Exercises on Ricci Flow, I

The following exercises are from Chapters 2, 3 and 4 of the book by Peng Lu, Lei Ni and the lecturer: *Hamilton's Ricci flow,* to be published by Science Press, China.

0.1 Chapter 2 exercises

Standard shrinking sphere.

Exercise 1 *Show that the metrics defined by*

$$g(t) \doteqdot \left(r_0^2 - 2(n-1)t\right) g_{S^n}$$

is a solution to the Ricci flow. Hint*: Use the scale-invariance of Ricci and* $\mathrm{Rc}(g_{S^n}) = (n-1) g_{S^n}$*. Show that the scalar curvature of the solution is given by*

$$R(g(t)) = \frac{n(n-1)}{r_0^2 - 2(n-1)t}.$$

In particular, the solution with $r_0^2 = n(n-1)$*, which is defined on* $(-\infty, n/2)$*, has scalar curvature* $R(g(t)) = \frac{1}{1-2t/n}$.

Homothetic Einstein solutions.

Exercise 2 *Suppose that* g_0 *is an Einstein metric, i.e.,* $\mathrm{Rc}(g_0) \equiv c g_0$ *for some* $c \in \mathbb{R}$*. Derive the explicit formula for the solution* $g(t)$ *of the Ricci flow with* $g(0) = g_0$*. Observe that* $g(t)$ *is homothetic to the initial metric* g_0 *and shrinks, is stationary, or expands depending on whether* c *is positive, zero, or negative, respectively.*

Norm of 2-tensor dominates trace.

Exercise 3 *By choosing coordinates where* $g_{ij} = \delta_{ij}$ *at a point, show that for any 2-tensor* a_{ij}

$$|a_{ij}|_g^2 \geq \frac{1}{n}\left(g^{ij} a_{ij}\right)^2.$$

Variation of the inverse of g.

Exercise 4 *By differentiating the formula* $g^{ij} g_{jk} = \delta_k^i$*, show that*

$$\frac{\partial}{\partial s} g^{ij} = -g^{ik} g^{j\ell} \frac{\partial}{\partial s} g_{k\ell}. \tag{1}$$

Exercise 5 *Given a metric* g_0*, let*

$$\mathcal{C} \doteqdot \{u g_0 : u > 0 \text{ and } \mathrm{Vol}(u g_0) = 1\}$$

be the space of unit volume metrics conformal to g_0*. Show that subject to the constraint of lying in* $\mathcal{C}$*, the critical points* E *have constant scalar curvature.*

Exercise 6 *Given* $\frac{\partial}{\partial s} g_{ij} = v_{ij}$, *compute* $\frac{\partial}{\partial s}\left(\Delta_{g(s)}\right).$

Commutator of $\frac{\partial}{\partial t} + \Delta_L$ and $\nabla\nabla$.

Exercise 7 *Using the formulas derived in the proof that under the Ricci flow, the hessian and the Lichnerowicz laplacian heat operator commute. That is, for any function* f *of space and time we have*

$$\nabla_i \nabla_j \left(\frac{\partial f}{\partial t} - \Delta f \right) = \left(\frac{\partial}{\partial t} - \Delta_L \right) \nabla_i \nabla_j f.$$

Establish under Ricci flow we have the identity

$$\nabla_i \nabla_j \left(\frac{\partial f}{\partial t} + \Delta f \right) = \left(\frac{\partial}{\partial t} + \Delta_L \right) \nabla_i \nabla_j f - 2\left(\nabla_i R_{j\ell} + \nabla_j R_{i\ell} - \nabla_\ell R_{ij} \right) \nabla_\ell f.$$

Bochner.

Exercise 8 *Show that if* X *is a 1-form, then*

$$\Delta X_i - R_{ij} X_j = \Delta_d X_i.$$

In particular, if the Ricci curvature of a closed manifold is positive, then there are no nontrivial harmonic 1-forms. By the Hodge theorem, this implies that the first Betti number $b_1(M)$ *is zero. This is a consequence of Myers' Theorem that the fundamental group of* M *is finite.*

Exercise 9 *Verify if* β *is a 2-form, then*

$$(\Delta_d \beta)_{ij} = \Delta \beta_{ij} + 2 R_{ik\ell j} \beta_{k\ell} - R_{ik}\beta_{kj} - R_{jk}\beta_{ik}$$

where $-(d\delta + \delta d)$. Hint*: using*

$$(d\beta)_{ijk} = \nabla_i \beta_{jk} - \nabla_j \beta_{ik} + \nabla_k \beta_{ij} \qquad (\delta\beta)_k = -\nabla_i \alpha_{ik},$$

show that

$$(\Delta_d \beta)_{jk} = \Delta \beta_{jk} + \left(\nabla_j \nabla_i - \nabla_i \nabla_j \right) \beta_{ik} + \left(\nabla_i \nabla_k - \nabla_k \nabla_i \right) \beta_{ij}$$

and apply the commutator formulas for covariant differentiation.

Exercise 10 *Show that under the Ricci flow, for any* 1*-form* X

$$\left(\frac{\partial}{\partial t} - \Delta_L \right) \left(\mathcal{L}_{X^\natural} g \right) = \mathcal{L}_{\left[\left(\frac{\partial}{\partial t} - \Delta_d\right) X\right]^\natural} g \;;$$

that is,

$$\left(\frac{\partial}{\partial t} - \Delta_L \right) \left(\nabla_i X_j + \nabla_j X_i \right) = \nabla_i Y_j + \nabla_j Y_i,$$

where $Y \doteqdot \left(\frac{\partial}{\partial t} - \Delta_d \right) X$*. Note that by taking* $X = df$*, we obtain*

$$\nabla_i \nabla_j \left(\frac{\partial f}{\partial t} - \Delta f \right) = \left(\frac{\partial}{\partial t} - \Delta_L \right) \nabla_i \nabla_j f.$$

since $\left(\frac{\partial}{\partial t} - \Delta_d \right) df = d \left(\frac{\partial}{\partial t} - \Delta \right) f$.

Exercise 11 *Show that if X is a Killing vector field, then*

$$\nabla_k \nabla_j X_i + R_{\ell kji} X_\ell = 0.$$

Exercise 12 *Calculate the evolution equation for $R_{ij} - \alpha R g_{ij}$, where $\alpha \in \mathbb{R}$.*

Exercise 13 *Verify formula*

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \left(-2R_{ij} + \nabla_i W_j + \nabla_j W_i\right) = \Delta_L v_{ij}.$$

where

$$W_j \doteqdot g_{jk} g^{pq} \left(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}\right)$$

by using

$$\frac{\partial}{\partial s} R_{ij} = -\frac{1}{2}\left(\Delta_L v_{ij} + \nabla_i \nabla_j V - \nabla_i (\operatorname{div} v)_j - \nabla_j (\operatorname{div} v)_i\right),$$

$$\frac{\partial}{\partial s} \Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left(\nabla_i v_{j\ell} + \nabla_j v_{i\ell} - \nabla_\ell v_{ij}\right).$$

and the fact that $W(g(0)) = 0$.

0.2 Chapter 3 exercises

Preservation of Ricci pinching.

Exercise 14

1. *Show that nonnegative sectional curvature $\frac{1}{2} R g_{ij} - R_{ij} \geq 0$ is preserved under the Ricci flow on a closed 3-manifold.*
2. *Show that if $R > 0$, then the inequality $R_{ij} \geq \varepsilon R g_{ij}$ is preserved for any $\varepsilon \geq 0$ (of course, $\varepsilon \leq 1/3$.)*

Variation of Rm.

Exercise 15 *Show that if $\frac{\partial}{\partial s} g_{ij} = h_{ij}$, then*

$$\frac{\partial}{\partial s} R^\ell_{ijk} = \frac{1}{2} g^{\ell p} \left\{ \begin{array}{c} \nabla_i \nabla_j h_{kp} + \nabla_i \nabla_k h_{jp} - \nabla_i \nabla_p h_{jk} \\ -\nabla_j \nabla_i h_{kp} - \nabla_j \nabla_k h_{ip} + \nabla_j \nabla_p h_{ik} \end{array} \right\}. \qquad (2)$$

$n = 3$ - principal sectional curvatures.

Exercise 16 *When $\dim M = 3$, show that at each point there exists an orthonormal frame $\{e_1, e_2, e_3\}$ such that the 2-forms $\varphi_1 \doteqdot e_2^* \wedge e_3^*$, $\varphi_2 \doteqdot e_3^* \wedge e_1^*$, $\varphi_3 \doteqdot e_1^* \wedge e_2^*$ are eigenvectors of Rm. In this case $\lambda = 2\langle \operatorname{Rm}(e_2, e_3) e_3, e_2 \rangle$, $\mu = 2\langle \operatorname{Rm}(e_1, e_3) e_3, e_1 \rangle$, $\nu = 2\langle \operatorname{Rm}(e_1, e_2) e_2, e_1 \rangle$ are twice the sectional curvatures.*

Exercise 17 *Show that if g has constant sectional curvature, then* $\operatorname{Rm} \equiv \frac{2R}{n(n-1)} \operatorname{Id}_{\wedge^2}$.

Exercise 18 *By examining the evolution of the Einstein tensor $\frac{1}{2}Rg_{ij} - R_{ij}$ (whose eigenvalues are the principal sectional curvatures), verify*

$$\frac{\partial}{\partial t} \operatorname{Rm} = \Delta \operatorname{Rm} + \operatorname{Rm}^2 + \operatorname{Rm}^{\#}$$

where

$$\operatorname{Rm}^2 + \operatorname{Rm}^{\#} = \begin{pmatrix} \lambda^2 + \mu\nu & 0 & 0 \\ 0 & \mu^2 + \lambda\nu & 0 \\ 0 & 0 & \nu^2 + \lambda\mu \end{pmatrix}$$

when $n = 3$. Note that one needs to apply Uhlenbeck's trick to the evolution equation for $\frac{1}{2}Rg - \operatorname{Rc}$ to get the exact correspondence.

3d trace-free part of Rc and Rm.

Exercise 19 *Show that when $n = 3$,*

$$\left| \operatorname{Rc} - \frac{1}{3} Rg \right|^2 = \left| \operatorname{Rm} - \frac{1}{3} R \operatorname{Id}_{\wedge^2} \right|^2.$$

Quicker proof with a worse constant.

Exercise 20 *By using the inequality*

$$\left| \nabla_i R_{jk} - \frac{1}{3} \nabla_i R g_{jk} \right|^2 \geq \frac{1}{3} \left| \operatorname{div} \left(\operatorname{Rc} - \frac{1}{3} Rg \right) \right|^2$$

and the contracted second Bianchi identity, show that

$$|\nabla_i R_{jk}|^2 \geq \frac{37}{108} |\nabla_i R|^2,$$

which is weaker than

$$|\nabla_i R_{jk}|^2 \geq \frac{7}{20} |\nabla_i R|^2.$$

∇R estimate again.

Exercise 21 *Let $(M^3, g(0))$ be a closed 3-manifold with positive Ricci curvature. Prove the following variant of the gradient of scalar curvature estimate. There exist constants $\beta_0 > 0$ and $\delta > 0$ depending only on $g(0)$ such that for all $\beta \in [0, \beta_0]$*

$$\frac{|\nabla R|^2}{R^3} \leq \beta R^{-\delta} + C R^{-3} \tag{3}$$

where $C < \infty$ depends only on β and $g(0)$. Hint : *Let*

$$V \doteqdot \frac{|\nabla R|^2}{R} + \frac{37}{2} \left(8\sqrt{3} + 1 \right) \left(|\operatorname{Rc}|^2 - \frac{1}{3} R^2 \right)$$

and show that

$$\frac{\partial}{\partial t}V \leq \Delta V - |\nabla \operatorname{Rc}|^2 + \frac{7400\sqrt{3}+925}{3} R \left(|\operatorname{Rc}|^2 - \frac{1}{3}R^2 \right)$$
$$\leq \Delta V - |\nabla \operatorname{Rc}|^2 + CR^{3-2\delta}$$

where we used

$$\left| \operatorname{Rc} - \frac{1}{3}Rg \right| \leq CR^{1-\delta}$$

to get the last inequality. Then use the equation

$$\frac{\partial}{\partial t}R^{2-\delta} = \Delta\left(R^{2-\delta}\right) - (2-\delta)(1-\delta)R^{-\delta}|\nabla R|^2 + 2(2-\delta)R^{1-\delta}|\operatorname{Rc}|^2$$

to derive

$$\frac{\partial}{\partial t}\left(V - \beta R^{2-\delta}\right) \leq \Delta\left(V - \beta R^{2-\delta}\right) + C$$

where C depends only on β and $g(0)$. Estimate (3) follows from this.

Degrees of tensors. We say that a tensor quantity X depending on the metric g has **degree** k in g if $X(cg) = c^k X(g)$ for any $c > 0$.

Exercise 22 *Show that* Rm *has degree* 1, Rc *has degree* 0, R *has degree* -1, $d\mu$ *has degree* $n/2$ *(if* $\dim M = n$.*)*

Exercise 23 *Show that if* $[0,\tilde{T})$ *is the maximal time interval of existence of the normalized Ricci flow, then* $\int_0^{\tilde{T}} \tilde{r}\left(\tilde{t}\right) d\tilde{t} = \infty$.

Exercise 24 *Suppose that we have the flow* $\frac{\partial x}{\partial t} = -\beta\nu$, *where* β *is some function. Compute the evolution equations for the metric, second fundamental form and mean curvature:* g_{ij}, h_{ij} *and* H.

0.3 Chapter 4 exercises

Exercise 25 *Show that* $\left(S^n \times \mathbb{R}^k, g(t)\right)$, $t \in (-\infty, 0)$, $n \geq 2$, *where*

$$g(t) = 2(n-1)|t|\, g_{S^{n-1}} + g_{\mathbb{E}^k},$$

is a shrinking gradient soliton with

$$f(\theta, x, t) \doteqdot \frac{|x|^2}{4|t|}, \quad \theta \in S^{n-1},\ x \in \mathbb{R},\ t < 0.$$

In a sense, the steady (stationary) euclidean metric is turned into a shrinking soliton by taking the product with a shrinking sphere.

Cigar form. One form of the cigar is:

$$g_\Sigma = ds^2 + \tanh^2 s\, d\theta^2.$$

Exercise 26 *By making the change of variables* $r \doteqdot M\cosh^2 s$*, show that the following is another form of the cigar metric:*

$$g = \left(1 - \frac{M}{r}\right) d\theta^2 + \left(1 - \frac{M}{r}\right)^{-1} \frac{dr^2}{4r^2},$$

where $r > M$*.*

Rosenau tends to round at singularity time. Let $\left(\mathbb{R}\times S^1(2), h\right)$ denote the flat cylinder, where $h = dx^2 + d\theta^2$ and $\theta \in S^1(2) = \mathbb{R}/4\pi\mathbb{Z}$. The Rosenau solution is the solution $g(t) = u(t) \cdot h$ to the Ricci flow defined for $t < 0$ by

$$u(x,t) = \frac{\sinh(-t)}{\cosh x + \cosh t}. \tag{4}$$

Exercise 27 *Show that since* $\lim_{t\to 0} \sinh(-t)\, u(x,t) = \frac{1}{\cosh x + 1}$*, the limit of* $\sinh(-t)\, g(t)$ *as* $t \to 0$ *is the round 2-sphere with scalar curvature* 1 *(radius* $\sqrt{2}$*).* Hint*: use the formula*

$$R_g = u^{-1}\left(R_h - \Delta_h \log u\right).$$

Exponential curvature decay of expanding 2d Ricci soliton.

Exercise 28 *Show that each of the metrics*

$$g(t) = t\left(F(r)^2\, dr^2 + r^2\, d\theta^2\right)$$

where

$$F(r) = \frac{2}{W\left(\left(\frac{2}{F(0)} - 1\right)\exp\left(\frac{2}{F(0)} - 1 - r^2\right)\right) + 1}$$

and $W : \mathbb{R}^+ \to \mathbb{R}^+$ *is the inverse of the function* $w(x) = xe^x$*, is asymptotic at infinity to a flat cone and the curvature decays exponentially as a function of the distance to the origin.* Hint*:* $\lim_{x\to 0_+} \frac{W(x)}{x} = 1$.