9 a. Supplement to lecture 9. Classification of ancient solutions on surfaces

The following is an edited version from the book 'Hamilton's Ricci flow' by B. Chow, P. Lu and L. Ni (in preparation).

A priori estimates for Ricci flow usually lead to geometric applications. The Harnack and entropy estimates are no exception. In this section we study ancient solutions on surfaces. Note that we study these ancient solutions in general and we do not assume they necessarily arise from dilating about a singularity of some solution to the Ricci flow. First we consider an application of the Harnack estimate.

Theorem 1 (2-d eternal solutions attaining the sup of their curvatures are cigars) *If* $\left(M^2, g(t)\right)$, $t \in (-\infty, \infty)$, *is a complete solution to the Ricci flow with curvature bounded on compact time intervals and such that* $\sup_{M \times (-\infty,\infty)} R$ *is attained at some point in space and time, then* $\left(M^2, g(t)\right)$ *is either flat or isometric to a constant multiple of the cigar soliton solution.*

Proof. By applying the maximum principle to $\frac{\partial}{\partial t} R = \Delta R + R^2$, we have either $g(t)$ is flat or has positive curvature. We claim that in the latter case $\left(M^2, g(t)\right)$ is a gradient Ricci soliton. The proposition then follows from this claim and the uniqueness of the cigar as the only complete steady gradient Ricci soliton on a noncompact surface with positive curvature.

To prove the claim, let

$$P \doteqdot \frac{\partial}{\partial t} \log R - |\nabla \log R|^2 = \Delta \log R + R.$$

This is the quantity defined in the proof of the Harnack estimate without the $1/t$ term. One can compute that

$$\frac{\partial}{\partial t} P = \Delta P + 2 \langle \nabla \log R, \nabla P \rangle + 2 \left| \nabla\nabla \log R + \frac{1}{2} R g \right|^2, \tag{1}$$

and hence

$$\frac{\partial}{\partial t} P \geq \Delta P + 2 \langle \nabla \log R, \nabla P \rangle + P^2. \tag{2}$$

Here we applied the inequality $|a_{ij}|^2 \geq \frac{1}{n} (\operatorname{tr} a)^2$ to $a = \nabla\nabla \log R + \frac{1}{2} R g$. Since the solution exists on the interval (α, ∞) for any $\alpha \in \mathbb{R}$, the maximum principle says

$$P = \Delta \log R + R \geq -\frac{1}{t - \alpha}$$

for all $t > \alpha$. Hence, at any $(x, t) \in M^2 \times (-\infty, \infty)$, by taking $\alpha \to -\infty$, we see that

$$\Delta \log R + R \geq 0.$$

Now by our hypothesis, there exists $(x_0, t_0) \in M^2 \times (-\infty, \infty)$ such that $R(x_0, t_0) = \sup_{M^2 \times (-\infty,\infty)} R$. At (x_0, t_0) we have $\frac{\partial R}{\partial t} = 0$ and $|\nabla R| = 0$, and hence $P(x_0, t_0) =$

0. Since $P \geq 0$, applying the strong maximum principle to (1) we see that $P \equiv 0$ and hence $\nabla\nabla \log R + \frac{1}{2} R g \equiv 0$ on $M^2 \times (-\infty, \infty)$, which says that $g(t)$ is a gradient Ricci soliton flowing along $\nabla \log R$. ■

Remark 2 *The above result extends to higher dimensions.*

Next we consider an application of the entropy estimate.

Theorem 3 (2-d Type I ancient solutions are round 2-spheres) *If $\left(M^2, g(t)\right)$, $t \in (-\infty, \omega)$, $\omega \leq \infty$, is a solution to the Ricci flow on a complete surface with curvature bounded compact time intervals and such that $\sup_{M \times (-\infty, \omega - 1)} |t| R < \infty$, then $\omega < \infty$ and the universal cover of $\left(M^2, g(t)\right)$ is isometric to either a round shrinking S^2 or the flat $\mathbb{R}^2$.*

Remark 4 *For the noncompact case, see (Theorem 26.1 of Hamilton's formation of singularities paper.)*

Proof. As in Lemma **??**, we have either $R \equiv 0$ or $R > 0$ everywhere; we assume the latter. By passing to the universal cover if necessary, we may assume $M^2 \cong S^2$.Since the area $\mathrm{A}(g(t))$ of M^2 evolves by

$$\frac{d}{dt} \mathrm{A}(g(t)) = -\int_{M^2} R \, d\mu = -8\pi,$$

we have $\mathrm{A}(g(t)) = 8\pi (T - t)$ for some constant T. The entropy is:

$$\begin{aligned} N(g(t)) &\doteqdot \int_{M^2} R \log (R \, \mathrm{A}) \, d\mu \\ &= \int_{M^2} R \log [R (T - t)] \, d\mu + 8\pi \log (8\pi), \end{aligned}$$

which is a nonincreasing function of time. By our assumption, we have

$$\sup_{M \times (-\infty, \omega - 1)} (T - t) R < \infty$$

and hence the limit

$$N_{-\infty} \doteqdot \lim_{t \to -\infty} N(g(t)) \tag{3}$$

exists and is finite.

Now take any sequence of points and times (x_i, t_i) with $R(x_i, t_i) = R_{\max}(t_i)$ and $t_i \to -\infty$, and consider the dilated solutions

$$g_i(t) \doteqdot K_i g\left(t_i + K_i^{-1} t\right), \tag{4}$$

where $K_i \doteqdot R(x_i, t_i)$. Since $0 < R(g_i(0)) \leq 1$, by Klingenberg's injectivity radius estimate (see Theorem 5.9 on p. 98 of [**?**]), $\mathrm{inj}(g_i(0)) \geq \sqrt{2}\pi$. Since $\mathrm{A}(g_i(0)) = 8\pi K_i (T - t_i) \leq C < \infty$, the diameters of $g_i(0)$ are uniformly bounded. Hence, by the Gromov type compactness Theorem **??**, there is a subsequence $\left(M^2, g_i(t), x_i\right)$ which limits to an ancient solution

$$\left(M_{-\infty}^2, g_{-\infty}(t), x_{-\infty}\right)$$

of the Ricci flow on a *closed* surface of positive curvature. The entropy of the limit satisfies

$$N\left(g_{-\infty}(t)\right) = \lim_{i\to\infty} N\left(g_i(t)\right) = \lim_{i\to\infty} N\left(g\left(t_i + K_i^{-1}t\right)\right) \equiv N_{-\infty},$$

since $\lim_{i\to\infty}\left(t_i + K_i^{-1}t\right) = -\infty$ for all $t \in (-\infty, 0]$. In particular, the entropy of the limit is independent of time. By Corollaries **??** and **??**, we conclude that $g_{-\infty}(t)$ is a shrinking round 2-sphere. Since the constant curvature metrics minimize entropy among all metrics on S^2, we have $N_{-\infty} \leq N(g(t))$. But since $N(g(t))$ is nonincreasing, we also have $N_{-\infty} \geq N(g(t))$. Thus $N(g(t)) \equiv N_{-\infty}$ and $g(t)$ is a shrinking 2-sphere of constant curvature. ■

We conclude this section with what happens in the complementary case where $\sup_{M\times(-\infty,\omega-1)} |t| R = \infty$:

Proposition 5 (2d - backward limit of Type II ancient solution is cigar) *If* $\left(M^2, g(t)\right)$, $t \in (-\infty, \omega)$, $\omega \leq \infty$, *is a solution to the Ricci flow on a complete surface with curvature bounded on compact time intervals and such that*

$$\sup_{M\times(-\infty,\omega-1)} |t| R = \infty,$$

then there exists a sequence of points and times (x_i, t_i) *with* $t_i \to -\infty$ *such that* $\left(M^2, g_i(t), x_i\right)$, *where* $g_i(t)$ *is given by (4), limits to a constant multiple of the cigar soliton solution.*

Proof. We first choose any times $T_i \to -\infty$ and positive numbers $\varepsilon_i \to 0$. Since the curvatures are uniformly bounded on compact time intervals, there exists a sequence (x_i, t_i) such that

$$|t_i| (t_i - T_i) R(x_i, t_i) \geq (1 - \varepsilon_i) \sup_{M\times[T_i,0]} |t| (t - T_i) R(x, t). \tag{5}$$

Define

$$\begin{aligned} \alpha_i &\doteqdot (t_i - T_i) R(x_i, t_i) \\ \omega_i &\doteqdot -t_i R(x_i, t_i). \end{aligned}$$

Then

$$\frac{1}{\alpha_i^{-1} + \omega_i^{-1}} = \frac{|t_i| (t_i - T_i) R(x_i, t_i)}{|T_i|} \to \infty$$

so that

$$\lim_{i\to\infty} \alpha_i = \infty = \lim_{i\to\infty} \omega_i.$$

Let $K_i \doteqdot R(x_i, t_i)$ and consider the sequence of dilated metrics

$$g_i(t) \doteqdot K_i g\left(t_i + K_i^{-1} t\right)$$

which are defined on the time intervasl $(-\infty, \omega_i]$. By (5) we have the curvature estimates

$$0 < R(g_i(t)) \leq \frac{\alpha_i \omega_i}{(1-\varepsilon_i)(\alpha_i + t)(\omega_i - t)} \doteqdot f_i(t).$$

Since M^2 is noncompact, by the Gromoll-Meyer injectivity radius estimate, since $0 < K(g_i(0)) \leq \frac{1}{2(1-\varepsilon_i)}$,

$$\operatorname{inj}(g_i(0)) \geq \sqrt{2(1-\varepsilon_i)}\pi.$$

Hence we can apply Hamilton's Cheeger-Gromov compactness theorem for solutions of Ricci flow to conclude that there exists a subsequence $\left(M^2, g_i(t), x_i\right)$ which converges to a limit solution $\left(M_\infty^2, g_\infty(t), x_\infty\right)$. Since on any compact time interval $I \subset (-\infty, \infty)$ we have $f_i(t) \to 1$ uniformly, this limit solution satisfies the curvature estimate

$$0 \leq R(g_\infty(t)) \leq 1$$

for all $t \in (-\infty, \infty)$. Since $R_{g_\infty(0)}(x_\infty) = \lim_{i\to\infty} R_{g_i(0)}(x_i) = 1 > 0$, the strong maximum principle implies $R(g_\infty(t)) > 0$. By Theorem 1, we conclude that $\left(M_\infty^2, g_\infty(t)\right)$ is a cigar soliton solution. ■

In any dimension, an ancient solution $(M^n, g(t))$ is called an **ancient κ-solution** if it is complete, nonflat with bounded nonnegative curvature operator and if for any metric ball $B(x,r) \subset M^n$ with $r > 0$ and such that $|\operatorname{Rm}| \leq r^{-2}$ on $B(x,r)$, we have $\frac{\operatorname{Vol} B(x,r)}{r^n} \geq \kappa$. Since the cigar soliton is not an ancient κ-solution for any $\kappa > 0$, from Propositions 3 and 5, we obtain the following.

Corollary 6 (Hamilton 1995 - $n = 2$, κ-ancient solutions are round S^2) *If $\left(M^2, g(t)\right)$ is an ancient κ-solution, then $\left(M^2, g(t)\right)$ is a round shrinking 2-sphere.*