_0(\xi).$$

In general the solution $v(t,x)$ above is denoted by $S(t)u_0$, where $S(t)$ is called the Schrödinger group. If we define, in the distributional sense,

$$K_t(x) = \frac{1}{(\pi i t)^{n/2}} e^{i\frac{|x|^2}{2t}}$$

then we have

$$S(t)u_0(x) = e^{it\Delta}u_0(x) = u_0 \star K_t(x) = \frac{1}{(\pi i t)^{n/2}}\int e^{i\frac{|x-y|^2}{2t}}u_0(y)\,dy \tag{8}$$

Problem 2.2. *Prove, in the sense of distributions, that the inverse Fourier transform of* $e^{-i\frac{1}{2}|\xi|^2 t}$ *is* $K_t(x) = \frac{1}{(\pi i t)^{n/2}}e^{i\frac{|x|^2}{2t}}$.

As mentioned already

$$\widehat{S(t)u_0}(\xi) = e^{-i\frac{1}{2}|\xi|^2 t}\hat{u}_0(\xi), \tag{9}$$

and this last one can be interpreted as saying that the solution $S(t)u_0$ above is the adjoint of the Fourier transform restricted on the paraboloid $P = \{(\xi, |\xi|^2) \text{ for } \xi \in \mathbb{R}^n\}$. This remark, strictly linked to (8) and (9), can be used to prove a variety of very deep estimates for $S(t)u_0$. For example from (8) we immediatelly have the so called *Dispersive Estimate*

$$\|S(t)u_0\|_{L^\infty} \lesssim \frac{1}{t^{n/2}}\|u_0\|_{L^1}. \tag{10}$$

From (9) instead we have the conservation of the homogeneous Sobolev norms[5]

$$\|S(t)u_0\|_{\dot H^s} = \|u_0\|_{\dot H^s}, \tag{11}$$

for all $s \in \mathbb{R}$. Interpolating (10) with (11) when $s = 0$ and using a so called TT^* argument one can prove the famous Strichartz estimates (see [19], [69] and [47] for some concise proofs):

Theorem 2.3. *[Strchartz Estimates for the Schrödinger operator] Fix* $n \geq 1$. *We call a pair* (q, r) *of exponents admissible if* $2 \leq q, r \leq \infty$, $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$ *and* $(q, r, n) \neq (2, \infty, 2)$. *Then for any admissible exponents* (q, r) *and* $(\tilde q, \tilde r)$ *we have the homogeneous Strichartz estimate*

$$\|S(t)u_0\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^n)} \tag{12}$$

and the inhomogeneous Strichartz estimate

$$\| \int_0^t S(t-t')F(t')\,dt'\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|F\|_{L^{\tilde q'} L^{\tilde r'}_x(\mathbb{R}\times\mathbb{R}^n)}, \tag{13}$$

where $\frac{1}{\tilde q} + \frac{1}{\tilde q'} = 1$ *and* $\frac{1}{\tilde r} + \frac{1}{\tilde r'} = 1$.

To finish this lecture we would like to present a *rifined* bilinear Strichartz estimate due originally to Bourgain in [9] (see also [12]).

Theorem 2.4. *Let* $n \geq 2$. *For any spacetime slab* $I_* \times \mathbb{R}^n$, *any* $t_0 \in I_*$, *and for any* $\delta > 0$, *we have*

$$\begin{aligned} \|uv\|_{L^2_t L^2_x(I_*\times\mathbb{R}^n)} \leq C(\delta)(\|u(t_0)\|_{\dot H^{-1/2+\delta}} + \|(i\partial_t + \Delta)u\|_{L^1_t \dot H_x^{-1/2+\delta}}) \\ \times (\|v(t_0)\|_{\dot H^{\frac{n-1}{2}-\delta}} + \|(i\partial_t + \Delta)v\|_{L^1_t \dot H_x^{\frac{n-1}{2}-\delta}}). \end{aligned} \tag{14}$$

This estimate is very useful when u is high frequency and v is low frequency, as it moves plenty of derivatives onto the low frequency term. This estimate shows in particular that there is little interaction between high and low frequencies. One can also check easily that when $n = 2$ one recovers the $L^4_t L^4_x$ Strichartz estimate contained in Theorem 2.3 above.

Proof. We fix δ, and allow our implicit constants to depend on δ. We begin by addressing the homogeneous case, with $u(t) := e^{it\Delta}\zeta$ and $v(t) := e^{it\Delta}\psi$ and consider the more general problem of proving

$$\|uv\|_{L^2_{t,x}} \lesssim \|\zeta\|_{\dot H^{\alpha_1}} \|\psi\|_{\dot H^{\alpha_2}}. \tag{15}$$

Scaling invariance for this estimate[6] demands that $\alpha_1 + \alpha_2 = \frac{n}{2} - 1$. Our first goal is to prove this for $\alpha_1 = -\frac{1}{2} + \delta$ and $\alpha_2 = \frac{n-1}{2} - \delta$. The estimate (15) may be recast using duality and renormalization as

$$\int g(\xi_1 + \xi_2, |\xi_1|^2 + |\xi_2|^2)|\xi_1|^{-\alpha_1}\widehat{\zeta}(\xi_1)|\xi_2|^{-\alpha_2}\widehat{\psi}(\xi_2)d\xi_1 d\xi_2 \lesssim \|g\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}\|\zeta\|_{L^2(\mathbb{R}^n)}\|\psi\|_{L^2(\mathbb{R}^n)}. \tag{16}$$

Since $\alpha_2 \geq \alpha_1$, we may restrict attention to the interactions with $|\xi_1| \geq |\xi_2|$. Indeed, in the remaining case we can multiply by $(\frac{|\xi_2|}{|\xi_1|})^{\alpha_2-\alpha_1} \geq 1$ to return to the case under consideration. In fact, we may further restrict attention to the case where $|\xi_1| > 4|\xi_2|$ since, in the other case, we can move the frequencies between the two factors and reduce to the case where $\alpha_1 = \alpha_2$, which can be treated by $L^4_{t,x}$ Strichartz estimates[7] when $n \geq 2$. Next, we decompose $|\xi_1|$ dyadically

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[5] We will see later that the L^2 norm is conserved also for the nonlinear problem (1).

[6] Here use the fact that if v is solution to the linear Schrödinger equation, then $v_\lambda(x,t) = v(\frac{x}{\lambda}, \frac{t}{\lambda^2}$ is also solution.

[7] In one dimension $n = 1$, Lemma 2.4 fails when u, v have comparable frequencies, but continues to hold when u, v have separated frequencies; see [24] for further discussion.

and $|\xi_2|$ in dyadic multiples of the size of $|\xi_1|$ by rewriting the quantity to be controlled as (N, Λ dyadic):

$$\sum_N \sum_\Lambda \int \int g_N(\xi_1 + \xi_2, |\xi_1|^2 + |\xi_2|^2)|\xi_1|^{-\alpha_1}\widehat{\zeta_N}(\xi_1)|\xi_2|^{-\alpha_2}\widehat{\psi_{\Lambda N}}(\xi_2)d\xi_1 d\xi_2.$$

Note that subscripts on g, ζ, ψ have been inserted to evoke the localizations to $|\xi_1 + \xi_2| \sim N, |\xi_1| \sim N, |\xi_2| \sim \Lambda N$, respectively. Note that in the situation we are considering here, namely $|\xi_1| \geq 4|\xi_2|$, we have that $|\xi_1 + \xi_2| \sim |\xi_1|$ and this explains why g may be so localized.

By renaming components, we may assume that $|\xi_1^1| \sim |\xi_1|$ and $|\xi_2^1| \sim |\xi_2|$. Write $\xi_2 = (\xi_2^1, \underline{\xi_2})$. We now change variables by writing $u = \xi_1 + \xi_2,\ v = |\xi_1|^2 + |\xi_2|^2$ and $dudv = Jd\xi_2^1 d\xi_1$. A calculation then shows that $J = |2(\xi_1^1 \pm \xi_2^1)| \sim |\xi_1|$. Therefore, upon changing variables in the inner two integrals, we encounter

$$\sum_N N^{-\alpha_1} \sum_{\Lambda \leq 1} (\Lambda N)^{-\alpha_2} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} g_N(u,v) H_{N,\Lambda}(u,v,\underline{\xi_2}) dudvd\underline{\xi_2}$$

where

$$H_{N,\Lambda}(u,v,\underline{\xi_2}) = \frac{\widehat{\zeta_N}(\xi_1)\widehat{\psi_{\Lambda N}}(\xi_2)}{J}.$$

We apply Cauchy-Schwarz on the u, v integration and change back to the original variables to obtain

$$\sum_N N^{-\alpha_1} \|g_N\|_{L^2} \sum_{\Lambda \leq 1} (\Lambda N)^{-\alpha_2} \int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} \int_{\mathbb{R}^n} \frac{|\widehat{\zeta_N}(\xi_1)|^2 |\widehat{\psi_{\Lambda N}}(\xi_2)|^2}{J} d\xi_1 d\xi_2^1 \right]^{\frac{1}{2}} d\underline{\xi_2}.$$

We recall that $J \sim N$ and use Cauchy-Schwarz in the $\underline{\xi_2}$ integration, keeping in mind the localization $|\xi_2| \sim \Lambda N$, to get

$$\sum_N N^{-\alpha_1 - \frac{1}{2}} \|g_N\|_{L^2} \sum_{\Lambda \leq 1} (\Lambda N)^{-\alpha_2 + \frac{n-1}{2}} \|\widehat{\zeta_N}\|_{L^2} \|\widehat{\psi_{\Lambda N}}\|_{L^2}.$$

Choose $\alpha_1 = -\frac{1}{2} + \delta$ and $\alpha_2 = \frac{n-1}{2} - \delta$ with $\delta > 0$ to obtain

$$\sum_N \|g_N\|_{L^2} \|\widehat{\zeta_N}\|_{L^2} \sum_{\Lambda \leq 1} \Lambda^\delta \|\widehat{\psi_{\Lambda N}}\|_{L^2}$$

which may be summed up, after using the Schwarz inequality, and the Plancherel theorem will give the claimed homogeneous estimate.

We turn our attention to the inhomogeneous estimate (14). For simplicity we set $F := (i\partial_t + \Delta)u$ and $G := (i\partial_t + \Delta)v$. Then we use Duhamel's formula (6) to write

$$u = e^{i(t-t_0)\Delta}u(t_0) - i\int_{t_0}^t e^{i(t-t')\Delta}F(t')\,dt', \quad v = e^{i(t-t_0)\Delta}v(t_0) - i\int_{t_0}^t e^{i(t-t')\Delta}G(t').$$

We obtain[8]

$$\begin{aligned}\|uv\|_{L^2} &\lesssim \left\|e^{i(t-t_0)\Delta}u(t_0)e^{i(t-t_0)\Delta}v(t_0)\right\|_{L^2}\\ &+\left\|e^{i(t-t_0)\Delta}u(t_0)\int_{t_0}^{t}e^{i(t-t')\Delta}G(t')\,dt'\right\|_{L^2}+\left\|e^{i(t-t_0)\Delta}v(t_0)\int_{t_0}^{t}e^{i(t-t')\Delta}F(t')dt'\right\|_{L^2}\\ &+\left\|\int_{t_0}^{t}e^{i(t-t')\Delta}F(t')dt'\int_{t_0}^{t}e^{i(t-t'')\Delta}G(x,t'')\,dt''\right\|_{L^2}\\ &:= I_1+I_2+I_3+I_4.\end{aligned}$$

The first term was treated in the first part of the proof. The second and the third are similar so we consider only I_2. Using the Minkowski inequality we have

$$I_2 \lesssim \int_{\mathbb{R}}\|e^{i(t-t_0)\Delta}u(t_0)e^{i(t-t')\Delta}G(t')\|_{L^2}\,dt',$$

and in this case the theorem follows from the homogeneous estimate proved above. Finally, again by Minkowski's inequality we have

$$I_4 \lesssim \int_{\mathbb{R}}\int_{\mathbb{R}}\|e^{i(t-t')\Delta}F(t')e^{i(t-t'')\Delta}G(t'')\|_{L^2_x}dt'dt'',$$

and the proof follows by inserting in the integrand the homogeneous estimate above. □

Remark 2.5. In the situation where the initial data are dyadically localized in frequency space, the estimate (15) is valid [9] at the endpoint $\alpha_1=-\frac{1}{2}, \alpha_2=\frac{n-1}{2}$. Bourgain's argument also establishes the result with $\alpha_1=-\frac{1}{2}+\delta, \alpha_2=\frac{n-1}{2}+\delta$, which is not scale invariant. However, the full estimate fails at the endpoint.

Problem 2.6. *Conider the following two questions:*

1. *Prove that the full estimate at the endpoint is false by calculating the left and right sides of* (16) *in the situation where* $\widehat{\zeta_1}=\chi_{R_1}$ *with* $R_1=\{\xi : \xi_1 = Ne^1+O(N^{\frac{1}{2}})\}$ *(where* e^1 *denotes the first coordinate unit vector),* $\widehat{\psi_2}(\xi_2)=|\xi_2|^{-\frac{n-1}{2}}\chi_{R_2}$ *where* $R_2=\{\xi_2 : 1 \ll |\xi_2| \ll N^{\frac{1}{2}}, \xi_2\cdot e^1 = O(1)\}$ *and* $g(u,v)=\chi_{R_0}(u,v)$ *with* $R_0=\{(u,v) : u = Ne^1+O(N^{\frac{1}{2}}), v=|u|^2+O(N)\}$.
2. *Use the same counterexample to show that the estimate*

   $$\|u\overline{v}\|_{L^2_{t,x}} \lesssim \|\zeta\|_{\dot{H}^{\alpha}_1}\|\psi\|_{\dot{H}^{\alpha}_2},$$

   where $u(t)=e^{it\Delta}\zeta,\ v(t)=e^{it\Delta}\psi$, *also fails at the endpoint.*

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[8] Alternatively, one can absorb the homogeneous components $e^{i(t-t_0)\Delta}u(t_0)$, $e^{i(t-t_0)\Delta}v(t_0)$ into the inhomogeneous term by adding an artificial forcing term of $\delta(t-t_0)u(t_0)$ and $\delta(t-t_0)v(t_0)$ to F and G respectively, where δ is the Dirac delta.

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