5a. Additional notes for Lecture 5. From the book Hamilton's Ricci flow by B. Chow, P. Lu and L. Ni (in preparation).

0.1 Local injectivity radius estimate

Definition 1 (κ-noncollapsed) *Given $\rho \in (0, \infty]$ and $\kappa > 0$, we say that a Riemannian manifold (M^n, g) is **κ-noncollapsed below the scale** ρ if for any metric ball $B(x,r)$ with $r < \rho$ satisfying $|\operatorname{Rm}(y)| \leq r^{-2}$ for all $y \in B(x,r)$, we have*

$$\frac{\operatorname{Vol} B(x,r)}{r^n} \geq \kappa.$$

*On the other hand, we say that g is **κ-collapsed** at the scale r if there exists $x \in M^n$ such that $|\operatorname{Rm}(y)| \leq r^{-2}$ for all $y \in B(x,r)$ and*

$$\frac{\operatorname{Vol} B(x,r)}{r^n} \leq \kappa.$$

Theorem 2 (No Local Collapsing) *Let $g(t)$, $t \in [0,T)$, be a smooth solution to the Ricci flow on a closed manifold M^n. If $T < \infty$, then for any $\rho > 0$ there exists $\kappa = \kappa(g(0), T, \rho) > 0$ such that $g(t)$ is κ-noncollapsed below the scale ρ for all $t \in [0,T)$.*

Corollary 3 (Local injectivity radius estimate) *Let $(M^n, g(t))$, $t \in [0,T)$, $T < \infty$, be a solution to the Ricci flow on a closed manifold. For every constant $C < \infty$, there exists a constant $a > 0$ depending only on C, $g(0)$ and T such that if (x_i, t_i) is a sequence of points and times such that*

$$|\operatorname{Rm}[g(t_i)]| \leq CK_i$$

in $B_{g(t_i)}\left(x_i, 1/\left(\sqrt{CK_i}\right)\right)$, where $K_i \doteqdot |\operatorname{Rm}[g(x_i,t_i)]|$, then

$$\operatorname{inj}_{g(t_i)}(x_i) \geq a/\sqrt{K_i}. \tag{1}$$

Proof. Suppose for some $C < \infty$ there does not exist an $a > 0$ such that (1) holds for all i. Then there exists a subsequence such that $|\operatorname{Rm}[g(t_i)]| \leq CK_i$ in $B_{g(t_i)}\left(x_i, (CK_i)^{-1/2}\right)$ and $(CK_i)^{1/2} \operatorname{inj}_{g(t_i)}(x_i) \to 0$. On the other hand, by Perelman's no local collapsing theorem, we have

$$\operatorname{Vol}_{g(t_i)}\left(B_{g(t_i)}(x_i, r)\right) \geq \kappa r^n \tag{2}$$

for all $r \leq 1/\left(\sqrt{CK_i}\right)$. Now let $\varepsilon_i \doteqdot \operatorname{inj}_{g_i}(x_i)$ and consider the corresponding sequence of pointed Riemannian manifolds $\left(B_{h_i}\left(x_i, \varepsilon_i^{-1}\right), h_i, x_i\right)$, where

$$h_i \doteqdot \varepsilon_i^{-2} CK_i g(t_i).$$

By the curvature bounds

$$|\operatorname{Rm}(h_i)| \leq \varepsilon_i^2 \text{ in } B_{h_i}\left(x_i, \varepsilon_i^{-1}\right),$$

the local higher derivative estimates and the injectivity radii being $\operatorname{inj}_{h_i}(x_i) = 1$, we can apply the Cheeger-Gromov compactness theorem for sequences of pointed Riemannian manifolds to obtain a complete flat Riemannian manifold $(M_\infty^n, h_\infty, x_\infty)$. The limit solution is complete since h_i is complete in compact subsets of $B_{h_i}\left(x_i, \varepsilon_i^{-1}\right)$ and $\varepsilon_i^{-1} \to \infty$. The limit is flat since the upper bound ε_i^2 for the norm of the curvature tends to zero. (2) translates to

$$\operatorname{Vol}_{h_i}\left(B_{h_i}(x_i, r)\right) \geq \kappa r^n$$

for all $r \leq \varepsilon_i^{-1}$. Since $\varepsilon_i^{-1} \to \infty$ and h_i converge to h_∞, we conclude

$$\operatorname{Vol}_{h_\infty}\left(B_{h_\infty}(x_\infty, r)\right) \geq \kappa r^n$$

for all $r < \infty$. This implies (M_∞^n, h_∞) is isometric to euclidean space. On the other hand, $\operatorname{inj}_{h_\infty}(x_\infty) = 1$ which is a contradiction. Note that for h_i the conjugate radius at x_i is at least $\pi \varepsilon_i^{-1}$ which is much larger than 1. ■

0.2 Cheeger-Gromov type compactness theorem

In the study of singular solutions, we are interested in taking limits of sequences of solutions of the Ricci flow. Such sequences are obtained by dilating about a sequence of points and times. Before we consider this case of time-dependent metrics, we first consider sequences of pointed Riemannian manifolds. Let $\{(M_i^n, g_i, O_i)\}_{i \in \mathbb{N}}$ be a sequence of complete Riemannian manifolds with a uniform bound on the curvatures over the whole manifolds and a lower bound on the injectivity radius at the origins:

1. $|\operatorname{Rm}(g_i)| \leq C$ on M_i^n
2. $\operatorname{inj}(g_i, O_i) \geq c > 0$.

Since we are interested in smooth convergence, we also assume bounds on the higher covariant derivatives of curvature:

3. $\left|\nabla^k \operatorname{Rm}(g_i)\right| \leq C_k$ on M_i^n, where ∇^k denotes the kth covariant derivative with respect to g_i.

The **Cheeger-Gromov compactness theorem** says the following.

Theorem 4 *There exists a subsequence which* ***converges in*** C^∞ *to a complete Riemannian manifold* $(M_\infty^n, g_\infty, O_\infty)$ *with* $|\operatorname{Rm}(g_\infty)| \leq C$ *on* M_∞^n, $\operatorname{inj}(g_\infty, O_\infty) \geq c > 0$, *and* $\left|\nabla^k \operatorname{Rm}(g_\infty)\right| \leq C_k$ *on* M_∞^n.

By **convergence in** C^∞ we mean that there exists an exhaustion $\{U_i\}_{i \in \mathbb{N}}$ of M_∞^n by open sets and diffeomorphisms $F_i : U_i \to M_i^n$ onto an open subset of M_i^n such that $F_i(O_\infty) = O_i$ and the sequence of metrics $\tilde{g}_i \doteqdot F_i^*(g_i)$ converges to g_∞ on each open set K with compact closure in each C^k-norm[1] (measured with respect to g_∞.) Note that for each K, we have $K \subset U_i$ for i large enough.

A local version of the compactness theorem for the Ricci flow is the following.

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[1] For sequences of solutions of the Ricci flow we give a more technical description of this below.

Theorem 5 (Compactness theorem - local) *Let* $\{(M_i^n, g_i(t), O_i)\}_{i\in\mathbb{N}}$, $t \in [0,T]$, *with* $T > 0$ *be a sequence of complete pointed solutions to the Ricci flow. Let* $p_0 \geq 4$ *be an integer and* $s_0 > 0$. *Suppose that we have:*

(i) the uniform (derivative of) curvature bounds

$$\sup_{B_{g_i(0)}(O_i,s_0)\times[0,T]} |\nabla^q \operatorname{Rm}(g_i(t))| \leq C_{q,s_0} < \infty$$

for all $0 \leq q \leq p_0$, *and*

(ii) an injectivity radius bound

$$\operatorname{inj}_{g_i(0)}(O_i) \geq \delta > 0$$

for all $i \in \mathbb{N}$.

Then there exists $c(n) < \infty$ *and a subsequence of*

$$\left\{\left(B_{g_i(0)}\left(O_i, e^{-c(n)TC_{0,s_0}} s_0\right), g_i(t), O_i\right)\right\}_{i\in\mathbb{N}}, \quad t \in [0,T],$$

which converges to an evolving pointed Riemannian manifold $\{B_\infty^n, g_\infty(t), O_\infty\}$, $t \in [0,T]$, *in the* $C^{p_0-2}(g_\infty(0))$*-topology and* $g_\infty(t)$ *is a solution of the Ricci flow.*

Furthermore, if we assume the global bounds

$$\sup_{M_i^n\times[0,T]} |\nabla^q \operatorname{Rm}(g_i(t))| \leq C_q < \infty$$

for all $0 \leq q \leq p_0$, *instead of (i), then there exists a subsequence of* $\{(M_i^n, g_i(t), O_i)\}_{i\in\mathbb{N}}$, $t \in [0,T]$, *which converges to an evolving pointed complete Riemannian manifold* $\{M_\infty^n, g_\infty(t), O_\infty\}$, $t \in [0,T]$ *in the* $C^{p_0-2}(g_\infty(0))$*-topology and* $g_\infty(t)$ *is a solution of the Ricci flow.*

0.3 Shorter proof of the classification of closed 3-manifolds with positive Ricci curvature

Using the compactness theorem we can shorten Hamilton's proof that closed 3-manifolds M^3 with positive Ricci curvature are diffeomorphic to spherical space forms. However we do not prove here, as Hamilton does, that the solution of the Ricci flow converges *exponentially fast in* C^∞ to a constant positive sectional curvature metric. Instead we prove the convergence after rescaling of a *sequence* of metrics $g(t_i)$ to a constant positive sectional curvature metric for some $t_i \to T$. This is enough to deduce that M^3 is a spherical space form. The basic estimates that the proof relies on are as follows. For a solution to the Ricci flow on a closed 3-manifold with positive Ricci curvature initially:

1. Nonnegative Ricci and positive scalar curvature are preserved.

2. The ‘pinching improves’ estimate.

3. The no local collapsing theorem and the consequent local injectivity radius estimate.

4. The strong maximum principle applied to solutions of the Ricci flow with nonnegative scalar curvature everywhere and positive scalar curvature at a point; this is applied on a limit solution.

5. Shi’s local derivative estimates).

6. The Cheeger-Gromov compactness theorem for sequences of pointed Riemannian manifolds (C^∞ version).

7. The contracted second Bianchi identity.

Theorem 6 *If* $\left(M^3, g_0\right)$ *is a closed 3-manifold with positive Ricci curvature, then* M^3 *is diffeomorphic to a spherical space form.*

Proof. By classical point picking methods, Corollary 3 of the no local collapsing Theorem 2, the derivative of curvature estimates, and the compactness theorem, there exists a sequence of points and times (x_i, t_i) with $t_i \to T$ such that

$$K_i \doteqdot |\mathrm{Rm}\,(x_i, t_i)| = \max_{M^3} |\mathrm{Rm}\,(t_i)|$$

and the rescaled solutions $g_i\,(t) \doteqdot K_i g\left(t_i + K_i^{-1} t\right)$, converge in C^∞ on compact sets to a complete solution $\left(M^3, g_\infty\,(t)\right)$, $t \in (-\infty, \omega)$, $\omega > 0$, with bounded nonnegative Ricci curvature (by the Hamilton-Ivey estimate we actually know that the limit solution has nonnegative sectional curvature, but we do not need this) and $|\mathrm{Rm}\,(x_\infty, 0)| = 1$. (At the moment we do not know whether M^3_∞ is compact or not, this will be proved later.) This implies $R\left(g_\infty\,(x_\infty, 0)\right) > 0$. Since $R\left(g_\infty\,(t)\right) \geq 0$, by the strong maximum principle, we have $R\left(g_\infty\,(t)\right) > 0$ on all of M_∞. Fix any $\rho > 0$ so that we have uniform positive lower bound $R\left(g_\infty\,(0)\right) \geq c > 0$ in $B_{g_\infty(0)}\,(x_\infty, \rho)$. We shall show that $\mathrm{Rc} \equiv \frac{1}{3} R g$ for the metric $g_\infty\,(0)$ in $B_{g_\infty(0)}\,(x_\infty, \rho)$. Recall by the ‘Ricci pinching improves’ estimate, that there exists $\delta > 0$ and $C < \infty$ such that for the original solution $g\,(t)$

$$\frac{\left|\mathrm{Rc} - \frac{1}{3} R g\right|}{R} \leq C R^{-\delta}. \tag{3}$$

Now $g_i\,(0) = K_i g\,(t_i)$ converges to $g_\infty\,(0)$ in C^∞ on compact sets. Hence for i large enough, we have

$$R\,(g_i)\,(x, 0) \geq \frac{1}{2} \inf_{B_{g_\infty(0)}(x_\infty, \rho)} R\left(g_\infty\,(0)\right) \geq \frac{c}{2}$$

for $x \in B_{g_i(0)}\,(x_i, \rho - 1)$. This implies

$$R\left(g\,(t_i)\right) = K_i R\left(g_i\,(0)\right) \geq \frac{c}{2} K_i \text{ in } B_{g_i(0)}\,(x_i, \rho - 1).$$

Now by (3), the scale-invariant measure of the difference from Einstein satisfies:

$$\begin{aligned}\frac{\left|\mathrm{Rc}-\frac{1}{3}Rg\right|}{R}\left(g_i\left(0\right)\right) &= \frac{\left|\mathrm{Rc}-\frac{1}{3}Rg\right|}{R}\left(g\left(t_i\right)\right) \\ &\leq CR\left(g\left(t_i\right)\right)^{-\delta} \leq C\left(\frac{c}{2}K_i\right)^{-\delta}\end{aligned}$$

in $B_{g_i(0)}\left(x_i, \rho-1\right)$. Again using the convergence of $g_i\left(0\right)$ to $g_\infty\left(0\right)$, we find that

$$\frac{\left|\mathrm{Rc}-\frac{1}{3}Rg\right|}{R}\left(g_\infty\left(0\right)\right) \leq 2C\left(\frac{c}{2}K_i\right)^{-\delta}$$

in $B_{g_\infty(0)}\left(x_\infty, \rho-2\right)$ for all i sufficiently large. Since $\lim_{i\to\infty} K_i = \infty$, we have $\left|\mathrm{Rc}-\frac{1}{3}Rg\right|\left(g_\infty\left(0\right)\right) \equiv 0$ in $B_{g_\infty(0)}\left(x_\infty, \rho-2\right)$. Since $\rho > 0$ is arbitrary, we conclude that $\mathrm{Rc} \equiv \frac{1}{3}Rg$ on M^3_∞ for the metric $g_\infty\left(0\right)$. Recall by the contracted second Bianchi identity, this implies $R\left(g_\infty\left(0\right)\right) \equiv \mathrm{const} > 0$. Hence (for example Myers' Theorem can used) M^3_∞ is compact and $\left(M^3_\infty, g_\infty\left(0\right)\right)$ is a spherical space form (in dimension 3 the Weyl tensor is zero so that Einstein metrics have constant sectional curvature). Finally, since M^3_∞ is compact and admits a complete metric which is a limit of a sequence of metrics on M^3, we conclude M^3_∞ is diffeomorphic to M^3. Therefore we have proved the existence of a C^∞ metric with constant positive sectional curvature on M^3. ■