Selected solutions to Exercises on Riemannian Geometry
From the book Hamilton's Ricci flow by B. Chow, P. Lu and L. Ni (in preparation).

EXER. 1 SOL. Let $\Phi_\alpha^i = \frac{\partial x^i}{\partial y^\alpha}$ so that $g_{\alpha\beta} = g_{ij}\Phi_\alpha^i \Phi_\beta^j$ and $g^{\gamma\delta} = g^{k\ell}\left(\Phi^{-1}\right)_k^\gamma \left(\Phi^{-1}\right)_\ell^\delta$. We compute

$$\begin{aligned}\Gamma_{\alpha\beta}^\gamma \Phi_\gamma^k &= \frac{1}{2} g^{k\ell}\left(\Phi^{-1}\right)_\ell^\delta \left(\frac{\partial}{\partial y^\alpha}\left(g_{jm}\Phi_\beta^j \Phi_\delta^m\right) + \frac{\partial}{\partial y^\beta}\left(g_{im}\Phi_\alpha^i \Phi_\delta^m\right) - \frac{\partial}{\partial y^\delta}\left(g_{ij}\Phi_\alpha^i \Phi_\beta^j\right)\right) \\ &= \Phi_\alpha^i \Phi_\beta^j \Gamma_{ij}^k + \frac{1}{2} g^{k\ell}\left(g_{j\ell}\frac{\partial}{\partial y^\alpha}\Phi_\beta^j + g_{i\ell}\frac{\partial}{\partial y^\beta}\Phi_\alpha^i - g_{ij}\left(\left(\Phi^{-1}\right)_\ell^\delta \frac{\partial}{\partial y^\delta}\Phi_\alpha^i\right)\Phi_\beta^j\right) \\ &+ \frac{1}{2} g^{k\ell}\left(\Phi^{-1}\right)_\ell^\delta \left(g_{jm}\Phi_\beta^j \frac{\partial}{\partial y^\alpha}\Phi_\delta^m + g_{im}\Phi_\alpha^i \frac{\partial}{\partial y^\beta}\Phi_\delta^m - g_{ij}\Phi_\alpha^i \frac{\partial}{\partial y^\delta}\Phi_\beta^j\right) \\ &= \Phi_\alpha^i \Phi_\beta^j \Gamma_{ij}^k + \frac{\partial^2 x^k}{\partial y^\alpha \partial y^\beta}.\end{aligned}$$

where we used

$$\frac{\partial}{\partial y^\alpha}\Phi_\beta^k = \frac{\partial}{\partial y^\beta}\Phi_\alpha^k = \frac{\partial^2 x^k}{\partial y^\alpha \partial y^\beta}, \quad \frac{\partial}{\partial y^\alpha}\Phi_\delta^i = \frac{\partial}{\partial y^\delta}\Phi_\alpha^i, \quad \frac{\partial}{\partial y^\beta}\Phi_\delta^j = \frac{\partial}{\partial y^\delta}\Phi_\beta^j.$$

EXER. 3 SOL.

$$\nabla_{\dot\gamma}|X|^2 = 2\langle \nabla_{\dot\gamma} X, X\rangle = 0.$$

EXER. 6 SOL. There exists an orthonormal basis $\{e_i\}_{i=1}^n$ such that $\alpha = \sum_{i=1}^n \lambda_i e_i^* \otimes e_i^*$. Furthermore, $\operatorname{Trace}_g(\alpha) = \sum_{i=1}^n \lambda_i$ and

$$\frac{1}{\omega_n}\int_{S^{n-1}} \langle V, e_i\rangle^2 \, d\sigma(V) = 1.$$

EXER. 11 SOL. Let $\psi(t)$ denote the 1-parameter group of diffeomorphisms generated by X

$$\begin{aligned}\varphi^*(\mathcal{L}_X \alpha) &= \varphi^*\left(\lim_{t\to 0}\frac{\psi(t)^*\alpha - \alpha}{t}\right) \\ &= \lim_{t\to 0}\frac{\left(\varphi^{-1}\circ\psi(t)\circ\varphi\right)^*\varphi^*\alpha - \varphi^*\alpha}{t} = \mathcal{L}_Y(\varphi^*\alpha)\end{aligned}$$

where Y is the vector field generating the 1-parameter group of diffeomorphisms $\varphi^{-1}\circ\psi(t)\circ\varphi$. Now

$$\begin{aligned}Y(x) &= \left.\frac{d}{dt}\right|_{t=0}\varphi^{-1}\circ\psi(t)\circ\varphi(x) = \left(\varphi^{-1}\right)_* \left.\frac{d}{dt}\right|_{t=0}\psi(t)\circ\varphi(x) \\ &= \left(\varphi^{-1}\right)_*(X(\varphi(x))) = (\varphi^* X)(x).\end{aligned}$$

For any $x \in M^n$ and $X \in T_{\varphi(x)}M^n$ we have

$$\begin{aligned}\left\langle \varphi^*\left(\operatorname{grad}_g f\right), \varphi^* X\right\rangle_{\varphi^* g}(x) &= \left\langle \operatorname{grad}_g f, X\right\rangle_g(\varphi(x)) \\ &= (Xf)(\varphi(x)) = (\varphi^* X)(f\circ\varphi)(x).\end{aligned}$$

Exer. 12 sol. 2a) We have

$$\nabla R \cdot X = DR_g\left(\mathcal{L}_X g\right) = -2\Delta\operatorname{div}(X) + \operatorname{div}\left(\operatorname{div}\mathcal{L}_X g\right) - \left\langle \mathcal{L}_X g, \mathrm{Rc}\right\rangle. \quad (1)$$

Now

$$\begin{aligned}\left(\operatorname{div}\mathcal{L}_X g\right)_i &= \nabla_j\left(\nabla_j X_i + \nabla_i X_j\right)\\ &= \Delta X_i + \nabla_i \operatorname{div}(X) + R_{ik}X_k\end{aligned}$$

and

$$\operatorname{div}\left(\operatorname{div}\mathcal{L}_X g\right) = \operatorname{div}(\Delta X) + \Delta\operatorname{div}(X) + \operatorname{div}(\mathrm{Rc}(X)).$$

The first term on the LHS may be rewritten as

$$\begin{aligned}\operatorname{div}(\Delta X) &= \nabla_i\nabla_j\nabla_j X_i\\ &= \nabla_j\nabla_i\nabla_j X_i\\ &= \Delta\operatorname{div}(X) + \operatorname{div}(\mathrm{Rc}(X))\end{aligned}$$

(check the second equality). Hence

$$\operatorname{div}\left(\operatorname{div}\mathcal{L}_X g\right) = 2\Delta\operatorname{div}(X) + 2\operatorname{div}(\mathrm{Rc}(X)).$$

Substituting this in (1) we obtain

$$\begin{aligned}\nabla R \cdot X &= 2\operatorname{div}(\mathrm{Rc}(X)) - \left\langle \mathcal{L}_X g, \mathrm{Rc}\right\rangle\\ &= 2\operatorname{div}(\mathrm{Rc}) \cdot X.\end{aligned}$$

Since X is arbitrary, we conclude that $\nabla R = 2\operatorname{div}(\mathrm{Rc})$.

2b) We compute

$$\begin{aligned}\frac{\partial}{\partial s}R_{ijk\ell} &= \frac{1}{2}\left(\nabla_i\nabla_k v_{j\ell} - \nabla_i\nabla_\ell v_{jk} - \nabla_j\nabla_k v_{i\ell} + \nabla_j\nabla_\ell v_{ik}\right)\\ &\quad + \frac{1}{2}\left(R_{ijkq}v_{q\ell} + R_{ijq\ell}v_{qk}\right)\end{aligned}$$

(note the slight change in the formula due to the lowering of an index in Rm.) Thus

$$\begin{aligned}&\frac{1}{2}\begin{pmatrix} \nabla_i\nabla_k\left(\nabla_j X_\ell + \nabla_\ell X_j\right) - \nabla_i\nabla_\ell\left(\nabla_j X_k + \nabla_k X_j\right)\\ -\nabla_j\nabla_k\left(\nabla_i X_\ell + \nabla_\ell X_i\right) + \nabla_j\nabla_\ell\left(\nabla_i X_k + \nabla_k X_i\right)\end{pmatrix}\\ &+ \frac{1}{2}\left(R_{ijkm}\left(\nabla_m X_\ell + \nabla_\ell X_m\right) + R_{ijm\ell}\left(\nabla_m X_k + \nabla_k X_m\right)\right)\\ &= D\,\mathrm{Rm}_g\left(\mathcal{L}_X g\right) = \mathcal{L}_X \mathrm{Rm} \qquad (2)\\ &= X^m\nabla_m R_{ijk\ell} + R_{mjk\ell}\nabla_i X^m + R_{imk\ell}\nabla_j X^m + R_{ijm\ell}\nabla_k X^m + R_{ijkm}\nabla_\ell X^m\end{aligned}$$

First we recognize $\nabla_m R_{ijk\ell}$ as a potential second Bianchi identity term. Next we look at the terms on the first two lines of the above equation. For example the first terms on the top two lines are

$$\begin{aligned}\nabla_i \nabla_k \nabla_j X_\ell - \nabla_j \nabla_k \nabla_i X_\ell &= \nabla_i \nabla_j \nabla_k X_\ell - \nabla_j \nabla_i \nabla_k X_\ell \\ &\quad - \nabla_i \left(R_{kj\ell m} X_m\right) + \nabla_j \left(R_{ki\ell m} X_m\right) \\ &= -R_{ijkm} \nabla_m X_\ell - R_{ij\ell m} \nabla_k X_m \\ &\quad - \nabla_i \left(R_{kj\ell m} X_m\right) + \nabla_j \left(R_{ki\ell m} X_m\right)\end{aligned}$$

Similarly (switch k and ℓ in the above)

$$\begin{aligned}-\nabla_i \nabla_\ell \nabla_j X_k + \nabla_j \nabla_\ell \nabla_i X_k &= R_{ij\ell m} \nabla_m X_k + R_{ijkm} \nabla_\ell X_m \\ &\quad + \nabla_i \left(R_{\ell jkm} X_m\right) - \nabla_j \left(R_{\ell ikm} X_m\right).\end{aligned}$$

Next we look at

$$\nabla_i \nabla_k \nabla_\ell X_j - \nabla_i \nabla_\ell \nabla_k X_j = -\nabla_i \left(R_{k\ell jm} X_m\right)$$

and

$$-\nabla_j \nabla_k \nabla_\ell X_i + \nabla_j \nabla_\ell \nabla_k X_i = \nabla_j \left(R_{k\ell im} X_m\right).$$

Substituting the above into (2), we get

$$\begin{aligned}&- R_{ijkm} \nabla_m X_\ell - R_{ij\ell m} \nabla_k X_m - \nabla_i \left(R_{kj\ell m} X_m\right) + \nabla_j \left(R_{ki\ell m} X_m\right) \\ &+ R_{ij\ell m} \nabla_m X_k + R_{ijkm} \nabla_\ell X_m + \nabla_i \left(R_{\ell jkm} X_m\right) - \nabla_j \left(R_{\ell ikm} X_m\right) \\ &- \nabla_i \left(R_{k\ell jm} X_m\right) + \nabla_j \left(R_{k\ell im} X_m\right) \\ &+ R_{ijkm} \left(\nabla_m X_\ell + \nabla_\ell X_m\right) + R_{ijm\ell} \left(\nabla_m X_k + \nabla_k X_m\right) \\ &= 2X^m \nabla_m R_{ijk\ell} + 2R_{mjk\ell} \nabla_i X^m + 2R_{imk\ell} \nabla_j X^m + 2R_{ijm\ell} \nabla_k X^m + 2R_{ijkm} \nabla_\ell X^m.\end{aligned}$$

Simplifying this yields

$$\begin{aligned}0 &= \nabla_i \left[\left(R_{jk\ell m} + R_{\ell jkm} + R_{k\ell jm}\right) X_m\right] + \nabla_j \left[\left(R_{ki\ell m} + R_{i\ell km} + R_{\ell kim}\right) X_m\right] \\ &\quad - 2 \left(\nabla_i R_{jmk\ell} + \nabla_m R_{ijk\ell} + \nabla_j R_{mik\ell}\right) X_m.\end{aligned}$$

Now we choose X so that at a point $X = 0$ and $\nabla_i X_j = \delta_{ij}$. Then

$$\begin{aligned}0 &= \left(R_{jk\ell i} + R_{\ell jki} + R_{k\ell ji}\right) + \left(R_{ki\ell j} + R_{i\ell kj} + R_{\ell kij}\right) \\ &= 2 \left(R_{jk\ell i} + R_{\ell jki} + R_{k\ell ji}\right)\end{aligned}$$

which implies the first Bianchi identity. Hence the first line vanishes and we conclude

$$\left(\nabla_i R_{jmk\ell} + \nabla_m R_{ijk\ell} + \nabla_j R_{mik\ell}\right) X_m = 0.$$

Since X is arbitrary, we obtain the second Bianchi identity.

EXER. 13 SOL. Using $\nabla_j X_i = -\nabla_i X_j$ and $\nabla_j Y_i = -\nabla_i Y_j$, we compute

$$\begin{aligned}&\nabla_i \left(X_k \nabla_k Y_j - Y_k \nabla_k X_j\right) + \nabla_j \left(X_k \nabla_k Y_i - Y_k \nabla_k X_i\right) \\ &= X_k \left(\nabla_i \nabla_k Y_j + \nabla_j \nabla_k Y_i\right) - Y_k \left(\nabla_i \nabla_k X_j + \nabla_j \nabla_k X_i\right) \\ &= -X_k Y_\ell \left(R_{ikj\ell} + R_{jki\ell}\right) + Y_k X_\ell \left(R_{ikj\ell} + R_{jki\ell}\right) = 0.\end{aligned}$$

Exer. 14 sol. There are only two possible types of nonzero components of W. Either there are 3 distinct indices such as W_{1231} or there are two distinct indices such as W_{1221}. First we compute, using the trace-free property,

$$W_{1231} = -W_{2232} - W_{3233} = 0.$$

Next, we have

$$W_{1221} = -W_{2222} - W_{3223} = -W_{3223} = W_{3113} = -W_{2112} = -W_{1221}$$

which implies $W_{1221} = 0$.

Exer. 20 sol. We compute

$$\begin{aligned}\Delta \left|\nabla f\right|^2 &= \nabla_i \nabla_i \left|\nabla_j f\right|^2 = 2\nabla_i \left(\nabla_i \nabla_j f \nabla_j f\right) \\ &= 2\left|\nabla_i \nabla_j f\right|^2 + 2\Delta \nabla_j f \nabla_j f.\end{aligned}$$

If $\mathrm{Rc} \geq 0$ and $\Delta f = 0$, then

$$\Delta \left|\nabla f\right|^2 \geq 2\left|\nabla\nabla f\right|^2 + 2\,\mathrm{Rc}\left(\nabla f, \nabla f\right) \geq 2\left|\nabla\nabla f\right|^2.$$

Integrating this over M we have

$$0 \geq \int_M \left[\left|\nabla\nabla f\right|^2 + \mathrm{Rc}\left(\nabla f, \nabla f\right)\right] d\mu.$$

Since $\left|\nabla\nabla f\right|^2 \geq 0$ and $\mathrm{Rc}\left(\nabla f, \nabla f\right) \geq 0$, we conclude $\nabla\nabla f \equiv 0$, and $\mathrm{Rc}\left(\nabla f, \nabla f\right) \equiv 0$.

Exer. 25 sol. If f is an eigenfunction with eigenvalue λ, then

$$\begin{aligned}(n-1)\,K \int_{M^n} \left|\nabla f\right|^2 d\mu &\leq \int_{M^n} \mathrm{Rc}\left(\nabla f, \nabla f\right) d\mu \\ &\leq \frac{n-1}{n} \int_{M^n} \left(\Delta f\right)^2 d\mu \\ &= \frac{n-1}{n} \lambda^2 \int_{M^n} f^2 d\mu\end{aligned}$$

and the estimate follows from

$$\int_{M^n} \left|\nabla f\right|^2 d\mu = \lambda \int_{M^n} f^2 d\mu > 0.$$

Exer. 30 sol. Given $s \geq 0$, let $r\left(x\right) \doteqdot d\left(\beta\left(s\right), x\right)$. We have $\nabla_i \nabla_j r \leq \frac{1}{r} g_{ij}$ in the C^2 sense wherever r is smooth and in the sense of support functions where r is not C^∞. From taking the limit as $s \to \infty$ in the definition, we would guess $\nabla_i \nabla_j b_\beta \geq 0$.

Exer. 49 sol. Let $\alpha_r : [0, A] \to M^n$, $r \in [0, r_0]$, be a 1-parameter family of paths with $\alpha_r\left(0\right) = p$, $\alpha_r\left(A\right) = \beta\left(r\right)$ and $\alpha_0 = \alpha$. Consider the smooth barrier function

$$h_\varepsilon\left(r\right) \doteqdot r^2 - \mathrm{L}\left(\alpha_r\right)^2 + A^2 - 2Ar\cos\theta + \varepsilon r.$$

Clearly $f_\varepsilon(r) \geq h_\varepsilon(r)$ and $h_\varepsilon(0) = 0$ since α is minimal. We compute

$$h'_\varepsilon(0) = -2A\cos\theta - 2A \left.\frac{d}{dr}\right|_{r=0} \mathrm{L}(\alpha_r) + \varepsilon = \varepsilon > 0$$

since $\left.\frac{d}{dr}\right|_{r=0} \mathrm{L}(\alpha_r) = \left\langle \dot\beta(0), \dot\alpha(A) \right\rangle = -\cos\theta$ by the first variation formula. Thus $f_\varepsilon(r) \geq h_\varepsilon(r) > 0$ for $r > 0$ sufficiently small.

Selected solutions to Exercises on Ricci Flow, I.

From Hamilton's Ricci flow by B. Chow, P. Lu and L. Ni.

Exer. 6 sol.

$$\frac{\partial}{\partial s}\left(\Delta_{g(s)}\right) = -v_{ij} \cdot \nabla_i \nabla_j - \frac{1}{2} g^{ij} g^{k\ell} \left(\nabla_i v_{j\ell} + \nabla_j v_{i\ell} - \nabla_\ell v_{ij}\right) \nabla_k.$$

In particular if $v_{ij} = \varphi g_{ij}$ for some function φ, then

$$\frac{\partial}{\partial s}\left(\Delta_{g(s)}\right) = -\varphi\Delta + \frac{n-2}{2} \nabla\varphi \cdot \nabla.$$

Exer. 8 sol. Recall $\Delta_d = -(d\delta + \delta d)$. We compute

$$(dX)_{ij} = \nabla_i X_j - \nabla_j X_i$$
$$(\delta dX)_j = -\nabla_i (dX)_{ij} = -\nabla_i (\nabla_i X_j - \nabla_j X_i)$$

and

$$\delta X = -\nabla_i X_i$$
$$(d\delta X)_j = -\nabla_j \nabla_i X_i$$

so that

$$\begin{aligned}(\Delta_d X)_i &= \nabla_i (\nabla_i X_j - \nabla_j X_i) + \nabla_j \nabla_i X_i \\ &= (\Delta X)_j + (\nabla_j \nabla_i - \nabla_i \nabla_j) X_i \\ &= (\Delta X)_j - R_{jk} X_k.\end{aligned}$$

Exer. 10 sol. From

$$\frac{\partial}{\partial t}(\nabla_i X_j) = \nabla_i \left(\frac{\partial}{\partial t} X_j\right) + (\nabla_i R_{jk} + \nabla_j R_{ik} - \nabla_k R_{ij}) X_k$$

and

$$\begin{aligned}\nabla_i (\Delta X_j - R_{jk} X_k) = {} & \Delta(\nabla_i X_j) + 2R_{kij\ell} \nabla_k X_\ell - R_{ik} \nabla_k X_j - R_{jk} \nabla_i X_k \\ & - (\nabla_i R_{jk} + \nabla_j R_{ik} - \nabla_k R_{ij}) X_k\end{aligned}$$

we conclude

$$\begin{aligned}&\nabla_i \left(\frac{\partial}{\partial t} X_j - (\Delta X_j - R_{jk} X_k)\right) + \nabla_j \left(\frac{\partial}{\partial t} X_i - (\Delta X_i - R_{ik} X_k)\right) \\ &= \left(\frac{\partial}{\partial t} - \Delta_L\right)(\nabla_i X_j + \nabla_j X_i).\end{aligned}$$