Exercises on Riemannian Geometry

The following 50 exercises are from Chapter 1, sections 1-10 of the book by Peng Lu, Lei Ni and the lecturer: *Hamilton's Ricci flow,* to be published by Science Press, China. There is some duplication with the exercises in the summer school lecture notes.

1. Effect of change of coordinates on the Christoffel symbols.

Exercise 1 *Let* $\{x^i\}$ *and* $\{y^\alpha\}$ *be coordinate functions on a common open set and define the components of the metric in these two coordinate systems by*

$$g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j) \quad \text{and} \quad g_{\alpha\beta} = g\left(\partial/\partial y^\alpha, \partial/\partial y^\beta\right).$$

Using $g_{\alpha\beta} = g_{ij}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}$ *show that*

$$\Gamma^\gamma_{\alpha\beta}\frac{\partial x^k}{\partial y^\gamma} = \Gamma^k_{ij}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta} + \frac{\partial^2 x^k}{\partial y^\alpha \partial y^\beta}.$$

The above exercise implies the following.

2. Effect of pull-back on the Christoffel symbols.

Exercise 2 *Verify that if* (M^n, g) *is a Riemannian manifold,* $\varphi : N^m \to M^n$ *is an immersion, and* $\{y^\alpha\}$ *and* $\{x^i\}$ *are local coordinates on* N *and* M *respectively, then*

$$\Gamma\left(\varphi^* g\right)^\gamma_{\alpha\beta}\frac{\partial \varphi^k}{\partial y^\gamma} = \left(\Gamma^k_{ij}\circ\varphi\right)\frac{\partial \varphi^i}{\partial y^\alpha}\frac{\partial \varphi^j}{\partial y^\beta} + \frac{\partial^2 \varphi^k}{\partial y^\alpha \partial y^\beta}$$

where $\varphi^i \doteqdot x^i \circ \varphi$.

3. Parallel vector fields have constant length.

Exercise 3 *Show that if a vector field* X *is parallel along a path* γ, *then* $|X|^2$ *is constant along* γ.

4. Inverse of a metric.

We can globally define a **metric** g^{-1} **on the cotangent bundle** by $g^{-1} \doteqdot g^{ij}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j}$ in any local coordinate system.

Exercise 4 *Show that* g^{-1} *is well-defined.*

5. Scaling properties of the curvatures.

Exercise 5 *Given a metric* g *and a positive constant* C, *show that*

$$\underset{(3,1)}{\mathrm{Rm}}(Cg) = \underset{(3,1)}{\mathrm{Rm}}(g)$$

(as a $(3,1)$*-tensor),*

$$\underset{(4,0)}{\mathrm{Rm}}(Cg) = C\underset{(4,0)}{\mathrm{Rm}}(g), \mathrm{Rc}(Cg) = \mathrm{Rc}(g), \text{ and } R(Cg) = C^{-1}R(g).$$

6. Geometric interpretation of tracing.

Exercise 6 *Show that the trace of a symmetric* 2*-tensor* α *is given by the following formula:*

$$\mathrm{Trace}_g(\alpha) = \frac{1}{\omega_n}\int_{S^{n-1}} \alpha(V,V)\,d\sigma(V)$$

where S^{n-1} *is the unit* $(n-1)$*-sphere,* $n\omega_n$ *its volume, and* $d\sigma$ *its volume form. From this show for any unit vector* U*, that* $\frac{1}{n-1}\mathrm{Rc}(U,U)$ *is the average of the sectional curvatures of planes containing the vector* U*. Similarly,* $\frac{1}{n}R(p)$ *is the average of* $\mathrm{Rc}(U,U)$ *over all unit vectors* $U \in S^{n-1} \subset T_pM^n$*.*

7. Once contracted 2nd Bianchi identity.

Exercise 7 *Show that by multiplying*

$$\nabla_i R_{jk\ell m} + \nabla_j R_{ki\ell m} + \nabla_k R_{ij\ell m} = 0 \tag{1}$$

by g^{im} *and summing (that is, contracting once), we have*

$$g^{im}\nabla_i R_{jk\ell m} = \nabla_j R_{k\ell} - \nabla_k R_{j\ell}. \tag{2}$$

That is, the divergence of Rm *is the exterior covariant derivative of* Rc *considered as a* 1*-form with values in the tangent bundle.*

8. Schur.

Exercise 8

1. *Using* $2g^{ij}\nabla_i R_{jk} = \nabla_k R$*, show that if* g *is an Einstein metric:* $R_{ij} = \frac{1}{n}Rg_{ij}$ *and* $n \geq 3$*, then* R *is a constant. Note that the condition* $R_{ij} = \frac{1}{n}Rg_{ij}$ *says that the Ricci curvatures depend only on the point and not on the line at the point. The result of this exercise says that in this case, if* $n \geq 3$*, then the Ricci curvatures also do not depend on the point.*

2. *Using the second Bianchi identity (1), show that if* $n \geq 3$ *and the sectional curvatures at each point are independent of the 2-plane, that is, if*

$$R_{ijk\ell} = \frac{R}{n(n-1)}\left(g_{i\ell}g_{jk} - g_{ik}g_{j\ell}\right),$$

then R *is a constant.*

9. Lie derivative, I.

Exercise 9 *Given a diffeomorphism* $\varphi : M^n \to M^n$*, we have* $\varphi^* : T^*_{\varphi(p)}M^n \to T^*_pM^n$*. The pull back acts on the tangent bundle by* $\varphi^* = \left(\varphi^{-1}\right)_* : T_{\varphi(p)}M^n \to T_pM^n$*. These actions extend to the tensor bundles of* M^n*. Show that definition*

$$\mathcal{L}_X\alpha \doteqdot \lim_{t\to 0}\frac{1}{t}\left(\alpha - (\varphi_t)_*\alpha\right)$$

is equivalent to

$$\mathcal{L}_X\alpha = \lim_{t\to 0}\frac{1}{t}\left(\varphi_t^*\alpha - \alpha\right) = \left.\frac{d}{dt}\right|_{t=0}\varphi_t^*\alpha.$$

10. Lie derivative of the metric.

Exercise 10 *Using*

$$\begin{aligned} (\mathcal{L}_X \alpha)(Y_1, \ldots, Y_r) &= X(\alpha(Y_1, \ldots, Y_r)) \\ &- \sum_{i=1}^{n} \alpha(Y_1, \ldots, Y_{i-1}, [X, Y_i], Y_{i+1}, \ldots, Y_r) \\ &= (\nabla_X \alpha)(Y_1, \ldots, Y_r) \\ &+ \sum_{i=1}^{n} \alpha(Y_1, \ldots, Y_{i-1}, \nabla_{Y_i} X, Y_{i+1}, \ldots, Y_r), \end{aligned} \tag{3}$$

show that the Lie derivative of the metric is given by

$$(\mathcal{L}_X g)(Y_1, Y_2) = g(\nabla_{Y_1} X, Y_2) + g(Y_1, \nabla_{Y_2} X) \tag{4}$$

and that in local coordinate this implies

$$(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i.$$

In particular, if f is a function, then

$$\left(\mathcal{L}_{\mathrm{grad}_g f}\, g\right)_{ij} = 2\nabla_i \nabla_j f. \tag{5}$$

11. Lie derivative, II.

Exercise 11 *Show that for any diffeomorphism $\varphi : M^n \to M^n$, tensor α, and vector field X*

$$\varphi^*(\mathcal{L}_X \alpha) = \mathcal{L}_{\varphi^* X}(\varphi^* \alpha), \tag{6}$$

and if $f : M^n \to \mathbb{R}$, then

$$\varphi^*\left(\mathrm{grad}_g f\right) = \mathrm{grad}_{\varphi^* g}(f \circ \varphi). \tag{7}$$

12. 2nd Bianchi identity from diffeomorphism invariance of curvature.

Exercise 12

1. *Prove the Jacobi identity*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for vector fields X, Y, Z as follows. Let $\varphi_t : M^n \to M^n$ be the one-parameter group of diffeomorphisms generated by X and take the time derivative at $t = 0$ of the 'invariance of the Lie bracket under diffeomorphism' equation:

$$\varphi_t^*[Y, Z] = [\varphi_t^* Y, \varphi_t^* Z].$$

2. (Hilbert, Kazdan) *Similarly, prove the (contracted) second Bianchi identities by considering the diffeomorphism invariance of the scalar curvature and Riemannian curvature tensor. More precisely, to obtain the contracted second Bianchi identity*

$$2g^{ij}\nabla_i R_{jk} = \nabla_k R$$

apply

$$\frac{\partial}{\partial s}R = -\Delta V + \operatorname{div}(\operatorname{div} v) - \langle v, \mathrm{Rc}\rangle$$

to the equation:

$$DR_g(\mathcal{L}_X g) = \mathcal{L}_X R = \nabla_i R X^i$$

where $DR_g(\mathcal{L}_X g)$ *denotes the linearization of* R_g *in the direction* $\mathcal{L}_X g$ *; to prove the second Bianchi identity (1) apply*

$$\frac{\partial}{\partial s}R_{ijk}^{\ell} = \frac{1}{2}g^{\ell p}\left\{\begin{array}{c} \nabla_i\nabla_j h_{kp} + \nabla_i\nabla_k h_{jp} - \nabla_i\nabla_p h_{jk} \\ -\nabla_j\nabla_i h_{kp} - \nabla_j\nabla_k h_{ip} + \nabla_j\nabla_p h_{ik} \end{array}\right\} \quad (8)$$

to:

$$D\,\mathrm{Rm}_g(\mathcal{L}_X g) = \mathcal{L}_X \mathrm{Rm}.$$

13. Killing vector fields is a Lie algebra.

Exercise 13 *Show directly using the Killing vector field equation that the vector space of Killing vector fields is a Lie algebra.*

14.

Exercise 14 *Show that the Weyl tensor vanishes when* $n = 3$.

15. Schouten and Bach tensors.

Exercise 15 *From*

$$\mathrm{C}\,M = \mathbb{R}g \odot g \oplus \left(S_0^2 M \odot g\right) \oplus \mathrm{W}\,M$$

we have the (reducible) decomposition:

$$CM \cong \left(S^2 M \odot g\right) \oplus \mathrm{W}\,M. \quad (9)$$

1. *Show that*

$$\mathrm{Rm} = \frac{1}{n-2}S \odot g + \mathrm{Weyl}$$

where

$$S \doteqdot \mathrm{Rc} - \frac{R}{2(n-1)}g$$

is the ***Schouten tensor***.

2. Show that if $n \geq 3$, then

$$\nabla^{\ell} W_{ijk\ell} = \frac{n-3}{n-2} B_{ijk}$$

where

$$\begin{aligned} B_{ijk} &\doteqdot \nabla_i S_{jk} - \nabla_j S_{ik} \\ &= \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)} \left(\nabla_i R g_{jk} - \nabla_j R g_{ik} \right) \end{aligned}$$

is the ***Bach tensor****.*

16. Formula for connection 1-forms.

Exercise 16 *Show that*

$$d\omega^k (e_i, e_j) = \omega_i^k (e_j) - \omega_j^k (e_i) .$$

Using this and the first structure equation

$$d\omega^i = \omega^j \wedge \omega_j^i$$

derive the formula for the connection 1-forms:

$$\omega_i^k (e_j) = \frac{1}{2} \left(d\omega^i (e_j, e_k) + d\omega^j (e_i, e_k) - d\omega^k (e_j, e_i) \right) .$$

Note the similarity between this and the formula for the Christoffel symbols

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell} \left(\frac{\partial}{\partial x^i} g_{j\ell} + \frac{\partial}{\partial x^j} g_{i\ell} - \frac{\partial}{\partial x^\ell} g_{ij} \right) .$$

17. 2nd Bianchi - exterior covariant derivative of Rm is zero.

Exercise 17 *Prove*

$$(d_\nabla \operatorname{Rm})_i^{\,j} \doteqdot d \operatorname{Rm}_i^{\,j} - \omega_i^k \wedge \operatorname{Rm}_k^{\,j} + \omega_k^j \wedge \operatorname{Rm}_i^{\,k} = 0.$$

Here $d_\nabla \operatorname{Rm}_i^{\,j}$ is the ***exterior covariant derivative*** *of Rm considered as a 2-form with values in $T^* M^n \otimes T M^n$. This is an equivalent formulation of the second Bianchi identity.*

18. Codazzi equations

Exercise 18 *Show that for X, Y, Z tangent to M^{n-1}*

$$(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = \langle \operatorname{Rm}_P (X, Y) Z, \nu \rangle .$$

19. Hessian and laplacian.

Exercise 19

1. *Show that the above two definitions of* Δ *are the same.* Hint*: show that for any function* f *and vectors* X *and* Y *at a point* p *we have the following formula for the* ***hessian*** $\nabla\nabla f$ *:*

$$\nabla\nabla f\,(X,Y) = X\,(Yf) - (\nabla_X Y)\,f$$

at p *independent of how one extends* X *and* Y *to a neighborhood of* p*. The laplacian is the trace of the hessian.*

2. *Also show that*

$$\Delta f = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial f}{\partial x^j} \right) \tag{10}$$

where $|g| \doteqdot \det g_{ij}$.

20. Bochner formula for $|\nabla f|^2$

Exercise 20 *Show that for any* C^3 *function* f

$$\Delta |\nabla f|^2 = 2\,|\nabla_i \nabla_j f|^2 + 2R_{ij}\nabla_i f \nabla_j f + 2\nabla_i f \nabla_i\,(\Delta f)\,.$$

Conclude from this that if $\mathrm{Rc} \geq 0$, $\Delta f \equiv 0$ *and* $|\nabla f| \equiv 1$, *then* ∇f *is parallel, i.e.,* $\nabla\nabla f \equiv 0$, *and* $\mathrm{Rc}\,(\nabla f, \nabla f) \equiv 0$. *As we shall see, the distance and Busemann functions satisfy* $|\nabla f| = 1$ *a.e.*

21.

Exercise 21 *Show that*

$$\Delta |\nabla f| = \frac{1}{|\nabla f|} \left(\nabla f \cdot \nabla\,(\Delta f) + \mathrm{Rc}\,(\nabla f, \nabla f) + |\nabla\nabla f|^2 - \left| \left\langle \nabla\nabla f, \frac{\nabla f}{|\nabla f|} \right\rangle \right|^2 \right)$$

wherever $|\nabla f| \neq 0$, *and conclude that if* $\mathrm{Rc} \geq 0$, *then*

$$\Delta |\nabla f| \geq \frac{\nabla f}{|\nabla f|} \cdot \nabla\,(\Delta f)\,.$$

In particular, if $\Delta f = 0$, *then*

$$\Delta |\nabla f| \geq 0.$$

22.

Exercise 22 *Show that if* $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ *(Ricci flow), then*

$$\left(\Delta - \frac{\partial}{\partial t} \right) |\nabla f|^2 = 2\,|\nabla_i \nabla_j f|^2 + 2\nabla_i f \nabla_i \left(\left(\Delta - \frac{\partial}{\partial t} \right) f \right).$$

23. Divergence theorem and integration by parts.

Exercise 23 *Derive the following consequences of the divergence theorem.*

1. *On a closed manifold,* $\int_{M^n} \Delta u d\mu = 0$.
2. (Green) *On a compact manifold,*
$$\int_{M^n} (u\Delta v - v\Delta u)\, d\mu = \int_{\partial M^n} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) d\sigma.$$
In particular, on a closed manifold
$$\int_{M^n} u\Delta v d\mu = \int_{M^n} v\Delta u d\mu.$$
3. *Show that if* f *is a function and* X *is a 1-form, then*
$$\int_{M^n} f \operatorname{div}(X)\, d\mu = -\int_{M^n} \langle \nabla f, X \rangle\, d\mu + \int_{\partial M^n} f \langle X, \nu \rangle\, d\sigma.$$

24.

Exercise 24 *Show that on a closed manifold*
$$\int_{M^n} |\nabla\nabla f|^2\, d\mu + \int_{M^n} \mathrm{Rc}(\nabla f, \nabla f)\, d\mu = \int_{M^n} (\Delta f)^2\, d\mu.$$
Since $|\nabla\nabla f|^2 \geq \frac{1}{n}(\Delta f)^2$, *this implies*
$$\int_{M^n} \mathrm{Rc}(\nabla f, \nabla f)\, d\mu \leq \frac{n-1}{n} \int_{M^n} (\Delta f)^2\, d\mu. \tag{11}$$

25. Lichnerowicz.

Exercise 25 *Suppose* f *is an eigenfunction of the laplacian with eigenvalue* λ*:*
$$\Delta f + \lambda f = 0.$$
Use (11) to show that if $\mathrm{Rc} \geq (n-1)\, Kg$, *where* $K > 0$ *is a constant, then*
$$\lambda \geq nK.$$
Equality is obtained by linear functions on the sphere of radius $1/\sqrt{K}$.

26. Busemann function, I.

Exercise 26 *Let* (M^n, g) *be euclidean space. Show that for any unit vector* $V \in \mathbb{R}^n$, *the Busemann function* b_{γ_V} *associated to the geodesic ray* $\gamma_V : [0,\infty) \to \mathbb{R}^n$ *defined by* $\gamma_V(s) \doteqdot sV$ *is the linear function given by*
$$b_{\gamma_V}(x) = \langle x, V \rangle$$
for all $x \in \mathbb{R}^n$.

27. Busemann function, II.

Exercise 27 *Show that given $x \in M^n$, the function $s \mapsto s - d(\beta(s), x)$ is nondecreasing and bounded above by $d(x, \beta(0))$.*

28. Busemann function is Lipschitz.

Exercise 28 *Show that for any Riemannian manifold (M^n, g) and geodesic ray β, the Busemann function b_β satisfies*

$$|b_\beta(x) - b_\beta(y)| \le d(x, y)$$

for all $x, y \in M^n$. I.e., b_β is Lipschitz with Lipschitz constant 1. Note that by Rademacher's theorem, b_β is C^1 a.e.

29. Busemann function, III.

Exercise 29 *Show that $|\nabla b_\beta| = 1$ at points where it is C^1.*

30. Busemann function, IV.

Exercise 30 *Show that if (M^n, g) is a complete Riemannian manifold with nonnegative sectional curvature and β is a ray, then the Busemann function b_β associated to β is convex.*

31. Expansion for volumes of balls.

Exercise 31 *Show that*

$$\mathrm{Vol}(B(p, r)) = \omega_n r^n \left(1 - \frac{R(p)}{6(n+2)} r^2 + O(r^3)\right).$$

32.

Exercise 32 *Show that in geodesic coordinates centered at a point $p \in M$, we have $g_{ij}(p) = \delta_{ij}$ and $\frac{\partial}{\partial x^i} g_{jk}(p) = 0$.*

33.

Exercise 33 *Suppose along a geodesic ray emanating from p we have*

$$\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^i}(p) = \lim_{r \to 0}\left(\frac{1}{r}\frac{\partial}{\partial \theta^i}\right) \doteqdot E_i \in T_p M^n$$

exists and is orthonormal (we shall often assume this in the sequel). Show that

$$\boxed{\lim_{r \to 0} \frac{\mathrm{J}}{r^{n-1}} = 1.} \tag{12}$$

Note that in this case $d\Theta = d\sigma_{S^{n-1}}$ is the volume form of the unit $(n-1)$-sphere. It is convenient to assume the normalization (12) when considering the Jacobian. We shall explicitly say this when we do this.

34.

Exercise 34 *Show that the Gauss Lemma is equivalent to the following statement. If $p \in M$, $V \in T_pM$, and $W \in T_{tV}(T_pM) \cong T_pM$ are such that $\langle W, V \rangle = 0$, then*

$$\left\langle \left(\exp_p\right)_* (W_{tV}), \left(\exp_p\right)_* (V_{tV}) \right\rangle = 0.$$

35.

Exercise 35 *One may think of the distance spheres $S(p,r)$ as evolving under the hypersurface flow $\frac{\partial x}{\partial r} = \nu$, where $\nu = \frac{\partial}{\partial r}$ is the unit outward normal. Show that more generally, if the hypersurfaces are evolving by $\frac{\partial x}{\partial r} = \beta \nu$ for some function β, then $\frac{\partial}{\partial r} d\sigma = \beta H d\sigma$, where $d\sigma = \sqrt{\det g_{ij}}\, d\theta^1 \wedge \cdots \wedge d\theta^{n-1}$ is the volume element of the hypersurface. Show also that*

$$\frac{\partial}{\partial r} g_{ij} = 2\beta h_{ij}.$$

36.

Exercise 36 *Show that for r small enough:*

$$h_{ij} = \frac{1}{r} g_{ij} + O(r)$$
$$H = \frac{n-1}{r} + O(r).$$

37.

Exercise 37 *Show that one can also derive*

$$\Delta = \frac{\partial^2}{\partial r^2} + H \frac{\partial}{\partial r} + \Delta_{S(p,r)} = \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} \log \sqrt{\det g} \frac{\partial}{\partial r} + \Delta_{S(p,r)} \tag{13}$$

directly from

$$\Delta f = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial f}{\partial x^j} \right).$$

38. Evolution of mean curvature for a hypersurface flow.

Exercise 38 *The above formula is a special case (where $\beta = 1$) of the fact that under the hypersurface flow $\frac{\partial x}{\partial r} = \beta \nu$, we have the equation*

$$\frac{\partial}{\partial r} H = -\Delta \beta - |h|^2 \beta - \mathrm{Rc}(\nu, \nu) \beta$$

where the laplacian is with respect to the induced metric on the hypersurface. Prove this. Note that when $\beta = -H$ (the mean curvature flow), we have a heat type equation for H:

$$\frac{\partial}{\partial r} H = \Delta H + |h|^2 H + \mathrm{Rc}(\nu, \nu) H.$$

39. Curvatures of a rotationally symmetric metric.

Exercise 39 *Use moving frames and the Cartan structure equations to derive*

$$K_{\mathrm{rad}} = -\frac{\phi''}{\phi}, \quad \textit{and } K_{\mathrm{sph}} = \frac{1-(\phi')^2}{\phi^2}$$

for $g = dr^2 + \phi(r)^2 g_{S^{n-1}}$.

40. Bishop volume comparison theorem.

Exercise 40 *By proving that*

$$\frac{\mathrm{Vol}(B(p,r))}{\mathrm{Vol}_K(B(p_K,r))}$$

is a nonincreasing function of r*, complete the proof of:*

Theorem. If (M^n, g) is a complete Riemannian manifold with $\mathrm{Rc} \geq (n-1)K$, then for any $p \in M^n$, the volume ratio

$$\frac{\mathrm{Vol}(B(p,r))}{\mathrm{Vol}_K(B(p_K,r))}$$

is a nonincreasing function of r, where p_K is a point in the n-dimensional simply-connected space form of constant curvature K and Vol_K denotes the volume in the space form. In particular

$$\mathrm{Vol}(B(p,r)) \leq \mathrm{Vol}_K(B(p_K,r)) \tag{14}$$

for all $r > 0$. Given p and $r > 0$, equality holds in (14) if and only if $B(p,r)$ is isometric to $B(p_K,r)$.

41.

Exercise 41 *Prove the Mean Value Inequality for* $\mathrm{sect} \leq H$.

Proposition. Suppose that (M^n, g) is a complete Riemannian manifold with $\mathrm{sect}(g) \leq H$ in a ball $B(x,r)$ where $r < \mathrm{inj}(g)$. If $f \in C^\infty(M^n)$ is subharmonic: $\Delta f \geq 0$ and if f is bounded from below on M^n, then

$$f(x) \leq \frac{1}{V_H(r)} \int_{B(x,r)} f \, d\mu$$

where $V_H(r)$ is the volume of a ball of radius r in the complete simply-connected manifold of constant sectional curvature H.

42. Let

$$\mathrm{AVR}(g) \doteqdot \lim_{r\to\infty} \frac{\mathrm{Vol}(B(p,r))}{\omega_n r^n}$$

be the **asymptotic volume ratio**.

Exercise 42 *Show that for* $s \leq r$

$$A(s) \geq n \frac{\operatorname{Vol} B(p,r)}{r^n} s^{n-1} \geq n\omega_n \operatorname{AVR}(g)\, s^{n-1}.$$

43.

Exercise 43 *Show that the Rauch Comparison Theorem may be used to prove the Hessian Comparison Theorem. Similarly, show that the Bishop Volume Comparison Theorem implies the Laplacian Comparison Theorem.*

44.

Exercise 44 *Show that if* $\operatorname{sect}(g) \leq K$*, then* $\nabla_i \nabla_j r \geq h_K(r)\, g_{ij}$.

45.

Exercise 45 *Under the assumption of* $\operatorname{sect}(g) \geq K$ *prove the following hessian comparisons:*

$$\begin{aligned}\nabla\nabla \cos\left(\sqrt{K} r\right) &\geq -K \cos\left(\sqrt{K} r\right) g \text{ if } K > 0\\ \nabla\nabla\left(r^2\right) &\leq 2g \text{ if } K = 0\\ \nabla\nabla \cosh\left(\sqrt{|K|} r\right) &\leq |K| \cosh\left(\sqrt{|K|} r\right) g \text{ if } K < 0.\end{aligned}$$

Here the comparisons are the same in any direction, not just the spherical directions as in the case of the distance function. Note that

$$r^2 = \lim_{r\to 0_+} \frac{2}{K}\left(1 - \cos\left(\sqrt{K} r\right)\right) = \lim_{r \to 0_-} \frac{2}{|K|}\left(\cosh\left(\sqrt{|K|} r\right) - 1\right).$$

The reason the sign in the equality is reversed when $K > 0$ *is because* $\frac{d}{dx}\cos x = -\sin x < 0$.

46.

Exercise 46 *Show that if* $u : M^n \to \mathbb{R}$ *satisfies* $\nabla_V \nabla_V u \leq 0$ *for every* $V \in TM$ *in the sense of support functions, then u is concave; that is, for every unit speed geodesic* $\beta : [a,b] \to M^n$ *we have*

$$u(\beta((1-s)a + sb)) \geq (1-s)\, u(\beta(a)) + s u(\beta(b))$$

for all $s \in [0,1]$.

47.

Exercise 47 *If the sectional curvatures are nonnegative and* $r(x) = d(x,p)$ *is the distance function, then*

$$\nabla_V \nabla_V r \leq \left.\frac{\partial^2}{\partial q^2}\right|_{q=0} \operatorname{L}(\gamma_q) \leq \frac{1}{r}\left|V^{\perp}\right|^2. \tag{15}$$

Generalize this to the case $\operatorname{sect} \geq K$*, where* $K \in \mathbb{R}$.

48.

Exercise 48 *Show that equation (15) implies*

$$\nabla_V \nabla_V \left(r^2\right) \leq 2\left|V\right|^2.$$

49. Toponogov - sect ≥ 0.

Theorem. Let (M^n, g) be a complete Riemannian manifold with nonnegative sectional curvature and $\alpha : [0, A] \to M^n$ be a unit speed minimal geodesic joining p to q. If $\beta : [0, B] \to M^n$ is a unit speed geodesic with $\beta(0) = q$ and if $\theta \in [0, \pi]$ is the angle between $\dot{\beta}(0)$ and $-\dot{\alpha}(A)$ (so that $\cos\theta = \left\langle \dot{\beta}(0), -\dot{\alpha}(A)\right\rangle$), then

$$d(\beta(r), p)^2 \leq r^2 + A^2 - 2Ar\cos\theta$$

for all $r > 0$. In particular of course,

$$d(\beta(B), p)^2 \leq A^2 + B^2 - 2AB\cos\theta. \tag{16}$$

By the law of cosines, equality is attained for euclidean space. That is, the RHS of (16) is the length squared of the side in the corresponding euclidean triangle with the same A, B and θ.

Proof. For $\varepsilon > 0$, let

$$f_\varepsilon(r) \doteqdot r^2 - d(\beta(r), p)^2 + A^2 - 2Ar\cos\theta + \varepsilon r.$$

By the previous lemma, f_ε is convex. We also have $f_\varepsilon(0) = 0$ and by a first variation argument, $f_\varepsilon(r) > 0$ for $r > 0$ and small enough, depending on ε. Note that at a point where the distance function to p is smooth, we actually have

$$f_\varepsilon'(0) = -2d(\beta(0), p)\left\langle \dot{\beta}(0), \dot{\alpha}(A)\right\rangle - 2A\cos\theta + \varepsilon = \varepsilon > 0.$$

Since f_ε is convex, we conclude that $f_\varepsilon(r) > 0$ for all $r > 0$. In particular, $\lim_{\varepsilon\to 0} f_\varepsilon(r) \geq 0$ for all $r > 0$, which proves the theorem. ■

Exercise 49 *Show that given $\varepsilon > 0$, indeed $f_\varepsilon(r) > 0$ for $r > 0$ and small enough.*

50.

Theorem. The heat kernel of hyperbolic space $\mathbb{H}^n$ is given in even dimensions by:

$$h_{2(m+1),-1}(x, y, t) = \left(\frac{-1}{2\pi}\right)^m \frac{\sqrt{2}e^{-(2m+1)^2 t/4}}{(4\pi t)^{3/2}} \left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^m \left(\int_r^\infty \frac{se^{-s^2/4t}}{\sqrt{\cosh s - \cosh r}} ds\right)$$

and in odd dimensions by:

$$h_{2m+1,-1}(x, y, t) = \left(\frac{-1}{2\pi}\right)^m \frac{e^{-m^2 t}}{(4\pi t)^{1/2}} \left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^m \left(e^{-r^2/4t}\right)$$

where $r \doteqdot d(x, y)$.

Exercise 50 *Using the fact that* $\Delta = \frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r}$ *acting on radial functions on* $\mathbb{H}^2$*, show that* $h_{2,-1}(x, y, t)$ *is a solution of the heat equation on* $\mathbb{H}^2$*. Furthermore, show that it is a fundamental solution.*

Note for the metric $g = dr^2 + \phi(r)^2 g_{S^{n-1}}$

$$\Delta = \frac{\partial^2}{\partial r^2} + (n-1)\frac{\phi'}{\phi}\frac{\partial}{\partial r} + \Delta_{S(p,r)}.$$