Stability and blow up for the non linear Schrödinger Equation

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Lecture #2

0.1 Orbital stability of the ground states in the subcritical case

We focus in this section onto the question of the stability of the ground state solitary wave $u(t,x) = Q(x)e^{it}$ where $Q > 0$ is the ground state solution. Let us first observe that two trivial instabilities are given by the symmetries of the equation:

- Scaling instability: $\forall \lambda > 0$, the solution to (NLS) with initial data $u_0(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x)$ is $u(t,x) = \lambda^{\frac{2}{p-1}} Q(\lambda x) e^{i\lambda^2 t}$.
- Galilean instability: $\forall \beta > 0$, the solution to (NLS) with initial data $u_0(x) = Q(x)e^{i\beta}$ is $u(t,x) = Q(x-\beta t)e^{it+\frac{\beta}{2}\cdot(x-\frac{\beta}{2}t)}$.

In both cases,

$$\sup_{t\in\mathbf{R}} |u(t,x) - Q(x)e^{it}| > |Q(x)|$$

and thus the solution does not stay uniformly close to Q, whatever close it was at initial time.

Cazenave and Lions [3] proved that these trivial instabilities are the only one. This is the socalled *orbital stability* of the ground state solitary wave.

Theorem 1 (Orbital stability of the ground state) *Let $N \geq 1$ and $1 < p < 1 + \frac{4}{N}$. For all $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that the following holds true. Let $u_0 \in H^1$ with*

$$\|u_0 - Q\|_{H^1} < \delta(\varepsilon),$$

then there exists a translation shift $x(t) \in \mathbf{R}^N$ and phase shift $\gamma(t) \in \mathbf{R}$ such that:

$$\forall t \geq 0, \quad \|u(t,x) - Q(x - x(t))e^{i\gamma(t)}\|_{H^1} < \varepsilon.$$

The strength -and the weakness- of the proof is that it relies only on the conservation laws and the *variational characterization of the ground state solitary wave.* This study falls into the classical sets of *concentration compactness techniques* as introduced by Lions in [9],[10]. Given $\lambda > 0$, we let

$$Q_\lambda(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x).$$

The following variational result immediately implies Theorem 1:

Proposition 1 (Description of the minimizing sequences) *Let* $N \geq 1$ *and* $1 < p < 1 + \frac{4}{N}$. *Let* $M > 0$ *be fixed.*
(i) Variational characterization of Q*: The minimization problem*

$$I(M) = \inf_{|u|_{L^2} = M} E(u) \tag{1}$$

is attained on the family

$$Q_{\lambda(M)}(\cdot - x_0) e^{i\gamma_0}, \quad x_0 \in \mathbf{R}^N, \gamma_0 \in \mathbf{R},$$

where $\lambda(M)$ *is the unique scaling such that* $|Q_{\lambda(M)}|_{L^2} = M$.
(ii) Description of the minimizing sequences: Any minimizing sequence v_n *to (1) is relatively compact in* H^1 *up to translation and phase shifts, that is up to a subsequence:*

$$v_n(\cdot + x_n) e^{i\gamma_n} \to Q_{\lambda(M)} \quad \text{in} \quad H^1.$$

The fact that Proposition 1 implies Theorem 1 is now a simple consequence of the conservation laws and is left to the reader. The next section is devoted to an outline of the proof of Proposition 1.

0.2 The concentration compactness argument

The first key to the proof of Proposition 1 is the description of the lack of compactness in $\mathbf{R}^N$ of the Sobolev injection $H^1 \hookrightarrow L^{p+1}$, $2 \leq p+1 < 2^*$. This description is a consequence of the so-called concentration compactness Lemma. Let us recall that the injection is compact on a smooth bounded domain. Note also that the injection is still compact when restricted to radial functions in dimension $N \geq 2$. Here one uses the estimate:

$$u^2(r) = -\int_r^{+\infty} u(s)u'(s)ds \ \text{ and thus } \ |u|_{L^\infty(r \geq R)} \leq \frac{C}{R^{\frac{N-1}{2}}} |\nabla u|_{L^2}^{\frac{1}{2}} |u|_{L^2}^{\frac{1}{2}}$$

so that any H^1 bounded sequence of radially symmetric functions is L^{p+1} compact. This would considerably simplify the proof of Proposition 1 when restricting the problem to radially symmetric functions.

In general, there holds the following:

Proposition 2 (Description of the lack of compactness of $H^1 \hookrightarrow L^q$) *Let a sequence $u_n \in H^1$ with*

$$|u_n|_{L^2} = M, \quad |\nabla u_n|_{L^2} \leq C, \tag{2}$$

Then there exists a subsequence u_{n_k} such that one of the following three scenari occurs:
(i) Compactness: $\exists y_k \in \mathbf{R}^N$ such that

$$\forall 2 \leq q < 2^*, \quad u_{n_k}(\cdot + y_k) \to u \quad in \quad L^q. \tag{3}$$

(ii) Vanishing:

$$\forall 2 < q < 2^*, \quad u_{n_k} \to 0 \quad in \quad L^q. \tag{4}$$

(iii) Dichotomy: $\exists v_k, w_k$, $\exists 0 < \alpha < 1$ such that $\forall 2 \leq q < 2^*$:

$$\left\{\begin{array}{l} Supp(v_k) \cap Supp(w_k) = \emptyset, \quad dist(Supp(v_k), Supp(w_k)) \to +\infty, \\ \|v_k\|_{H^1} + \|w_k\|_{H^1} \leq C, \\ \|v_k\|_{L^2} \to \alpha M, \quad \|w_k\|_{L^2} \to (1-\alpha)M, \\ \lim_{k\to+\infty} |\int |u_{n_k}|^q - \int |v_k|^q - \int |w_k|^q|| = 0, \\ \liminf_{k\to+\infty} \int |\nabla u_{n_k}|^2 - \int |\nabla v_k|^2 - \int |\nabla w_k|^2 \geq 0. \end{array}\right. \tag{5}$$

Remark 1 *Observe that the key in the dichotomy case is that there is no loss of potential energy during the splitting in space of u_{n_k} into two bumps v_k, w_k which support go away from each other, while on the other hand only a lower semi continuity bound can be derived for the kinetic energy,*

Remark 2 *The case dichotomy corresponds to the localization of the first bubble of concentration. One can then continue the extraction iteratively and obtain the socalled profile decomposition of the sequence u_n, see P. Gerard [5], Hmidi, Keraani [7] for a very elegant proof.*

Proof of Proposition 2

We follow Cazenave [2]. Let $u_n \in H^1$ be as in the hypothesis of Proposition 2.

step 1 Concentration function.

Let the sequence of concentration functions:

$$\rho_n(R) = \sup_{y \in \mathbf{R}^N} \int_{B(y,R)} |u_n(x)|^2 dx.$$

The following facts are elementary and left to the reader:

- Monotonicity: $\forall n \geq 0$, $\rho_n(R)$ is a nondecreasing function of R.

- The concentration point is attained:

$$\forall R>0, \quad \forall n \geq 0, \quad \exists y_n(R) \in \mathbf{R}^N \text{ such that } \rho_n(R) = \int_{B(y_n(R),R)} |u_n(x)|^2 dx.$$

- Uniform Hölder continuity: $\exists C, \alpha > 0$ independent of n such that

$$\forall R_1, R_2 > 0, \quad \forall n \geq 0, \quad |\rho_n(R_1) - \rho_n(R_2)| \leq C|R_1^N - R_2^N|^\alpha. \tag{6}$$

  This last fact is a simple consequence of the H^1 bound (2).

step 2 Limit of concentration functions.

From (6) and Ascoli's theorem, there exists a subsequence $n_k \to +\infty$ and a nondecreasing limit ρ such that

$$\forall R > 0, \quad \lim_{k \to +\infty} \rho_{n_k}(R) = \rho(R). \tag{7}$$

Let now

$$\mu = \lim_{R \to +\infty} \liminf_{n \to +\infty} \rho_n(R).$$

By definition, there exists $R_k \to +\infty$ such that

$$\lim_{k \to +\infty} \rho_{n_k}(R_k) = \mu.$$

We now claim some stability of the sequence R_k which is a very general and simple fact but crucial for the rest of the argument:

$$\mu = \lim_{k \to +\infty} \rho_{n_k}(R_k) = \lim_{k \to +\infty} \rho_{n_k}(\frac{R_k}{2}) = \lim_{R \to +\infty} \rho(R). \tag{8}$$

Proof of (8): First oberve from the monotonicity of ρ_{n_k} that

$$\limsup_{k \to +\infty} \rho_{n_k}(\frac{R_k}{2}) \leq \limsup_{k \to +\infty} \rho_{n_k}(R_k) = \mu. \tag{9}$$

For the other sense, we argue as follows. For every $R > 0$, there holds:

$$\rho(R) = \liminf_{k \to +\infty} \rho_{n_k}(R) \geq \liminf_{n \to +\infty} \rho_n(R)$$

and thus:

$$\lim_{R \to +\infty} \rho(R) \geq \mu. \tag{10}$$

Eventually, for any $R > 0$, we have $\frac{R_k}{2} \geq R$ for k large enough and thus:

$$\rho_{n_k}(\frac{R_k}{2}) \geq \rho_{n_k}(R).$$

Letting $k \to +\infty$ implies:

$$\forall R > 0, \quad \lim_{k \to +\infty} \rho_{n_k}(\frac{R_k}{2}) \geq \rho(R).$$

Letting now $R > 0$ yields:

$$\lim_{k \to +\infty} \rho_{n_k}(\frac{R_k}{2}) \geq \lim_{R \to +\infty} \rho(R) \geq \mu$$

where we used (10) in the last step. This together with (9) concludes the proof of (8). The proof now proceed by making an hypothesis on μ.

Step 3: $\mu = 0$ is vanishing.

Assume $\mu = 0$. Then from (8), $\lim_{R \to +\infty} \rho(R) = 0$. But ρ is nondecreasing positive so: $\forall R > 0$, $\rho(R) = 0$. In particular, $\rho(1) = 0$ and thus

$$\lim_{k \to +\infty} \rho_{n_k}(1) = \lim_{k \to +\infty} \sup_{y \in \mathbf{R}^N} \int_{B(y,1)} |u_{n_k}|^2 = 0. \tag{11}$$

We claim that this together with the H^1 bound on u_{n_k} implies (4). There is a slight difficulty here which is that we need to pass from a local information -vanishing on every ball- to a global information -vanishing of the global L^q norm-. This relies on a refinement of the Gagliardo Nirenberg interpolation inequality. Indeed, we claim that

$$\forall u \in H^1, \quad \int |u|^{2+\frac{4}{N}} \leq C \|u\|_{H^1}^2 \|u\|_{L^2}^{\frac{4}{N}} \tag{12}$$

can be refined for:

$$\forall u \in H^1, \quad \int |u|^{2+\frac{4}{N}} \leq C \|u\|_{H^1}^2 \left[\sup_{y \in \mathbf{R}^N} \int_{B(y,1)} |u|^2 \right]^{\frac{2}{N}}. \tag{13}$$

This together with (11) implies

$$u_{n_k} \to 0 \quad \text{in} \quad L^{2+\frac{4}{N}} \quad \text{as} \quad k \to +\infty$$

and (4) follows by interpolation using the global H^1 bound.
Proof of (13): Take a partition of $\mathbf{R}^N$ with balls of radius $\frac{1}{2}$ and smooth cut off functions χ_k adapted to this partition. Then from (12):

$$\begin{aligned} \int |u|^{2+\frac{4}{N}} & \leq C \Sigma_{k \geq 0} \int |\chi_k u|^{2+\frac{4}{N}} \leq C \Sigma_{k \geq 0} \left(\int |\nabla(\chi_k u)|^2 \right) \left(\int |\chi_k u|^2 \right)^{\frac{2}{N}} \\ & \leq C \left[\sup_{y \in \mathbf{R}^N} \int_{B(y,1)} |u|^2 \right]^{\frac{2}{N}} \Sigma_{k \geq 0} \left[\int |\chi_k \nabla u|^2 + |\nabla \chi_k|^2 |u|^2 \right] \\ & \leq C \left[\sup_{y \in \mathbf{R}^N} \int_{B(y,1)} |u|^2 \right]^{\frac{2}{N}} \|u\|_{H^1}^2. \end{aligned}$$

Step 4: $\mu = M$ is compactness.

Let n_k be the sequence satisfying (7). For $R > 0$, let $y_k(R)$ such that

$$\rho_{n_k}(R) = \int_{B(y_k(R),R)} |u_{n_k}(x)|^2 dx. \tag{14}$$

Pick $\varepsilon > 0$. Then from (8), there exist $R_0, R(\varepsilon)$ such that

$$\rho(R_0) > \frac{M}{2}, \quad \rho(R(\varepsilon)) > M - \varepsilon.$$

Hence there exists $k_0(\varepsilon)$ such that $\forall k \geq k_0(\varepsilon)$,

$$\rho_{n_k}(R_0) = \int_{B(y_k(R_0),R_0)} |u_{n_k}|^2 > \frac{M}{2}, \quad \rho_{n_k}(R(\varepsilon)) = \int_{B(y_k(R(\varepsilon)),R(\varepsilon))} |u_{n_k}|^2 > M - \varepsilon.$$

But the total L^2 mass being M, this implies that the balls $B(y_k(R_0), R_0)$ and $B(y_k(R(\varepsilon)), R(\varepsilon))$ cannot be disjoint. Hence -draw a picture- we can find $R_1(\varepsilon)$ such that:

$$\forall \varepsilon > 0, \quad \forall k \geq k_0(\varepsilon), \quad \int_{B(y_k(R_0),R_1(\varepsilon))} |u_{n_k}|^2 \geq M - \varepsilon.$$

By possibly raising the value of $R_1(\varepsilon)$ for the values $k \in [1, k_0(\varepsilon)]$, this implies that the sequence $v_k = u_{n_k}(\cdot + y_k(R_0))$ is L^2 compact:

$$\forall \varepsilon > 0, \quad \exists R_2(\varepsilon) > 0 \text{ such that } \forall k \geq 1, \quad \int_{|y| \geq R_2(\varepsilon)} |v_k(y)|^2 dy < \varepsilon.$$

The compactness of the embedding $H^1 \hookrightarrow L^2(B(0, R(\varepsilon)))$ then implies that v_k a Cauchy sequence in L^2, and the H^1 boundedness now implies (3) by interpolation.

Step 5: $0 < \mu < M$ is dichotomy.

Let again (n_k, R_k) satisfying (7), (8). Then we can write:

$$u_{n_k} = v_k + w_k + z_k$$

with

$$v_k = u_{n_k} \mathbf{1}_{|y - y_k(\frac{R_k}{2})| \leq \frac{R_k}{2}}, \quad w_k = u_{n_k} \mathbf{1}_{|y - y_k(\frac{R_k}{2})| \geq R_k}, \quad z_k = u_{n_k} \mathbf{1}_{\frac{R_k}{2} < |y - y_k(\frac{R_k}{2})| < R_k}.$$

The key is to observe from (14) and (8) that:

$$\begin{aligned} \int |z_k|^2 &= \int_{B(y_k(\frac{R_k}{2}),R_k)} |u_{n_k}|^2 - \int_{B(y_k(\frac{R_k}{2}),\frac{R_k}{2})} |u_{n_k}|^2 \\ &\leq \rho_{n_k}(R_k) - \int_{B(y_k(\frac{R_k}{2}),\frac{R_k}{2})} |u_{n_k}|^2 = \rho_{n_k}(R_k) - \rho_{n_k}(\frac{R_k}{2}) \\ &\to 0 \text{ as } k \to +\infty. \end{aligned}$$

The claim dichotomy now follows by taking smooth cut off in the localization. The L^p norm of z_k will go to zero using the vanishing of the L^2 norm and the global H^1 bound, and the error introduced by localization will be treated using $R_k \to +\infty$. This is left to the reader.
This concludes the proof of Proposition 2.

We now show how the description of the Sobolev injection given by Proposition 2 is a powerful tool to study variational problems.

Proof of Proposition 1

step1 Computation of $I(M)$.

let $I(M)$ be given by (1). We claim that

$$-\infty < I(M) = M^{\frac{2(1-s_c)}{|s_c|}} I(1) < 0. \tag{15}$$

Indeed, $I(M) > -\infty$ follows directly form the Gagliardo-Nirenberg inequality and the subcriticality condition $1 < p < 1 + \frac{4}{N}$. The computation of the nonpositive value of the infimum follows from the scaling properties of the problem. First, given $u \in H^1$ with $\|u\|_{L^2} = 1$, we use the L^2 scaling

$$v_\lambda(x) = \lambda^{\frac{N}{2}} u(\lambda x)$$

to get:

$$E(v_\lambda) = \lambda^2 \left[\frac{1}{2} \int |\nabla u|^2 - \frac{1}{(p+1)\lambda^{(p-1)|s_c|}} \int |u|^{p+1} \right].$$

Letting $\lambda \to 0$ yields $I(1) < 0$. The homogeneity in M of $I(M)$ is derived by using the scaling of the equation

$$v_\lambda(x) = \lambda^{\frac{2}{p-1}} u(\lambda x),$$

for then

$$\|v_\lambda\|_{L^2} = \lambda^{|s_c|} \|u\|_{L^2}, \quad E(v_\lambda) = \lambda^{2(1-s_c)} E(u),$$

which yields the claim.

step 2 Vanishing cannot occur.

Let now u_n be a minimizing sequence for $I(M)$. Then u_n is bounded in H^1 from the Gagliardo-Nirenberg inequality and satisfies the assumptions of Proposition 1. We claim that vanishing cannot occur. Indeed, up to a sequence, we would have from (4):

$$I(M) = \lim_{k \to +\infty} E(u_{n_k}) \geq \liminf_{k \to +\infty} \frac{1}{2} \int |\nabla u_{n_k}|^2 \geq 0$$

which contradicts (15).

step 3 Dichotomy cannot occur.

We now claim that dichotomy cannot occur. Indeed, if it did, then from (5), we would have sequences v_k, w_k and $0 < \alpha < 1$ such that

$$\|v_k\|_{L^2} = \alpha M, \quad \|w_k\|_{L^2} = (1-\alpha)M$$

and

$$I(M) \geq \liminf_{k\to+\infty} E(v_k) + \liminf_{k\to+\infty} E(w_k).$$

In particular, this implies:

$$I(M) \geq I(\alpha M) + I((1-\alpha)M)$$

and thus from (15):

$$1 \leq \alpha^{\frac{2(1-s_c)}{|s_c|}} + (1-\alpha)^{\frac{2(1-s_c)}{|s_c|}} \quad \text{for some} \quad 0 < \alpha < 1.$$

Now a straightforward convexity argument implies from $\frac{2(1-s_c)}{|s_c|} > 1$ that this implies $\alpha = 0$ or $\alpha = 1$, a contradiction.

step 4 Conclusion.

We conclude that only compactness occurs ie

$$u_{n_k}(\cdot + x_k) \to u \quad \text{in} \quad L^{p+1}.$$

Observe then from the strong L^{p+1} convergence and the lower semicontinuity of the $\dot{H}^1$ norm that u attains the infimum:

$$\|u\|_{L^2} = M, \quad E(u) = I(M).$$

It thus remains to characterize the infimum. We claim that:

$$u(x) = Q_{\lambda(M)}(\cdot + x_0)e^{i\gamma_0} \tag{16}$$

which concludes the proof of Proposition 2.
Proof of (16): First observe from $\int |\nabla |u||^2 \leq \int |\nabla u|^2$ that $v = |u|$ is a minimizer with $v \geq 0$. From standard Euler Lagrange theory, v solves

$$\Delta v + v|v|^{p-1} = \mu v, \quad v \in H^1.$$

Note that the Lagrange multiplier μ a priori depends on v. In fact, we claim that is does not. Multiplying the equation by v and then $y \cdot \nabla v$ (Pohozaev integration), we compute:

$$\mu = \frac{N+2-p(N-2)}{2M\left(\frac{N(p-1)}{4}-1\right)} I(M) > 0.$$

We now observe by rescaling that $w(x) = \lambda^{\frac{2}{p-1}} v(\lambda x)$ with $\lambda = \sqrt{\mu}$ satisfies

$$\Delta w - w + w|w|^{p-1} = 0, \quad w \in H^1(\mathbf{R}^N), \quad w \geq 0,$$

and w non zero. From the uniqueness of the ground states, this yields:

$$w(x) = Q(x - x_0),$$

and hence $v(x) = Q_{\lambda(M)}(x - x_0)$. This implies in particular that v does not vanish, and thus $\int |\nabla u|^2 = \int |\nabla |u||^2$ -because they both are minimizers- with u that never vanishes implies

$$u(x) = |u(x)|e^{i\gamma_0} = Q_{\lambda(M)}(x - x_0)e^{i\gamma_0}.$$

This concludes the proof of (16).

Furthers comments

1. More general nonlinearities: The proof we have presented reproduces the original argument by Cazenave, Lions [3] and heavily relies onto the specific scaling properties of the nonlinearity. More general nonlinearities are treated together with a general criterion for the orbital stability of the ground state in Grillakis, Shatah, Strauss [6].

2. Other problems: This strategy of proof of stability is extremely robust and applies to a large class of nonlinear dispersive systems. See for example [8] for an extension to nonlinear kinetic problems arising in gravitation.

3. Asymptotic stability: An important question is to know whether, when stability holds, asymptotic stability also holds, that is do solutions asymptotically converge to the ground state in some local norm in space as $t \to +\infty$? This kind of property corresponds to a form of asymptotic irreversibility of the flow. This is an extremely delicate problem which has attracted a considerable amount of work for the past ten years. For some specific type of nonlinearities, asymptotic stability holds due to a fine tuning mechanism known as the "Fermi Golden Rule", see Soffer, Weinstein [13], Rodnianski, Soffer, Schlag [12], Sulem, Buslaev [1], Sigal, Zhou [4]. However, the case of pure power is still open because essentially small solitons are delicate to deal with. Indeed, in the pure power case, a soliton Q_λ can be made arbitrarily small in H^1 and not disperse. Moreover, one should keep in mind that the asymptotic stability is *false* in the completely integrable case

$N = 1$, $p = 3$, see [14].

4. General dynamics: In general, one expects the long time behavior of the solution to correspond to a splitting of the solution into a non dispersive part corresponding to a sum of decoupled solitary waves moving at different speeds and a radiative part which disperses -ie goes to 0 in L^∞ say-. Such a general behavior has been claimed in the integrable case for the KdV system

$$(KdV) \quad \begin{cases} u_t + (u_{xx} + u^2)_x = 0, \quad (t, x) \in [0, T) \times \mathbf{R}, \\ u(0, x) = u_0(x), \quad u_0 : \mathbf{R}^N \to \mathbf{R}, \end{cases}$$

but complete integrability plays a very specific role here. See Rodnianski, Soffer, Schlag [12], Martel, Merle, Tsai [11], for the case of nonintegrable (NLS) systems but with specific nonlinearities. One should think here that in general, even the simpler question of the orbital stability of the multisolitary wave in the pure power case for (NLS) is open.

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