Stability and blow up for the non linear Schrödinger Equation

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We continue in these notes the study of the power nonlinearity Schrödinger equation

$$(NLS) \quad \begin{cases} iu_t = -\Delta u - |u|^{p-1}u, & (t,x) \in [0,T) \times \mathbf{R}^N \\ u(0,x) = u_0(x), & u_0 : \mathbf{R}^N \to \mathbf{C} \end{cases} \tag{1}$$

with $u_0 \in H^1 = \{u, \nabla u \in L^2(\mathbf{R}^N)\}$ in dimension $N \geq 1$ and for energy subcritical nonlinearities:

$$1 < p < +\infty \quad \text{for} \quad N = 1, 2, \quad 1 < p < 2^* - 1 \quad \text{for} \quad N \geq 3. \tag{2}$$

Here $2^* = \frac{2N}{N-2}$ is the Sobolev exponent of the injection $\dot{H}^1 \hookrightarrow L^{2^*}$. Let us recall that the case $p = 3$ appears in various areas of physics: for the propagation of waves propagating in non linear media and optical fibers for $N = 1$, the focusing of laser beams for $N = 2$, the Bose-Einstein condensation phenomenon for $N = 3$, see the monograph [63] for a more systematic introduction to this physical aspect of the problem.

Our aim is to focus onto the description of the long time behavior or the singularity formation of solutions in the energy space H^1. The possibility of finite time blow up corresponding to a self focusing of the nonlinear wave will be of particular interest to us. Note that (NLS) is an infinite dimensional Hamiltonian system without any space localization property and infinite speed of propagation. It is in this context together with the critical generalized KdV equation one of the few examples where blow up is known to occur. For (NLS), an elementary proof of existence of blow up solutions is known since the 60's but is based on energy constraints and is not constructive. In particular, *no qualitative information of any type on the blow up dynamics is obtained this way.*

The theory of global existence or blow up for (NLS) as known up to now is intimately connected to the theory of ground states or solitons which are special periodic solutions to the Hamiltonian system. A central question is the stability of these solutions and the description of the flow around them which has attracted a considerable amount of work for the past thirty years. Here we shall introduce a new angle of attack for these problems based on *Liouville* type theorems and the *dynamical* classification of the soliton solution among the solutions to the Hamiltonian system.

These notes are organized as follows.

In the first section, we recall the main standard results about subcritical non linear Schrödinger equations and in particular the existence and orbital stability of soliton like solutions. In the second section, we focus onto the critical blow up problem and recall the few general results known on the singularity formation in this case. Section 3 is devoted to an exposition of the series of results obtained in collaboration with F.Merle in [44], [45], [46], [47], [48] and [58] and which allow in particular a complete description of the so-called stable log-log blow up dynamics. In section 4, we present a detailed proof of the first of these results which provides a sharp upper bound on blow up rate for a suitable class of initial data. In section 5, we outline the main steps of the proof of the sharp lower bound on the blow up rate and the mass quantization theorem which rely on fine dispersive properties of the flow. We expect the presentation to be essentially self contained provided the prior knowledge of standard tools in the study of non linear PDE's.

1 The subcritical problem

In this section, we recall the main classical facts regarding the global well posedness in the energy space of (NLS). We will in particular introduce a fundamental object for the study of (1): the ground state solitary wave.

1.1 Global well posedness in the subcritical case

Let us consider the general non linear Schrödinger equation:

$$\begin{cases} iu_t = -\Delta u - |u|^{p-1}u \\ u(0,x) = u_0(x) \in H^1 \end{cases} \tag{3}$$

with p satisfying the energy subcriticality assumption (2). The local well posedness of (3) in H^1 is a result of Ginibre, Velo, [15], see also [22]. Thus, for $u_0 \in H^1$, there exists $0 < T \leq +\infty$ such that $u(t) \in \mathcal{C}([0,T), H^1)$. Moreover, the life time of the solution can be proved to be lower bounded by a function depending on the H^1 size of the solution only, $T(u_0) \geq f(\|u_0\|_{H^1})$, and hence there holds the blow up alternative:

$$T < +\infty \ \text{ implies } \ \lim_{t\to T} \|u(t)\|_{H^1} = +\infty. \tag{4}$$

To prove the global existence of the solution, it thus suffices to control the size of the solution in H^1. This is achieved in some cases using the invariants of the flow. Indeed, the following H^1 quantities are conserved by the flow:

- L^2-norm:
$$\int |u(t,x)|^2 = \int |u_0(x)|^2; \tag{5}$$

- Energy -or Hamiltonian-:
$$E(u(t,x)) = \frac{1}{2}\int |\nabla u(t,x)|^2 - \frac{1}{p+1}\int |u(t,x)|^{p+1} = E(u_0); \tag{6}$$

- Momentum:
$$Im\left(\int \nabla u \overline{u}(t,x)\right) = Im\left(\int \nabla u_0 \overline{u_0}(x)\right). \tag{7}$$

Note that the growth condition on the non linearity (2) ensures from Sobolev embedding that the energy is well defined, and this is why H^1 is referred to as the energy space.

These invariants are related to the group of symmetry of (3) in H^1:

- Space-time translation invariance: if $u(t,x)$ solves (3), then so does $u(t+t_0, x+x_0)$, $t_0 \in \mathbf{R}$, $x_0 \in \mathbf{R}^N$.
- Phase invariance: if $u(t,x)$ solves (3), then so does $u(t,x)e^{i\gamma}$, $\gamma \in \mathbf{R}$.
- Scaling invariance: if $u(t,x)$ solves (3), then so does $u_\lambda(t,x) = \lambda^{\frac{2}{p-1}}u(\lambda^2 t, \lambda x)$, $\lambda > 0$.
- Galilean invariance: if $u(t,x)$ solves (3), then so does $u(t, x-\beta t)e^{i\frac{\beta}{2}\cdot(x-\frac{\beta}{2}t)}$, $\beta \in \mathbf{R}^N$.

Let us point out that this group of H^1 symmetries is the same like for the *linear* Schrödinger equation -up to the conformal invariance to which we will come back later-.

The *critical space* is defined to be the Sobolev space which is invariant by the scaling symmetry:
$$|u_\lambda(t)|_{\dot{H}^{s_c}} = |u(\lambda^2 t)|_{\dot{H}^{s_c}} \quad \text{for} \quad s_c = \frac{N}{2} - \frac{2}{p-1}. \tag{8}$$
Note that $s_c < 1$ from (2).

As an outcome, we have the following result:

Theorem 1 (Global wellposedness in the subcritical case) *Let $N \geq 1$ and $1 < p < 1 + \frac{4}{N}$ -equivalently $s_c < 0$-, then all solutions to (3) are global and bounded in H^1.*

Proof of Theorem 1

The proof relies on an a priori bound on the H^1 norm of the solution. Indeed, $|u(t)|_{L^2} = |u_0|_{L^2}$. Next, there holds the Gagliardo-Nirenberg interpolation estimate:

$$\forall v \in H^1, \quad \int |v|^{p+1} \leq C(N,p) \left(\int |\nabla v|^2\right)^{\frac{N(p-1)}{4}} \left(\int |v|^2\right)^{\frac{p+1}{2} - \frac{N(p-1)}{4}}. \tag{9}$$

Applying this with $v = u(t)$, we get from the conservation of the energy and the L^2 norm:

$$\forall t \in [0,T), \quad E_0 \geq \frac{1}{2}[\int |\nabla v|^2 - C(u_0)\left(\int |\nabla v|^2\right)^{\frac{N(p-1)}{4}}].$$

The subcriticality assumption $p < 1 + \frac{4}{N}$ now implies an a priori bound on the H^1 norm which concludes the proof of Theorem 1.

The critical exponent

$$p = 1 + \frac{4}{N} \quad \text{ie} \quad s_c = 0$$

arises from this analysis and corresponds to the so-called L^2 *critical case.* It is the smallest power nonlinearity for which blow up can occur and corresponds to an exact balance between the kinetic and potential energies under the constraint of conserved L^2 mass. The L^2 supercritical -and energy subcritical cases- corresponds to

$$1 + \frac{4}{N} < p < 2^* - 1 \quad \text{ie} \quad 0 < s_c < 1.$$

1.2 The solitary wave

We focus in this subsection onto the subcritical case

$$1 < p < 1 + \frac{4}{N} \tag{10}$$

and aim at understanding the long time dynamics of the flow.
A fundamental feature of the focusing (NLS) problem is the existence of periodic solutions. Indeed,

$$u(t,x) = \phi(x)e^{it}$$

is an H^1 solution to (3) iff ϕ solves the nonlinear elliptic equation:

$$\Delta\phi - \phi + \phi|\phi|^{p-1} = 0, \quad \phi \in H^1(\mathbf{R}^N) \tag{11}$$

There are plenty of ways to construct solutions to (11), the simplest of which being to look for radial solutions and using a shooting method, see [3].

Proposition 1 (Existence of radial profiles to (11)) *Let $N \geq 1$ and p satisfy (2), then there exists radially symmetric solutions to (11). In fact, the following holds:*
(i) For $N = 1$, all solutions to (11) are translates of

$$Q(x) = \left(\frac{p+1}{2\cosh^2(\frac{(p-1)x}{2})} \right)^{p-1}. \tag{12}$$

(ii) For $N \geq 2$, there exist a sequence of radial solutions $(Q_n)_{n\geq 0}$ with increasing L^2 norm such that Q_n vanishes n times on $\mathbf{R}^N$.

The exact structure of the set of solutions to (11) is not known in dimension $N \geq 2$. An important rigidity property however which combines nonlinear elliptic techniques and ODE techniques is *the uniqueness of the nonnegative* solution $Q = Q_0$ to (11).

Proposition 2 (Uniqueness of the ground state) *All solutions to*

$$\Delta\phi - \phi + \phi|\phi|^{p-1} = 0, \quad \phi \in H^1(\mathbf{R}^N), \quad \phi(x) > 0 \tag{13}$$

are a translate of an exponentially decreasing $\mathcal{C}^2$ radial profile $Q(r)$ ([14]) which is the unique nonnegative radially symmetric solution to (11) ([26]). Q is the socalled ground state solution.

The uniqueness is thus the consequence of two facts: a positive decaying at infinity solution to (13) is necessarily radially symmetric with respect to a point. This is a very deep and non trivial result due to Gidas, Ni, Nirenberg [14] and which relies on the maximum principle. Then there is uniqueness of the radial decaying positive solution in the ODE sense. The original -and delicate- proof of this last fact by Kwong [26] has been revisited by MacLeod [32] and is very nicely presented in the Appendix of Tao [64].

Let us now observe that we may let the full group of symmetries of (3) act on the solitary wave $u(t,x) = Q(x)e^{it}$ to get a $2N+2$ parameters family of solitary waves:

$$u(t,x) = \lambda^{\frac{2}{p-1}} Q(\lambda(x+x_0) - \lambda^2\beta t) e^{i\lambda^2 t} e^{i\gamma_0} e^{i\frac{\beta}{2}\cdot(\lambda(x+x_0)-\lambda^2\beta t)}, \ \ (\lambda, x_0, \gamma_0, \beta) \in \mathbf{R}^*_+ \times \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N.$$

Observe that these waves are moving according to the free Galilean motion and oscillating at a phase related to their size λ: the larger the λ, the wilder the oscillations in time. Observe also that the fact the problem is subcritical in both L^2 and H^1 implies that a solitary wave can be made *arbitrarily small* in H^1.

Let us finish this section by introducing the Korteweg de Vries equation for which the structure of the solitary wave family is somehow more enlightening. The KdV system corresponds to the description of waves propagating at the surface of water in certain regimes. It is the case $p = 2$ of the generalized KdV equations:

$$(gKdV) \quad \begin{cases} u_t + (u_{xx} + u^p)_x = 0, \quad (t,x) \in [0,T) \times \mathbf{R}, \\ u(0,x) = u_0(x), \quad u_0 : \mathbf{R} \to \mathbf{R}. \end{cases} \tag{14}$$

This system is also a nonlinear dispersive equation and admits the same two conservation laws like the (NLS):

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2},$$

$$E(u) = \frac{1}{2}\int |u_x|^2 - \frac{1}{p+1}\int |u|^{p+1} = E(u_0).$$

The Cauchy problem is moreover subcritical in H^1 and thus all solutions are global and bounded in H^1 for $p < 5$. We now look for traveling waves propagating at speed $c > 0$: $u(t,x) = Q_c(x-ct)$ solves (14) iff

$$(Q_c)_{xx} - cQ_c + Q_c^p = 0, \quad Q_c \in H^1.$$

By rescaling, this implies :

$$Q_c(x) = c^{\frac{1}{p-1}} Q(\sqrt{c}x)$$

where Q is the H^1 solution to (11) which is in fact explicit and given by (12). An explicit computation then yields:

$$\|Q_c\|_{L^2} = c^{\frac{5-p}{8(p-1)}}\|Q\|_{L^2}, \quad \|\nabla Q_c\|_{L^2} = c^{\frac{3p+1}{8(p-1)}}\|\nabla Q\|_{L^2}.$$

The outcome is that small solitons travel slowly while large solitons are fast.

1.3 Orbital stability of the ground states in the subcritical case

We focus in this section onto the question of the stability of the ground state solitary wave $u(t,x) = Q(x)e^{it}$ where $Q > 0$ is the ground state solution to (11). Let us first observe that two trivial instabilities are given by the symmetries of the equation:

- Scaling instability: $\forall \lambda > 0$, the solution to (3) with initial data $u_0(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x)$ is $u(t,x) = \lambda^{\frac{2}{p-1}} Q(\lambda x) e^{i\lambda^2 t}$.
- Galilean instability: $\forall \beta > 0$, the solution to (3) with initial data $u_0(x) = Q(x)e^{i\beta}$ is $u(t,x) = Q(x-\beta t)e^{it + \frac{\beta}{2}\cdot(x - \frac{\beta}{2}t)}$.

In both cases,

$$\sup_{t\in\mathbf{R}} |u(t,x) - Q(x)e^{it}| > |Q(x)|$$

and thus the solution does not stay uniformly close to Q, whatever close it was at initial time.

Cazenave and Lions [10] proved that these trivial instabilities are the only one. This is the socalled *orbital stability* of the ground state solitary wave.

Theorem 2 (Orbital stability of the ground state) *Let $N \geq 1$ and p satisfy (10). For all $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that the following holds true. Let $u_0 \in H^1$ with*

$$\|u_0 - Q\|_{H^1} < \delta(\varepsilon),$$

then there exists a translation shift $x(t) \in \mathbf{R}^N$ and phase shift $\gamma(t) \in \mathbf{R}$ such that:

$$\forall t \geq 0, \quad \|u(t,x) - Q(x - x(t))e^{i\gamma(t)}\|_{H^1} < \varepsilon.$$

The strength -and the weakness- of the proof is that it relies only on the conservation laws and the *variational characterization of the ground state solitary wave*. This study falls into the classical sets of *concentration compactness techniques* as introduced by Lions in [30],[31]. Given $\lambda > 0$, we let

$$Q_\lambda(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x).$$

The following variational result immediately implies Theorem 2:

Proposition 3 (Description of the minimizing sequences) *Let $N \geq 1$ and p satisfy (10). Let $M > 0$ be fixed.*
(i) Variational characterization of Q: The minimization problem

$$I(M) = \inf_{|u|_{L^2} = M} E(u) \tag{15}$$

is attained on the family

$$Q_{\lambda(M)}(\cdot - x_0)e^{i\gamma_0}, \quad x_0 \in \mathbf{R}^N, \gamma_0 \in \mathbf{R},$$

where $\lambda(M)$ is the unique scaling such that $|Q_{\lambda(M)}|_{L^2} = M$.
(ii) Description of the minimizing sequences: Any minimizing sequence v_n to (15) is relatively compact in H^1 up to translation and phase shifts, that is up to a subsequence:

$$v_n(\cdot + x_n)e^{i\gamma n} \to Q_{\lambda(M)} \quad \text{in} \quad H^1.$$

The fact that Proposition 3 implies Theorem 2 is now a simple consequence of the conservation laws and is left to the reader. The next section is devoted to an outline of the proof of Proposition 3.

1.4 The concentration compactness argument

The first key to the proof of Proposition 3 is the description of the lack of compactness in $\mathbf{R}^N$ of the Sobolev injection $H^1 \hookrightarrow L^{p+1}$, $2 \leq p+1 < 2^*$. This description is a consequence of the so-called concentration compactness Lemma. Let us recall that the injection is compact on a smooth bounded domain. Note also that the injection is still compact when restricted to radial functions in dimension $N \geq 2$. Here one uses the estimate:

$$u^2(r) = -\int_r^{+\infty} u(s)u'(s)ds \ \text{ and thus } \ |u|_{L^\infty(r\geq R)} \leq \frac{C}{R^{\frac{N-1}{2}}}|\nabla u|_{L^2}^{\frac{1}{2}}|u|_{L^2}^{\frac{1}{2}}$$

so that any H^1 bounded sequence of radially symmetric functions is L^{p+1} compact. This would considerably simplify the proof of Proposition 3 when restricting the problem to radially symmetric functions.

In general, there holds the following:

Proposition 4 (Description of the lack of compactness of $H^1 \hookrightarrow L^q$) *Let a sequence $u_n \in H^1$ with*

$$|u_n|_{L^2} = M, \quad |\nabla u_n|_{L^2} \leq C, \tag{16}$$

Then there exists a subsequence u_{n_k} such that one of the following three scenari occurs:
(i) Compactness: $\exists y_k \in \mathbf{R}^N$ such that

$$\forall 2 \leq q < 2^*, \quad u_{n_k}(\cdot + y_k) \to u \quad in \quad L^q. \tag{17}$$

(ii) Vanishing:

$$\forall 2 < q < 2^*, \quad u_{n_k} \to 0 \quad in \quad L^q. \tag{18}$$

(iii) Dichotomy: $\exists v_k, w_k$, $\exists 0 < \alpha < 1$ such that $\forall 2 \leq q < 2^*$:

$$\left\{ \begin{array}{l} Supp(v_k) \cap Supp(w_k) = \emptyset, \quad dist(Supp(v_k), Supp(w_k)) \to +\infty, \\ \|v_k\|_{H^1} + \|w_k\|_{H^1} \leq C, \\ \|v_k\|_{L^2} \to \alpha M, \quad \|w_k\|_{L^2} \to (1-\alpha)M, \\ \lim_{k\to+\infty} |\int |u_{n_k}|^q - \int |v_k|^q - \int |w_k|^q| = 0, \\ \liminf_{k\to+\infty} \int |\nabla u_{n_k}|^2 - \int |\nabla v_k|^2 - \int |\nabla w_k|^2 \geq 0. \end{array} \right. \tag{19}$$

Remark 1 *Observe that the key in the dichotomy case is that there is no loss of potential energy during the splitting in space of u_{n_k} into two bumps v_k, w_k which support go away from each other, while on the other hand only a lower semi continuity bound can be derived for the kinetic energy,*

Remark 2 *The case dichotomy corresponds to the localization of the first bubble of concentration. One can then continue the extraction iteratively and obtain the socalled profile decomposition of the sequence u_n, see P. Gerard [13], Hmidi, Keraani [20] for a very elegant proof.*

Proof of Proposition 4

We follow Cazenave [9]. Let $u_n \in H^1$ be as in the hypothesis of Proposition 4.

step 1 Concentration function.

Let the sequence of concentration functions:

$$\rho_n(R) = \sup_{y \in \mathbf{R}^N} \int_{B(y,R)} |u_n(x)|^2 dx.$$

The following facts are elementary and left to the reader:

- Monotonicity: $\forall n \geq 0$, $\rho_n(R)$ is a nondecreasing function of R.
- The concentration point is attained:

$$\forall R > 0, \quad \forall n \geq 0, \quad \exists y_n(R) \in \mathbf{R}^N \text{ such that } \rho_n(R) = \int_{B(y_n(R),R)} |u_n(x)|^2 dx.$$

- Uniform Hölder continuity: $\exists C, \alpha > 0$ independent of n such that

$$\forall R_1, R_2 > 0, \quad \forall n \geq 0, \quad |\rho_n(R_1) - \rho_n(R_2)| \leq C|R_1^N - R_2^N|^\alpha. \tag{20}$$

  This last fact is a simple consequence of the H^1 bound (16).

step 2 Limit of concentration functions.

From (20) and Ascoli's theorem, there exists a subsequence $n_k \to +\infty$ and a nondecreasing limit ρ such that

$$\forall R > 0, \quad \lim_{k \to +\infty} \rho_{n_k}(R) = \rho(R). \tag{21}$$

Let now

$$\mu = \lim_{R \to +\infty} \liminf_{n \to +\infty} \rho_n(R).$$

By definition, there exists $R_k \to +\infty$ such that

$$\lim_{k \to +\infty} \rho_{n_k}(R_k) = \mu.$$

We now claim some stability of the sequence R_k which is a very general and simple fact but crucial for the rest of the argument:

$$\mu = \lim_{k \to +\infty} \rho_{n_k}(R_k) = \lim_{k \to +\infty} \rho_{n_k}(\frac{R_k}{2}) = \lim_{R \to +\infty} \rho(R). \tag{22}$$

Proof of (22): First oberve from the monotonicity of ρ_{n_k} that

$$\limsup_{k \to +\infty} \rho_{n_k}(\frac{R_k}{2}) \leq \limsup_{k \to +\infty} \rho_{n_k}(R_k) = \mu. \tag{23}$$

For the other sense, we argue as follows. For every $R > 0$, there holds:

$$\rho(R) = \liminf_{k\to+\infty} \rho_{n_k}(R) \geq \liminf_{n\to+\infty} \rho_n(R)$$

and thus:

$$\lim_{R\to+\infty} \rho(R) \geq \mu. \tag{24}$$

Eventually, for any $R > 0$, we have $\frac{R_k}{2} \geq R$ for k large enough and thus:

$$\rho_{n_k}(\frac{R_k}{2}) \geq \rho_{n_k}(R).$$

Letting $k \to +\infty$ implies:

$$\forall R > 0, \quad \lim_{k\to+\infty} \rho_{n_k}(\frac{R_k}{2}) \geq \rho(R).$$

Letting now $R > 0$ yields:

$$\lim_{k\to+\infty} \rho_{n_k}(\frac{R_k}{2}) \geq \lim_{R\to+\infty} \rho(R) \geq \mu$$

where we used (24) in the last step. This together with (23) concludes the proof of (22). The proof now proceed by making an hypothesis on μ.

Step 3: $\mu = 0$ is vanishing.

Assume $\mu = 0$. Then from (22), $\lim_{R\to+\infty} \rho(R) = 0$. But ρ is nondecreasing positive so: $\forall R > 0$, $\rho(R) = 0$. In particular, $\rho(1) = 0$ and thus

$$\lim_{k\to+\infty} \rho_{n_k}(1) = \lim_{k\to+\infty} \sup_{y\in\mathbf{R}^N} \int_{B(y,1)} |u_{n_k}|^2 = 0. \tag{25}$$

We claim that this together with the H^1 bound on u_{n_k} implies (18). There is a slight difficulty here which is that we need to pass from a local information -vanishing on every ball- to a global information -vanishing of the global L^q norm-. This relies on a refinement of the Gagliardo Nirenberg interpolation inequality. Indeed, we claim that

$$\forall u \in H^1, \quad \int |u|^{2+\frac{4}{N}} \leq C\|u\|_{H^1}^2 \|u\|_{L^2}^{\frac{4}{N}} \tag{26}$$

can be refined for:

$$\forall u \in H^1, \quad \int |u|^{2+\frac{4}{N}} \leq C\|u\|_{H^1}^2 \left[\sup_{y\in\mathbf{R}^N} \int_{B(y,1)} |u|^2\right]^{\frac{2}{N}}. \tag{27}$$

This together with (25) implies

$$u_{n_k} \to 0 \ \text{ in } \quad L^{2+\frac{4}{N}} \ \text{ as } \ k \to +\infty$$

and (18) follows by interpolation using the global H^1 bound.
Proof of (27): Take a partition of $\mathbf{R}^N$ with balls of radius $\frac{1}{2}$ and smooth cut off functions χ_k adapted to this partition. Then from (26):

$$\begin{aligned}
\int |u|^{2+\frac{4}{N}} &\leq C\Sigma_{k\geq 0}\int |\chi_k u|^{2+\frac{4}{N}} \leq C\Sigma_{k\geq 0}\left(\int |\nabla(\chi_k u)|^2\right)\left(\int |\chi_k u|^2\right)^{\frac{2}{N}} \\
&\leq C\left[\sup_{y\in\mathbf{R}^N}\int_{B(y,1)}|u|^2\right]^{\frac{2}{N}}\Sigma_{k\geq 0}\left[\int |\chi_k\nabla u|^2 + |\nabla\chi_k|^2|u|^2\right] \\
&\leq C\left[\sup_{y\in\mathbf{R}^N}\int_{B(y,1)}|u|^2\right]^{\frac{2}{N}}\|u\|_{H^1}^2.
\end{aligned}$$

Step 4: $\mu = M$ is compactness.

Let n_k be the sequence satisfying (21). For $R > 0$, let $y_k(R)$ such that

$$\rho_{n_k}(R) = \int_{B(y_k(R),R)} |u_{n_k}(x)|^2 dx. \tag{28}$$

Pick $\varepsilon > 0$. Then from (22), there exist $R_0, R(\varepsilon)$ such that

$$\rho(R_0) > \frac{M}{2}, \quad \rho(R(\varepsilon)) > M - \varepsilon.$$

Hence there exists $k_0(\varepsilon)$ such that $\forall k \geq k_0(\varepsilon)$,

$$\rho_{n_k}(R_0) = \int_{B(y_k(R_0),R_0)} |u_{n_k}|^2 > \frac{M}{2}, \quad \rho_{n_k}(R(\varepsilon)) = \int_{B(y_k(R(\varepsilon)),R(\varepsilon))} |u_{n_k}|^2 > M - \varepsilon.$$

But the total L^2 mass being M, this implies that the balls $B(y_k(R_0), R_0)$ and $B(y_k(R(\varepsilon)), R(\varepsilon))$ cannot be disjoint. Hence -draw a picture- we can find $R_1(\varepsilon)$ such that:

$$\forall \varepsilon > 0, \quad \forall k \geq k_0(\varepsilon), \quad \int_{B(y_k(R_0),R_1(\varepsilon))} |u_{n_k}|^2 \geq M - \varepsilon.$$

By possibly raising the value of $R_1(\varepsilon)$ for the values $k \in [1, k_0(\varepsilon)]$, this implies that the sequence $v_k = u_{n_k}(\cdot + y_k(R_0))$ is L^2 compact:

$$\forall \varepsilon > 0, \quad \exists R_2(\varepsilon) > 0 \ \text{ such that } \ \forall k \geq 1, \quad \int_{|y|\geq R_2(\varepsilon)} |v_k(y)|^2 dy < \varepsilon.$$

The compactness of the embedding $H^1 \hookrightarrow L^2(B(0,R(\varepsilon)))$ then implies that v_k a Cauchy sequence in L^2, and the H^1 boundedness now implies (17) by interpolation.

Step 5: $0<\mu<M$ is dichotomy.

Let again (n_k,R_k) satisfying (21), (22). Then we can write:

$$u_{n_k}=v_k+w_k+z_k$$

with

$$v_k=u_{n_k}\mathbf{1}_{|y-y_k(\frac{R_k}{2})|\le\frac{R_k}{2}},\quad w_k=u_{n_k}\mathbf{1}_{|y-y_k(\frac{R_k}{2})|\ge R_k},\quad z_k=u_{n_k}\mathbf{1}_{\frac{R_k}{2}<|y-y_k(\frac{R_k}{2})|<R_k}.$$

The key is to observe from (28) and (22) that:

$$\begin{aligned}\int|z_k|^2 &= \int_{B(y_k(\frac{R_k}{2}),R_k)}|u_{n_k}|^2-\int_{B(y_k(\frac{R_k}{2}),\frac{R_k}{2})}|u_{n_k}|^2\\ &\le \rho_{n_k}(R_k)-\int_{B(y_k(\frac{R_k}{2}),\frac{R_k}{2})}|u_{n_k}|^2=\rho_{n_k}(R_k)-\rho_{n_k}(\frac{R_k}{2})\\ &\to 0\ \text{ as }\ k\to+\infty.\end{aligned}$$

The claim dichotomy now follows by taking smooth cut off in the localization. The L^p norm of z_k will go to zero using the vanishing of the L^2 norm and the global H^1 bound, and the error introduced by localization will be treated using $R_k\to+\infty$. This is left to the reader.
This concludes the proof of Proposition 4.

We now show how the description of the Sobolev injection given by Proposition 4 is a powerful tool to study variational problems.

Proof of Proposition 3

step1 Computation of $I(M)$.

let $I(M)$ be given by (15). We claim that

$$-\infty<I(M)=M^{\frac{2(1-s_c)}{|s_c|}}I(1)<0. \tag{29}$$

Indeed, $I(M)>-\infty$ follows directly form the Gagliardo-Nirenberg inequality (9) and the subcriticality condition (10). The computation of the nonpositive value of the infimum follows from the scaling properties of the problem. First, given $u\in H^1$ with $\|u\|_{L^2}=1$, we use the L^2 scaling

$$v_\lambda(x)=\lambda^{\frac{N}{2}}u(\lambda x)$$

to get:

$$E(v_\lambda) = \lambda^2 \left[\frac{1}{2}\int |\nabla u|^2 - \frac{1}{(p+1)\lambda^{(p-1)|s_c|}}\int |u|^{p+1}\right].$$

Letting $\lambda \to 0$ yields $I(1) < 0$. The homogeneity in M of $I(M)$ is derived by using the scaling of the equation

$$v_\lambda(x) = \lambda^{\frac{2}{p-1}} u(\lambda x),$$

for then

$$\|v_\lambda\|_{L^2} = \lambda^{|s_c|}\|u\|_{L^2}, \quad E(v_\lambda) = \lambda^{2(1-s_c)}E(u),$$

which yields the claim.

step 2 Vanishing cannot occur.

Let now u_n be a minimizing sequence for $I(M)$. Then u_n is bounded in H^1 from (9) and satisfies the assumptions of Proposition 3. We claim that vanishing cannot occur. Indeed, up to a sequence, we would have from (18):

$$I(M) = \lim_{k\to+\infty} E(u_{n_k}) \geq \liminf_{k\to+\infty} \frac{1}{2}\int |\nabla u_{n_k}|^2 \geq 0$$

which contradicts (29).

step 3 Dichotomy cannot occur.

We now claim that dichotomy cannot occur. Indeed, if it did, then from (19), we would have sequences v_k, w_k and $0 < \alpha < 1$ such that

$$\|v_k\|_{L^2} = \alpha M, \quad \|w_k\|_{L^2} = (1-\alpha)M$$

and

$$I(M) \geq \liminf_{k\to+\infty} E(v_k) + \liminf_{k\to+\infty} E(w_k).$$

In particular, this implies:

$$I(M) \geq I(\alpha M) + I((1-\alpha)M)$$

and thus from (29):

$$1 \leq \alpha^{\frac{2(1-s_c)}{|s_c|}} + (1-\alpha)^{\frac{2(1-s_c)}{|s_c|}} \quad \text{for some} \quad 0 < \alpha < 1.$$

Now a straightforward convexity argument implies from $\frac{2(1-s_c)}{|s_c|} > 1$ that this implies $\alpha = 0$ or $\alpha = 1$, a contradiction.

step 4 Conclusion.

We conclude that only compactness occurs ie

$$u_{n_k}(\cdot + x_k) \to u \ \text{ in } \ L^{p+1}.$$

Observe then from the strong L^{p+1} convergence and the lower semicontinuity of the $\dot{H}^1$ norm that u attains the infimum:

$$\|u\|_{L^2} = M, \quad E(u) = I(M).$$

It thus remains to characterize the infimum. We claim that:

$$u(x) = Q_{\lambda(M)}(\cdot + x_0)e^{i\gamma_0} \tag{30}$$

which concludes the proof of Proposition 4.
Proof of (30): First observe from $\int |\nabla |u||^2 \leq \int |\nabla u|^2$ that $v = |u|$ is a minimizer with $v \geq 0$. From standard Euler Lagrange theory, v solves

$$\Delta v + v|v|^{p-1} = \mu v, \quad v \in H^1.$$

Note that the Lagrange multiplier μ a priori depends on v. In fact, we claim that is does not. Multiplying the equation by v and then $y \cdot \nabla v$ (Pohozaev integration), we compute:

$$\mu = \frac{N+2-p(N-2)}{2M\left(\frac{N(p-1)}{4} - 1\right)} I(M) > 0.$$

We now observe by rescaling that $w(x) = \lambda^{\frac{2}{p-1}} v(\lambda x)$ with $\lambda = \sqrt{\mu}$ satisfies

$$\Delta w - w + w|w|^{p-1} = 0, \quad w \in H^1(\mathbf{R}^N), \quad w \geq 0,$$

and w non zero. From the uniqueness statement of Proposition 2, this yields:

$$w(x) = Q(x - x_0),$$

and hence $v(x) = Q_{\lambda(M)}(x - x_0)$. This implies in particular that v does not vanish, and thus $\int |\nabla u|^2 = \int |\nabla |u||^2$ -because they both are minimizers- with u that never vanishes implies

$$u(x) = |u(x)|e^{i\gamma_0} = Q_{\lambda(M)}(x - x_0)e^{i\gamma_0}.$$

This concludes the proof of (30).

Furthers comments

1. More general nonlinearities: The proof we have presented reproduces the original argument by Cazenave, Lions [10] and heavily relies onto the specific scaling properties of the nonlinearity. More general nonlinearities are treated together with a general criterion for the orbital stability of the ground state in Grillakis, Shatah, Strauss [18].

2. Other problems: This strategy of proof of stability is extremely robust and applies to a large class of nonlinear dispersive systems. See for example [28] for an extension to nonlinear kinetic problems arising in gravitation.

3. Asymptotic stability: An important question is to know whether, when stability holds, asymptotic stability also holds, that is do solutions asymptotically converge to the ground state in some local norm in space as $t \to +\infty$? This kind of property corresponds to a form of asymptotic irreversibility of the flow. This is an extremely delicate problem which has attracted a considerable amount of work for the past ten years. For some specific type of nonlinearities, asymptotic stability holds due to a fine tuning mechanism known as the "Fermi Golden Rule", see Soffer, Weinstein [62], Rodnianski, Soffer, Schlag [59], Sulem, Buslaev [8], Sigal, Zhou [12]. However, the case of pure power is still open because essentially small solitons are delicate to deal with. Indeed, in the pure power case, a soliton Q_λ can be made arbitrarily small in H^1 and not disperse. Moreover, one should keep in mind that the asymptotic stability is *false* in the completely integrable case $N = 1$, $p = 3$, see [68].

4. General dynamics: In general, one expects the long time behavior of the solution to correspond to a splitting of the solution into a non dispersive part corresponding to a sum of decoupled solitary waves moving at different speeds and a radiative part which disperses -ie goes to 0 in L^∞ say-. Such a general behavior has been claimed in the integrable case for the KdV system

$$(KdV) \quad \begin{cases} u_t + (u_{xx} + u^2)_x = 0, & (t, x) \in [0, T) \times \mathbf{R}, \\ u(0, x) = u_0(x), & u_0 : \mathbf{R}^N \to \mathbf{R}, \end{cases}$$

but complete integrability plays a very specific role here. See Rodnianski, Soffer, Schlag [59], Martel, Merle, Tsai [37], for the case of nonintegrable (NLS) systems but with specific nonlinearities. One should think here that in general, even the simpler question of the orbital stability of the multisolitary wave in the pure power case for (NLS) is open.

1.5 Existence of blow up solutions: the virial law

Let us now consider (NLS) in the critical and super critical range $p \geq 1 + \frac{4}{N}$. The Cauchy theory ensures global existence for small data in H^1 but for large data, the Gagliardo Nirenberg inequality (9) does not suffice anymore to ensure global existence. A well known global obstructive argument known as the viriel law allows one to very easily prove indeed the existence of finite time blow up solutions.

Theorem 3 (Virial blow up for $E_0 < 0$) *Let $u_0 \in \Sigma = H^1 \cap \{xu \in L^2\}$ with*

$$E_0 < 0,$$

then the corresponding solution to (3) blows up in finite time $0 < T < +\infty$.

Proof of Theorem 3

Integrating by parts in (3), we find:

$$\frac{d^2}{dt^2}\int |x|^2|u(t,x)|^2dx = 4N(p-1)E_0 - \frac{16s_c}{N-2s_c}\int |\nabla u|^2 \leq 4N(p-1)E_0 \tag{31}$$

from $s_c \geq 0$. Hence from $E_0 < 0$, the positive quantity $\int |x|^2|u(t,x)|^2dx$ lies below an inverted parabola and hence the solution cannot exist for all times. This concludes the proof of Theorem 3.

This blow up argument is extraordinary because it provides a blow up criterion based essentially on a pure Hamiltonian information $E_0 < 0$ which applies to arbitrarily large initial data in H^1. In particular, it exhibits an *open* region of the energy space -up to extra integrability condition- where blow up is proven to be a stable phenomenon. While it may seem at first hand to be very specific to the (NLS) problem, this kind of convexity argument is very common for parabolic or wave type problems, see for example [21], kinetic problems, see [17], or even compressible Euler equations, see [61].

However, it has has two major weaknesses:
(i) It heavily relies on a very specific algebra and hence is very unstable by perturbation of the equation. It thus is completely unable to predict blow up even in situations where it is strongly expected. A typical case is for example (NLS) on a domain with Dirichlet boundary conditions, see [57].
(ii) More fundamentally, this argument is *purely obstructive* in nature and says very little a priori on the singularity formation. More dramatically, the blow up time formally predicted which is the time of vanishing of the variance $\int |x|^2|u|^2$ is in fact almost never correct.

2 The L^2 critical problem

We focus in this section onto the L^2 critical case

$$\begin{cases} iu_t = -\Delta u - |u|^{\frac{4}{N}}u, & (t,x) \in \mathbf{R}\times\mathbf{R}^N \\ u(0,x) = u_0(x) \in H^1 \end{cases} \tag{32}$$

which is the smallest power like nonlinearity for which blow up occurs. Our aim is at first to show that a large part of the orbital stability theory developed for subcritical problems still applies in some generalized sense and provides some information on the structure of the singularity formation. The next section will be devoted to further dynamical classification results which in particular provide a complete description of a stable blow up regime which is among the very few understood problems so far.

2.1 Variational characterization of the ground state

The minimization problem (15) is no longer adapted to the critical problem due to the L^2 scaling invariance

$$u_\lambda(t,x) = \lambda^{\frac{N}{2}} u(\lambda^2 t, \lambda x). \tag{33}$$

Indeed, one easily proves that $I(M) = 0$ for $M << 1$ and $I(M) = -\infty$ for $M >> 1$. In fact, this may be easily rectified by remarking that the L^2 criticality of (32) corresponds to an exact balance between the kinetic and the potential energy. This may be quantified in a sharp way from the knowledge of the exact constant in the Gagliardo-Nirenberg inequality (9) as observed by Weinstein, [67].

Proposition 5 (Minimizers of the energy) *Let the H^1 functional:*

$$J(v) = \frac{(\int |\nabla v|^2)(\int |v|^2)^{\frac{2}{N}}}{\int |v|^{2+\frac{4}{N}}}. \tag{34}$$

The minimization problem

$$\min_{v \in H^1, \ v \neq 0} J(v)$$

is attained on the three parameters family:

$$\lambda_0^{\frac{N}{2}} Q(\lambda_0 x + x_0) e^{i\gamma_0}, \quad (\lambda_0, x_0, \gamma_0) \in \mathbf{R}_*^+ \times \mathbf{R}^N \times \mathbf{R},$$

where Q is the unique positive radial solution to the system:

$$\left\{ \begin{array}{l} \Delta Q - Q + Q^{1+\frac{4}{N}} = 0 \\ Q(r) \to 0 \quad as \quad r \to +\infty. \end{array} \right. \tag{35}$$

In particular, there holds the following Gagliardo-Nirenberg inequality with best constant:

$$\forall v \in H^1, \quad E(v) \geq \frac{1}{2}\int |\nabla v|^2 \left(1 - \left(\frac{|v|_{L^2}}{|Q|_{L^2}}\right)^{\frac{4}{N}}\right). \tag{36}$$

While $E(Q) = I(M) < 0$ in the subcritical case, here in the critical case there holds

$$E(Q) = 0.$$

-This can be seen for example by multiplying the Q equation by $\frac{N}{2}Q + y \cdot \nabla Q$ and integrating by parts-. A reformulation of (36) which is very useful is the following variational characterization of Q:

Proposition 6 (Variational characterization of the ground state) *Let $v \in H^1$ such that*

$$\int |v|^2 = \int Q^2 \quad \text{and} \quad E(v) = 0,$$

then

$$v(x) = \lambda_0^{\frac{N}{2}} Q(\lambda_0 x + x_0) e^{i\gamma_0},$$

for some parameters $\lambda_0 \in \mathbf{R}_+^*$, $x_0 \in \mathbf{R}^N$, $\gamma_0 \in \mathbf{R}$.

To sum up, the situation is as follows: let $v \in H^1$, then if $|v|_{L^2} < |Q|_{L^2}$ ie for "small" v, the kinetic energy dominates the potential energy and (36) yields $E(v) > C(v) \int |\nabla v|^2$ and the energy is in particular non negative; at the critical mass level $|v|_{L^2} = |Q|_{L^2}$, the only zero energy function, ie for which the kinetic and the potential energies exactly balance, is Q up to the symmetries of scaling, phase and translation which generate the three dimensional manifold of minimizers of (34). For $|v|_{L^2} > |Q|_{L^2}$, the sign of the energy is no longer prescribed.

Remark 3 *Remark that on the contrary to the subcritical case, the scaling (33) leaves the L^2 norm invariant and hence there are no small solitary waves in the critical case.*

2.2 The sharp global wellposedness criterion

A generalization of Theorem 1 has been obtained by Weinstein [67]:

Theorem 4 (Global well posedness for subcritical mass) *Let $u_0 \in H^1$ with $|u_0|_{L^2} < |Q|_{L^2}$, the corresponding solution $u(t)$ to (32) is global and bounded in H^1.*

Proof of Theorem 4

From the conservation of the L^2 norm, $\|u(t)\|_{L^2} < \|Q\|_{L^2}$ for all $t \in [0, T)$, and thus an a priori bound on $\|u(t)\|_{H^1}$ follows from the conservation of the energy and the sharp Gagliardo-Nirenberg inequality (36) applied to $v = u(t)$. This concludes the proof of Theorem 4.

A spectacular feature is that Weinstein's criterion for global existence is sharp. On the one hand, from (35),

$$W(t,x) = Q(x)e^{it}$$

is a solution to (32) with critical mass $\|W\|_{L^2} = \|Q\|_{L^2}$. Note that W keeps its shape in time and *does not disperse*. A contrario, one should know that for initial data $u_0 \in \Sigma$ with $\|u_0\|_{L^2} < \|Q\|_{L^2}$, the solution can be proved to scatter as $t \to +\infty$. One should thus think of $\|Q\|_{L^2}$ as the minimal amount of mass required to avoid complete dispersion of the wave. The solitary wave is thus the *minimal object* in L^2 sense for which dispersion -measured by the kinetic term- and concentration -measured by the potential term- exactly

compensate.
The H^1 group of symmetries of (32) generates in fact a $N+2$ parameter family of solitary waves -we forget here about Galilean invariance-:

$$W_{\lambda_0,x_0,\gamma_0}(t,x)=\lambda_0^{\frac{N}{2}}Q(\lambda_0 x+x_0)e^{i(\gamma_0+\lambda_0^2 t)},\ \ (\lambda_0,x_0,\gamma_0)\in\mathbf{R}_*^+\times\mathbf{R}^N\times\mathbf{R}. \tag{37}$$

Now observe that in the L^2 critical case, *all the H^1 symmetries of (32) are L^2 isometries.* All the solitary waves (37) thus have critical L^2 mass:

$$|W_{\lambda_0,x_0,\gamma_0}|_{L^2}=|Q|_{L^2}.$$

Moreover, $E(Q)=0$ and $Im(\int\nabla Q\overline{Q})=0$ imply:

$$E(W_{\lambda_0,x_0,\gamma_0})=0,\ \ Im(\int\nabla W_{\lambda_0,x_0,\gamma_0}\overline{W_{\lambda_0,x_0,\gamma_0}})=0.$$

In other words, the L^2 criticality of the equation implies the existence of a $N+2$ parameters family of solitary waves with arbitrary size in H^1 but equal Hamiltonian invariants. The consideration of these invariants only is thus no longer enough to estimate the size of the solution nor to separate within these different solitary waves. -as it would be in subcritical regimes-

The L^2 scaling invariance of the solitary waves is a known criterion of instability, see [65] and [18]. In our case, it may be precised by exhibiting an *explicit blow up solution.* The existence of this object is based on the pseudo-conformal symmetry of (32) which is not in the energy space H^1 but in the virial space $\Sigma=H^1\cap\{xu\in L^2\}$. It has the following form: if $u(t,x)$ is a solution to (32), then so is

$$v(t,x)=\frac{1}{|t|^{\frac{N}{2}}}\overline{u}(\frac{1}{t},\frac{x}{t})e^{i\frac{|x|^2}{4t}}. \tag{38}$$

An equivalent but more enlightening way of seeing this symmetry is the following: for any parameter $a\in\mathbf{R}$, the solution to (32) with initial data $v_a(0,x)=u(0,x)e^{ia\frac{|x|^2}{4}}$ is

$$v_a(t,x)=\frac{1}{(1+at)^{\frac{N}{2}}}u(\frac{t}{1+at},\frac{x}{1+at})e^{ia\frac{|x|^2}{4(1+at)}}. \tag{39}$$

Note that this symmetry is also a symmetry of the linear equation. Nevertheless, the fundamental difference between the linear and the non linear equation is that all solutions to the linear equation are dispersive and go to zero for example in L^2_{loc} as $t\to+\infty$, whereas the non linear problem admits non dispersive solutions: the solitary waves. The pseudo-conformal transformation applied to the non dispersive solution now yields an *explicit* finite time blow up solution:

$$S(t,x)=\frac{1}{|t|^{\frac{N}{2}}}Q(\frac{x}{t})e^{-i\frac{|x|^2}{4t}+\frac{i}{t}}. \tag{40}$$

This solution should be viewed as the solution to (32) with Cauchy data at $t=-1$:

$$S(-1,x)=Q(x)e^{i\frac{|x|^2}{4}-i}.$$

It blows up at time $T=0$ with the following explicit properties:

- First observe that the pseudo-conformal symmetry (38) is again an L^2 isometry and thus $|S|_{L^2}=|Q|_{L^2}$ implies that the global wellposedness criterion of Theorem 4 is sharp.
- From explicit computation, S has non negative energy:

$$E(S)>0 \tag{41}$$

  and thus does not belong to the class of blow up solutions described by the virial Theorem 3.
- The blow up speed, measured by the L^2 norm of the gradient -as the L^2 norm itself is conserved-, is given by:

$$|\nabla S(t)|_{L^2}\sim\frac{1}{|t|}. \tag{42}$$

- The solution leaves the Cauchy space H^1 by forming a Dirac mass in L^2:

$$|S(t)|^2\rightharpoonup\left(\int Q^2\right)\delta_{x=0}\ \ \text{as}\ \ t\to 0. \tag{43}$$

Like the solitary wave is a non dispersive global solution, $S(t)$ is a non dispersive blow up solution in the sense that *it accumulates all its L^2 mass into blow up*: no L^2 mass is lost in the focusing process.

Remark 4 *Note that the existence of critical mass blow up solutions seems to be a delicate phenomenon. It may be proved to hold in situations when the pseudo-conformal invariance is lost, typically on a domain with Dirichlet boundary condition, [7]. It may also be proved not to hold in some cases, see [41], [36].*

2.3 Scaling lower bound on blow up rate

In the setting of arbitrarily large initial data, little is known regarding the description of the singularity formation. This is mainly a consequence of the fact that the virial blow up argument does not provide any insight into the blow up dynamics. In particular, there holds *no known a priori bound* on the blow up speed $|\nabla u(t)|_{L^2}$, not even of exponential type, and it thus becomes difficult to analyze the interactions between the regular and the singular part of the solution in the blow up dynamics. In fact, only a lower bound on the blow up rate is known in general as a very simple consequence of the scaling invariance of the problem:

Proposition 7 (Scaling lower bound on blow up rate) *Let $u_0 \in H^1$ such that the corresponding solution $u(t)$ to (32) blows up in finite time $0 < T < +\infty$, then there holds for some constant $C > 0$:*

$$\forall t \in [0, T), \quad |\nabla u(t)|_{L^2} \geq \frac{C}{\sqrt{T-t}} \tag{44}$$

Proof of Proposition 7

The proof is elementary and based on the scaling invariance of the equation and the local well posedness theory in H^1. Indeed, consider for fixed $t \in [0, T)$

$$v^t(\tau, z) = |\nabla u(t)|_{L^2}^{-\frac{N}{2}} u\left(t + |\nabla u(t)|_{L^2}^{-2}\tau, |\nabla u(t)|_{L^2}^{-1} z\right).$$

v^t is a solution to (32) by scaling invariance. We have $|\nabla v^t(0)|_{L^2} = 1$, $|v^t|_{L^2} = |u_0|_{L^2}$, and thus by the resolution of the Cauchy problem locally in time by fixed point argument, there exists $\tau_0 > 0$ independent of t such that v^t is defined on $[0, \tau_0]$. Therefore, $t + |\nabla u(t)|_{L^2}^{-2}\tau_0 \leq T$ which is the desired result. This concludes the proof of Proposition 7.

One of the main open problem in the field is to understand which are the possible blow up speeds. In particular, a striking open problem is to understand whether a self similar blow up corresponding to a speed somehow dictated by the scaling invariance of the equation

$$|\nabla u(t)|_{L^2} \sim \frac{C}{\sqrt{T-t}} \tag{45}$$

is possible. We shall later come back to this important problem which is connected in some cases to very fine dispersive properties of the flow.

2.4 The L^2 concentration phenomenon

A general L^2 concentration result of blow up solutions has been obtained by Merle, Tsutsumi, [49], in the radial case, and generalized by Nawa, [53]. It provides a preliminary description of the singularity formation: at blow up time, the solution leaves the Cauchy space H^1 by forming a Dirac mass in L^2 and accumulating at least *a minimal and universal quantum of L^2 mass.*

Theorem 5 (L^2 concentration phenomenon) *Let $u_0 \in H^1$ such that the corresponding solution $u(t)$ to (32) blows up in finite time $0 < T < +\infty$. Then there exists some continuous function of time $x(t) \in \mathbf{R}^N$ such that:*

$$\forall R > 0, \quad \liminf_{t \to T} \int_{|x-x(t)| \leq R} |u(t,x)|^2 dx \geq \int Q^2. \tag{46}$$

Proof of Theorem 5

The proof is again purely variational. We prove the result in the radial case for $N \geq 2$. The general case follows from concentration compactness techniques, see [52], [20].
Let $u_0 \in H^1$ radial and assume that the corresponding solution $u(t)$ to (32) blows up at time $0 < T < +\infty$, or equivalently:

$$\lim_{t \to T} \|\nabla u(t)\|_{L^2} = +\infty. \tag{47}$$

We need to prove (46) and argue by contradiction: assume that for some $R > 0$ and $\varepsilon > 0$, there holds on some sequence $t_n \to T$,

$$\lim_{n \to +\infty} \int_{|y| \leq R} |u(t_n, y)|^2 dy \leq \int Q^2 - \varepsilon. \tag{48}$$

Let us rescale the solution by its size and set:

$$\lambda(t_n) = \frac{1}{\|\nabla u(t_n)\|_{L^2}}, \quad v_n(y) = \lambda^{\frac{N}{2}}(t_n) u(t_n, \lambda(t_n) y),$$

then from explicit computation:

$$\|\nabla v_n\|_{L^2} = 1 \ \text{ and } \ E(v_n) = \lambda^2(t_n) E(u). \tag{49}$$

First observe that v_n is H^1 bounded and we may assume on a sequence $n \to +\infty$:

$$v_n \rightharpoonup V \ \text{ in } \ H^1.$$

We first claim that V is non zero. Indeed, from (47), (49) and the conservation of the energy for $u(t)$, $E(v_n) \to 0$ as $n \to +\infty$, and thus:

$$\frac{1}{2 + \frac{4}{N}} \int |v_n|^{2+\frac{4}{N}} = \frac{1}{2} \int |\nabla v_n|^2 - E(v_n) = \frac{1}{2} - E(v_n) \to \frac{1}{2} \ \text{ as } \ n \to +\infty.$$

Now from the compact embedding of $H^1_{radial} \hookrightarrow L^{2+\frac{4}{N}}$, $v_n \to V$ in $L^{2+\frac{4}{N}}$ up to a subsequence, and thus $\frac{1}{2+\frac{4}{N}} \int |V|^{2+\frac{4}{N}} \geq \frac{1}{2}$ and V is non zero. Moreover, from the weak H^1 convergence and the strong $L^{2+\frac{4}{N}}$ convergence,

$$E(V) \leq \liminf_{n \to +\infty} E(v_n) = 0.$$

Last, we have from (47), (48) and the weak H^1 convergence: $\forall A > 0$

$$\begin{aligned} \int_{|y| \leq A} |V(y)|^2 dy &\leq \liminf_{n \to +\infty} \int_{|y| \leq A} |v_n(y)|^2 dy \leq \lim_{n \to +\infty} \int_{|y| \leq \frac{R}{\lambda(t_n)}} |v(t_n, y)|^2 dy \\ &= \lim_{n \to +\infty} \int_{|x| \leq R} |u(t_n, x)|^2 dx \leq \int Q^2 - \varepsilon. \end{aligned}$$

Thus $\int |V|^2 \leq \int Q^2 - \varepsilon$ which together with V non zero and $E(V) \leq 0$ contradicts the sharp Gagliardo-Nirenberg inequality (36). This concludes the proof of Theorem 5.

The proof in the nonradial case has been simplified by Hmidi, Keraani [20], which derived the following optimal result from concentration compactness - more precisely profile decomposition- techniques:

Lemma 1 *Let a sequence $u_n \in H^1$ with*

$$\limsup_{n\to+\infty} |\nabla u_n|_{L^2} \leq |\nabla Q|_{L^2}, \quad \limsup_{n\to+\infty} |u_n|_{L^{2+\frac{4}{N}}} \geq |Q|_{L^{2+\frac{4}{N}}},$$

then there exists $x_n \in \mathbf{R}^N$ and $V \in H^1$ such that up to a subsequence:

$$v_n(\cdot + x_n) \rightharpoonup V \quad \textit{weakly in} \quad H^1$$

with

$$\|V\|_{L^2} \geq \|Q\|_{L^2}.$$

Two natural questions following Theorem 5 are still open in the general case:
(i) Does the function $x(t)$ have a limit as $t \to T$ defining then at least one exact blow up point in space where L^2 concentration takes place?
(i) Which is the exact amount of mass which is focused by the blow up dynamic?

An explicit construction of blow up solutions due to Merle, [38], is the following: let k points $(x_i)_{1\leq i\leq k} \in \mathbf{R}^N$, then there exists a blow up solution $u(t)$ which blows up in finite time $0 < T < +\infty$ exactly at these k points and behaves locally near x_i like $S(t)$ given by (40). In particular, it satisfies:

$$|u(t)|^2 \rightharpoonup \Sigma_{1\leq i\leq k} |Q|^2_{L^2} \delta_{x=x_i} \quad \text{as} \quad t \to T,$$

in the sense of measures. Let us observe first that from the construction, one could place at x_i instead of $S(t)$ any pseudo-conformal transformation of an excited ground state solution Q_i solution to (35). The solution focuses then at x_i exactly the mass $|Q_i|_{L^2}$ which is quantized but arbitrarily large for $N \geq 2$. Second, similarly as for $S(t)$, such a solution is non dispersive as it accumulates all its initial L^2 mass into blow up.

A general conjecture concerning L^2 concentration is formulated in [48] and states that a blow up solution focuses a quantized and universal amount of mass at a finite number of points in $\mathbf{R}^N$, the rest of the L^2 mass being purely dispersed. The exact statement is the following:

Conjecture (*): *Let $u(t) \in H^1$ be a solution to (32) which blows up in finite time $0 < T < +\infty$. Then there exist $(x_i)_{1 \le i \le L} \in \mathbf{R}^N$ with $L \le \frac{\int |u_0|^2}{\int Q^2}$, and $u^* \in L^2$ such that: $\forall R > 0$,*

$$u(t) \to u^* \quad in \quad L^2(\mathbf{R}^N - \bigcup_{1 \le i \le L} B(x_i, R))$$

$$and \quad |u(t)|^2 \rightharpoonup \Sigma_{1 \le i \le L} m_i \delta_{x = x_i} + |u^*|^2 \quad with \quad m_i \in [\int Q^2, +\infty).$$

2.5 Orbital stability of the ground state

We shall from now on restrict ourselves to considering *small super critical mass initial data*, that is initial data with L^2 mass just above the critical mass required for blow up:

$$u_0 \in \mathcal{B}_{\alpha^*} = \{u_0 \in H^1 \ \text{ with } \int Q^2 \le \int |u_0|^2 \le \int Q^2 + \alpha^*\} \tag{50}$$

for some parameter $\alpha^* > 0$ small enough. This situation is conjectured to locally describe the generic blow up dynamic around one blow up point.

Let us recall that $E(Q) = 0$ together with the virial blow up result of Theorem 3 imply the instability of the solitary wave $Q(x)e^{it}$. We claim however that the orbital stability of Q may be retrieved in some sense. The following theorem generalizes Theorem 2:

Theorem 6 (Orbital stability in the critical case) *Let $N \ge 1$. For all $\alpha^* > 0$ small enough, there exists $\delta(\alpha^*)$ with $\delta(\alpha^*) \to 0$ as $\alpha^* \to 0$ such that the following holds true. Let $u_0 \in H^1$ with*

$$\int |u_0|^2 \le \int Q^2 + \alpha^*, \quad E(u) \le \alpha^* \int |\nabla u|^2, \tag{51}$$

and let $u(t)$ be the corresponding solution to (32) with life time $0 < T \le +\infty$. Then there exists a translation shift $x(t) \in \mathbf{R}^N$ and a phase shift $\gamma(t) \in \mathbf{R}$ such that:

$$\forall t \in [0, T), \quad \|\lambda^{\frac{N}{2}}(t) u(t, \lambda(t)x + x(t)) e^{-i\gamma(t)} - Q\|_{H^1} < \delta(\alpha^*). \tag{52}$$

Note that a finite time blow up solution with small super critical mass automatically satisfies (51) near blow up time, and hence it is closed to the ground state in H^1 up to the set of H^1 symmetries. This property is again purely based on the conservation laws and the variational characterization of Q, and not on refined properties of the flow.

Proof of Theorem 6

Equivalently, we need to prove the following: let a sequence $u_n \in H^1$ with

$$\|u_n\|_{L^2} \to \|Q\|_{L^2}, \quad \limsup_{n\to+\infty} \frac{E(u_n)}{\|\nabla u_n\|_{L^2}^2} \leq 0, \tag{53}$$

let

$$v_n = \lambda_n^{\frac{N}{2}} u(\lambda_n x) \quad \text{with} \quad \lambda_n = \frac{\|\nabla Q\|_{L^2}}{\|\nabla u_n\|_{L^2}}, \tag{54}$$

then there exist $x_n \in \mathbf{R}^N$, $\gamma_n \in \mathbf{R}$ such that:

$$v_n(\cdot + x_n)e^{i\gamma_n} \to Q \quad \text{in} \quad H^1 \quad \text{as} \quad n \to +\infty. \tag{55}$$

Indeed, observe from (53) and (54) that

$$\|v_n\|_{L^2} \to \|Q\|_{L^2}, \quad \|\nabla v_n\|_{L^2} = \|\nabla Q\|_{L^2}, \quad \limsup_{n\to+\infty} E(v_n) \leq 0.$$

We now apply Proposition 4 to v_n. If vanishing occurs, then up to a subsequence, we have for n large enough:

$$E(v_n) \geq \frac{\|\nabla Q\|_{L^2}^2}{4}$$

which contradicts $\limsup_{n\to+\infty} E(v_n) \leq 0$. If dichotomy occurs, then there exist w_k, z_k and $0 < \alpha < 1$ such that

$$\|w_k\|_{L^2} \to \alpha\|Q\|_{L^2}, \quad \|z_k\|_{L^2} \to (1-\alpha)\|Q\|_{L^2} \quad \text{and} \quad 0 \geq \limsup_{k\to+\infty}(E(w_k) + E(z_k)).$$

But from the sharp Gagliardo-Nirenberg inequality (36) applied to w_k and z_k, this implies

$$\|\nabla w_k\|_{L^2} + \|\nabla z_k\|_{L^2} \to 0 \quad \text{as} \quad k \to +\infty$$

and thus

$$\|v_{n_k}\|_{L^{2+\frac{4}{N}}} \to 0 \quad \text{as} \quad k \to +\infty,$$

and we are back to the vanishing case. Hence compactness occurs and

$$v_n(\cdot + x_n) \to V \quad \text{strongly in} \quad L^{2+\frac{4}{N}}, L^2$$

up to a subsequence. But then $E(v) \leq 0$ and $\|V\|_{L^2} = \|Q\|_{L^2}$ imply from (36) and Proposition 6 that $V(x) = Q(x + x_0)e^{i\gamma_0}$. This in turns implies $E(V) = 0$ and thus $|\nabla v_n(\cdot + x_n)|_{L^2}^2 \to |\nabla Q|_{L^2}^2$ which implies (55) follows. This concludes the proof of Theorem 6.

2.6 Classification of the critical mass blow up solution

We shall present in this subsection a result which was proved by Merle in 93 and which has long remained an isolated point in the theory. It gives a characterization of the critical mass blow up solution $S(t)$ given by (40) as the unique minimal blow up solution and is our first result which goes beyond variational informations:

Theorem 7 (Classification of the critical mass blow up solution) *Let* $u_0 \in H^1$ *with*

$$|u_0|_{L^2} = |Q|_{L^2}.$$

Assume that the corresponding solution to (32) blows up in finite time $0 < T < +\infty$. *Then*

$$u(t) = S(t)$$

up to the symmetries.

This result should be understood as a Liouville type result. As we shall see, what is special with a critical mass blow up solution is that it does not disperse in the sense that it focuses all its mass at blow up time:

$$|u|^2(t) \rightharpoonup \left(\int |u_0|^2\right) \delta_{x_0} \quad \text{as} \quad t \to T$$

in the weak sense of measures. In fact, Weinstein observed in 83 that the flow of critical mass blow up solutions is compact up to the symmetries in the sense that there exist $\lambda(t), x(t), \gamma(t)$ such that

$$\lambda^{\frac{N}{2}}(t) u(t, \lambda(t)x + x(t)) e^{i\gamma(t)} \to Q \quad \text{in} \quad L^2 \quad \text{as} \quad t \to T. \tag{56}$$

Let us say that the understanding of the non dispersive objects is at the heart of many recent breakthroughs in the description of singularity formations, see for example [51], [35], [46]. It is somehow the simplest problem to understand which then governs the intuition. See Theorem 16 below for a super critical mass version of Theorem 7 -which proof is much more delicate-.

In a recent work [24] on the energy critical (NLS), Kenig and Merle also understood the role played by this kind of classification result to prove global well posedness and scattering. There is indeed a deep conceptual and technical connection between the proof of the Liouville theorem in [24] and the proof of Theorem 7 as we now present it.

Proof of Theorem 7

The original proof by Merle [40] has been further simplified by Banica [1] and Hmidi, Keraani [19], and it is the proof we present now.

step 1 Compactness of the flow in H^1 up to scaling.

Let u as in the hypothesis of the Theorem with blow up time $0 < T < \infty$. Let

$$\lambda(t) = \frac{|\nabla Q|_{L^2}}{|\nabla u(t)|_{L^2}} \to 0 \ \text{ as } \ t \to T.$$

Then

$$v(t,x) = \lambda^{\frac{N}{2}}(t) u(t, \lambda(t)x + x(t))$$

satisfies:

$$|\nabla v(t)|_{L^2} = |\nabla Q|_{L^2}, \ \lim_{t\to T} E(v) = 0, \ |v(t)|_{L2} = |Q|_{L^2}.$$

Arguing as for the proof of Theorem 6, we conclude from standard concentration compactness techniques and the variational characterization of the ground state that:

$$v(t, x + x(t))e^{i\gamma(t)} \to Q \ \text{ in } \ H^1 \ \text{ as } \ t \to T.$$

Note that this automatically implies (56).

step 2 A refined Cauchy-Schwarz for critical mass functions.

For $|w|_{L^2} < |Q|_{L^2}$, the energy controls the kinetic energy from (36). This controls fails for $|w|_{L^2} = |Q|_{L^2}$ but can be retrieved in some weak sense. Indeed, Banica observed the following: let a smooth real valued ψ and $w \in H^1$ with $|w|_{L^2} = |Q|_{L^2}$, then:

$$|Im(\nabla\psi \cdot \nabla w\overline{w})|^2 \leq C\sqrt{E(w)}\left(\int |\nabla\psi|^2|w|^2\right)^{\frac{1}{2}}. \tag{57}$$

Indeed, for any $a > 0$,

$$|we^{ia\psi}|_{L^2} = |Q|_{L^2} \ \text{ and thus } \ E(we^{ia\psi}) \geq 0$$

and the result follows by expanding in a.

step 3 L^2 compactness of u and control of the concentration point.

We now claim that u is L^2 compact: $\forall\varepsilon > 0$, $\exists R > 0$ such that

$$\forall t \in [0,T), \ \int_{|x|\geq R} |u(t,x)|^2 dx < \varepsilon. \tag{58}$$

Indeed, pick ε large enough, For $R > 0$, let $\chi_R(x) = \chi(\frac{x}{R})$ where χ is a smooth radial cut off function with $\chi(r) = 0$ for $r \leq \frac{1}{2}$, $\chi(r) = 1$ for $r \geq 1$. Then integrating by parts in (32) and using (57), we get:

$$\left|\frac{1}{2}\frac{d}{dt}\int \chi_R|u|^2\right| = |Im(\nabla\chi_R \cdot \nabla u\overline{u})| \leq C\sqrt{E(u)}\left(\int |\nabla\chi_R|^2|u|^2\right)^{\frac{1}{2}} \leq \frac{C}{R}\sqrt{E_0}|u_0|_{L^2}$$

where we used the conservation of energy and L^2 norm in the last step. Integrating in time on $[0,T]$ and using $T<+\infty$ yields (58).
Now observe that (56) and (58) automatically imply a localization of the concentration point:

$$\forall t\in[0,T),\quad |x(t)|\leq C(u_0). \tag{59}$$

step 4 $u\in\Sigma$.

From (59) and up to a translation in space, we may consider a sequence of times $t_n\to T$ such that

$$x(t_n)\to 0\in\mathbf{R}^N.$$

From (56), (58):

$$|u(t_n,x)|^2\rightharpoonup\left(\int|Q|^2\right)\delta_0 \quad\text{as}\quad t_n\to T. \tag{60}$$

This means that at time T, all the mass is at the origin. Even though there is no finite speed of propagation for (NLS), the idea is to integrate backwards from the singularity to conclude that this implies that there was not much mass initially at infinity, that is

$$u_0\in\Sigma=H^1\cap\{xu\}\in L^2. \tag{61}$$

This step is very important and corresponds to a non trivial *gain of regularity* for the asymptotic object which is a direct consequence of its non dispersive behavior.
Let a smooth radial cut off function $\psi(r)=r^2$ for $r\leq 1$, $\psi(r)=8$ for $r\geq 2$ and such that $|\nabla\psi|^2\leq C\psi$. Let $A>0$ and $\psi_A(r)=A^2\psi(\frac{r}{A})$, then:

$$|\nabla\psi_A|^2\leq C\psi_A. \tag{62}$$

Then integrating by parts in (32), we have using (57) and (62):

$$\left|\frac{1}{2}\frac{d}{dt}\int\psi_A|u|^2\right|=|Im(\nabla\psi_A\cdot\nabla u\overline{u})|\leq C\sqrt{E_0}\left(\int|\nabla\psi_A|^2|u|^2\right)^{\frac{1}{2}}\leq C\sqrt{E_0}\left(\int\psi_A|u|^2\right)^{\frac{1}{2}}$$

or equivalently:

$$\left|\frac{d}{dt}\sqrt{\int\psi_A|u|^2}\right|\leq C\sqrt{E_0}. \tag{63}$$

Now observe from (60) that

$$\int\psi_A|u(t_n)|^2\to 0\quad\text{as}\quad t_n\to T.$$

Integrating (63) on $[t,t_n]$ and letting $t_n\to T$, we thus get:

$$\forall t\in[0,T),\quad\sqrt{\int\psi_A|u(t)|^2}\leq C(E_0)(T-t).$$

Note that the right hand side of the above expression is independent of A. We may thus let $A \to \infty$ and conclude to an even more precise version of (61):

$$\forall t \in [0,T), \quad u(t) \in \Sigma \quad \text{with} \quad \int |x|^2 |u(t,x)|^2 dx \to 0 \quad \text{as} \quad t \to T. \tag{64}$$

step 5 Pseudo-conformal transformation.

The conclusion of the proof is pure magic. It relies on the following completely general fact. Let $u(t)$ be a solution to (32) leaving on $[0,T)$, then

$$v(t,x) = \left(\frac{T}{T+t}\right)^{\frac{N}{2}} u\left(\frac{tT}{T+t}, \frac{Tx}{T+t}\right) e^{i\frac{|x|^2}{4(T+t)}}$$

is a solution to (32) with

$$\|v\|_{L^2} = \|u\|_{L^2} \quad \text{and} \quad E(v) = \frac{1}{8} \lim_{t \to T} \int |x|^2 |u(t,x)|^2 dx.$$

Applying this to u and using (64), this implies that

$$\|v\|_{L^2} = \|u\|_{L^2} = |Q|_{L^2} \quad \text{and} \quad E(v) = 0.$$

From Proposition 6, $v = Q$ up to the symmetries of the flow, and this concludes the proof of Theorem 7.

Let us conclude this section by introducing a result by Merle which is a consequence of the uniqueness of the critical mass blow up solution.
Consider the critical mass blow up solution $S(t)$ given by (40). Observe that it blows up at $t = 0$ but is perfectly well defined for $t \neq 0$. If we view $S(t)$ as the solution to (32) with nice and smooth initial data $S(-1)$, we can ask whether we can define a continuation of the solution after blow up time. This kind of question is connected to the stability of the singularity formation.

A way of reformulating the problem is as follows. Pick $\varepsilon > 0$ and consider the sequence of initial data for $\varepsilon > 0$:

$$u_\varepsilon(-1) = \left(1 - \frac{\varepsilon}{\|Q\|_{L^2}}\right)^{\frac{1}{2}} S(-1), \quad \varepsilon \to 0.$$

Then from $|u_\varepsilon(-1)|_{L^2} < |Q|_{L^2}$, the corresponding solution $u_\varepsilon(t)$ to (NLS) is global. We ask for the behavior of $u_\varepsilon(1)$ as $\varepsilon \to 0$. The striking result proved by Merle [39] shows a chaotic behavior of $u_\varepsilon(1)$ as $\varepsilon \to 0$.

Theorem 8 (Chaotic behavior of the continuation of $S(t)$) *There holds for some phase shift θ_ε:*

$$\left\| u_\varepsilon(1) - e^{i\theta_\varepsilon} S(1) \right\|_{H^1} \to 0 \quad as \quad \varepsilon \to 0.$$

Moreover, for any $\theta \in \mathbf{R}$, there exists a subsequence $\varepsilon_k \to 0$ such that:

$$\theta_{\varepsilon_k} \to \theta \quad as \quad k \to +\infty.$$

It would be very interesting to be able to extend this kind of result to super critical mass blow up solution and understand there the possible continuation properties of the solution.

2.7 Explicit construction of blow up solutions

Let us now ask the question of the construction of blow up solutions with prescribed blow up dynamics for super critical mass initial data. The question of which dynamics are possible is wide open in general. Observe however that for *small* super critical mass initial data $u_0 \in \mathcal{B}(\alpha^*)$, the conservation of the Hamiltonian implies that all blow up solutions satisfy (51) of Theorem 6 near blow up time. We may thus rewrite (52) by saying that u admit a decomposition

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}}(Q+\varepsilon)(t, \frac{x-x(t)}{\lambda(t)})e^{i\gamma(t)}, \tag{65}$$

where

$$|\varepsilon(t)|_{H^1} \leq \delta(\alpha^*)$$

and

$$\lambda(t) \sim \frac{1}{|\nabla u(t)|_{L^2}}. \tag{66}$$

The main point of this non linear decomposition is that it now allows a perturbative analysis by studying the equation governing the H^1 small excess of mass $\varepsilon(t)$. Indeed, to describe the blow up dynamic is now equivalent to understand in the perturbative regime how to extract from the infinite dimensional dynamic of (32) a finite dimensional and possibly universal dynamic for the evolution of the geometrical parameters $(\lambda(t), x(t), \gamma(t))$ which is coupled to the dispersive dynamic which drives the small excess of mass $\varepsilon(t)$. To estimate for example the blow up speed is now equivalent to estimating the size of $\lambda(t)$, or to prove the existence of the blow up point is equivalent to proving the existence of a strong limit $x(t) \to x(T) \in \mathbf{R}^N$ as $t \to T$. Similarly, the structure in space of the singularity relies on the dispersive behavior of ε as t approaches blow up time.

As it allows a perturbative approach of the blow up problem, the existence of the geometrical decomposition (65) is a first step for the construction of blow up solutions to (32). We already mentioned a blow up construction by Merle, [38], which build non L^2

dispersive blow up solutions. There are two other fundamental results of construction of blow up solutions.

A first natural question is the existence in the super critical case of a blow up dynamic similar to the one of the explicit critical mass blow up solution $S(t)$. In [6], Bourgain and Wang construct indeed in dimension $N = 1, 2$ solutions $u(t)$ to (32) which blow up in finite time and behave locally like the explicit blow up solution $S(t)$ given by (40). More precisely, given a limiting profile $u^* \in H^1$ sufficiently decaying at infinity -for technical reason- and *flat near zero* -this is not a technical point...- in the sense that for some $A > 0$ large enough,

$$\frac{d^i}{dx^i}u^*(0) = 0, \quad 1 \leq i \leq A, \tag{67}$$

they build a solution to (32) which blows up in finite time $0 < T < +\infty$ at $x = 0$ and satisfies:

$$u(t) - S(t - T) \to u^* \ \text{ in } \ H^1 \ \text{ as } \ t \to T. \tag{68}$$

Note that the flatness assumption (67) is not open in H^1, and this statement ensures thus the stability of the $S(t)$ dynamic on a finite codimensional manifold -we refer to [25] for a discussion on the finite codimensional nature of this set of blow solutions-.The meaning of this flatness assumption is to decouple in space the regular part of the solution which will evolve to u^*, and the singular part which will consist of $S(t)$ only. As a corollary, these solutions have the same blow up speed like $S(t)$:

$$|\nabla u(t)|_{L^2} \sim \frac{1}{T-t}. \tag{69}$$

Now this rate of blow up turns out not to be the "generic" one. They are never observed numerically and are thus believe to be unstable. In fact, these solutions are obtained by constructing a global solution to (NLS) such that

$$v(t,x) \to Q \ \text{ in } \ L^2_{loc} \ \text{ as } \ t \to +\infty$$

and then applying the pseudo-conformal transformation. The outcome is a form of *finite codimensional stability* of the ground state solution, see Krieger and Schlag [25] for a further discussion of this approach.

Now because it is related to the scaling symmetry of the problem, on which in other instances formal arguments indeed rely to derive the correct blow up speed, the scaling lower bound (44) has long been conjectured to be optimal. Yet, in the mid 80's, numerical simulations and very clever formal arguments, see Landman, Papanicolaou, Sulem, Sulem, [27], have suggested that the correct and stable blow up speed is a slight correction to the scaling law:

$$|\nabla u(t)|_{L^2} \sim \sqrt{\frac{\log|\log(T-t)|}{T-t}}, \tag{70}$$

which is referred to as *the log-log law*. Solutions blowing up with this speed indeed appeared to be stable with respect to perturbation of the initial data. Quite an amount of formal work has been devoted to understanding the exact nature of the double log correction to the scaling estimate. We refer to the excellent monograph by Sulem, Sulem, [63], for further discussions on this subject. In this frame, for $N = 1$, Perelman in [56] rigorously proves the existence of one solution which blows up according to (70) and its stability in some space strictly included in H^1.

These two constructions of blow up solutions thus imply the following: there are at least two blow up dynamics for (32) with two different speeds, one which is a continuation of the explicit $S(t)$ blow up dynamic with the $1/(T-t)$ speed (69), and which is suspected to be unstable because it is not seen numerically; one with the log-log speed (70) which is conjectured to be stable from numerics.

2.8 Structural instability of the log-log law

We have so far exhibited two important features of the blow up dynamics for (NLS):
(i) there exists a critical mass blow up solution;
(ii) there are at least two blow up speeds.

These two facts are somehow fundamental difficulties for the analysis. The existence of the critical mass blow up solution implies that the set of initial data which yields a finite time blow up solution is not open, and thus blow up is not a stable phenomenon a priori. On the contrary, only one blow up regime is from numerics expected to be stable.

These facts are somehow believed to be closely related to the very specific algebraic structure of (32), and in particular to the existence of the pseudo-conformal symmetry.

An important result in this direction is the so called *structural instability* of the log-log law in the following sense. Consider in dimension $N = 2$ the Zakharov system:

$$\begin{cases} iu_t = -\Delta u + nu \\ \frac{1}{c_0^2} n_{tt} = \Delta n + \Delta |u|^2 \end{cases} \tag{71}$$

for some fixed constant $0 < c_0 < +\infty$. In the limit $c_0 \to +\infty$, we formally recover (32) -see [63] for an introduction to the Zakharov equations-. This system displays a variational structure like (32). In particular, a virial law in the spirit of (31) holds and yields finite time blow up for radial non positive energy initial data, see Merle, [43]. Moreover, a one parameter family of blow up solution may be constructed and should be understood as a continuation of the exact $S(t)$ solution for (32), see Glangetas, Merle, [16]. These explicit solutions have blow up speed:

$$|\nabla u(t)|_{L^2} \sim \frac{C(u_0)}{T-t}.$$

They moreover appear to be stable from numerics, see Papanicolaou, Sulem, Sulem, Wang, [55]. Now from Merle, [42], *all finite time blow up solutions to (71) satisfy*

$$|\nabla u(t)|_{L^2} \geq \frac{C(u_0)}{T-t}.$$

In particular, there will be no log-log blow up solutions for (71). This fact suggests that in some sense, the Zakharov system provides a much more stable and robust blow up dynamics than its asymptotic limit (NLS). This fact enlightens the belief that the log-log law heavily relies on the specific algebraic structure of (32), and some non linear degeneracy properties will indeed be at the heart of our understanding of the blow up dynamics.

3 Blow up dynamics of small super critical mass initial data

In this section, we restrict ourselves to initial data with small super critical mass, that is:

$$u_0 \in \mathcal{B}_{\alpha^*} = \{u_0 \in H^1 \ \text{ with } \ \int Q^2 \leq \int |u_0|^2 \leq \int Q^2 + \alpha^*\},$$

for some parameter $\alpha^* > 0$ small enough. We present the results on the blow up dynamics obtained in the series of papers [44], [45], [58], [46], [47], [48] and which allow a precise understanding of the blow up dynamics in this setting. The description of the blow up dynamic involves two different type of questions:

- In [44], [45], we consider non positive energy initial data and address the question of an upper bound on the blow up rate. In [58], the dynamically richer case of non negative energy is addressed together with the issue of the stability of the blow up regimes.
- In [46], using as a starting point the point of view and the estimates in [44], [45], [58], we investigate the question of the shape of the solution in space and the existence of a universal asymptotic profile which attracts blow up solutions. These questions rely on Liouville type of theorems to classify the non dispersive dynamics of solitary waves. Further understanding of these issues will then allow one as in [47], [48], to prove sharp lower bounds on the blow up rate related to the expected log-log law and then quantization results on the focused mass -or equivalently Conjecture (*) for data $u_0 \in \mathcal{B}_{\alpha^*}$-.

First, we introduce for notational purpose the following invariant which sign is preserved by the H^1 symmetries:

$$E_G(u) = E(u) - \frac{1}{2}\left(\frac{Im(\int \nabla u \overline{u})}{|u|_{L^2}}\right)^2. \tag{72}$$

Next, we will assume in all our results a *Spectral Property* which amounts counting the number of negative directions of an explicit Schrödinger operator $-\Delta + V$ where the well localized potential V is stationary and build from the ground state. This property was proved in dimension $N=1$ in [44] using the explicit formula for the ground state Q, and checked numerically in dimension $N=2,3,4$ to which we will thus restrict ourselves, see Proposition 9. Note that this property is the only part of the proof where the restriction on the dimension is needed.

3.1 Finite time blow up for non positive energy initial data

In this subsection, we address the question of the blow up dynamics for non positive energy solutions. The result is the following:

Theorem 9 ([44],[45]) *Let $N=1,2,3,4$. There exist universal constants $\alpha^*, C^*>0$ such that the following holds true. Given $u_0 \in \mathcal{B}_{\alpha^*}$ with*

$$E_G(u_0)<0,$$

the corresponding solution $u(t)$ to (32) blows up in finite time $0<T<+\infty$ and there holds for t close to T:

$$|\nabla u(t)|_{L^2} \leq C^* \left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}}. \tag{73}$$

Theorem 9 provides the first known upper bound on the singularity formation which is the key towards the decomposition of the solution into a singular part and a regular part. Moreover, it removes for non positive energy solutions the possibility of $S(t)$ type of blow up as the log-log upper bound (73) is below the $1/(T-t)$ speed. We will indeed later prove that there is in this case only one blow up regime which speed is given by the exact log-log law (70). An important corollary of Theorem 9 obtained using the pseudo-conformal transformation is the instability of $S(t)$ in a strong sense. $S(t)$ is the critical mass blow up solution, so it is unstable in a trivial sense: any H^1 neighborhood of $S(-1)$ contains initial data $u(-1)$ which solution $u(t)$ is global in time, it suffices to take subcritical mass initial data. We claim a much stronger statement which is that the blow up dynamic of $S(t)$ itself is unstable in the following sense: any H^1 neighborhood of $S(-1)$ contains initial data $u(-1)$ which solution $u(t)$ blows up in finite time but with the log-log speed.
Indeed, let the initial data at time $t=1$: $u_\eta(1,x)=(1+\eta)Q(x)$ for $\eta>0$ and small, and $u_\eta(t)$ the corresponding solution to (32). From explicit computation, $E(u_\eta)<0$ and thus u_η satisfies the hypothesis of Theorem 9. It thus blows up in finite time $1<T_\eta<+\infty$. We now apply the pseudo-conformal symmetry and consider the solutions

$$v_\eta(t)=\frac{1}{|t|^{\frac{N}{2}}}u_\varepsilon\left(\frac{-1}{t},\frac{x}{t}\right)e^{-i\frac{|x|^2}{4t}-i}.$$

First observe that

$$v_\eta(-1)\to S(-1) \text{ as } \eta\to 0$$

in some strong sense. Next, from its definition, $v_\eta(t)$ blows up in finite time $T'_\eta = \frac{-1}{T_\eta} < 0$. Now $T'(\eta) < 0$ and the uniform space time bound on $|xu_\varepsilon(t)|_{L^2}$ given by the virial law (31) ensure that $v_\eta(t)$ satisfies upper bound (73) for t close enough to T'_η as desired.

3.2 H^1 stability of the log-log law

Let us now investigate the dynamics for positive energy initial data. In this case, three different dynamics are known to possibly occur:

- $S(t)$ behavior: the results in [6], [25], yield the existence of finite time blow up solutions $u(t)$ satisfying $u_0 \in \mathcal{B}_{\alpha^*}$, $E_0^G > 0$ and $|\nabla u(t)|_{L^2} \sim \frac{1}{T-t}$ near blow up time.
- log-log behavior: Using the pseudo-conformal symmetry and Theorem 9, one can exhibit strictly positive energy solutions satisfying the log-log upper bound (73), see [58].
- Global solutions: Given $u(t) \in \Sigma$ a solution to (32) which blows up at $0 < T < +\infty$, the pseudo conformal symmetry (39) applied with parameter $a = \frac{1}{T}$ yields a solution $v(t)$ to (32) globally defined on $[0, +\infty)$.

There certainly is a poor understanding in general of which conditions on the initial data are enough to select one of the above dynamics. Nevertheless, we have the following:

Theorem 10 ([58]) *Let $N = 1, 2, 3, 4$. There exist universal constants $C^*, C_1^* > 0$ such that the following is true:*

(i) Rigidity of blow up rate: Let $u_0 \in \mathcal{B}_{\alpha^*}$ with

$$E_G(u_0) > 0,$$

and assume that the corresponding solution $u(t)$ to (32) blows up in finite time $T < +\infty$, then there holds for t close to T either

$$|\nabla u(t)|_{L^2} \leq C^* \left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}}$$

or

$$|\nabla u(t)|_{L^2} \geq \frac{C_1^*}{(T-t)\sqrt{E_G(u_0)}}. \tag{74}$$

(ii) Stability of the log-log law: Moreover, the set of initial data $u_0 \in \mathcal{B}_{\alpha^*}$ such that $u(t)$ blows up in finite time with the upper bound (73) is open in H^1.

Comments on Theorem 10

1. *Size of the log-log set*: Let us denote $\mathcal{O}$ the set of log-log blow up solutions. It is known to contain non positive energy initial data from Theorem 9. Then the pseudo-conformal invariance allows one to obtain non negative energy solutions in $\mathcal{O}$. One can prove that this procedure does not describe all $\mathcal{O}$, and that there exist initial data $u_0 \in \Sigma \cap \mathcal{O}$ with non negative energy which cannot be obtained using the pseudo-conformal symmetry from a non positive energy initial data, see [58].

2. *Stability versus instability*: Let us recall that the existence of critical mass blow up solution $S(t)$ implies that the set of initial data which lead to a finite time blow up solution is not open in H^1. In this setting, the fact that the blow up speed is a sufficient criterion of stability in the energy space is a new feature in the non linear dispersive setting. Now if stability of the log-log regime is proved, instability in the strong sense of solutions satisfying the lower bound (74) is proved only for $S(t)$ itself. The dynamical instability in this sense of these solutions is open. A simpler result would be to prove the strong instability of the solutions build in [6], this is also open.

3. *Universal upper bound on the blow up speed*: The upper bound (73) corresponds to the stable blow up dynamic, while the lower bound (74) is obtained from the assumption that the solution is escaping the stable blow up regime. In this sense, these estimates correspond to two different asymptotic blow up regimes which each require a specific analysis. This is why no general upper bound on the blow up rate of any type holds so far. Let us recall that the solutions build in [6] satisfy the exact law

$$|\nabla u(t)|_{L^2} \sim \frac{C(u_0)}{T-t}.$$

Whether these two regimes are the only ones is open. Let us remark that this would imply from the pseudo-conformal symmetry that blow up in Σ always occurs in finite time, this is also an open problem.

We have not addressed so far the question of the blow up dynamics of zero energy initial data. This question turns out to be very fundamental but requires different type of ideas. Note that the solitary wave $Q(x)e^{it}$ is a global in time zero energy solution to (32). We will come back later to the issue of classifying this dynamic among the set of zero energy solutions.

3.3 Universality of the blow up profile

We now turn to the question of the dispersive properties of the blow up solutions to (32). Recall from (65) that blow up solutions admit near blow up time a decomposition:

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}}(Q+\varepsilon)(t, \frac{x-x(t)}{\lambda(t)})e^{i\gamma(t)}$$

for some H^1 small excess of mass $\varepsilon(t)$. We ask the question of the dispersive behavior of $\varepsilon(t)$ as $t \to T$. Equivalently, we ask whether asymptotic stability holds. This type of questions goes beyond blow up issues and is related to a wide range of problems regarding the asymptotic stability of solitary waves in non linear dispersive PDE's. Let us recall that in the subcritical case, this is a very delicate issue connected to the socalled Fermi Golden Rule dispersive effect. A similar phenomenon will be at hand here, but of degenerate type. One should also have in mind an important simplification of the problem in the L^2 critical case: even though we are dealing with a pure power nonlinearity, there are *no small solitary waves* because a solitary wave must have at least L^2 mass $|Q|_{L^2}$.

Our angle of approach towards this problem is to first walk in the steps of the breakthrough analysis by Martel and Merle, [35], for the study of a similar problem for the generalized critical equation:

$$(Critical \quad KdV) \quad \begin{cases} u_t + (u_{xx} + u^5)_x = 0, & (t,x) \in [0,T) \times \mathbf{R}, \\ u(0,x) = u_0(x), \in H^1 & u_0 : \mathbf{R} \to \mathbf{R}. \end{cases} \tag{75}$$

This equation shares a lot of the variational structure of (32), and in particular finite time blow up solutions admit a geometrical decomposition similar to (65). In [35], Martel and Merle prove the asymptotic stability of Q as the blow up profile, but not in a dynamical way. The proof is based on an argument by contradiction: assuming that asymptotic stability does not holds, they are able using monotonicity arguments to extract from the problem an asymptotic object which generates a *non dispersive solution to (32). They then prove that this object does not exist.*

This Liouville type strategy turns out to be a very powerful tool to study the qualitative properties of the long time dynamics. This approach has been recently used by Kenig, Merle [24], to prove global well posedness and scattering for the energy critical (NLS) in the focusing case.

In the setting of the L^2 critical (NLS), the result is the following:

Theorem 11 ([46]) *Let $N = 1, 2, 3, 4$. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution $u(t)$ blows up in finite time $0 < T < +\infty$. Then there exist parameters $\lambda_0(t) = \frac{|\nabla Q|_{L^2}}{|\nabla u(t)|_{L^2}}$, $x_0(t) \in \mathbf{R}^N$ and $\gamma_0(t) \in \mathbf{R}$ such that*

$$e^{i\gamma_0(t)} \lambda_0^{\frac{N}{2}}(t) u(t, \lambda_0(t)x + x_0(t)) \to Q \quad in \quad L^2_{loc} \quad as \quad t \to T.$$

In the variables of the decomposition (65), this means:

$$\varepsilon(t) \to 0 \quad \text{as} \quad t \to T \quad \text{in} \quad L^2_{loc}.$$

Let us observe that this is the typical dispersive behavior for Schrödinger group: the L^2 mass is conserved, so L^2 convergence to zero is forbidden, but it happens locally in space meaning that the excess of mass is dispersed away. This theorem thus asserts that in rescaled variables, blow up solutions admit a universal asymptotic profile in space which is given by the ground state Q itself.

The question of the dispersive behavior of ε is deeply connected to the one of the existence of self similar blow up solutions. Such an object is ruled out near the ground state by the asymptotic stability:

Theorem 12 ([46]) *Let $N = 1, 2, 3, 4$. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution $u(t)$ blows up in finite time $0 < T < +\infty$. Then:*

$$|\nabla u(t)|_{L^2}\sqrt{T-t} \to +\infty \quad as \quad t \to T. \tag{76}$$

An important fact which will enlighten our further analysis is that there do exist self similar solutions, but they never belong to L^2. More precisely, a standard way of exhibiting self similar solutions is to look for a blow up solution with the form:

$$U_b(t,x) = \frac{1}{(2b(T-t))^{\frac{N}{4}}} Q_b\left(\frac{x}{\sqrt{2b(T-t)}}\right) e^{-i\frac{\log(T-t)}{2b}}$$

for some fixed parameter $b > 0$ and a fixed profile Q_b solving the elliptic ODE:

$$\Delta Q_b - Q_b + ib\left(\frac{N}{2}Q_b + y \cdot \nabla Q_b\right) + Q_b|Q_b|^{\frac{4}{N}} = 0. \tag{77}$$

Now from [60], solutions Q_b *never belong to* L^2 from a logarithmic divergence at infinity:

$$|Q_b(y)| \sim \frac{C(b)}{|y|^{\frac{N}{2}}} \quad \text{as} \quad |y| \to +\infty$$

and thus always miss the energy space. Nevertheless, for any given parameter $b > 0$ small enough, one can exhibit a solution to (77) which will be in $\dot{H}^1$ and will satisfy:

$$Q_b \to Q \quad \text{as } b \to 0 \quad \text{in} \quad \dot{H}^1 \cap L^2_{loc}.$$

In other words, one can build self similar solutions to (32) which on compacts sets will look like a smooth solution, but then display an oscillatory behavior at infinity in space which induces *a non L^2 tale escaping the soliton core.*

Now to prove Theorems 11 or 12, one needs to understand how to use the information that the solution we consider is in L^2, and one thus needs to exhibit L^2 *dispersive estimates on the solution.* Recall that L^2 is the scaling invariant space for this equation, and thus any dispersive information in L^2 is in fact a global information in space. Now the only given global information in L^2 is the conservation of the L^2 norm, and somehow the task here is *to be able to use the conservation of the L^2 norm in a dynamical way.*

We will precisely describe in section 5.1 the reduction to the Liouville theorem and the role played in particular by *zero energy solutions.*

Remark 5 *The existence or non existence of self similar solutions seems to be a quite delicate issue. Astrophysical models like the gravitational relativistic Vlasov-Poisson system -[29]- or the compressible Euler equations -[33]- indeed exhibit a critical type variational structure and admit explicit self similar blow up solutions. The stability of these objects is even known sometimes, see [29].*

3.4 Exact log-log law and the mass quantization conjecture

The L^2 dispersive estimates needed for the proof of Theorem 11 are exhibited for the asymptotic object in the setting of a Liouville theorem. In this sense, these estimates are not proved for a "true" blow up solution but are exhibited as specific properties of a non dispersive blow up solution. In other words, Theorem 11 relies on a *qualitative* approach to asymptotic stability but does not provide any *quantitative* estimate on the rate of dispersion.

A more refined study of the structure of the flow near Q will in fact allow us to obtain direct dispersive estimates in L^2 on a blow up solution. More specifically, let a finite time blow up solution $u(t) \in \mathcal{O}$. We exhibit a global in space information on the solution by proving that in rescaled variables, the space divides in three specific regions:
(i) on compact sets, the solution looks like Q in a strong sense;
(ii) a radiative regime then takes place where the solution looks like the *non L^2 tale* of the explicit self similar solutions to (77);
(iii) this regime cannot last forever in space because the tale of self similar solutions is not L^2, while the solution is. We then exhibit a third regime further away in space -a form of free boundary in some sense- where a purely linear dispersive dynamic takes place.

Another way of viewing the picture is the following: the non linear dynamic represented by Q on compact set is connected to a linear dispersive dynamic at infinity in space by a universal radiative regime given by the tale of explicit self similar solutions. This radiation is the mechanism which takes the L^2 mass out of the soliton core on compact sets to disperse it to infinity in space: this is an "outgoing radiation" process corresponding to the so-called dynamical metastability of self similar profiles Q_b solutions to (77). Such a picture is typical to the so-called Fermi-Golden rule which provides a phenomenological description of the coupling between finite and infinite dimensional systems, see [62], [12]. Now the rate at which the mass is extracted is submitted to one global constraint in time: the conservation of the L^2 norm. Moreover, this mechanism quantifies how the L^2 constraint implies the non persistence of the self similar regime, that is how the radiation is connected to the dispersive dynamic at infinity. And we have seen that this is related to obtaining lower bounds on the blow up rate.

The outcome of this analysis is the following sharp lower bound on blow up rate.

Theorem 13 ([47]) *Let $N = 1,2,3,4$. There exists a universal constants $C_3^* > 0$ such that the following holds true. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution $u(t)$ blows up in finite time $0 < T < +\infty$, then one has the following lower bound on blow up rate for t close to T:*

$$|\nabla u(t)|_{L^2} \geq C_3^* \left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}}. \tag{78}$$

In the log-log regime, the blow up speed may in fact be exactly evaluated according to:

$$\frac{|\nabla u(t)|_{L^2}}{|\nabla Q|_{L^2}}\left(\frac{T-t}{\log|\log(T-t)|}\right)^{\frac{1}{2}} \to \frac{1}{\sqrt{2\pi}} \quad \text{as} \quad t \to T. \tag{79}$$

It is a surprising fact somehow that the analysis needed to obtain lower and upper bounds in the log-log regime requires different type of informations:
(i) The proof of the upper bound on blow up rate (73) requires only local in space information on the soliton core, and global in $\dot{H}^1$. But nothing is needed in L^2 and indeed explicit self similar profiles solutions to (77) in $\dot{H}^1$ would fit into this analysis.
(ii) The proof of the lower bound on blow up rate (78) requires global in space dispersive informations in L^2, that is estimates on the solution in the different regimes in space. One may then estimate the flux of L^2 norm in between these different regimes which is submitted to the L^2 conservation constraint. This yields the exact log-log law.

Moreover, and this certainly is the main motivation to go through the whole log-log analysis, the precise understanding of the L^2 structure in space of the solution in rescaled variables now allows us to investigate the behavior of the solution in the original non rescaled variables.

Indeed, let us make the following simple observation. From (65), we may write a blow up solution $u(t)$ near blow up time:

$$u(t,x) = Q_{sing}(t,x) + \tilde{u}(t,x)$$

with

$$Q_{sing}(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}} Q(t, \frac{x-x(t)}{\lambda(t)}) e^{i\gamma(t)}, \quad \tilde{u}(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}} \varepsilon(t, \frac{x-x(t)}{\lambda(t)}) e^{i\gamma(t)}.$$

Q_{sing} is the singular part of the solution. We address the following natural question: does the excess of mass $\tilde{u}(t,x)$ remain smooth up to blow up time? A first answer to this question has been obtained in rescaled variables. Indeed, Theorem 11 asserts:

$$\varepsilon(t) \to 0 \ \text{ as } \ t \to T \ \text{ in } \ L^2_{loc}.$$

But as $\lambda(t) \to 0$ as $t \to T$, this is very far from obtaining regularity control on $\tilde{u}(t)$. In particular, it does not prevent a priori the excess of mass $\tilde{u}$ from focusing some small mass at blow up time. The regularity of $\tilde{u}$ is thus deeply related both to the shape in space of $\varepsilon(t)$ and the rate at which it is dispersed. In other words, we need to describe the scattering structure of ε as the rescaled time $s \to +\infty$. This issue is now precisely the one addressed in the proof of Theorem 13. Further use of the obtained estimates allow one to prove the following result.

Theorem 14 ([48]) *Let $N = 1,2,3,4$. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution to (32) blows up in finite time $0 < T < +\infty$. Then there exist parameters $(\lambda(t), x(t), \gamma(t)) \in \mathbf{R}^*_+ \times \mathbf{R}^N \times \mathbf{R}$ and an asymptotic profile $u^* \in L^2$ such that*

$$u(t) - \frac{1}{\lambda(t)^{\frac{N}{2}}} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to u^* \ \ \text{in} \ \ L^2 \ \ \text{as} \ \ t \to T. \tag{80}$$

Moreover, the blow up point is finite in the sense that

$$x(t) \to x(T) \in \mathbf{R}^N \ \ \text{as} \ \ t \to T.$$

In other words, up to a singular part which has a universal space time structure, blow up solutions remain smooth in L^2 at blow up time.

A fundamental corollary is the so called *quantization phenomenon* for (32): blow up solutions in $\mathcal{B}_{\alpha^*}$ focus the universal amount of mass $\int Q^2$ into blow up, the rest is purely dispersed, or in other words:

$$|u(t)|^2 \rightharpoonup \left(\int Q^2\right) \delta_{x=x(T)} + |u^*|^2 \ \text{ as } \ t \to T \ \text{ with } \ \int |u_0|^2 = \int Q^2 + \int |u^*|^2.$$

This is in contrast with the Zakharov model (71) where explicit blow up solutions build by Glangetas, Merle, [16], accumulate a continuum of mass into blow up, see also [29].

A second outcome of Theorem 14 is the fact that the formation of the singularity is a well localized in space phenomenon. Indeed, blow up occurs at a well defined blow up point $x(T)$ where a fixed amount of mass is focused, but outside $x(T)$, the solution has a strong L^2 limit. It means in particular that the phase of the solution is not oscillatory outside blow up point, whereas the phase $\gamma(t)$ of the singularity is known to satisfy $\gamma(t) \to +\infty$ as $t \to T$. This strong regularity of the solution outside the blow up point was not expected. From the proof also, one can prove that the blow up point $x(T)$ and the asymptotic profile u^* are in the log-log regime continuous functions of the initial data.

Observe now that Theorem 14 includes both blow up regimes which would in particular be characterized by a different law for $\lambda(t)$ in the singular part of the solution. We now claim that the difference between the two blow up regimes may be seen on the asymptotic profile u^* which in fact *connects in a universal way depending on the blow up regime the regular and singular parts of the solution.*

Theorem 15 ([48]) *Let $N = 1, 2, 3, 4$. There exists a universal constant $C^* > 0$ such that the following holds true. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution $u(t)$ to (32) blows up in finite time $0 < T < +\infty$. Let $x(T)$ its blow up point and $u^* \in L^2$ its profile given by Theorem 14, then for $R > 0$ small enough, we have:*
(i) Log-log case: if $u_0 \in \mathcal{O}$, then

$$\frac{1}{C^*(\log|\log(R)|)^2} \leq \int_{|x-x(T)|\leq R} |u^*(x)|^2 dx \leq \frac{C^*}{(\log|\log(R)|)^2}, \tag{81}$$

and in particular:

$$u^* \notin H^1 \quad and \quad u^* \notin L^p \quad for \quad p > 2. \tag{82}$$

(ii) $S(t)$ case: if $u(t)$ satisfies (74), then

$$\int_{|x-x(T)|\leq R} |u^*|^2 \leq C^* E_0 R^2, \tag{83}$$

and

$$u^* \in H^1.$$

The fact that one can separate within the two blow up dynamics and see the different blow up speeds on the asymptotic profile u^* is a new unexpected feature for (NLS). This results strengthens our belief that $S(t)$ type of solutions are in some sense on the boundary of the set of finite time blow up solutions:

- The stable log-log blow up scenario is based on the ejection of a radiative mass which strongly couples the singular and the regular parts of the solution and induces the singular behavior (81) of the profile at blow up point. The universal singular behavior (81) is the "trace" of the radiative regime in the rescaled variables which couples the blow up dynamic on compact sets to the dispersive dynamic at infinity.

- On the contrary, the $S(t)$ regime corresponds to the formation of a minimal mass blow up bubble very decoupled from the regular part which indeed remains smooth in the Cauchy space. This blow up scenario somehow corresponds to the "minimal" blow up configuration. Observe that in this last regime, (83) in dimension $N = 1$ implies $u^*(0) = 0$. Now in [6], Bourgain and Wang construct for a given radial profile u^* smooth with $\frac{d^i}{dr^i}u^*(r)_{|r=0} = 0$, $1 \leq i \leq A$, a solution to (32) with blow up point $x = 0$ and asymptotic profile u^*. In their proof, A is very large, what is used to decouple the regular and the singular parts of the solution. In this sense, the estimate (83) proves in general a decoupling of this kind for the $S(t)$ dynamic. It is an open problem to estimate the exact degeneracy of u^*.

4 Log-log upper bound on the blow up rate

This section is devoted to a presentation of the main results needed for the proof of the log-log upper bound on blow up rate in the non positive energy case, that is Theorem 9. We will in particular focus onto the proof of the key dispersive controls in $\dot{H}^1$ which are at the heart of the control from above on the blow up speed. More detailed proofs are to be found in [44], [45].

The heart of our analysis will be to exhibit as a consequence of dispersive properties of (32) close to Q strong rigidity constraints for the dynamics of non positive energy solutions. These will in turn imply monotonicity properties, that is the existence of a Lyapounov function. The corresponding estimates will then allow us to prove blow up in a dynamical way and the sharp upper bound on the blow up speed will follow.

In the whole section, we consider a data

$$u_0 \in \mathcal{B}_{\alpha^*}$$

for some small universal $\alpha^* > 0$ and let $u(t)$ the corresponding solution to (32) with maximal time interval existence $[0, T)$ in H^1, $0 < T \leq +\infty$. We further assume $E_G(u_0) < 0$. According to Comment 1 of Theorem 9, we equivalently have up to a fixed Galilean transformation:

$$E_0 < 0 \ \text{ and } \ Im\left(\int \nabla u_0 \overline{u_0}\right) = 0. \tag{84}$$

For a given function f, we will note

$$\Lambda f = \frac{N}{2} f + y \cdot \nabla f.$$

Note that from integration by parts:

$$(\Lambda f, g) = -(f, \Lambda g).$$

4.1 Existence of the geometrical decomposition

Let an initial data $u_0 \in \mathcal{B}(\alpha^*)$ with $E_G(u_0) < 0$. First observe that up to a fixed Galilean transform, we may equivalently assume

$$E(u_0) < 0 \ \text{ and } \ Im(\nabla u \overline{u_0}) = 0. \tag{85}$$

Proposition 6 thus applies and implies for $t \in [0,T)$ the existence of a geometrical decomposition

$$u(t,x) = \frac{1}{\lambda_0^{\frac{N}{2}}(t)}(Q+\varepsilon_0)(t, \frac{x-x_0(t)}{\lambda_0(t)})e^{i\gamma_0(t)}, \quad \|\varepsilon_0\|_{H^1} \leq \delta(\alpha^*).$$

Let us observe that this geometrical decomposition is by no mean unique. Nevertheless, one can freeze and regularize this decomposition by choosing a set of orthogonality conditions on the excess of mass: this is the so-called modulation theory which will be examined later on. Let us so far assume that we have a smooth decomposition of the solution: $\forall t \in [0,T)$,

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}}(Q+\varepsilon)(t, \frac{x-x(t)}{\lambda(t)})e^{i\gamma(t)} \tag{86}$$

with

$$\lambda(t) \sim \frac{C}{|\nabla u(t)|_{L^2}} \quad \text{and} \quad |\varepsilon(t)|_{H^1} \leq \delta(\alpha^*) \to 0 \quad \text{as} \quad \alpha^* \to 0.$$

To study the blow up dynamic is now equivalent to understanding the coupling between the finite dimensional dynamic which governs the evolution of the geometrical parameters $(\lambda(t), \gamma(t), x(t))$ and the infinite dimensional dispersive dynamic which drives the excess of mass $\varepsilon(t)$.

To enlighten the main issues, let us rewrite (32) in the so-called rescaled variables. Let us introduce the rescaled time:

$$s(t) = \int_0^t \frac{d\tau}{\lambda^2(\tau)}.$$

It is elementary to check that whatever is the blow up behavior of $u(t)$, one always has:

$$s([0,T)) = \mathbf{R}^+.$$

Let us set:

$$v(s,y) = e^{i\gamma(t)}\lambda(t)^{\frac{N}{2}}u(\lambda(t)x + x(t)),$$

then from direct computation, $u(t,x)$ solves (32) on $[0,T)$ iff $v(s,y)$ solves: $\forall s \geq 0$,

$$iv_s + \Delta v - v + v|v|^{\frac{4}{N}} = i\frac{\lambda_s}{\lambda}\left(\frac{N}{2}v + y\cdot\nabla v\right) + i\frac{x_s}{\lambda}\cdot\nabla v + \tilde{\gamma}_s v, \tag{87}$$

where $\tilde{\gamma} = -\gamma - s$. Now $v(s,y) = Q(y) + \varepsilon(s,y)$ and we linearize (87) close to Q. The obtained system has the form:

$$i\varepsilon_s + L\varepsilon = i\frac{\lambda_s}{\lambda}\left(\frac{N}{2}Q + y\cdot\nabla Q\right) + \gamma_s Q + i\frac{x_s}{\lambda}\cdot\nabla Q + R(\varepsilon), \tag{88}$$

$R(\varepsilon)$ formally quadratic in ε, and $L = (L_+, L_-)$ is the matrix linearized operator closed to Q which has components:

$$L_+ = -\Delta + 1 - \left(1 + \frac{4}{N}\right)Q^{\frac{4}{N}}, \quad L_- = -\Delta + 1 - Q^{\frac{4}{N}}.$$

A standard approach is to think of equation (88) in the following way: it is essentially a linear equation forced by terms depending on the law for the geometrical parameters. The classical study of this kind of system relies on the understanding of the dispersive properties of the propagator e^{isL} of the linearized operator close to Q. In particular, one needs to exhibit its spectral structure. This has been done by Weinstein, [66], using the variational characterization of Q. The result is the following: L is a non self adjoint operator with a generalized eigenspace at zero. The eigenmodes are explicit and generated by the symmetries of the problem:

$$L_+\left(\frac{N}{2}Q + y\cdot\nabla Q\right) = -2Q \quad \text{(scaling invariance)},$$

$$L_+(\nabla Q) = 0 \quad \text{(translation invariance)},$$

$$L_-(Q) = 0 \quad \text{(phase invariance)}, \quad L_-(yQ) = -2\nabla Q \quad \text{(Galilean invariance)}.$$

An additional relation is induced by the pseudo-conformal symmetry:

$$L_-(|y|^2Q) = -4(\frac{N}{2}Q + y\cdot\nabla Q),$$

and this in turns implies the existence of an additional mode ρ solution to

$$L_+\rho = -|y|^2Q.$$

These explicit directions induce "growing" solutions to the homogeneous linear equation $i\partial_s\varepsilon + L\varepsilon = 0$. More precisely, there exists a (2N+3) dimensional space S spanned by the above directions such that $H^1 = M \oplus S$ with $|e^{isL}\varepsilon|_{H^1} \leq C$ for $\varepsilon \in M$ and $|e^{isL}\varepsilon|_{H^1} \sim s^3$ for $\varepsilon \in S$. As each symmetry is at the heart of a growing direction, a first idea is to use the symmetries from modulation theory to a priori ensure that ε is orthogonal to S. Roughly speaking, the strategy to construct blow up solutions is then: chose the parameters λ, γ, x so as to get good a priori dispersive estimates on ε in order to build it from a fixed point scheme. Now the fundamental problem is that one has (2N+2) symmetries, but (2N+3) bad modes in the set S. Both constructions in [6] and [56] develop non trivial strategies to overcome this intrinsic difficulty of the problem.

Our strategy will be more non linear. On the basis of the decomposition (86), we will prove dispersive estimates on ε induced by the virial structure (31). The proof will rely on non linear degeneracies of the structure of (32) around Q. Using then the Hamiltonian information $E_0 < 0$, we will inject these estimates into the finite dimensional dynamic which governs $\lambda(t)$ -which measures the size of the solution- and prove rigidity properties of Lyapounov type. This will then allow us to prove finite time blow up together with the control of the blow up speed.

4.2 Choice of the blow up profile

Before exhibiting the modulation theory type of arguments, we present in this subsection a formal discussion regarding explicit solutions of equation (87) which is inspired from a discussion in [63]. This corresponds to a finite dimensional reduction of the problem which actually computes the leading order terms of the solution.

First, let us observe that the key geometrical parameter is λ which measures the size of the solution. Let us then set

$$-\frac{\lambda_s}{\lambda} = b$$

and look for solutions to a simpler version of (87):

$$iv_s + \Delta v - v + ib\left(\frac{N}{2}v + y\cdot\nabla v\right) + v|v|^{\frac{4}{N}} = 0.$$

From the orbital stability property, we want solutions which remain close to Q in H^1. Let us look for solutions of the form $v(s,y) = Q_{b(s)}(y)$ where the mappings $b \to Q_b$ and the law for $b(s)$ are the unknown. We think of b as remaining uniformly small and $Q_{b=0} = Q$. Injecting this ansatz into the equation, we get:

$$i\frac{db}{ds}\left(\frac{\partial \overline{Q}_b}{\partial b}\right) + \Delta\overline{Q}_{b(s)} - \overline{Q}_{b(s)} + ib(s)\left(\frac{N}{2}\overline{Q}_{b(s)} + y\cdot\nabla\overline{Q}_{b(s)}\right) + \overline{Q}_{b(s)}|\overline{Q}_{b(s)}|^{\frac{4}{N}} = 0.$$

To handle the linear group, we let $\overline{P}_{b(s)} = e^{i\frac{b(s)}{4}|y|^2}\overline{Q}_{b(s)}$ and solve:

$$i\frac{db}{ds}\left(\frac{\partial \overline{P}_b}{\partial b}\right) + \Delta\overline{P}_{b(s)} - \overline{P}_{b(s)} + \left(\frac{db}{ds} + b^2(s)\right)\frac{|y|^2}{4}\overline{P}_{b(s)} + \overline{P}_{b(s)}|\overline{P}_{b(s)}|^{\frac{4}{N}} = 0. \qquad (89)$$

A remarkable fact related to the specific algebraic structure of (32) around Q is that (89) admits three solutions:

- The first one is $(b(s), \overline{P}_{b(s)}) = (0, Q)$, that is the ground state itself. This is just a consequence of the scaling invariance.
- The second one is $(b(s), \overline{P}_{b(s)}) = (\frac{1}{s}, Q)$. This non trivial solution is a rewriting of the explicit critical mass blow up solution $S(t)$ and is induced by the pseudo-conformal symmetry.
- The third one is given by $(b(s), \overline{P}_{b(s)}) = (b, \overline{P}_b)$ for some fixed non zero constant b and $\overline{P}_b$ satisfies:

$$\Delta\overline{P}_b - \overline{P}_b + \frac{b^2}{4}|y|^2\overline{P}_b + \overline{P}_b|\overline{P}_b|^{\frac{4}{N}} = 0. \qquad (90)$$

The solutions to this non linear elliptic equation are those who produce the explicit self similar profiles solutions to (77). A simple way to see this is to recall that we have set $b = -\frac{\lambda_s}{\lambda}$, so if b is frozen, we have from $\frac{ds}{dt} = \frac{1}{\lambda^2}$:

$$b = -\frac{\lambda_s}{\lambda} = -\lambda\lambda_t \ \text{ ie } \ \lambda(t) = \sqrt{2b(T-t)},$$

this is the scaling law for the blow up speed.

Now a crucial point again is -[60]- that the solutions to (90) never belong to L^2 from a logarithmic divergence at infinity:

$$|P_b(y)| \sim \frac{C(P_b)}{|y|^{\frac{N}{2}}} \ \text{ as } \ |y| \to +\infty.$$

This behavior is a consequence of the oscillations induced by the linear group after the turning point $|y| \geq \frac{2}{|b|}$. Nevertheless, in the ball $|y| < \frac{2}{|b|}$, the operator $-\Delta + 1 - \frac{b^2|y|^2}{4}$ is coercive, and no oscillations will take place in this zone.

Because we track a log-log correction to the self similar law as an upper bound on the blow up speed, the profiles $\overline{Q}_b = e^{-i\frac{b}{4}|y|^2}\overline{P}_b$ with $\overline{P}_b$ solving (90) are natural candidates as refinements of the Q profile in the geometrical decomposition (65). Nevertheless, as they are not in L^2, we need to build a smooth localized version avoiding the non L^2 tale, what according to the above discussion is doable in the coercive zone $|y| < \frac{2}{|b|}$.

Proposition 8 (Localized self similar profiles) *There exist universal constants $C > 0$, $\eta^* > 0$ such that the following holds true. For all $0 < \eta < \eta^*$, there exist constants $\nu^*(\eta) > 0$, $b^*(\eta) > 0$ going to zero as $\eta \to 0$ such that for all $|b| < b^*(\eta)$, let*

$$R_b = \frac{2}{|b|}\sqrt{1-\eta}, \quad R_b^- = \sqrt{1-\eta}R_b,$$

$B_{R_b} = \{y \in \mathbf{R}^N, \ \ |y| \leq R_b\}$. *Then there exists a unique radial solution Q_b to*

$$\begin{cases} \Delta Q_b - Q_b + ib\left(\frac{N}{2}Q_b + y\cdot\nabla Q_b\right) + Q_b|Q_b|^{\frac{4}{N}} = 0, \\ P_b = Q_b e^{i\frac{b|y|^2}{4}} > 0 \ \ in \ \ B_{R_b}, \\ Q_b(0) \in (Q(0) - \nu^*(\eta), Q(0) + \nu^*(\eta)), \quad Q_b(R_b) = 0. \end{cases}$$

Moreover, let a smooth radially symmetric cut-off function $\phi_b(x) = 0$ for $|x| \geq R_b$ and $\phi_b(x) = 1$ for $|x| \leq R_b^-$, $0 \leq \phi_b(x) \leq 1$ and set

$$\tilde{Q}_b(r) = Q_b(r)\phi_b(r),$$

then

$$\tilde{Q}_b \to Q \ \ as \ \ b \to 0$$

in some very strong sense, and $\tilde{Q}_b$ *satisfies*

$$\Delta\tilde{Q}_b - \tilde{Q}_b + ib(\tilde{Q}_b)_1 + \tilde{Q}_b|\tilde{Q}_b|^{\frac{4}{N}} = -\Psi \tag{91}$$

with

$$Supp(\Psi) \subset \{R_b^- \leq |y| \leq R_b\} \quad and \quad |\Psi_b|_{\mathcal{C}^1} \leq e^{-\frac{C}{|b|}}.$$

Eventually, $\tilde{Q}_b$ *has supercritical mass:*

$$\int |\tilde{Q}_b|^2 = \int Q^2 + c_0 b^2 + o(b^2) \quad as \quad b \to 0 \tag{92}$$

for some universal constant $c_0 > 0$*.*

The meaning of this proposition is that one can build localized profiles $\tilde{Q}_b$ on the ball B_{R_b} which are a smooth function of b and approximate Q in a very strong sense as $b \to 0$, and these profiles satisfy the self similar equation up to an exponentially small term Ψ_b supported around the turning point $\frac{2}{b}$. The proof of this Proposition uses standard variational tools in the setting of non linear elliptic problems. In fact, the implicit function theorem would do the job as well, see [56].

Now one can think of making a formal expansion of $\tilde{Q}_b$ in terms of b, and the first term is non zero:

$$\frac{\partial\tilde{Q}_b}{\partial b}_{|b=0} = -\frac{i}{4}|y|^2 Q.$$

However, the energy of $\tilde{Q}_b$ is degenerated in b at all orders:

$$|E(\tilde{Q}_b)| \leq e^{-\frac{C}{|b|}}, \tag{93}$$

for some universal constant $C > 0$.

The existence of a one parameter family of profiles satisfying the self similar equation up to an exponentially small term and having an exponentially small energy is an algebraic property of the structure of (32) around Q which is at the heart of the existence of the log-log regime.

4.3 Modulation theory

We now are in position to exhibit the sharp decomposition needed for the proof of the log-log upper bound. From Theorem 6 and the proximity of $\tilde{Q}_b$ to Q in H^1, the solution $u(t)$ to (32) is for all time close to the four dimensional manifold

$$\mathcal{M} = \{e^{i\gamma}\lambda^{\frac{N}{2}}\tilde{Q}_b(\lambda y + x), \quad (\lambda, \gamma, x, b) \in \mathbf{R}_+^* \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}\}.$$

We now sharpen the decomposition according to the following Lemma. In what follows, we let

$$\varepsilon = \varepsilon_1 + i\varepsilon_2$$

be the real and imaginary parts decomposition.

Lemma 2 (Non linear modulation of the solution close to $\mathcal{M}$) *There exist $\mathcal{C}^1$ functions of time $(\lambda, \gamma, x, b) : [0,T) \to (0,+\infty) \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}$ such that:*

$$\forall t \in [0,T), \quad \varepsilon(t,y) = e^{i\gamma(t)} \lambda^{\frac{N}{2}}(t) u(t, \lambda(t)y + x(t)) - \tilde{Q}_{b(t)}(y) \tag{94}$$

satisfies:
(i)

$$\left(\varepsilon_1(t), \Lambda\Sigma_{b(t)}\right) + \left(\varepsilon_2(t), \Lambda\Theta_{b(t)}\right) = 0, \tag{95}$$

$$\left(\varepsilon_1(t), y\Sigma_{b(t)}\right) + \left(\varepsilon_2(t), y\Theta_{b(t)}\right) = 0, \tag{96}$$

$$-\left(\varepsilon_1(t), \Lambda^2\Theta_{b(t)}\right) + \left(\varepsilon_2(t), \Lambda^2\Sigma_{b(t)}\right) = 0, \tag{97}$$

$$-\left(\varepsilon_1(t), \Lambda\Theta_{b(t)}\right) + \left(\varepsilon_2(t), \Lambda\Sigma_{b(t)}\right) = 0, \tag{98}$$

where $\varepsilon = \varepsilon_1 + i\varepsilon_2$, $\tilde{Q}_b = \Sigma_b + i\Theta_b$ in terms of real and imaginary parts;

$$\textit{(ii)} \quad |1 - \lambda(t)\frac{|\nabla u(t)|_{L^2}}{|\nabla Q|_{L^2}}| + |\varepsilon(t)|_{H^1} + |b(t)| \leq \delta(\alpha^*) \quad \textit{with} \quad \delta(\alpha^*) \to 0 \quad \textit{as} \quad \alpha^* \to 0.$$

Let us insist onto the fact that the reason for this precise choice of orthogonality conditions is a fundamental issue which will be addressed in the next section.

Proof of Lemma 2

This Lemma follows the standard frame of modulation theory and is obtained from Theorem 6 using the implicit function theorem.
From Theorem 6, there exist parameters $\gamma_0(t) \in \mathbf{R}$ and $x_0(t) \in \mathbf{R}^N$ such that with $\lambda_0(t) = \frac{|\nabla Q|_{L^2}}{|\nabla u(t)|_{L^2}}$,

$$\forall t \in [0,T), \quad \left| Q - e^{i\gamma_0(t)} \lambda_0(t)^{\frac{N}{2}} u(\lambda_0(t)x + x_0(t)) \right|_{H^1} < \delta(\alpha^*)$$

with $\delta(\alpha^*) \to 0$ as $\alpha^* \to 0$. Now we sharpen this decomposition using the fact that $\tilde{Q}_b \to Q$ in H^1 as $b \to 0$, i.e. we chose $(\lambda(t), \gamma(t), x(t), b(t))$ close to $(\lambda_0(t), \gamma_0(t), x_0(t), 0)$ such that

$$\varepsilon(t,y) = e^{i\gamma(t)} \lambda^{1/2}(t) u(t, \lambda(t)y + x(t)) - \tilde{Q}_{b(t)}(y)$$

is small in H^1 and satisfies suitable orthogonality conditions (95), (96), (97) and (98). The existence of such a decomposition is a consequence of the implicit function Theorem. For $\delta > 0$, let $V_\delta = \{v \in H^1(\mathbf{C});\quad |v - Q|_{H^1} \le \delta\}$, and for $v \in H^1(\mathbf{C})$, $\lambda_1 > 0$, $\gamma_1 \in \mathbf{R}$, $x_1 \in \mathbf{R}^N$, $b \in \mathbf{R}$ small, define

$$\varepsilon_{\lambda_1,\gamma_1,x_1,b}(y) = e^{i\gamma_1}\lambda_1^{\frac{N}{2}} v(\lambda_1 y + x_1) - \tilde{Q}_b. \tag{99}$$

We claim that there exists $\overline{\delta} > 0$ and a unique C^1 map : $V_{\overline{\delta}} \to (1-\overline{\lambda}, 1+\overline{\lambda}) \times (-\overline{\gamma}, \overline{\gamma}) \times B(0, \overline{x}) \times (-\overline{b}, \overline{b})$ such that if $v \in V_{\overline{\delta}}$, there is a unique $(\lambda_1, \gamma_1, x_1, b)$ such that $\varepsilon_{\lambda_1,\gamma_1,x_1,b} = (\varepsilon_{\lambda_1,\gamma_1,x_1,b})_1 + i(\varepsilon_{\lambda_1,\gamma_1,x_1,b})_2$ defined as in (99) satisfies

$$\rho^1(v) = ((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_1, \Lambda\Sigma_b) + ((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_2, \Lambda\Theta_b) = 0,$$

$$\rho^2(v) = ((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_1, y\Sigma_b) + ((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_2, y\Theta_b) = 0,$$

$$\rho^3(v) = -\Big((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_1, \Lambda^2\Theta_b\Big) + \Big((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_2, \Lambda^2\Sigma_b\Big) = 0,$$

$$\rho^4(v) = ((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_1, \Lambda\Theta_b) - ((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_2, \Lambda\Sigma_b) = 0.$$

Moreover, there exists a constant $C_1 > 0$ such that if $v \in V_{\overline{\delta}}$, then $|\varepsilon_{\lambda_1,\gamma_1,x_1}|_{H^1} + |\lambda_1 - 1| + |\gamma_1| + |x_1| + |b| \le C_1\overline{\delta}$. Indeed, we view the above functionals $\rho^1, \rho^2, \rho^3, \rho^4$ as functions of $(\lambda_1, \gamma_1, x_1, b, v)$. We first compute at $(\lambda_1, \gamma_1, x_1, b, v) = (1, 0, 0, 0, v)$:

$$\frac{\partial \varepsilon_{\lambda_1,\gamma_1,x_1,b}}{\partial x_1} = \nabla v, \quad \frac{\partial \varepsilon_{\lambda_1,\gamma_1,x_1,b}}{\partial \lambda_1} = \frac{N}{2}v + x\cdot\nabla v, \quad \frac{\partial \varepsilon_{\lambda_1,\gamma_1,x_1,b}}{\partial \gamma_1} = iv, \quad \frac{\partial \varepsilon_{\lambda_1,\gamma_1,x_1,b}}{\partial b} = -\left(\frac{\partial \tilde{Q}_b}{\partial b}\right)_{|b=0}.$$

Now recall that $(\tilde{Q}_b)_{|b=0} = Q$ and $\left(\frac{\partial \tilde{Q}_b}{\partial b}\right)_{|b=0} = -i\frac{|y|^2}{4}Q$. Therefore, we obtain at the point $(\lambda_1, \gamma_1, x_1, b, v) = (1, 0, 0, 0, Q)$,

$$\begin{aligned}
&\frac{\partial \rho^1}{\partial \lambda_1} = |\Lambda Q|_2^2, \quad \frac{\partial \rho^1}{\partial \gamma_1} = 0, \quad \frac{\partial \rho^1}{\partial x_1} = 0, \frac{\partial \rho^1}{\partial b} = 0,\\
&\frac{\partial \rho^2}{\partial \lambda_1} = 0, \quad \frac{\partial \rho^2}{\partial \gamma_1} = 0, \quad \frac{\partial \rho^2}{\partial x_1} = -\frac{1}{2}|Q|_2^2, \frac{\partial \rho^2}{\partial b} = 0,\\
&\frac{\partial \rho^3}{\partial \lambda_1} = 0, \quad \frac{\partial \rho^3}{\partial \gamma_1} = -|\Lambda Q|_2^2, \quad \frac{\partial \rho^3}{\partial x_1} = 0, \frac{\partial \rho^3}{\partial b} = 0,\\
&\frac{\partial \rho^4}{\partial \lambda_1} = 0, \quad \frac{\partial \rho^4}{\partial \gamma_1} = 0, \quad \frac{\partial \rho^4}{\partial x_1} = 0, \frac{\partial \rho^4}{\partial b} = \frac{1}{4}|yQ|_2^2.
\end{aligned}$$

The Jacobian of the above functional is non zero, thus the implicit function Theorem applies and conclusion follows. This concludes the proof of Lemma 2.

Let us now write down the equation satisfied by ε in rescaled variables. To simplify notations, we note

$$\tilde{Q}_b = \Sigma + \Theta$$

in terms of real and imaginary parts. We have: $\forall s\in\mathbf{R}_+$, $\forall y\in\mathbf{R}^N$,

$$\begin{aligned}
b_s\frac{\partial\Sigma}{\partial b}+\partial_s\varepsilon_1-M_-(\varepsilon)+b\Lambda\varepsilon_1 &= \left(\frac{\lambda_s}{\lambda}+b\right)\Lambda\Sigma+\tilde{\gamma}_s\Theta+\frac{x_s}{\lambda}\cdot\nabla\Sigma \\
&+ \left(\frac{\lambda_s}{\lambda}+b\right)\Lambda\varepsilon_1+\tilde{\gamma}_s\varepsilon_2+\frac{x_s}{\lambda}\cdot\nabla\varepsilon_1 \\
&+ Im(\Psi)-R_2(\varepsilon)
\end{aligned} \tag{100}$$

$$\begin{aligned}
b_s\frac{\partial\Theta}{\partial b}+\partial_s\varepsilon_2+M_+(\varepsilon)+b\Lambda\varepsilon_2 &= \left(\frac{\lambda_s}{\lambda}+b\right)\Lambda\Lambda\Theta-\tilde{\gamma}_s\Sigma+\frac{x_s}{\lambda}\cdot\nabla\Theta \\
&+ \left(\frac{\lambda_s}{\lambda}+b\right)\Lambda\varepsilon_2-\tilde{\gamma}_s\varepsilon_1+\frac{x_s}{\lambda}\cdot\nabla\varepsilon_2 \\
&- Re(\Psi)+R_1(\varepsilon),
\end{aligned} \tag{101}$$

with $\tilde{\gamma}(s)=-s-\gamma(s)$. The linear operator close to $\tilde{Q}_b$ is now a deformation of the linear operator L close to Q and is $M=(M_+,M_-)$ with

$$M_+(\varepsilon)=-\Delta\varepsilon_1+\varepsilon_1-\left(\frac{4\Sigma^2}{N|\tilde{Q}_b|^2}+1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_1-\left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_2,$$

$$M_-(\varepsilon)=-\Delta\varepsilon_2+\varepsilon_2-\left(\frac{4\Theta^2}{N|\tilde{Q}_b|^2}+1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_2-\left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_1.$$

The formally quadratic in ε interaction terms are:

$$R_1(\varepsilon)=(\varepsilon_1+\Sigma)|\varepsilon+\tilde{Q}_b|^{\frac{4}{N}}-\Sigma|\tilde{Q}_b|^{\frac{4}{N}}-\left(\frac{4\Sigma^2}{N|\tilde{Q}_b|^2}+1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_1-\left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_2,$$

$$R_2(\varepsilon)=(\varepsilon_2+\Theta)|\varepsilon+\tilde{Q}_b|^{\frac{4}{N}}-\Theta|\tilde{Q}_b|^{\frac{4}{N}}-\left(\frac{4\Theta^2}{N|\tilde{Q}_b|^2}+1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_2-\left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_1.$$

Two natural estimates may now be performed:

- First, we may rewrite the conservation laws in the rescaled variables and linearize the obtained identities close to Q. This will give crucial degeneracy estimates on some specific order one in ε scalar products.

- Next, we may inject the orthogonality conditions of Lemma 2 into the equations (100), (101). This will compute the geometrical parameters in their differential form $\frac{\lambda_s}{\lambda},\tilde{\gamma}_s,\frac{x_s}{\lambda},b_s$ in terms of ε: these are the so called modulation equations. This step requires estimating the non linear interaction terms. A crucial point here is to use the fact that the ground state Q is exponentially decreasing in space.

The outcome is the following:

Lemma 3 (First estimates on the decomposition) *We have for all $s \geq 0$:*
(i) Estimates induced by the conservation of the energy and the momentum:

$$|(\varepsilon_1, Q)| \leq \delta(\alpha^*)\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)^{\frac{1}{2}} + e^{-\frac{C}{|b|}} + C\lambda^2|E_0|, \tag{102}$$

$$|(\varepsilon_2, \nabla Q)| \leq C\delta(\alpha^*)\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)^{\frac{1}{2}}. \tag{103}$$

(ii) Estimate on the geometrical parameters in differential form:

$$\left|\frac{\lambda_s}{\lambda} + b\right| + |b_s| + |\tilde{\gamma}_s| \leq C\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)^{\frac{1}{2}} + e^{-\frac{C}{|b|}}, \tag{104}$$

$$\left|\frac{x_s}{\lambda}\right| \leq \delta(\alpha^*)\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)^{\frac{1}{2}} + e^{-\frac{C}{|b|}}, \tag{105}$$

where $\delta(\alpha^*) \to 0$ as $\alpha^* \to 0$.

Remark 6 *The exponentially small term in the degeneracy estimate (102) is in fact related to the value of $E(\tilde{Q}_b)$, so we use here in a fundamental way the non linear degeneracy estimate (93).*

Comments on Lemma 3:

1. $\dot{H}^1$ norm: The norm which appears in the estimates of Lemma 3 is essentially a local norm in space. The conservation of the energy indeed relates the $\int |\nabla\varepsilon|^2$ norm with the local norm. These two norms will turn out to play an equivalent role in the analysis. A key is that no global L^2 norm is needed so far.

2. Degeneracy of the translation shift: Comparing estimates (104) and (105), we see that the term induced by translation invariance is smaller than the ones induced by scaling and phase invariances. This non trivial fact is an outcome of our use of the Galilean transform to ensure the zero momentum condition (84).

4.4 The virial type dispersive estimate

Our aim in this subsection is to exhibit the dispersive virial type inequality at the heart of the proof of the log-log upper bound. This information will be obtained as a consequence of the virial structure of (32) in Σ.

Let us first recall that the virial identity (31) corresponds to two identities:

$$\frac{d^2}{dt^2}\int |x|^2|u|^2 = 4\frac{d}{dt}Im(\int x \cdot \nabla u\overline{u}) = 16E_0. \tag{106}$$

We want to understand what information can be extracted from this dispersive information in the variables of the geometrical decomposition.

To clarify the claim, let us consider an ε solution to the linear homogeneous equation

$$i\partial_s\varepsilon + L\varepsilon = 0 \tag{107}$$

where $L = (L_+, L_-)$ is the linearized operator close to Q. A dispersive information on ε may be extracted using a similar virial law like (31):

$$\frac{1}{2}\frac{d}{ds}Im(\int y\cdot\nabla\varepsilon\overline{\varepsilon}) = H(\varepsilon,\varepsilon), \tag{108}$$

where $H(\varepsilon,\varepsilon) = (\mathcal{L}_1\varepsilon_1,\varepsilon_1) + (\mathcal{L}\varepsilon_2,\varepsilon_2)$ is a Schrödinger type quadratic form decoupled in the real and imaginary parts with explicit Schrödinger operators:

$$\mathcal{L}_1 = -\Delta + \frac{2}{N}\left(\frac{4}{N}+1\right)Q^{\frac{4}{N}-1}y\cdot\nabla Q \ \ , \ \ \mathcal{L}_2 = -\Delta + \frac{2}{N}Q^{\frac{4}{N}-1}y\cdot\nabla Q.$$

Note that both these operators are of the form $-\Delta + V$ for some smooth well localized time independent potential $V(y)$, and thus from standard spectral theory, they both have a finite number of negative eigenvalues, and then continuous spectrum on $[0,+\infty)$. A simple outcome is then that given an $\varepsilon \in H^1$ which is orthogonal to all the bound states of $\mathcal{L}_1, \mathcal{L}_2$, then $H(\varepsilon,\varepsilon)$ is coercive, that is

$$H(\varepsilon,\varepsilon) \geq \delta_0\left(\int|\nabla\varepsilon|^2 + \int|\varepsilon|^2e^{-|y|}\right)$$

for some universal constant $\delta_0 > 0$. Now assume that for some reason -it will be in our case a consequence of modulation theory and the conservation laws-, ε is indeed for all times orthogonal to the bound states -and resonances...-, then injecting the coercive control of $H(\varepsilon,\varepsilon)$ into (108) yields:

$$\frac{1}{2}\frac{d}{ds}Im(\int y\cdot\nabla\varepsilon\overline{\varepsilon}) \geq \delta_0\left(\int|\nabla\varepsilon|^2 + \int|\varepsilon|^2e^{-|y|}\right). \tag{109}$$

Integrating this in time yields a standard dispersive information: a space time norm is controlled by a norm in space.

We want to apply this strategy to the full ε equation. There are two main obstructions.

First, it is not reasonable to assume that ε is orthogonal to the exact bound states of H. In particular, due to the right hand side in the ε equation, other second order terms will appear which will need be controlled. We thus have to exhibit a set of orthogonality conditions which ensures both the coercivity of the quadratic form H and the control of these other second order interactions. Note that the number of orthogonality conditions we can ensure on ε is the number of symmetries plus the one from b. A first key is the following Spectral Property which has been proved in dimension $N=1$ in [44] using the explicit value of Q and checked numerically for $N=2,3,4$.

Proposition 9 (Spectral Property) *Let $N=1,2,3,4$. There exists a universal constant $\delta_0>0$ such that $\forall\varepsilon=\varepsilon_1+i\varepsilon_2\in H^1$,*

$$\begin{aligned} H(\varepsilon,\varepsilon) &\geq \delta_0\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right)-\frac{1}{\delta_0}\Big\{(\varepsilon_1,Q)^2+(\varepsilon_1,\Lambda Q)^2+(\varepsilon_1,yQ)^2 \\ &+ (\varepsilon_2,\Lambda Q)^2+(\varepsilon_2,\Lambda^2Q)^2+(\varepsilon_2,\nabla Q)^2\Big\}. \end{aligned} \tag{110}$$

To prove this property amounts first counting exactly the number of negative eigenvalues of each Schrödinger operator, and then prove that the specific chosen set of orthogonality conditions, which is not exactly the set of the bound states, is enough to ensure the coercivity of the quadratic form. Both these issues appear to be non trivial when Q is not explicit.

Then, the second major obstruction is the fact that the right hand side $Im(\int y\cdot\nabla\varepsilon\overline{\varepsilon})$ in (109) is an unbounded function of ε in H^1. This is a priori a major obstruction to the strategy, *but an additional non linear algebra inherited from the virial law (31) rules out this difficulty.*

The formal computation is as follows. Given a function $f\in\Sigma$, we let $\Phi(f)=Im(\int y\cdot\nabla f\overline{f})$. According to (108), we want to compute $\frac{d}{ds}\Phi(\varepsilon)$. Now from (106) and the conservation of the energy:

$$\forall t\in[0,T),\quad \Phi(u(t))=4E_0t+c_0$$

for some constant c_0. The key observation is that the quantity $\Phi(u)$ is scaling, phase and also translation invariant from zero momentum assumption (84). Using (94), we get:

$$\forall t\in[0,T),\quad \Phi(\varepsilon+\tilde{Q}_b)=4E_0t+c_0.$$

We now expand this according to:

$$\Phi(\varepsilon+\tilde{Q}_b)=\Phi(\tilde{Q}_b)-2(\varepsilon_2,\Lambda\Sigma)+2(\varepsilon_1,\Lambda\Theta)+\Phi(\varepsilon).$$

A simple algebra yields:

$$\Phi(\tilde{Q}_b)=-\frac{b}{2}|y\tilde{Q}_b|_2^2\sim -Cb$$

for some universal constant $C>0$. Next, from the choice of orthogonality condition (98),

$$(\varepsilon_2,\Lambda\Sigma)-(\varepsilon_1,\Lambda\Theta)=0.$$

We thus get using $\frac{dt}{ds}=\lambda^2$:

$$(\Phi(\varepsilon))_s \sim 4\lambda^2 E_0 + Cb_s.$$

In other words, to compute the a priori unbounded quantity $(\Phi(\varepsilon))_s$ *for the full non linear equation* is from the virial law equivalent to computing the time derivative of b_s, what of course makes now perfectly sense in H^1.

The virial dispersive structure on $u(t)$ in Σ thus induces a dispersive structure in $L^2_{loc}\cap \dot{H}^1$ on $\varepsilon(s)$ for the full non linear equation.

The key dispersive virial estimate is now the following.

Proposition 10 (Local viriel estimate in ε) *There exist universal constants $\delta_0>0$, $C>0$ such that for all $s\geq 0$, there holds:*

$$b_s \geq \delta_0\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) - \lambda^2 E_0 - e^{-\frac{C}{|b|}}. \tag{111}$$

Proof of Proposition 10

Using the heuristics, we can compute in a suitable way b_s using the orthogonality condition (98). The computation -see Lemma 5 in [45]- yields:

$$\begin{aligned} &\frac{1}{4}|yQ|_2^2 b_s = H(\varepsilon,\varepsilon) + 2\lambda^2|E_0| - \frac{x_s}{\lambda}\cdot\{(\varepsilon_2,\nabla\Lambda\Sigma)-(\varepsilon_1,\nabla\Lambda\Theta)\} \qquad (112)\\ - &\left(\frac{\lambda_s}{\lambda}+b\right)\left\{(\varepsilon_2,\Lambda^2\Sigma)-(\varepsilon_1,\Lambda^2\Theta)\right\} - \tilde{\gamma}_s\{(\varepsilon_1,\Lambda\Sigma)+(\varepsilon_2,\Lambda\Theta)\}\\ - &(\varepsilon_1, Re\Lambda\Psi)) - (\varepsilon_2, Im(\Lambda\Psi)) + (l.o.t), \end{aligned}$$

where the lower order terms may be estimated from the smallness of ε in H^1:

$$|l.o.t| \leq \delta(\alpha^*)\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right).$$

We now explain how the choice of orthogonality conditions and the conservation laws allow us to deduce (111).

step 1 Modulation theory for phase and scaling.

The choice of orthogonality conditions (97), (95) has been made to cancel the two second order in ε scalar products in (112):

$$\left(\frac{\lambda_s}{\lambda}+b\right)\left\{(\varepsilon_2,\Lambda^2\Sigma)-(\varepsilon_1,\Lambda^2\Theta)\right\} + \tilde{\gamma}_s\{(\varepsilon_1,\Lambda\Sigma)+(\varepsilon_2,\Lambda\Theta)\}=0.$$

step 2 Elliptic estimate on the quadratic form H.

We now need to control the negative directions in the quadratic form as given by Proposition 9. The directions $(\varepsilon_1, \Lambda Q)$, (ε_1, yQ), $(\varepsilon_2, \Lambda^2 Q)$ and $(\varepsilon_2, \Lambda Q)$ are treated thanks to the choice of orthogonality conditions and the closeness of $\tilde{Q}_b$ to Q for $|b|$ small. For example,

$$\begin{aligned}(\varepsilon_2, \Lambda Q)^2 &= |\{(\varepsilon_2, \Lambda Q - \Lambda\Sigma) + (\varepsilon_1, \Lambda\Theta)\} + (\varepsilon_2, \Lambda\Sigma) - (\varepsilon_1, \Lambda\Theta)|^2 \\ &= |(\varepsilon_2, \Lambda Q - \Lambda\Sigma) + (\varepsilon_1, \Lambda\Theta)|^2\end{aligned}$$

so that

$$(\varepsilon_2, \Lambda Q)^2 \leq \delta(\alpha^*)(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}).$$

Similarly, we have:

$$(\varepsilon_1, yQ)^2 + (\varepsilon_2, \Lambda^2 Q)^2 + (\varepsilon_1, \Lambda Q)^2 \leq \delta(\alpha^*)(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}). \tag{113}$$

The negative direction $(\varepsilon_1, Q)^2$ is treated from the conservation of the energy which implied (102). The direction $(\varepsilon_2, \nabla Q)$ is treated from the zero momentum condition which ensured (103). Putting this together yields:

$$(\varepsilon_1, Q)^2 + (\varepsilon_2, \nabla Q)^2 \leq \delta(\alpha^*)\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} + \lambda^2 |E_0|\right) + e^{-\frac{C}{|b|}}.$$

step 3 Modulation theory for translation and use of Galilean invariance.

The Galilean invariance has been used to ensure the zero momentum condition (84) which in turn led together with the choice of orthogonality condition (96) to the degeneracy estimate (105):

$$|\frac{x_s}{\lambda}| \leq C\delta(\alpha^*)(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|})^{\frac{1}{2}} + e^{-\frac{C}{|b|}}.$$

Therefore, we estimate the term induced by translation invariance in (112) as

$$\left|\frac{x_s}{\lambda} \cdot \{(\varepsilon_2, \nabla\Lambda\Sigma) - (\varepsilon_1, \nabla\Lambda\Theta)\}\right| \leq C\delta(\alpha^*)\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) + e^{-\frac{C}{|b|}}.$$

step 4 Conclusion.

Injecting these estimates into the elliptic estimate (110) yields so far:

$$b_s \geq \delta_0\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) - 2\lambda^2 E_0 - e^{-\frac{C}{|b|}} - \frac{1}{\delta_0}(\lambda^2 E_0)^2.$$

We now use in a crucial way *the sign of the energy* $E_0 < 0$ and the smallness $\lambda^2 |E_0| \leq \delta(\alpha^*)$ which is a consequence of the conservation of the energy to conclude. This ends the proof of Proposition 10.

4.5 Monotonicity and control of the blow up speed

The virial dispersive estimate (111) means a control of the excess of mass ε by an exponentially small correction in b in time averaging sense. More specifically, this means that in rescaled variables, the solution writes $\tilde{Q}_b + \varepsilon$ where $\tilde{Q}_b$ is the regular deformation of Q and the rest is in a suitable norm exponentially small in b. This is thus an expansion of the solution with respect to an internal parameter in the problem: b.

This virial control is the first dispersive estimate for the infinite dimensional dynamic driving ε. Observe that it means little by itself if nothing is known about $b(t)$. We shall now inject this information into the finite dimensional dynamic driving the geometrical parameters. The outcome will be *a rigidity property for the parameter $b(t)$ which will in turn imply the existence of a Lyapounov functional in the problem.* This step will again heavily rely on the conservation of the energy.

We start with exhibiting the rigidity property which proof is a maximum principle type of argument.

Proposition 11 (Rigidity property for b) *$b(s)$ vanishes at most once on $\mathbf{R}_+$.*

Note that the existence of a quantity with prescribed sign in the description of the dynamic is unexpected. Indeed, b is no more then the projection of some a priori highly oscillatory function onto a prescribed direction. It is a very specific feature of the blow up dynamic that this projection has a fixed sign.

Proof of Proposition 11

Assume that there exists some time $s_1 \geq 0$ such that $b(s_1) = 0$ and $b_s(s_1) \leq 0$, then from (111), $\varepsilon(s_1) = 0$. Thus from the conservation of the L^2 norm and $\tilde{Q}_{b(s_1)} = Q$, we conclude $\int |u_0|^2 = \int Q^2$ what contradicts the strictly negative energy assumption. This concludes the proof of Proposition 11.

The next step is to get the exact sign of b. This is done by injecting the virial dispersive information (111) into the modulation equation for the scaling parameter what will yield

$$-\frac{\lambda_s}{\lambda} \sim b. \tag{114}$$

The key rigidity property is the following:

Proposition 12 (Rigidity of the flow) *There exists a time $s_0 \geq 0$ such that*

$$\forall s > s_0, \quad b(s) > 0.$$

Moreover, the size of the solution is in this regime an almost Lyapounov functional in the sense that:

$$\forall s_2 \geq s_1 \geq s_0, \quad \lambda(s_2) \leq 2\lambda(s_1). \tag{115}$$

Proof of Proposition 12

step 1 Equation for the scaling parameter.

The modulation equation for the scaling parameter λ inherited from choice of orthogonality condition (95) implied control (104):

$$\left|\frac{\lambda_s}{\lambda} + b\right| \leq C\left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)^{\frac{1}{2}} + e^{-\frac{C}{|b|}},$$

which implies (114) in a weak sense. Nevertheless, this estimate is not good enough to possibly use the virial estimate (111). We claim using extra degeneracies of the equation that (104) can be improved for:

$$\left|\frac{\lambda_s}{\lambda} + b\right| \leq C\left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) + e^{-\frac{C}{|b|}} \tag{116}$$

step 2 Use of the virial dispersive relation and the rigidity property.

We now inject the virial dispersive relation (111) into (116) to get:

$$\left|\frac{\lambda_s}{\lambda} + b\right| \leq C b_s + e^{-\frac{C}{|b|}}.$$

We integrate this inequality in time to get: $\forall 0 \leq s_1 \leq s_2$,

$$\left|\log\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right) + \int_{s_1}^{s_2} b(s)ds\right| \leq \frac{1}{4} + \int_{s_1}^{s_2} e^{-\frac{C}{|b(s)|}} ds. \tag{117}$$

The key is now to use the rigidity property of Proposition 11 to ensure that $b(s)$ has a fixed sign for $s \geq \tilde{s}_0$, and thus: $\forall s \geq \tilde{s}_0$,

$$\left|\int_{s_1}^{s_2} e^{-\frac{C}{|b(s)|}} ds\right| \leq \frac{1}{2}\left|\int_{s_1}^{s_2} b(s)ds\right|. \tag{118}$$

step 3 b is positive for s large enough.

Assume that $\left|\int_0^{+\infty} b(s)ds\right| < +\infty$, then b has a fixed sign for $s \geq \tilde{s}_0$ and $|b_s| \leq C$, and thus: $b(s) \to 0$ as $s \to +\infty$. Now from (117) and (118), this implies that $|\log(\lambda(s))| \leq C$ as $s \to +\infty$, and in particular $\lambda(s) \geq \lambda_0 > 0$ for s large enough. Injecting this into virial control (111) for s large enough yields:

$$b_s \geq \frac{1}{2}|E_0|\lambda_0^2.$$

Integrating this on large time intervals contradicts the uniform boundedness of b. Here we have used again the assumption $E_0 < 0$.
We thus have proved: $\left|\int_0^{+\infty} b(s)ds\right| = +\infty$. Now assume that $b(s) < 0$ for all $s \geq \tilde{s}_1$, then from (117) and (118) again, we conclude that $\log(\lambda(s)) \to 0$ as $s \to +\infty$. Now from $\lambda(t) \sim \frac{1}{|\nabla u(t)|_{L^2}}$, this yields $|\nabla u(t)|_{L^2} \to 0$ as $t \to T$. But from Gagliardo-Nirenberg inequality and the conservation of the energy and the L^2 mass, this implies $E_0 = 0$, contradicting again the assumption $E_0 < 0$.

step 4 Almost monotonicity of the norm.

We now are in position to prove (115). Indeed, injecting the sign of b into (117) and (118) yields in particular: $\forall s_0 \leq s_1 \leq s_2$,

$$\frac{1}{4} + \frac{1}{2}\int_{s_1}^{s_2} b(s)ds \leq -\log\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right) \leq \frac{1}{4} + 2\int_{s_1}^{s_2} b(s)ds, \tag{119}$$

and thus:

$$\forall s_0 \leq s_1 \leq s_2, \quad -\log\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right) \geq \frac{1}{4},$$

what yields (115). This concludes the proof of Proposition 12.

Note that from the above proof, we have obtained $\int_0^{+\infty} b(s)ds = +\infty$, and thus from (119):

$$\lambda(s) \to 0 \quad \text{as} \quad s \to \infty, \tag{120}$$

that is finite or infinite time blow up. *On the contrary to the virial argument, the blow up proof is no longer obstructive but completely dynamical, and relies mostly on the rigidity property of Proposition 11.*

Let us now conclude the proof of Theorem 9. We need to prove finite time blow up together with the log-log upper bound (73) on blow up rate. The proof goes as follows.

step 1 Lower bound on $b(s)$.

We claim: there exist some universal constant $C > 0$ and some time $s_1 > 0$ such that $\forall s \geq s_1$,

$$Cb(s) \geq \frac{1}{\log|\log(\lambda(s))|}. \tag{121}$$

Indeed, first recall (111). Now that we know the sign of $b(s)$ for $s \geq s_0$ from Proposition 12, and we may thus view (111) as a differential inequality for b for $s > s_0$:

$$b_s \geq -e^{-\frac{C}{b}} \geq -b^2 e^{-\frac{C}{2b}} \quad \text{ie} \quad -\frac{b_s}{b^2} e^{\frac{C}{2b}} \leq 1.$$

We integrate this inequality from the non vanishing property of b and get for $s \geq \tilde{s}_1$ large enough:

$$e^{\frac{C}{b(s)}} \leq s + e^{\frac{C}{b(1)}} \leq 2Cs \quad \text{ie} \quad b(s) \geq \frac{C}{\log(s)}. \tag{122}$$

We now recall (119) on the time interval $[\tilde{s}_1, s]$:

$$\frac{1}{2}\int_{\tilde{s}_1}^{s} b \leq -\log\left(\frac{\lambda(s)}{\lambda(\tilde{s}_1)}\right) + \frac{1}{4} \leq -2\log(\lambda(s))$$

for $s \geq \tilde{s}_2$ large enough from $\lambda(s) \to 0$ as $s \to +\infty$. Inject (122) into the above inequality, we get for $s \geq \tilde{s}_3$

$$C\frac{s}{\log(s)} \leq \int_{\tilde{s}_2}^{s} \frac{Cd\tau}{\log(\tau)} \leq \frac{1}{4}\int_{\tilde{s}_2}^{s} b \leq -\log(\lambda(s)) \quad \text{ie} \quad |\log(\lambda(s))| \geq C\frac{s}{\log(s)}$$

for some universal constant $C > 0$, and thus for s large

$$\log|\log(\lambda(s))| \geq \log(s) - \log(\log(s)) \geq \frac{1}{2}\log(s)$$

and conclusion follows from (122). This concludes the proof of (121).

step 2 Finite time blow up and control of the blow up speed.

We first use the finite or infinite time blow up result (120) to consider a sequence of times $t_n \to T \in [0, +\infty]$ defined for n large such that

$$\lambda(t_n) = 2^{-n}.$$

Let $s_n = s(t_n)$ the corresponding sequence and $\bar{t}$ such that $s(\bar{t}) = s_0$ given by Proposition 12. Note that we may assume $n \geq \overline{n}$ such that $t_n \geq \bar{t}$. Remark that $0 < t_n < t_{n+1}$ from (115), and so $0 < s_n < s_{n+1}$. Moreover, there holds from (115)

$$\forall s \in [s_n, s_{n+1}], \quad 2^{-n-1} \leq \lambda(s) \leq 2^{-(n-1)}. \tag{123}$$

We now claim that (73) follows from a control from above of the size of the intervals $[t_n, t_{n+1}]$ for $n \geq \overline{n}$.
Let $n \geq \overline{n}$. (121) implies

$$\int_{s_n}^{s_{n+1}} \frac{ds}{\log|\log(\lambda(s))|} \leq C\int_{s_n}^{s_{n+1}} b(s)ds.$$

(119) with $s_1 = s_n$ and $s_2 = s_{n+1}$ yields:

$$\frac{1}{2}\int_{s_n}^{s_{n+1}} b(s) \leq \frac{1}{4} - |yQ|_{L^2}^2 \log\left(\frac{\lambda(s_{n+1})}{\lambda(s_n)}\right) \leq C.$$

Therefore,

$$\forall n \geq \overline{n}, \quad \int_{s_n}^{s_{n+1}} \frac{ds}{\log|\log(\lambda(s))|} \leq C.$$

Now we change variables in the integral at the left of the above inequality according to $\frac{ds}{dt} = \frac{1}{\lambda^2(s)}$ and estimate with (123):

$$C \geq \int_{s_n}^{s_{n+1}} \frac{ds}{\log|\log(\lambda(s))|} = \int_{t_n}^{t_{n+1}} \frac{dt}{\lambda^2(t)\log|\log(\lambda(t))|} \geq \frac{1}{10\lambda^2(t_n)\log|\log(\lambda(t_n))|} \int_{t_n}^{t_{n+1}} dt$$

so that

$$t_{n+1} - t_n \leq C\lambda^2(t_n)\log|\log(\lambda(t_n))|.$$

From $\lambda(t_n) = 2^{-n}$ and summing the above inequality in n, we first get

$$T < +\infty$$

and

$$\begin{aligned}
C(T - t_n) &\leq \sum_{k \geq n} 2^{-2k}\log(k) = \sum_{n \leq k \leq 2n} 2^{-2k}\log(k) + \sum_{k \geq 2n} 2^{-2k}\log(k) \\
&\leq C2^{-2n}\log(n) + 2^{-4n}\log(2n) \sum_{k \geq 0} 2^{-2k} \frac{\log(2n+k)}{\log(2n)} \\
&\leq C2^{-2n}\log(n) + C2^{-4n}\log(n) \leq C2^{-2n}\log(n) \leq C\lambda^2(t_n)\log|\log(\lambda(t_n))|.
\end{aligned}$$

From the monotonicity of λ (115), we extend the above control to the whole sequence $t \geq \overline{t}$. Let $t \geq \overline{t}$, then $t \in [t_n, t_{n+1}]$ for some $n \geq \overline{n}$, and from $\frac{1}{2}\lambda(t_n) \leq \lambda(t) \leq 2\lambda(t_n)$, we conclude

$$\lambda^2(t)\log|\log(\lambda(t))| \geq C\lambda^2(t_n)\log|\log(\lambda(t_n))| \geq C(T - t_n) \geq C(T - t).$$

Now remark that the function $f(x) = x^2\log|\log(x)|$ is non decreasing in a neighborhood at the right of $x = 0$, and moreover

$$f\left(\frac{C}{2}\sqrt{\frac{T-t}{\log|\log(T-t)|}}\right) = \frac{C^2}{4}\frac{(T-t)}{\log|\log(T-t)|}\log\left|\log\left(C\sqrt{\frac{T-t}{\log|\log(T-t)|}}\right)\right| \leq C(T-t)$$

for t close enough to T, so that we get for some universal constant C^*:

$$f(\lambda(t)) \geq f\left(C^*\sqrt{\frac{T-t}{\log|\log(T-t)|}}\right) \quad \text{ie} \quad \lambda(t) \geq C^*\sqrt{\frac{T-t}{\log|\log(T-t)|}}$$

and (73) is proved.

This concludes the proof of Theorem 9.

5 Log-log lower bound on the blow up rate

We outline in this section the proof of the log-log lower bound on blow up rate (78). We will first prove a weaker result which is the nonexistence of self similar solutions which relies on an obstructive argument and the reduction to a Liouville theorem. We shall stress here the specific role played by *zero energy* solutions. We then outline the main steps of the proof of the sharp log-log lower bound which relies onto a dynamical understanding of the L^2 conservation law.

5.1 Reduction to a Liouville theorem

We outline in this subsection the proof of Theorem 11. We in particular stress the role of Liouville theorems which amounts classifying the *non dispersive dynamics of the flow.* The reference for more details is [46]

For simplicity, let us consider an initial data $u_0 \in H^1$ with

$$Im(\int \nabla u_0 \overline{u_0}) = 0, \quad E(u_0) < 0. \tag{124}$$

Then from Theorem 9, $u(t)$ blows up in finite time $0 < T < +\infty$ with a log-log upper bound on blow up rate. Let us introduce the geometrical decomposition

$$u(t,x) = \frac{1}{\lambda_u^{\frac{N}{2}}}(Q + \varepsilon_u)(t, \frac{x - x_u(t)}{\lambda_u(t)})e^{i\gamma_u(t)}$$

where $\varepsilon_u(t)$ satisfies the orthogonality conditions (95), (96), (97) with $b = 0$. At this stage we do not need the refined modulation on b. We need to prove

$$\varepsilon_u(t) \to 0 \text{ as } t \to T \text{ in } L^2_{loc}.$$

We argue by contradiction and assume that there exists a sequence $t_n \to T$ such that

$$\lambda_u(t_n)^{\frac{N}{2}} u(t_n, \lambda_u(t_n)x + x_u(t_n))e^{-i\gamma_u(t_n)} \rightharpoonup v(0) \text{ in } H^1 \text{ and } v(0) \neq Q.$$

The whole problem reduces to the following: what information can we extract from the knowledge that $v(0)$ is obtained as a limit of a renormalized sequence of a blow up solution? The answer is somewhat universal: $v(0)$ generates *a non dispersive solution* to the (NLS) flow. The Liouville theorem to prove then becomes roughly: the only non dispersive solution of the flow near Q is the solitary wave, and the contradiction will follow.

step1 Stability of weak convergence.

Let the renormalized object

$$u_n(0,x) = \lambda_u(t_n)^{\frac{N}{2}} u(t_n, \lambda_u(t_n)x + x_u(t_n))e^{-i\gamma_u(t_n)},$$

then

$$|\nabla u_n(0)|_{L^2} \sim |\nabla Q|_{L^2}$$

and our assumption is

$$u_n(0) \rightharpoonup v(0) \quad \text{in} \quad H^1 \quad \text{and} \quad v(0) \neq Q.$$

Let us then consider $u_n(\tau, x)$ and $v(\tau)$ respectively the solutions to (NLS) with initial data $u_n(0)$ and $v(0)$. Then first by rescaling

$$u_n(\tau, x) = \lambda_u(t_n)^{\frac{N}{2}} u(t_n + \lambda_u^2(t_n)\tau, \lambda_u(t_n)x + x_u(t_n))e^{-i\gamma_u(t_n)}. \tag{125}$$

In particular, the life time of u_n is

$$T_n = \frac{T - t_n}{\lambda_u^2(t_n)}. \tag{126}$$

Let $(T_v^-, T_v^+) \in [-\infty, 0] \times [0, +\infty]$ be the lifetimes of v respectively on the left and on the right in time. A very simple general Lemma -which has nothing to do with blow up but simply with the H^1 local Cauchy theory- is the stability of the weak convergence in the following sense:

$$T_v \leq \liminf_{n \to +\infty} T_n \tag{127}$$

and

$$\forall \tau \in [0, T_v), \quad u_n(\tau) \rightharpoonup v(\tau) \quad \text{in} \quad H^1. \tag{128}$$

Equivalently, if we introduce the geometrical decompositions

$$u_n(\tau, x) = \frac{1}{\lambda_n^{\frac{N}{2}}}(Q + \varepsilon_n)(\tau, \frac{x - x_n(\tau)}{\lambda_n(\tau)})e^{i\gamma_n(\tau)}, \quad v(\tau, x) = \frac{1}{\lambda_v^{\frac{N}{2}}}(Q + \varepsilon_v)(\tau, \frac{x - x_v(\tau)}{\lambda_v(\tau)})e^{i\gamma_v(\tau)},$$

then: $\forall \tau \in (T_v^- 0, T_v^+)$,

$$(\lambda_n(\tau), x_n(\tau), \gamma_n(\tau)) \to (\lambda_v(\tau), x_v(\tau), \gamma_v(\tau)) \tag{129}$$

and

$$\varepsilon_n(\tau) \rightharpoonup \varepsilon_v(\tau) \quad \text{in} \quad H^1.$$

The whole question is now to understand what specific *qualitative dynamical* informations hold on v as a consequence of the fact that is extracted as a limit of a blow up dynamic.

step 2 Control of the invariants of v.

A first very simple observation is that

$$\int Q^2 \leq \int |v(0)|^2 \leq \int |Q|^2 + \alpha^*, \quad E(v(0)) \leq 0. \tag{130}$$

First observe that $v(0)$ is non zero. Indeed, from $\lambda_n(0) \to 0$ as $n \to +\infty$, we can find $A > 0$ such that:

$$\forall n \geq 0, \quad \int_{|x|\geq A} |u_n(0)|^2 = \int_{|\lambda_n(0)y+x_n(0)|\geq A} |\varepsilon_n(0) + Q|^2(y),$$

so that for $\alpha^* > 0$ small enough: $\forall n \geq 0$, $\int_{|x|\leq A} |u_n(0)|^2 \geq \frac{3}{4}\int Q^2$, and by (128):

$$\frac{3}{4}\int Q^2 \leq \int |v(0)|^2 \leq \int Q^2 + \alpha^*, \tag{131}$$

and $v(0)$ is non zero. To control the energy, let us first observe from (124) that

$$E(u_n(0)) = \lambda^2(t_n)E(u(0)) \leq 0. \tag{132}$$

Let us now define a radial cut-off function ρ such that

$$0 \leq \rho \leq 1, \ \rho(x) = 1 \text{ for } |x| \leq 1, \ \rho(x) = 0 \text{ for } |x| \geq 2, \ \sqrt{\rho}, \sqrt{1-\rho} \in \mathcal{C}^2,$$

and for $k \in \mathbf{N}$, $\rho_k(x) = \rho(\frac{|x|}{k})$. First by direct computation:

$$E(u_n(0)) = E(u_n(0)\sqrt{\rho_k}) + E(u_n(0)\sqrt{1-\rho_k}) + R_{n,k}$$

whit $R_{n,k} = -\frac{1}{8}\int |u_n(0)|^2|\nabla\rho_k|^2\left(\frac{1}{\rho_k} + \frac{1}{1-\rho_k}\right) + \frac{1}{2+\frac{4}{N}}\int |u_n(0)|^{2+\frac{4}{N}}(\rho_k^{1+\frac{2}{N}} + (1-\rho_k)^{1+\frac{2}{N}} - 1)$, so that $R_{n,k} \to R_k = -\frac{1}{8}\int |v(0)|^2|\nabla\rho_k|^2\left(\frac{1}{\rho_k} + \frac{1}{1-\rho_k}\right) + \frac{1}{2+\frac{4}{N}}\int |v(0)|^{2+\frac{4}{N}}(\rho_k^{1+\frac{2}{N}} + (1-\rho_k)^{1+\frac{2}{N}} - 1)$ as $n \to +\infty$. Second, observe that $\forall n \geq 0$, $|u_n(0) - Q|_{L^2} \leq \frac{1}{2}|Q|_{L^2}$, and thus, there exists $k_0 > 0$ such that $\forall k > k_0$ and $\forall n \geq 0$, $|u_n(0)\sqrt{1-\rho_k}|_{L^2} \leq |Q|_{L^2}$. This implies by Gagliardo–Nirenberg inequality $E(u_n(0)\sqrt{1-\rho_k}) \geq 0$. We now pass to the limit $n \to +\infty$ using (132) and obtain

$$\overline{\lim}_{n\to\infty}E(u_n(0)\sqrt{\rho_k}) + R_k \leq 0.$$

But $\overline{\lim}_{n\to\infty}E(u_n(0)\sqrt{\rho_k}) \geq E(v(0)\sqrt{\rho_k})$ so that $E(v(0)\sqrt{\rho_k}) + R_k \leq 0$. Letting $k \to +\infty$ yields $E(v(0)) \leq 0$. (130) now follows from the variational characterization of Q and (131).

step 3 v has zero energy and momentum.

We now claim the first nontrivial information on the asymptotic object v:

$$E(v) = 0 \ \text{ and } \ Im(\int \nabla v\overline{v}) = 0. \tag{133}$$

Indeed, we compute:

$$\lambda_n(\tau) = \frac{\lambda_u(t_n + \lambda_u^2(t_n)\tau)}{\lambda_u(t_n)}.$$

We then use the *monotonicity* of λ_u in the log-log regime, see (115), which implies:

$$\forall \tau \leq 0, \quad \lambda_n(\tau) \geq \frac{1}{2}.$$

Passing to the limit $n \to +\infty$ in (129), we get:

$$\forall \tau \in (T_v^-, 0], \quad \lambda_v(\tau) \geq \frac{1}{2} \text{ and thus } |\nabla v(\tau)|_{L^2} \leq C.$$

This implies that v is globally defined on the left in time

$$T_v^- = -\infty. \tag{134}$$

But from Theorem 9 and (72), this implies:

$$E_G(u) = E(u) - \frac{1}{2}\left(\frac{Im(\int \nabla u \overline{u})}{|u|_{L^2}}\right)^2 \geq 0$$

and now (130) implies (133).
Observe that as announced, we are reduced to understanding zero energy solutions.

step 4 v blows up in finite time on the right in time.

We now reach the core of the argument and claim that v blows up on the right in time:

$$T_v^+ < +\infty. \tag{135}$$

This is a completely non trivial information which relies on the results of section 3. The following preliminary Liouville type theorem may indeed be extracted from this analysis, see [46], and is very much a direct consequence of the rigidity of the flow provided by Proposition 11.

Proposition 13 (First Liouville Theorem) *Let $v(0) \in \mathcal{B}(\alpha^*)$ with*

$$E(v_0) = 0 \quad \textit{and} \quad Im(\int \nabla v_0 \overline{v_0}) = 0.$$

If $v(0) \neq Q$, then v blows up on the right **or** *on the left in time.*

(134) and (133) now yield (135).

step 5 Continuity of the blow up time.

We may now apply the log-log analysis to v which blows up in finite time and in particular obtain a geometrical decomposition

$$v(\tau, x) = \frac{1}{\lambda_v^{\frac{N}{2}}}(Q_{b(\tau)} + \varepsilon_v)(\tau, \frac{x - x_v(\tau)}{\lambda_v(\tau)})e^{i\gamma_v(\tau)}.$$

Proposition 11 will then yield a time τ_0 such that

$$b_v(\tau_0) > 0.$$

But then from the stability of weak convergence, we will conclude that

$$\forall n \geq n(\tau_0), \quad \forall \tau \geq \tau_0, \quad b_n(\tau) > 0.$$

But then again the control of the singularity formation provided by the log-log analysis will allow us to control the blow up time T_n of u_n with respect to the blow up time of T_v. In other words, because the log-log analysis provides a control of the blow up dynamics by quantities which are local in space and pass to the weak limit, we can prove that *the blow up time is a continuous function of the initial data* in a strong enough sense. We may thus sharpen (127) and prove:

$$T_n \to T_v \quad \text{as} \quad n \to +\infty. \tag{136}$$

Let us insist again that this claim is non trivial and uses the whole log-log machinery.

Remark 7 *From (126) and (136), we conclude that*

$$\lambda_u(t_n) \geq C_v \sqrt{T_u - t_n}$$

and thus together with the scaling lower bound (44):

$$|\nabla u(t_n)|_{L^2} \sim \frac{C}{\lambda_u(t_n)} \sim \frac{C}{\sqrt{T - t_n}}.$$

We thus see the intimate link between asymptotic objects and self similar blow up dynamics.

step 6 Existence of the limiting focusing measure.

Let us introduce the limiting measure of a blow up solution which is the object which will allow us to measure in which sense v is a non dispersive object:

Lemma 4 (Existence of a weak focusing measure at blow-up time) *Let $u_0 \in \mathcal{B}(\alpha^*)$ with $E(u_0) \leq 0$ and assume that the corresponding solution $u(t)$ blows up in finite time $0 < T < +\infty$. Then there exists a continuous function $x(t) : [0, T[\to \mathbf{R}$ such that the limiting focusing measure*

$$\mu(u) = \lim_{t \to T} |u|^2(t, x + x(t))$$

is well defined, and the concentration point has a finite limit at blow-up time, ie $x(t) \to x(T)$ as $t \to T$ for some finite $x(T) \in \mathbf{R}^N$.

Proof of Lemma 4

Let an initial condition $u_0 \in \mathcal{B}(\alpha^*)$ with $E(u_0) \leq 0$ and assume that the corresponding solution blows up in finite time T. Note that the upper bound (73) holds from Theorem 9. Let the geometrical decomposition $\varepsilon(t,y) = e^{i\gamma(t)}\lambda^{\frac{N}{2}}(t)u(t,\lambda(t)y + x(t)) - Q(y)$. Let $\Psi \in \mathcal{C}_0^\infty(\mathbf{R})$ real valued and compute

$$\begin{aligned}\left|\frac{d}{dt}\int \Psi(x)|u|^2(t,x+x(t))dx\right| &= \left|-x_t \cdot \int \nabla\Psi(x)|u|^2(t,x+x(t))dx\right.\\ &+ \left. 2Im\left(\int \nabla\Psi(x)\cdot\nabla u\overline{u}(t,x+x(t))dx\right)\right|\\ &\leq C_\Psi\left(|x_t| + |\nabla u(t)|_{L^2}\right).\end{aligned}$$

Now observe from (105) that $\forall s \in [0,+\infty)$, $|\frac{x_s}{\lambda}| \leq C$ for some universal constant C, so that from $\frac{ds}{dt} = \frac{1}{\lambda^2}$, there holds for t close enough to T: $|x_t| \leq C|\nabla u(t)|_{L^2}$. It thus suffices to observe from (73) that

$$\int_0^T |\nabla u(\tau)|_{L^2}d\tau < +\infty.$$

This concludes the proof of Lemma 4.

step 7 v does not disperse.

We now come to the very heart of the proof. The main information that we can extract from the fact that v is asymptotic to a blow up dynamic is that *its limiting measure is a Dirac mass*, or in other words:

$$\mu(v) = \left(\int |v(0)|^2\right)\delta_{x=0}. \tag{137}$$

Proof of (137): The key is to observe that the limiting focusing measure is again a continuous function of the initial data which passes to the weak limit. In other words, let μ_n be the limiting focusing measure associated to u_n, then:

$$\mu(u_n) \rightharpoonup \mu(v) \tag{138}$$

in the weak sense of measures. Indeed, fix $\phi \in \mathcal{C}_0^\infty(\mathbf{R})$ real valued and let $x_n(T_n)$ be the finite concentration point of $u_n(t)$. First estimate as for the proof of Proposition 4: $\forall t \in [0,T_n)$,

$$|(x_n)_t| + \left|\frac{d}{dt}\int \phi(x)|u_n|^2(t,x+x_n(t))dx\right| \leq C_\phi|\nabla u_n(t)|_{L^2}.$$

Thus, from the stability of weak convergence and the continuity of blow time (136), we get: $\forall t \in [0,T_v)$

$$\begin{aligned}\limsup_{n\to+\infty}|x_n(T_n) - x_v(T_v)| &\leq \limsup_{n\to+\infty}\left(|x_n(T_n) - x_n(t)| + |x_n(t) - x_v(t)| + |x_v(t) - x_v(T_v)|\right)\\ &\leq 2C_v(T_v - t)^{\frac{1}{4}}.\end{aligned}$$

Letting $t \to T_v$ ensures $x_n(T_n) \to x_v(T_v)$ as $n \to +\infty$. The continuity of $\mu_n(\phi)$ follows similarily.
It now remains to identify the limit of μ_n, and here we use the fact that u_n is just a rescaled version of a blowing up u. Indeed, we claim:

$$\forall R > 0, \quad \forall n \geq 0, \quad \mu_n(1_{|x|\leq R}) = \mu_u(1_{|x|\leq \lambda_u(t_n)R}). \tag{139}$$

To wit, first compute from (125): $\forall \tau \in (-T_n^-, T_n^+)$,

$$x_n(\tau) = \frac{1}{\lambda_u(t_n)} \left\{ x_u(t_n + \lambda_u(t_n)^2\tau) - x_u(t_n) \right\},$$

so that:

$$\begin{aligned}
\int_{|x|\leq R} |u_n|^2(\tau, x + x_n(\tau)) &= \int_{|x|\leq R} \lambda_u(t_n)|u|^2(t_n + \lambda_u^2(t_n)\tau, \lambda_u(t_n)(x + x_n(\tau)) + x_u(t_n)) \\
&= \int_{|x|\leq R} \lambda_u(t_n)|u|^2(t_n + \lambda_u^2(t_n)\tau, \lambda_u(t_n)x + x_u(t_n + \lambda_u^2(t_n)\tau)) \\
&= \left[\int_{|x|\leq \lambda_u(t_n)R} |u|^2(t, x + x_u(t)) \right] \left(t = t_n + \lambda_u^2(t_n)\tau \right) \\
&\to \mu_u(1_{\lambda_u(t_n)R}) \quad \text{as} \quad \tau \to T_n^+
\end{aligned}$$

from $T_n^+ = \frac{T_u^+ - t_n}{\lambda_u^2(t_n)}$, and (139) follows.
Observe now that the function $f(r) = \mu_u(1_{|x|\leq r})$ is a nondecreasing positive function of $r > 0$, and let $m = lim_{r\to 0} f(r)$, then (139), $\lambda_u(t_n) \to 0$ as $n \to +\infty$ and (138) imply:

$$\mu(v) = m\delta_{x=0}. \tag{140}$$

Now using that for any function ψ, there holds:

$$\left| \frac{d}{d\tau} \int \psi |v|^2 \right| \leq C_\psi |\nabla v|_{L^2} \quad \text{and} \quad \int_0^T |\nabla v(\tau)|_{L^2} < +\infty,$$

we conclude that $v(\tau)$ is L^2 compact in the sense that

$$\forall \eta > 0, \quad \exists A > 0 \quad \text{such that} \quad \forall \tau \in [0, T_v^+), \quad \int_{|x - x_v(t)|\geq A} |v(\tau)|^2 \leq \eta.$$

Thus (140) and the conservation of the L^2 norm force $v(\tau)$ to accumulate its whole L^2-mass into blow-up and (137) follows.

5.2 An overview of the Liouville theorem

(137) expresses the fact that the asymptotic object v does not disperse because it accumulates all its L^2 mass into blow up. Another solution with this behavior is actually the

critical mass blow up solution. Our Liouville theorem which finishes the proof of Theorem 11 states that this solution is actually the only one in $\mathcal{B}(\alpha^*)$. In other words, it is the super critical mass version of the classification of the critical mass blow up solution by Merle [40]:

Theorem 16 (Liouville theorem) *Let $v(t) \in \mathcal{B}(\alpha^*)$ be a finite time blow up solution to (32) which is non dispersive in the sense that*

$$\mu(v) = \left(\int |v(0)|^2\right)\delta_{x=0}, \tag{141}$$

then $v(t) = S(t)$ the critical mass blow up solution up to the symmetries of the flow.

This concludes the proof of Theorem 11 because $E(S) > 0$ while $E(v) \leq 0$, a contradiction.

We shall not give the details of the proof of Theorem 16 and refer to [46], part B. We simply want to reformulate the problem in a more manageable frame.

Let v as in the hypothesis of Theorem 16. Up to a translation in space, we may assume that v blows up at $x(T) = 0$. First one can easily deduce from (141) by integrating the L^2 fluxes of mass backwards from the singularity -using the upper bound (73)-:

$$v(t) \in \Sigma \ \text{ with } \int |x|^2|v|^2 \to 0 \ \text{ as } t \to T. \tag{142}$$

This corresponds to a gain of regularity on the asymptotic object. From this, one can use the *pseudo conformal symmetry* to map v to a *global in time zero energy solution to (32).* This is a purely algebraic computation where the variance estimate (142) plays a crucial role. Then the proof reduces to the following dynamical classification of the solitary wave:

Theorem 17 (Dynamical classification of the solitary wave) *Let $u_0 \in \mathcal{B}(\alpha^*) \cap \Sigma$ with*

$$E(u_0) = 0.$$

Assume that

$$u_0 \neq Q,$$

then the corresponding solution to (32) blows up in finite time both on the left **and** *on the right in time.*

In other words, the solitary wave is in $\mathcal{B}(\alpha^*)$ the only zero energy solution to (32) in Σ which leaves on an infinite time interval.

As we have seen, the zero energy condition naturally emerges for asymptotic objects. The main difficulty with zero energy solutions and that there may exist a dynamic where the solution is global and obeys a nonlinear vanishing

$$|\nabla u(t)|_{L^2} \to 0 \ \text{ as } \ t \to +\infty.$$

Such a dynamic can be proved to necessarily occur at the scaling rate

$$|\nabla u(t)|_{L^2} \sim \frac{C}{\sqrt{t}}$$

but is very delicate to rule out. This relies on very fine dispersive properties of the flow. Actually, it is the comprehension of this Liouville theorem which allows one to sharpen the analysis once more and derive the exact log-log blow up speed.

5.3 The sharp log-log law

This section is devoted to the proof of the sharp log-log law (79) of Theorem 13. For the sake of simplicity, we restrict ourselves to the case

$$E_0 < 0, \quad Im(\nabla u_0 \overline{u_0}) = 0.$$

Our analysis is just in the straight continuation of section 4.
Let us summarize the results of section 4 as follows. We have a decomposition

$$u(t,x) = \frac{1}{\lambda^{\frac{N}{2}}(t)}(Q_{b(t)} + \varepsilon)(t, \frac{x - x(t)}{\lambda(t)})e^{i\gamma(t)}$$

with the following bounds:

- Dispersive control of ε in time averaging sense:

$$b_s \geq C\left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) - e^{-\frac{C}{|b|}}. \tag{143}$$

- Lower bound on b for s large enough:

$$b(s) \geq \frac{C}{\log s}. \tag{144}$$

- The asymptotic stability of Q as a blow up profile, Theorem 11, may be rephrased as follows:

$$b(s) \to 0 \ \text{ as } \ s \to +\infty. \tag{145}$$

To prove the sharp exact log-log law, we need two new estimates:

- A pointwise lower bound on b which quantifies (145):

$$b(s) \leq \frac{C}{\log s}. \tag{146}$$

- A pointwise upper bound on ε which improves the averaged control (143):

$$\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \leq e^{-\frac{C}{b}}. \tag{147}$$

5.4 The outgoing radiation

The basic intuition is that we need to refine the dispersive estimate (143) to sharpen our understanding of the law for b. In particular, the first question we may ask is: where does the negative contribution $-e^{-\frac{C}{b}}$ in (143) come from?

This is easily answered by looking at the equation for ε which roughly speaking has a structure:

$$i\partial_s \varepsilon + L\varepsilon + ib\Lambda\varepsilon \sim -\Psi.$$

The right hand side Ψ is the leading term which contains the exponentially small error in b. Let us recall that this term is generated by the cutting procedure in the construction of the profiles $\tilde{Q}_b$, Proposition 8, and is a reflection of the fact that the solutions to the self similar equation

$$\Delta Q_b - Q_b + ib\Lambda Q_b + Q_b|Q_b|^{\frac{4}{N}} = 0$$

never belong to L^2.

A natural expectation is then that away from the soliton core $|y| \geq \frac{2}{b}$ where nonlinear effects and the influence of the ground state should be negligible, the solution ε will want to look like the solution to the linear equation:

$$\Delta \zeta_b - \zeta_b + ib\Lambda \zeta_b = \Psi.$$

Let us first solve this linear ODE. The following Lemma is elementary and relies on the so called WKB method, see [46], [47]

Lemma 5 (Linear outgoing radiation) *See Lemma 15 in [46]. There exist universal constants $C > 0$ and $\eta^* > 0$ such that $\forall 0 < \eta < \eta^*$, there exists $b^*(\eta) > 0$ such that $\forall 0 < b < b^*(\eta)$, the following holds true: let Ψ given by (91), there exists a unique radial solution ζ_b to*

$$\begin{cases} \Delta \zeta_b - \zeta_b + ib\Lambda \zeta_b = \Psi \\ \int |\nabla \zeta_b|^2 < +\infty. \end{cases} \tag{148}$$

Moreover, let

$$\Gamma_b = \lim_{|y| \to +\infty} |y|^N |\zeta_b(y)|^2, \tag{149}$$

then:

$$\Gamma_b \sim e^{-\frac{2\pi(1\pm\eta)}{b}} \quad as \quad b \to 0. \tag{150}$$

Let us give a brief insight into the proof of Lemma 5.

Proof of Lemma 5

Let us first note that letting

$$\xi_b = \zeta_b e^{i\frac{b|y|^2}{4}},$$

we equivalently solve:

$$\xi_b'' + \frac{N-1}{r}\xi_b' - \xi_b + \frac{b^2r^2}{4}\xi_b = \tilde{\Psi} \quad \text{and} \quad \int \left|\nabla\left(\xi_b e^{-ib\frac{r^2}{4}}\right)\right|^2 < +\infty \tag{151}$$

where $\tilde{\Psi} = \Psi e^{i\frac{b|y|^2}{4}}$. We view this linear equation as an ODE on $\xi_b = \xi_b(r)$, $r \in [0, +\infty)$. Consider first the linear equation for $Z = Z(r)$ on $(0, +\infty)$:

$$Z'' + \frac{N-1}{r}Z' - Z + \frac{b^2r^2}{4}Z = 0. \tag{152}$$

Let Z_{rad} be the solution to (152) with $Z_{rad}(0) = 1$, $Z'_{rad}(0) = 0$. Using a simple fixed point argument, we can construct a solution after the turning point $\frac{2}{b}$ with:

$$Z_{\pm}(r) \sim \frac{1}{r^{\frac{N-1}{2}}\left(1 - \frac{b^2r^2}{4}\right)} e^{\frac{\Theta_{\pm}(br)}{b}} \quad \text{for} \quad r \geq \frac{2}{b} \tag{153}$$

where $\Theta_{\pm}(r) \sim \pm\int_0^r \sqrt{1 - \frac{s^2}{4}}ds$ is the WKB phase. We then pick the outgoing behavior

$$Z_{out}(r) = Z_+(r) \tag{154}$$

and use the variation of the constant to construct a solution to (151) with

$$\xi_b(r) \sim C_1(b)(Z_{out}, \tilde{\Psi})Z_{rad}(r) \quad \text{for} \quad r \leq \frac{1}{b},$$

$$\xi_b(r) \sim C_2(b)(Z_{rad}, \tilde{\Psi})Z_{out}(r) \quad \text{for} \quad r \geq \frac{4}{b} \tag{155}$$

where (f, g) denotes the $L^2(\mathbf{R}^N)$ inner product. Observe from (153) and (154) that

$$\zeta_b(r) = e^{-i\frac{b|y|^2}{4}}\xi_b(r) \in \dot{H}^1$$

and thus using also (155):

$$\Gamma_b = \lim_{r\to+\infty} r^N|\zeta_b(r)|^2 = |C_2(b)|^2(Z_{rad}, \tilde{\Psi})^2.$$

The constant $C_2(b)$ is a *linear* constant and corresponds to the non vanishing Wronskian between Z_{rad} and Z_{out}, It can easily be proved to satisfy

$$C_2(b) \sim e^{-\frac{\pi(1\pm\eta)}{b}} \quad \text{as} \quad b \to 0.$$

The key is now that

$$\lim_{b\to 0} |(Z_{rad}, \tilde{\Psi})| > 0. \tag{156}$$

Indeed, let $\tilde{Z}_{rad}(r) = e^{-i\frac{b|y|^2}{4}}\xi_b(r)$, then from the construction of $\tilde{Q}_b$, Proposition 8, both Z_{rad} and $\tilde{\Psi}$ are real valued and thus:

$$\begin{aligned}(Z_{rad}, \tilde{\Psi}) &= Re(\tilde{Z}_{rad}, \overline{\Psi}) = Re(\overline{\tilde{Z}_{rad}}, \Delta\tilde{Q}_b - \tilde{Q}_b + ib\Lambda\tilde{Q}_b + \tilde{Q}_b|\tilde{Q}_b|^{\frac{4}{N}}) \\ &= (\overline{\tilde{Z}_{rad}}, \tilde{Q}_b|\tilde{Q}_b|^{\frac{4}{N}}) \to (J, Q^{1+\frac{4}{N}}) \quad \text{as} \quad b \to 0\end{aligned}$$

where J is the solution to $\Delta J - J = 0$, $J(0) = 1$, $J'(0) = 0$. Hence $J > 0$ and

$$(J, Q^{1+\frac{4}{N}}) > 0,$$

cqfd. This concludes the main steps of the proof of Lemma 5.

Remark 8 *The meaning of (156) is a nonlinear nonvanishing property. It states that some singular projection related to Q on the continuous spectrum of the operator $-L + ib\Lambda$ is non vanishing, and the corresponding constant is driving the anomalously slow convergence of the solution to Q. This structure is reminiscent to the socalled Fermi Golden Rule for metastability of periodic structures, see [62], [12]. This way of computing the log-log law with the beautiful observation that the constant (156) is non trivial is due to Galina Perelman, [56].*

5.5 The refined virial estimate

We now aim at refining the dispersive virial estimate by giving a more precise description of the nonpositive contribution $-e^{-\frac{C}{b}}$. A natural idea would be to consider a new variable $\tilde{\varepsilon} = \varepsilon - \zeta_b$ and to try and get better estimate for $\tilde{\varepsilon}$. The difficulty is now the same like for self similar solutions: ζ_b is not in L^2.

What we are facing here is a form of free boundary problem. We aim at finding the right size of $A = A(b)$ such that for $|y| < cA(b)$, the radiation ζ_b is a good approximation of ε, but no longer for $|y| > CA(b)$ because our solution is L^2 why ζ_b is not. It turns out that a correct choice for A -which can be proved at the end of the day to be essentially sharp, [47]- is

$$A(b) = e^{\frac{a}{b}}, \quad 0 < a << 1.$$

Note that similar kind of free boundary problems already appeared for the study of the singularity formation for the nonlinear heat equation, see Merle and Zaag [51].

We thus introduce a localized version of the radiation: let a radial cut off function $\chi_A(r) = \chi\left(\frac{r}{A}\right)$ with $\chi(r) = 1$ for $0 \leq r \leq 1$ and $\chi(r) = 0$ for $r \geq 2$, we let

$$\tilde{\zeta} = \chi_A \zeta_b = \tilde{\zeta}_{re} + i\tilde{\zeta}_{im}.$$

The equation satisfied by $\tilde{\zeta}$ is :

$$\Delta\tilde{\zeta} - \tilde{\zeta} + ib\Lambda\tilde{\zeta} = \Psi + F_A$$

with

$$F_A = (\Delta\chi_A)\zeta_b + 2\nabla\chi_A \cdot \nabla\zeta_b + iby \cdot \nabla\chi_A\zeta_b. \tag{157}$$

We introduce the new variable

$$\tilde{\varepsilon} = \varepsilon - \tilde{\zeta}$$

and claim the following refined virial dispersion:

Lemma 6 (Virial dispersion in the radiative regime) *There holds for some universal constants $\delta_1 > 0$, $c > 0$ and $s \geq 0$:*

$$\{f_1(s)\}_s \geq \delta_1 \left(\int |\nabla\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|}\right) + c\Gamma_b - C\lambda^2 E_0 - \frac{1}{\delta_1}\int_A^{2A} |\varepsilon|^2, \tag{158}$$

with

$$f_1 \sim \frac{b}{4}|yQ|_{L^2}^2. \tag{159}$$

Let us compare the two dispersive relations (143) and (158). The main difference is the presence of the Γ_b term -to the power one- *with the good sign.* The price to pay is the presence of the boundary term $\int_A^{2A} |\varepsilon|^2$ which cannot be directly estimated. Why this is a gain will be discussed in the next subsection.

The proof of Lemma 6 relies on tedious integration by parts and algebraic formulas, and we refer to [47] for a proof. Let us simply outline the main issues.
Recall that (143) is a consequence of the virial identity (112) which schematically is:

$$b_s = H(\varepsilon, \varepsilon) - Re(\varepsilon, \overline{\Lambda\Psi}) + \text{lot}. \tag{160}$$

The leading order term in this relation is the *linear term* $-Re(\varepsilon, \overline{\Lambda\Psi})$. The radiation has precisely been introduced to remove this term. But it cannot be removed completely, all we can do is replace Ψ which was the error to the self similar equation for $\tilde{Q}_b$ localized around $\frac{2}{b}$ by F_A given by (157) which is the error to equation for the radiation and is localized around A. Schematically, (160) becomes for $\tilde{\varepsilon}$:

$$b_s = H(\tilde{\varepsilon}, \tilde{\varepsilon}) - Re(\tilde{\varepsilon}, \overline{\Lambda F_A}) + \text{lot}.$$

Let us focus onto the linear term and try to estimate it. The simplest thing to do is Cauchy-Schwarz:

$$|(\tilde{\varepsilon}, F_A)| \leq C\left(\int |\tilde{\varepsilon}|^2\right)^{\frac{1}{2}}\left(\int |F_A|^2\right)^{\frac{1}{2}}.$$

Now a simple computation from (149) and (157) reveals that

$$\int_A^{2A} |F_A|^2 \sim \Gamma_b \tag{161}$$

independently of A. This is a reflection of the fact that the radiation just misses L^2 by a log and seems to indicate that there is not much to gain a priori by taking larger and larger A -unless one has uniform decay estimates on ε like in the setting of the proof of the Liouville theorem (17)...-.

The key is now to remember that A is the zone where ζ_b is no longer a good approximation of the solution. We thus come back to ε and split:

$$-Re(\tilde{\varepsilon}, \overline{\Lambda F_A}) = -Re(\varepsilon, \overline{\Lambda F_A}) + Re(\tilde{\zeta}, \overline{\Lambda F_A}).$$

The first term is explicit from (149) and is *the flux term*:

$$Re(\tilde{\zeta}, \overline{\Lambda F_A}) \sim \Gamma_b$$

thanks to the fact that $A >> \frac{2}{b}$. We then treat the second term from Cauchy-Schwarz and (161), and (158) follows.

5.6 Estimate of the outgoing flux of L^2 norm

We now need to control the term $\int_A^{2A} |\varepsilon|^2$ in the RHS of (158). The key is a localization in space of the L^2 conservation law.

Let a radial non negative cut off function $\phi(r)$ such that $\phi(r) = 0$ for $r \leq \frac{1}{2}$, $\phi(r) = 1$ for $r \geq 3$, $\frac{1}{4} \leq \phi'(r) \leq \frac{1}{2}$ pour $1 \leq r \leq 2$, $\phi'(r) \geq 0$. Let $\phi_A(s,r) = \phi\left(\frac{r}{A(s)}\right)$, then:

Lemma 7 (L^2 dispersion at infinity in space) *There holds for some universal constant $C > 0$ and $s \geq 0$:*

$$\left\{\int \phi_A |\varepsilon|^2\right\}_s \geq \frac{b}{400}\int_A^{2A} |\varepsilon|^2 - \Gamma_b^{1+C} - \Gamma_b^{\frac{a}{2}} \int |\nabla \varepsilon|^2. \tag{162}$$

The meaning of (162) is that *in the blow regime* where $b(s) > 0$, some L^2 mass is ejected to infinity from the soliton core. This process is measurable around A.

Proof of Lemma 7

Take the inner product of (100) with $\phi_A\varepsilon_1$ and of (101) with $\phi_A\varepsilon_2$ and integrate by parts. Note that the supports of $(\tilde{Q}_b,\Psi)$ and ϕ_A are disjoints. We thus get a linear identity decoupled from the non linear dynamic $|y|\leq\frac{2}{b}$:

$$\begin{aligned}\frac{1}{2}\left\{\int\phi_A|\varepsilon|^2\right\}_s &= \frac{1}{2}\int\frac{\partial\phi_A}{\partial s}|\varepsilon|^2+\frac{b}{2}\int y\cdot\nabla\phi_A|\varepsilon|^2+Im\left(\int\nabla\phi_A\cdot\nabla\varepsilon\overline{\varepsilon}\right) \qquad (163)\\ &- \frac{1}{2}\left(\frac{\lambda_s}{\lambda}+b\right)\int y\cdot\nabla\phi_A|\varepsilon|^2-\frac{1}{2}\frac{x_s}{\lambda}\cdot\int\nabla\phi_A|\varepsilon|^2.\end{aligned}$$

First observe from the choice of ϕ:

$$10\int\phi'\left(\frac{y}{A}\right)|\varepsilon|^2\geq\frac{1}{A}\int y\cdot\nabla\phi\left(\frac{y}{A}\right)|\varepsilon|^2\geq\frac{1}{10}\int\phi'\left(\frac{y}{A}\right)|\varepsilon|^2\geq\frac{1}{40}\int_A^{2A}|\varepsilon|^2. \qquad (164)$$

The main term in (163) is:

$$\frac{b}{2}\int y\cdot\nabla\phi_A|\varepsilon|^2\geq\frac{b}{20}\int\phi'\left(\frac{y}{A}\right)|\varepsilon|^2.$$

The $H^{\frac{1}{2}}$ coming from the linear Shrödinger flow is treated thanks to the choice of a large enough A:

$$\begin{aligned}&\left|Im\left(\int\nabla\phi_A\cdot\nabla\varepsilon\overline{\varepsilon}\right)\right| = \left|Im\left(\int\frac{1}{A}\nabla\phi\left(\frac{y}{A}\right)\cdot\nabla\varepsilon\overline{\varepsilon}\right)\right|\\ \leq\ & \frac{1}{A}\left(\int|\nabla\varepsilon|^2\right)^{\frac{1}{2}}\left(\int\phi'\left(\frac{y}{A}\right)|\varepsilon|^2\right)^{\frac{1}{2}}\leq\frac{40}{bA}\int|\nabla\varepsilon|^2+\frac{b}{40}\int\phi'\left(\frac{y}{A}\right)|\varepsilon|^2,\\ \leq\ & \frac{b}{100}\int\phi'\left(\frac{y}{A}\right)|\varepsilon|^2+\Gamma_b^{\frac{a}{2}}\int|\nabla\varepsilon|^2. \qquad (165)\end{aligned}$$

The other terms are estimated using the control of the modulation parameters (104), (105), and (162) follows. This concludes the proof of Lemma 7.

5.7 Derivation of the Lyapounov function and computation of the log-log law

The virial estimate (158) corresponds to non linear interactions on compact sets. The L^2 linear estimate (162) measures the interactions with the linear dynamic at infinity and quantifies the mechanism of mass ejection. We now couple these two informations with a global information in space: the conservation of the L^2 norm. The outcome is the derivation of a Lyapounov function:

Proposition 14 (Lyapounov functional in H^1) *There holds for some universal constant $C,c_0>0$ and for $s\geq 0$:*

$$\{\mathcal{J}\}_s\leq -Cb\left(\Gamma_b+\int|\nabla\tilde{\varepsilon}|^2+\int|\tilde{\varepsilon}|^2e^{-|y|}+\int_A^{2A}|\varepsilon|^2\right), \qquad (166)$$

with
$$\mathcal{J}(s) = c_0b^2 + o(b^2) + 2(\varepsilon_1,\Sigma) + 2(\varepsilon_2,\Theta) + \int (1-\phi_A)|\varepsilon|^2. \tag{167}$$

Proof of Proposition 14

Multiply (158) by $\frac{\delta_1 b}{800}$ and sum with (162). We get:

$$\begin{aligned}\left\{\int \phi_A|\varepsilon|^2\right\}_s + \frac{\delta_1 b}{800}\{f_1\}_s \ &\geq \ \frac{\delta_1^2 b}{800}\left(\int |\nabla\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|}\right) + \frac{b}{800}\int_A^{2A}|\varepsilon|^2 \\ &+ \ \frac{c\delta_1 b}{1000}\Gamma_b - \frac{C}{b^2}\lambda^2 E_0 - \Gamma_b^{\frac{a}{2}}\int |\nabla\varepsilon|^2, \end{aligned} \tag{168}$$

f_1 given by (159). Integrating by parts in time and using (104) yields:

$$b\{f_1\}_s = \{bf_1(b)\}_s + \mathrm{lot}.$$

We now inject the conservation of the L^2 norm:

$$\int |\varepsilon|^2 + \int |\tilde{Q}_b|^2 + 2(\varepsilon_1,\Sigma) + 2(\varepsilon_2,\Theta) = \int |u_0|^2.$$

Writing $\int \phi_A|\varepsilon|^2 = \int |\varepsilon|^2 - \int (1-\phi_A)|\varepsilon|^2$, we compute:

$$\left\{\int \phi_A|\varepsilon|^2\right\}_s = -\left\{\left(\int |\tilde{Q}_b|^2 - \int Q^2\right) + 2(\varepsilon_1,\Sigma) + 2(\varepsilon_2,\Theta) + \int (1-\phi_A)|\varepsilon|^2\right\}_s.$$

Now recall from (92) that $\tilde{Q}_b$ has super critical mass:

$$\int |\tilde{Q}_b|^2 - \int Q^2 = c_0 b^2 + o(b^2) \ \text{ as } \ b \to 0.$$

We thus get from (168):

$$\begin{aligned}\{-\mathcal{J}\}_s \ &\geq \ \frac{\delta_1^2 b}{800}\left(\int |\nabla\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2 e^{-|y|} + \int_A^{2A}|\varepsilon|^2\right) + \frac{c\delta_1 b}{100}\Gamma_b - \frac{C}{b^2}\lambda^2 E_0 \\ &- \ \Gamma_b^{\frac{a}{2}}\int |\nabla\varepsilon|^2 + \mathrm{lot}\end{aligned} \tag{169}$$

where $\mathcal{J}$ is given by (167).
We now have:

$$\Gamma_b^{\frac{a}{2}}\int |\nabla\varepsilon|^2 \leq \Gamma_b^{\frac{a}{2}}\left(\Gamma_b^{1-C\eta} + \int |\nabla\tilde{\varepsilon}|^2\right) \leq \Gamma_b^{1+\frac{a}{4}} + \Gamma_b^{\frac{a}{2}}\int |\nabla\tilde{\varepsilon}|^2$$

for $a > C\eta$. Injecting this into (169) yields (166). This concludes the proof of Proposition 14.

Let us explain the meaning of (167). Recall from (143) that

$$|\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \leq e^{-\frac{C}{b}} \tag{170}$$

in time averaging sense. If we could derive such an estimate pointwise, then (167) would imply

$$\mathcal{J}(s) \sim b(s)$$

and hence (166) becomes a pointwise differential inequation for b:

$$b_s \leq -Cb\Gamma_b,$$

and the upper bound (146) immediately follows by integration in time.

In other words, the log-log law corresponds to the dispersive mechanism which balances the mass ejection process described by Lemma 7 with the global constraint of the conservation of the L^2 norm.

From the technical point of view, to derive a pointwise bound on ε at the H^1 level is not obvious. There is an algebraic miracle which will save us here and which is hidden in the exact structure of the Lyapounov function (167).

Proof of the exact log-log law (79)

step 1: Asymptotic stability.

We first claim (145):

$$b(s) \to 0 \quad \text{as} \quad s \to +\infty.$$

Note that this could in fact be assumed from Theorem 11. Indeed, from (166), one has $\forall s \geq s_0$:

$$\int_{s_0}^{s} b\Gamma_b \leq \mathcal{J}(s_0) - \mathcal{J}(s).$$

Remark then that from its explicit value (167),

$$|\mathcal{J}| \leq C$$

and thus

$$\int_{s_0}^{+\infty} b\Gamma_b < +\infty.$$

Since $|b_s| \leq C$ from (104) , (145) follows.

step 2 Estimate on $\mathcal{J}$.

From energy type of estimates, we claim the following control on $\mathcal{J}$: $\forall s \geq s_1$,

$$\mathcal{J}(s) - f_2(b(s)) \quad \begin{cases} \geq -\Gamma_b^{1-Ca} + \frac{1}{C}\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right), \\ \leq CA^2 \log A \left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) + \Gamma_b^{1-Ca}, \end{cases} \tag{171}$$

where $f_2(b) \sim b^2$ satisfies

$$0 < \frac{df_2}{db^2}_{|b^2=0} < +\infty. \tag{172}$$

Remark 9 *The main point is that the dominant term in ε in (167) is $2(\varepsilon_1, \Sigma) + 2(\varepsilon_2, \Theta) + \int (1-\phi_A)|\varepsilon|^2$ which also appears in the conservation of the energy. Now the orthogonality conditions on ε, which in our analysis are related to the virial relation that is dispersion on compact sets, are also adapted to the coercive structure of the linearized energy, see also [44]. Indeed, let*

$$L_+ = -\Delta + 1 - \left(1 + \frac{4}{N}\right) Q^{\frac{4}{N}}, \quad L_- = -\Delta + 1 - Q^{\frac{4}{N}}$$

the linear operator close to Q, recall that the quadratic form which appears when linearizing the energy close to Q is $(L_+\varepsilon_1, \varepsilon_1) + (L_-\varepsilon_2, \varepsilon_2) - \int |\varepsilon|^2$. There holds from [65], [34] and [32]:

Lemma 8 *[65], [34], [32]] There exists a universal constant $\delta_2 > 0$ such that the following holds true. Let $\mu_+ < 0$ the lowest eigenvalue of L_+ and ϕ_+ a corresponding eigenvector with $\|\phi_+\|_{L^2} = 1$, then for all $v = v_1 + iv_2 \in H^1$,*

$$(L_+v_1, v_1) + (L_-v_2, v_2) \geq \delta_2 |v|_{H^1}^2 - \frac{1}{\delta_2}\left\{(v_1, \phi_+)^2 + (v_1, \nabla Q)^2 + (v_2, Q)^2\right\} \tag{173}$$

A simple localization argument then implies: $\exists \delta_3 > 0$ such that:

$$\begin{aligned} &(L_+\varepsilon_1, \varepsilon_1) + (L_-\varepsilon_2, \varepsilon_2) - \int \phi_A |\varepsilon|^2 \geq \delta_3 (\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}) \\ - &\frac{1}{\delta_3}\left\{(\varepsilon_1, Q)^2 + (\varepsilon_1, |y|^2 Q)^2 + (\varepsilon_1, yQ)^2 + (\varepsilon_2, Q_2)^2\right\}. \end{aligned} \tag{174}$$

It now suffices to inject the conservation of the energy:

$$2(\varepsilon_1, \Sigma) + 2(\varepsilon_2, \Theta) + \int (1-\phi_A)|\varepsilon|^2 = \int (1-\phi_A)|\varepsilon|^2 + (L_+\varepsilon_1, \varepsilon_1) + (L_-\varepsilon_2, \varepsilon_2) + \text{lot}$$

into the definition of $\mathcal{J}$ (167). The upper bound in (171) follows from the following general Hardy type estimate:

$$\int (1-\phi_A)|\varepsilon|^2 \leq CA^2 \log A \left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right).$$

The lower bound in (171) follows from from the coercivity property (174) and the fact tha our choice of orthogonality conditions and the conservation laws allow one to treat all the scalar product terms in (174). This concludes the proof of (171).

step 3: Pointwise control of ε by b.

We now turn to the core of the proof and the derivation of the pointwise bound on ε (147). Let $s \geq 0$, we consider two cases.
Case 1: $b_s(s) \leq 0$, then (147) follows from (143).
Case 2: $b_s(s) > 0$. First observe from $b(s) \to 0$ as $s \to +\infty$ that we may assume without loss of generality that $s \in (s_1^*, s_2^*)$ with:

$$b_s(s_1^*) = b_s(s_2^*) = 0 \quad \text{and} \quad b_s \geq 0 \quad \text{in} \quad [s_1^*, s_2^*],$$

and thus:

$$b(s_1^*) \leq b(s) \leq b(s_2^*). \tag{175}$$

Case 1 applies at $s = s_1^*$:

$$(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|})(s_1) \leq \Gamma_{b(s_1)}^{1-Ca}. \tag{176}$$

Next, from the Lyapounov monotonicity property (166):

$$\mathcal{J}(s_2^*) \leq \mathcal{J}(s) \leq \mathcal{J}(s_1^*),$$

and injecting (171) into this inequality, we get:

$$\begin{aligned} & f_2(b(s)) + \frac{1}{C}\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)(s) \leq \mathcal{J}(s) + \Gamma_{b(s)}^{1-Ca} \\ \leq \quad & f_2(b(s_1^*)) + \Gamma_{b(s)}^{1-Ca} + \Gamma_{b(s_1^*)}^{1-Ca} \leq f_2(b(s_1^*)) + 2\Gamma_{b(s)}^{1-Ca}, \end{aligned}$$

where we used (175) for the two last steps. Now from the monotonicity property (172) of $f_2(b)$,

$$f_2(b(s_1^*)) \leq f_2(b(s)),$$

and (147) follows.

step 4 Pointwise uniform control of the scaling parameter.

The proof of the sharp log-log law is now a simple consequence of (166) and (147). We first claim the following uniform estimate for $s \geq s_2$ large enough:

$$Cb(s) \leq \frac{1}{\log|\log\lambda(s)|}, \tag{177}$$

for some universal constant $C > 0$. Indeed, from (167), (147) and (171), we have for $s \geq s_2$:

$$\frac{b^2}{C} \leq \mathcal{J} \leq Cb^2. \tag{178}$$

Thus from (166) , $g = \sqrt{\mathcal{J}}$ satisfies the following differential inequation:

$$g_s = \frac{(\mathcal{J})_s}{2\sqrt{\mathcal{J}}} \leq -\frac{C}{g} g e^{-\frac{C}{g}} \leq -\frac{1}{2C} g^2 e^{-\frac{2C}{g}} \quad \text{ie} \quad \left(e^{\frac{2C}{g}}\right)_s \geq 1,$$

and thus we have for $s \geq s_2$ large enough:

$$e^{\frac{C}{b(s)}} \geq s \quad \text{ie} \quad b(s) \leq Cg(s) \leq \frac{C}{\log(s)}. \tag{179}$$

Now observe from (104) and (147) that:

$$\left|\frac{\lambda_s}{\lambda} + b\right| \leq \Gamma_b^{\frac{1}{2}}$$

and thus

$$\frac{b}{2} \leq -\frac{\lambda_s}{\lambda} \leq 2b. \tag{180}$$

We integrate this in time on $[s_2, s]$ using $b > 0$ to derive for $s \geq s_2$ large enough:

$$-\log(\lambda(s)) \leq 2\int_{s_2}^{s} b(\tau)d\tau \leq C\int_{s_2}^{s} \frac{1}{\log(\tau)} d\tau \leq Cs.$$

Taking the log of this inequality and injecting (179) yields:

$$\log|\log(\lambda(s))| \leq C\log(s) \leq \frac{C}{b(s)},$$

and (177) is proved.

step 2 Conclusion.

We differ here from the strategy of integration on doubling time intervals of the norm. We claim that the pointwise bound on ε (147) yields a pointwise differential inequality for $\lambda(t)$: $\forall t \geq t_2$,

$$\frac{1}{C} \leq -\left(\lambda^2 \log|\log(\lambda)|\right)_t \leq C, \tag{181}$$

for some universal constant $C > 0$.
Indeed, we compute:

$$\begin{aligned} -\left(\lambda^2 \log|\log(\lambda)|\right)_t &= -\lambda\lambda_t \log|\log(\lambda)| \left(2 + \frac{1}{|\log(\lambda)| \quad \log|\log(\lambda)|}\right) \\ &= -\frac{\lambda_s}{\lambda} \log|\log(\lambda)| \left(2 + \frac{1}{|\log(\lambda)| \quad \log|\log(\lambda)|}\right). \end{aligned}$$

Thus from (180), we have:

$$\frac{b}{4}\log|\log(\lambda)| \leq -\left(\lambda^2\log|\log(\lambda)|\right)_t \leq 4b\log|\log(\lambda)|,$$

and (181) now follows from (121) and (177). Integrating (181) in time t yields::

$$\forall t \geq t_2, \quad \frac{T-t}{C} \leq \lambda^2(t)\log|\log(\lambda(t))| \leq C(T-t).$$

The lower bound

$$|\nabla u(t)|_{L^2} \geq C\sqrt{\frac{\log|\log(T-t)|}{T-t}}$$

now easily follows for t close enough to T. The proof of the exact convergence (79) is now a simple technical issue, see [47]. This concludes the proof of the exact log-log law (79).

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