EQUIDISTRIBUTION AND L-FUNCTIONS

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Abstract. These are notes for two lectures given at the 2007 summer school "Homogeneous Flows, Moduli Spaces and Arithmetic" in Pisa, Italy. The first lecture introduces Heegner points and closed geodesics on the modular surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ and highlights some of their arithmetic significance. The second lecture discusses how subconvex bounds for certain automorphic L-functions yield quantitative equidistribution results for Heegner points and closed geodesics.

1. Lecture One

Let us start the discussion with the equivalence of integral binary quadratic forms. The concept was introduced and studied in a systematic fashion in Gauss's *Disquisitiones Arithmeticae* (1801).

An *integral binary quadratic form* is a homogeneous polynomial

$$\langle a,b,c\rangle := ax^2+bxy+cy^2 \in \mathbb{Z}[x,y]$$

with associated *discriminant*

$$d := b^2-4ac \in \mathbb{Z}.$$

The possible discriminants are the integers congruent to 0 or 1 mod 4. We shall assume that the form $\langle a,b,c\rangle$ is not a product of linear factors in $\mathbb{Z}[x,y]$, then d is not a square, hence $ac\neq 0$. If $d<0$ then $ac>0$ and we shall assume that we are in the *positive definite* case $a,c>0$. Furthermore, we shall assume that d is a *fundamental discriminant* which means that it cannot be written as $d'e^2$ for some smaller discriminant d'. Then $\langle a,b,c\rangle$ is a *primitive* form which means that a,b,c are relatively prime. The possible fundamental discriminants are the square-free numbers congruent to 1 mod 4 and 4 times the square-free numbers congruent to 2 or 3 mod 4.

Example 1. The first few negative fundamental discriminants are: -3, -4, -7, -8, -11. The first few positive fundamental discriminants are: 5, 8, 12, 13, 17.

Gauss discovered that every form $\langle a,b,c\rangle$ with a given discriminant d can be reduced by some integral unimodular substitution

$$(x,y)\mapsto(\alpha x+\beta y,\gamma x+\delta y),\qquad \begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}),$$

to some form with the same discriminant that lies in a finite set depending only on d. Forms that are connected by such a substitution are called *equivalent*. It is easiest to understand this reduction by looking at the simple substitutions

$$(x,y)\overset{T}{\mapsto}(x-y,y)\qquad\text{and}\qquad(x,y)\overset{S}{\mapsto}(-y,x). \tag{1}$$

The induced actions on forms are given by

$$\langle a,b,c\rangle \overset{T}{\mapsto} \langle a, b-2a, c+a-b\rangle \qquad \text{and} \qquad \langle a,b,c\rangle \overset{S}{\mapsto} \langle c,-b,a\rangle.$$

Now a given form $\langle a,b,c\rangle$ can always be taken to some $\langle a,b',c'\rangle$ with $|b'|\leqslant |a|$ by applying T or T^{-1} a few times. If $|a|\leqslant |c'|$ then we stop our reduction. Otherwise we apply S to get some $\langle a'',b'',c''\rangle$ with $|a''|<|a|$ and we start over with this form. In this algorithm we cannot apply S infinitely many times because $|a|$ decreases at each such step. Hence in a finite number of steps we arrive at an equivalent form $\langle a,b,c\rangle$ whose coefficients satisfy

$$|b|\leqslant |a|\leqslant |c|, \qquad b^2-4ac=d. \tag{2}$$

These constraints are satisfied by finitely many triples (a,b,c). Indeed, we have

$$|d|=|b^2-4ac|\geqslant 4|ac|-b^2\geqslant 3b^2, \tag{3}$$

so there are only $\ll |d|^{1/2}$ choices for b and for each such choice there are only $\ll_\varepsilon d^\varepsilon$ choices for a and c since the product ac is determined by b. We have shown that the number of equivalence classes of integral binary quadratic forms of fundamental discriminant d, denoted $h(d)$, satisfies the inequality

$$h(d)\ll_\varepsilon |d|^{1/2+\varepsilon}. \tag{4}$$

In the case $d<0$ it is straightforward to compile a maximal list of inequivalent forms satisfying (2). There is an algorithm for $d>0$ as well but it is less straightforward. Note that for $d>0$ (3) implies that $4ac=b^2-d<0$, hence by an extra application of S we can always arrange for a reduced form $\langle a,b,c\rangle$ that $a>0$.

Example 2. The equivalence classes for $d=-23$ are represented by the forms $\langle 1,1,6\rangle$, $\langle 2,\pm 1,3\rangle$. Hence $h(-23)=3$. The equivalence classes for $d=21$ are represented by the forms $\langle 1,1,-5\rangle$, $\langle -1,1,5\rangle$. Hence $h(21)=2$.

To obtain a geometric picture of equivalence classes of forms we shall think of $\mathbb{Q}(\sqrt{d})$ as embedded in $\mathbb{C}$ such that $\sqrt{d}/i>0$ for $d<0$ and $\sqrt{d}>0$ for $d>0$, and for $q_1,q_2\in\mathbb{Q}$ we shall consider the conjugation

$$\overline{q_1+q_2\sqrt{d}} := q_1-\sqrt{d}.$$

Each form $\langle a,b,c\rangle$ decomposes as

$$ax^2+bxy+cy^2=a(x-zy)(x-\bar{z}y),$$

where

$$z:=\frac{-b+\sqrt{d}}{2a}, \qquad \bar{z}:=\frac{-b-\sqrt{d}}{2a}. \tag{5}$$

Using (1) we can see that the action of $\mathrm{SL}_2(\mathbb{Z})$ on z and $\bar{z}$ is the usual one given by fractional linear transformations:

$$z\overset{T}{\mapsto} z+1 \qquad \text{and} \qquad z\overset{S}{\mapsto} -1/z.$$

Therefore in fact we are looking at the standard action of $\mathrm{SL}_2(\mathbb{Z})$ on certain conjugate pairs of points of $\mathbb{Q}(\sqrt{d})$ embedded in $\mathbb{C}$. For $d<0$ we consider the points $z\in\mathcal{H}$ and obtain $h(d)$ points on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$. These are the *Heegner points* of discriminant $d<0$. For $d>0$ we consider the geodesics $G_{\bar{z},z}\subseteq\mathcal{H}$ connecting the real points $\{\bar{z},z\}$ and obtain $h(d)$ geodesics on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$.

It is a remarkable fact that for $d > 0$ any geodesic $G_{\bar{z},z}$ as above becomes closed when projected to $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$, and its length is an important arithmetic quantity associated with the number field $\mathbb{Q}(\sqrt{d})$. To see this take any matrix $M \in \mathrm{GL}_2^+(\mathbb{R})$ which takes 0 to $\bar{z}$ and ∞ to z, for example[1]

$$M := \begin{pmatrix} z & \bar{z} \\ 1 & 1 \end{pmatrix}, \tag{6}$$

then M takes the real segment (resp. geodesic) connecting $\{0, \infty\}$ to the real segment (resp. geodesic) connecting $\{\bar{z}, z\}$. In particular, using that M is a conformal automorphism of the Riemann sphere, we see that $G_{\bar{z},z}$ is the Thales semicircle of the real segment $[\bar{z}, z]$ contained in $\mathcal{H}$, parametrized as

$$G_{\bar{z},z} = \{g(\lambda)i : \ \lambda > 0\}, \qquad \text{where} \qquad g(\lambda) := M \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Moreover, the unique isometry of $\mathcal{H}$ fixing the geodesic $G_{\bar{z},z}$ and taking $g(1)i$ to $g(\lambda)i$ is given by the matrix

$$M \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} M^{-1} \in \mathrm{SL}_2(\mathbb{R}). \tag{7}$$

Therefore we want to see that for some $\lambda > 1$ the matrix

$$M \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} M^{-1} = \frac{1}{z - \bar{z}} \begin{pmatrix} z\lambda - \bar{z}\lambda^{-1} & z\bar{z}(\lambda^{-1} - \lambda) \\ \lambda - \lambda^{-1} & z\lambda^{-1} - \bar{z}\lambda \end{pmatrix} \tag{8}$$

is in $\mathrm{SL}_2(\mathbb{Z})$, and then the projection of $G_{\bar{z},z}$ to $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ has length

$$\int_1^{\lambda^2} \frac{dy}{y} = 2\ln(\lambda)$$

for the smallest such $\lambda > 1$. A necessary condition for λ is that the trace and anti-diagonal elements of the matrix (8) are in $\mathbb{Z}$, and so are twice the diagonal elements as well. Using that

$$z - \bar{z} = \frac{\sqrt{d}}{a}, \qquad z\bar{z} = \frac{c}{a}$$

this is equivalent to:

$$\lambda + \lambda^{-1} \in \mathbb{Z}, \qquad \{a, b, c\}\frac{\lambda - \lambda^{-1}}{\sqrt{d}} \subseteq \mathbb{Z}.$$

As $(a, b, c) = 1$ we can simplify this to

$$\lambda + \lambda^{-1} \in \mathbb{Z}, \qquad \text{and} \qquad \frac{\lambda - \lambda^{-1}}{\sqrt{d}} \in \mathbb{Z}.$$

In other words, there are integers m, n such that

$$\lambda = \frac{m + n\sqrt{d}}{2} \qquad \text{and} \qquad \lambda^{-1} = \frac{m - n\sqrt{d}}{2}. \tag{9}$$

As $\lambda > 1$ the integers m, n are positive and they satisfy the diophantine equation

$$m^2 - dn^2 = 4. \tag{10}$$

[1] we assume here that $a > 0$ which is legitimate as we have seen

The equations (9)–(10) are not only necessary but also sufficient for (8) to lie in $\mathrm{SL}_2(\mathbb{Z})$. Namely, (8)–(10) imply that

$$M\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}M^{-1}=\begin{pmatrix}\frac{m-bn}{2} & -nc\\ na & \frac{m+bn}{2}\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}) \tag{11}$$

since

$$m\pm bn\equiv m^2-dn^2\equiv 0\pmod 2.$$

The λ's given by (9)–(10) are exactly the totally positive[2] units in the ring of integers $\mathcal{O}_d$ of $\mathbb{Q}(\sqrt{d})$. These units form a group isomorphic to $\mathbb{Z}$ by Dirichlet's theorem, therefore there is a smallest $\lambda=\lambda_d>1$ among them (which generates the group). In other words, the sought $\lambda=\lambda_d>1$ exists and comes from the smallest positive solution of (10). In classical language, the matrices (11) are the *automorphs* of the form $\langle a,b,c\rangle$.

To summarize, the $\mathrm{SL}_2(\mathbb{Z})$-orbits of forms $\langle a,b,c\rangle$ with given fundamental discriminant d give rise to $h(d)$ Heegner points on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ for $d<0$ and $h(d)$ closed geodesics of length $2\ln(\lambda_d)$ for $d>0$ where $\lambda_d=(m+n\sqrt{d})/2$ is the smallest totally positive unit of $\mathcal{O}_d$ greater than 1. This geometric picture is even more interesting in the light of the following refinement of (4) which is a consequence of Dirichlet's class number formula and Siegel's theorem:

$$\begin{aligned}|d|^{1/2-\varepsilon}\ll_\varepsilon h(d)\ll_\varepsilon |d|^{1/2+\varepsilon},&\qquad d<0,\\ d^{1/2-\varepsilon}\ll_\varepsilon h(d)\ln(\lambda_d)\ll_\varepsilon d^{1/2+\varepsilon},&\qquad d>0.\end{aligned}\tag{12}$$

This shows that the set of Heegner points of discriminant $d<0$ has cardinality about $|d|^{1/2}$, while the set of closed geodesics of discriminant $d>0$ has total length about $d^{1/2}$.

2. Lecture Two

In the light of (12) the natural question arises if the set Λ_d of Heegner points (resp. closed geodesics) of fundamental discriminant d becomes equidistributed in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ as $d\to-\infty$ (resp. $d\to+\infty$). That is, given a smooth and compactly supported weight function $g:\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}\to\mathbb{C}$ do we have

$$\begin{aligned}\frac{1}{h(d)}\sum_{z\in\Lambda_d}g(z)\to\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}}g(z)\,d\mu(z),&\qquad d\to-\infty,\\ \frac{1}{h(d)\,2\ln(\lambda_d)}\sum_{G\in\Lambda_d}\int_G g(z)\,ds(z)\to\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}}g(z)\,d\mu(z),&\qquad d\to+\infty,\end{aligned}\tag{13}$$

where $d\mu(z)$ abbreviates the $\mathrm{SL}_2(\mathbb{R})$-invariant probability measure on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ and $ds(z)$ abbreviates the hyperbolic arc length? Duke proved in 1988 that the answer is yes in the sharper form that the difference of the two sides is $\ll_g|d|^{-\delta}$ for some fixed $\delta>0$. Earlier in 1968 Linnik established the above limits with error term $\ll_g(\log|d|)^{-A}$ for all $A>0$ under the condition that $\left(\frac{d}{p}\right)=1$ for a fixed odd prime p.

We shall discuss Duke's quantitative result and a refinement of it from the modern perspective of subconvex bounds for automorphic L-functions. Our first step is

[2] i.e. positive under both embeddings $\mathbb{Q}(\sqrt{d})\hookrightarrow\mathbb{R}$

to decompose spectrally the weight function considered in (13) as

$$g(z) = \langle g, 1\rangle + \sum_{j=1}^{\infty} \langle g, u_j\rangle u_j(z) + \frac{1}{4\pi}\int_{-\infty}^{\infty} \langle g, E(\cdot, \tfrac{1}{2}+it)\rangle E(z, \tfrac{1}{2}+it)\, dt,$$

where

$$\langle f_1, f_2\rangle := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} f_1 \overline{f_2}\, d\mu(z),$$

and $\{u_j\}$ are Hecke–Maass cusp forms on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ with $\langle u_j, u_j\rangle = 1$. The above decomposition converges in $L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H})$ and also uniformly on compact sets. If

$$\Delta := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

denotes the hyperbolic Laplacian and we use the notation

$$\Delta u_j(z) = \left(\tfrac{1}{4} + t_j^2\right) u_j(z), \qquad \Delta E(z, \tfrac{1}{2}+it) = \left(\tfrac{1}{4}+t^2\right) E(z, \tfrac{1}{2}+it),$$

then for any smooth and compactly supported $g(z)$ and for any $B > 0$ we have

$$\langle g, u_j\rangle \ll_{g,B} (1 + |t_j|)^{-B}, \qquad \langle g, E(\cdot, \tfrac{1}{2}+it)\rangle \ll_{g,B} (1+|t|)^{-B}. \tag{14}$$

Therefore in order to establish Duke's theorem with error term $\ll_g |d|^{-\delta}$ it suffices to show that if g is a Hecke–Maass cusp form with $\langle g, g\rangle = 1$ or a standard Eisenstein series $E(\cdot, \frac{1}{2}+it)$ then for some fixed $\delta > 0$ and $A > 0$ the sums considered in (13) satisfy

$$\sum_{\Lambda_d} \cdots \ll (1+|t|)^A \, |d|^{\frac{1}{2}-\delta}, \tag{15}$$

where $t = t_g$ is the spectral parameter of g, i.e.

$$\Delta g(z) = \left(\tfrac{1}{4} + t^2\right) g(z).$$

At this point we remark that g has a Fourier decomposition of the form

$$g(x+iy) = c_1 y^{\frac{1}{2}+it} + c_2 y^{\frac{1}{2}-it} + \sqrt{y}\sum_{n\neq 0} \rho_g(n) K_{it}(2\pi|n|y) e^{2\pi i n x},$$

where $c_{1,2}$ are some constants[3] and K_{it} is a Bessel function. The Fourier coefficients $\rho_g(n)$ are proportional to the Hecke eigenvalues of g and we have the uniform bound

$$\rho_g(1) \ll_\varepsilon (1+|t|)^\varepsilon e^{\frac{\pi}{2}|t|}. \tag{16}$$

We note that this inequality is sharp for $|t| \gg 1$.

Now we quote a formula which can be attributed to several people[4] and relates the sums in (15) to central values of automorphic L-functions:

$$\left|\sum_{\Lambda_d} \cdots\right|^2 = c_d \, |d|^{\frac{1}{2}} \, |\rho_g(1)|^2 \, \Lambda\left(\tfrac{1}{2}, g\right) \Lambda\left(\tfrac{1}{2}, g\otimes\left(\tfrac{d}{\cdot}\right)\right), \tag{17}$$

where the factor c_d is nonzero and takes only finitely many different values. In this formula $\Lambda(s, \Pi)$ denotes the completed L-function. The finite part $L(s, \Pi)$ of the L-function is defined in terms of Hecke eigenvalues. The infinite part of the L-function is a product of exponential and gamma factors whose contribution in

[3] $c_1 = c_2 = 0$ if g is a cusp form, $c_1 = |c_2| = 1$ if g is an Eisenstein series

[4] Dirichlet, Hecke, Siegel, Maass, Shimura, Waldspurger, Kohnen–Zagier, Duke, Katok–Sarnak

(17) *exactly* kills the exponential factor in (16). We conclude that (15) follows by a subconvex bound of the form

$$L\left(\tfrac{1}{2}, g\otimes\left(\tfrac{d}{\cdot}\right)\right) \ll (1+|t|)^A\,|d|^{\frac{1}{2}-\delta},$$

where $\delta>0$ and $A>0$ are some fixed constants (different from those in (15)). In the case when g is a cusp form such a bound was proved by Duke–Friedlander–Iwaniec (1993) for any $\delta<\frac{1}{22}$, by Bykovskiĭ (1998) for any $\delta<\frac{1}{8}$, and by Conrey–Iwaniec (2000) for any $\delta<\frac{1}{6}$. In the case when g is an Eisenstein series $E(\cdot,\frac{1}{2}+it)$, the above becomes

$$\left|L\left(\tfrac{1}{2}+it,\left(\tfrac{d}{\cdot}\right)\right)\right|^2 \ll (1+|t|)^A\,|d|^{\frac{1}{2}-\delta},$$

and this was established by Burgess (1963) for any $\delta<\frac{1}{8}$, and by Conrey–Iwaniec (2000) for any $\delta<\frac{1}{6}$.

We shall now formulate a refinement of (13) using the natural action of the narrow ideal class group H_d of $\mathbb{Q}(\sqrt{d})$ on Λ_d. This action comes from the natural bijection $H_d\leftrightarrow\Lambda_d$ which we describe in the Appendix. Note in particular that $|H_d|=h(d)$ by this bijection. Now given some $z_0\in\Lambda_d$ when $d<0$ and some $G_0\in\Lambda_d$ when $d>0$ and given some subgroup $H\leqslant H_d$ one can ask if

$$\begin{aligned}\frac{1}{|H|}\sum_{\sigma\in H} g(z_0^\sigma) &\to \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} g(z)\,d\mu(z), \qquad d\to-\infty,\\ \frac{1}{|H|\,2\ln(\lambda_d)}\sum_{\sigma\in H}\int_{G_0^\sigma} g(z)\,ds(z) &\to \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} g(z)\,d\mu(z), \qquad d\to+\infty.\end{aligned}\tag{18}$$

Using characters of the abelian group H_d we can decompose the sums over H into twisted sums over H_d:

$$\begin{aligned}\frac{1}{|H|}\sum_{\sigma\in H}\dots &= \frac{1}{|H|}\sum_{\sigma\in H_d}\frac{1}{(H_d:H)}\sum_{\substack{\psi\in\widehat{H}_d\\ \psi_{|H}\equiv 1}}\psi(\sigma)\dots\\ &= \sum_{\substack{\psi\in\widehat{H}_d\\ \psi_{|H}\equiv 1}}\frac{1}{|H_d|}\sum_{\sigma\in H_d}\psi(\sigma)\dots\end{aligned}$$

Note that the number of characters of H_d restricting to the identity character on H is $(H_d:H)$. Therefore, by the same discussion as above, the limits (18) follow with the strong error term $\ll_g |d|^{-\delta}$ as long as

$$(H_d:H)\ll|d|^\eta$$

for a fixed constant $0<\eta<\delta$ and, for some fixed constant $A>0$ and for any L^2-normalized Hecke–Maass cusp form or standard Eisenstein series in place of g, we can prove

$$\begin{aligned}\sum_{\sigma\in H_d}\psi(\sigma)g(z_0^\sigma) &\ll (1+|t|)^A\,|d|^{\frac{1}{2}-\delta}, \qquad d<0,\\ \sum_{\sigma\in H_d}\psi(\sigma)\int_{G_0^\sigma} g(z)\,ds(z) &\ll (1+|t|)^A\,|d|^{\frac{1}{2}-\delta}, \qquad d>0.\end{aligned}\tag{19}$$

The twisted sums here can be related to central automorphic L-values similarly as in (17). The formula is based on the deep work of Waldspurger (1985) and was carefully derived by Zhang (2002) when $d<0$ and by Popa (2006) when $d>0$:

$$\left|\sum_{\sigma\in H_d}\overline{\psi(\sigma)}\dots\right|^2 = c_d\,|d|^{\frac12}\,|\rho_g(1)|^2\,\Lambda\left(\tfrac12,g\otimes f_\psi\right), \tag{20}$$

where f_ψ is the so-called Jacquet–Langlands lift of ψ, discovered by Hecke (1926) and Maass (1949) in this special case: it is a modular form on $\mathcal{H}$ of level $|d|$ and nebentypus $\left(\frac{d}{\cdot}\right)$ whose L-function agrees with that of ψ.

If the character $\psi : H_d \to \mathbb{C}^\times$ is real-valued then it is one of the genus characters discovered by Gauss (1801). In this case

$$\Lambda(s,\psi)=\Lambda(s,(\tfrac{d_1}{\cdot}))\Lambda(s,(\tfrac{d_2}{\cdot})),$$

where $d=d_1d_2$ is a factorization of d into fundamental discriminants d_1 and d_2, whence (20) simplifies to

$$\left|\sum_{\sigma\in H_d}\overline{\psi(\sigma)}\dots\right|^2 = c_d\,|d|^{\frac12}\,|\rho_g(1)|^2\,\Lambda\left(\tfrac12,g\otimes(\tfrac{d_1}{\cdot})\right)\Lambda\left(\tfrac12,g\otimes(\tfrac{d_2}{\cdot})\right).$$

In fact (17) is the special case of this formula when ψ is the trivial character ($d_1=1$, $d_2=d$). The necessary estimate (19) follows by the subconvex bound discussed before:

$$L\left(\tfrac12,g\otimes(\tfrac{d_i}{\cdot})\right)\ll(1+|t|)^A\,|d_i|^{\frac12-\delta},\qquad i=1,2.$$

If the character $\psi : H_d\to\mathbb{C}^\times$ is not real-valued then f_ψ is a cusp form of level $|d|$, and we need a subconvex bound of the form

$$L\left(\tfrac12,g\otimes f_\psi\right)\ll(1+|t|)^A\,|d|^{\frac12-\delta}.$$

In the case when g is a cusp form such a bound was proved by Harcos–Michel (2006) with $\delta=\frac{1}{3000}$. In the case when g is an Eisenstein series the above becomes

$$\left|L\left(\tfrac12+it,\psi\right)\right|^2\ll(1+|t|)^A\,|d|^{\frac12-\delta},$$

and this was established by Duke–Friedlander–Iwaniec (2002) with $\delta=\frac{1}{12000}$ and by Blomer–Harcos–Michel (2006) with $\delta=\frac{1}{1000}$.

3. Appendix

We shall show that the equivalence classes of forms of fundamental discriminant d can be mapped bijectively to narrow ideal classes of the quadratic number field $\mathbb{Q}(\sqrt{d})$ in a natural fashion. As the latter classes form an abelian group under multiplication this will exhibit a natural multiplication law on the equivalence classes of forms. This law is called *composition* in the classical theory, it was also discovered by Gauss.

Recall that a fractional ideal of $\mathbb{Q}(\sqrt{d})$ is an $\mathcal{O}_d$ module contained in $\mathbb{Q}(\sqrt{d})$ and two nonzero fractional ideals are equivalent (in the narrow sense) if their quotient is a principal fractional ideal generated by a totally positive element of $\mathbb{Q}(\sqrt{d})$. Here "totally positive element" can clearly be changed to "element of positive norm" where the norm of $\mu\in\mathbb{Q}(\sqrt{d})$ is given by $N(\mu)=\mu\bar\mu$. Recall also that we can represent equivalence classes of forms of fundamental discriminant d by some

$$Q_i(x,y)=a_ix^2+b_ixy+c_iy^2=a_i(x-z_iy)(x-\bar z_iy),\qquad i=1,\dots,h(d),$$

with

$$a_i > 0, \qquad z_i := \frac{-b_i + \sqrt{d}}{2a_i}, \qquad \bar{z}_i := \frac{-b_i - \sqrt{d}}{2a_i}.$$

It will suffice to show that each fractional ideal I of $\mathbb{Q}(\sqrt{d})$ is equivalent to some fractional ideal

$$I_i := \mathbb{Z} + \mathbb{Z}z_i, \qquad i = 1, \dots, h(d),$$

and that the fractional ideals I_i are pairwise inequivalent.

Any fractional ideal I can be written as

$$I = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \qquad \text{with} \qquad \frac{\overline{\omega}_1\omega_2 - \omega_1\overline{\omega}_2}{\sqrt{d}} > 0.$$

We associate to I (and ω_1, ω_2) the binary quadratic form

$$Q_I(x, y) := \frac{(x\omega_1 - y\omega_2)(x\bar{\omega}_1 - y\bar{\omega}_2)}{N(I)},$$

where $N(I) > 0$ is the absolute norm of I, i.e. the multiplicative function that agrees with $(\mathcal{O}_d : I)$ for integral ideals I. We claim first that $Q_I(x, y)$ has integral coefficients and discriminant d. To see the claim we can assume that I is an integral ideal since $Q_I(x, y)$ does not change if we replace I by nI (and ω_i by $n\omega_i$) for some positive integer n. Then ω_1, ω_2 and their conjugates are in $\mathcal{O}_d$ and the claim amounts to:

- $N(I) \mid \omega_1\bar{\omega}_1,\ \omega_1\bar{\omega}_2 + \bar{\omega}_1\omega_2,\ \omega_2\bar{\omega}_2;$
- $(\omega_1\bar{\omega}_2 - \bar{\omega}_1\omega_2)^2 = N(I)^2 d.$

The first statement follows from the fact that $\omega_1, \omega_2, \omega_1 + \omega_2$ are elements of I, hence their norms are divisible by $N(I)$. The second statement follows by writing $\mathcal{O}_d$ as $\mathbb{Z} + \mathbb{Z}\omega$ and then noting that

$$\begin{vmatrix} \omega_1 & \bar{\omega}_1 \\ \omega_2 & \bar{\omega}_2 \end{vmatrix}^2 = (\mathcal{O}_d : I)^2 \begin{vmatrix} 1 & 1 \\ \omega & \bar{\omega} \end{vmatrix}^2 = N(I)^2 d.$$

The claim implies that there is a unique i and a unique $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$Q_I(\alpha x + \beta y, \gamma x + \delta y) = Q_i(x, y).$$

We can write this as

$$\frac{N(\alpha\omega_1 - \gamma\omega_2)}{N(I)}(x - zy)(x - \bar{z}y) = a_i(x - z_i y)(x - \bar{z}_i y),$$

where

$$z := \frac{-\beta\omega_1 + \delta\omega_2}{\alpha\omega_1 - \gamma\omega_2}. \tag{21}$$

This implies immediately that

$$N(\alpha\omega_1 - \gamma\omega_2) = a_i N(I) > 0. \tag{22}$$

Then a straightforward calculation yields

$$\frac{z - \bar{z}}{\sqrt{d}} = \frac{\alpha\delta - \beta\gamma}{N(\alpha\omega_1 - \gamma\omega_2)} \frac{\overline{\omega}_1\omega_2 - \omega_1\overline{\omega}_2}{\sqrt{d}} > 0$$

which by

$$\frac{z_i - \bar{z}_i}{\sqrt{d}} = \frac{1}{a_i} > 0$$

forces that $z = z_i$. But then (21)–(22) imply that

$$I = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \mathbb{Z}(\alpha\omega_1 - \gamma\omega_2) + \mathbb{Z}(-\beta\omega_1 + \delta\omega_2)$$

is equivalent to

$$\mathbb{Z} + \mathbb{Z}z = \mathbb{Z} + \mathbb{Z}z_i = I_i.$$

Now assume that I_i and I_j are equivalent, i.e. there is some $\mu \in \mathbb{Q}(\sqrt{d})$ such that

$$\mu(\mathbb{Z} + \mathbb{Z}z_i) = \mathbb{Z} + \mathbb{Z}z_j, \qquad N(\mu) > 0.$$

Then we certainly have some $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ such that

$$\mu = \alpha + \beta z_j, \qquad \mu z_i = \gamma + \delta z_j.$$

In particular,

$$z_i = \frac{\gamma + \delta z_j}{\alpha + \beta z_j} \qquad \text{with} \qquad N(\alpha + \beta z_j) > 0.$$

By a straightforward calculation as before,

$$\frac{z_i - \bar{z}_i}{\sqrt{d}} = \frac{\alpha\delta - \beta\gamma}{N(\alpha + \beta z_j)} \frac{z_j - \bar{z}_j}{\sqrt{d}},$$

which shows that

$$\alpha\delta - \beta\gamma = 1 \qquad \text{and} \qquad N(\alpha + \beta z_j) = \frac{z_j - \bar{z}_j}{z_i - \bar{z}_i} = \frac{a_i}{a_j}.$$

Now we obtain

$$a_i(x - z_i y)(x - \bar{z}_i y) = a_j\big((\alpha + \beta z_j)x - (\gamma + \delta z_j)y\big)\big((\alpha + \beta\bar{z}_j)x - (\gamma + \delta\bar{z}_j)y\big),$$

i.e.

$$Q_i(x, y) = Q_j(\alpha x - \gamma y, -\beta x + \delta y), \qquad \begin{pmatrix} \alpha & -\gamma \\ -\beta & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

This clearly implies that $i = j$, since otherwise the forms Q_i and Q_j are inequivalent.

Incidentally, we see that the equivalence class of the associated form $Q_I(x, y)$ only depends on the narrow class of I (in particular, it is independent of the choice of ordered basis of I) and two fractional ideals I and J are in the same narrow class if and only if $Q_I(x, y)$ and $Q_J(x, y)$ are equivalent.

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