12 a. Supplement to lecture 12. Curvature estimates for the Ricci flow on 3-manifolds with positive Ricci curvature via the maximum principle for systems

The following is an edited version from the book 'Hamilton's Ricci flow' by B. Chow, P. Lu and L. Ni (in preparation).

Recall from Lecture 4 the following result.

Theorem 1 (Maximum principle systems applied to the curvature operator) *Let $g(t)$, $t \in [0,T)$, be a solution to the Ricci flow on a closed manifold M^n. Let $K \subset E \doteqdot \wedge^2 M^n \otimes_S \wedge^2 M^n$ be a subset which is invariant under parallel translation and whose intersection $K_x \doteqdot K \cap E_x$ with each fiber is closed and convex. Suppose the* ODE

$$\frac{d}{dt}\mathbf{M} = \mathbf{M}^2 + \mathbf{M}^{\#}$$

has the property that for any $\mathbf{M}(0) \in K$, we have $\mathbf{M}(t) \in K$ for all $t \in [0,T)$. If $\mathrm{Rm}(0) \in K$, then $\mathrm{Rm}(t) \in K$ for all $t \in [0,T)$.

We now apply the maximum principles to obtain pinching estimates for the Ricci flow on closed 3-manifolds with positive Ricci curvature.

1. Given $C \geq 1/2$, let

$$K = \{\mathbf{M} : \lambda_3(\mathbf{M}) \leq C(\lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}))\}.$$

Since λ_3 is convex ($\lambda_3(\mathbf{M}) = \max_{|V|=1} \mathbf{M}(V,V)$) and $\lambda_1 + \lambda_2$ is concave, we have K_x is convex for all $x \in M$. That each K_x is preserved by the ODE follows from the calculation:

$$\frac{d}{dt}[\lambda_3 - C(\lambda_1 + \lambda_2)] = \lambda_3(\lambda_3 - C(\lambda_1 + \lambda_2)) - C\left(\lambda_1^2 - \frac{1}{C}\lambda_1\lambda_2 + \lambda_2^2\right).$$

In particular, if $\lambda_3 - C(\lambda_1 + \lambda_2) = 0$ and $C \geq 1/2$, then

$$\frac{d}{dt}[\lambda_3 - C(\lambda_1 + \lambda_2)] \leq 0.$$

Suppose $\mathrm{Rc}(g(0)) > 0$. Since M^3 is compact, there exists $C \geq 1/2$ such that at $t = 0$

$$\lambda_3(\mathrm{Rm}) \leq C(\lambda_1(\mathrm{Rm}) + \lambda_2(\mathrm{Rm})). \tag{1}$$

That is, $\mathrm{Rm}(g(0)) \subset K$. By the maximum principle for tensors, $\mathrm{Rm}(g(t)) \subset K$ and inequality (1) is true for all $t \geq 0$. Now (1) implies $\mathrm{Rc} \geq C^{-1}\lambda_3(\mathrm{Rm})\, g \geq \frac{1}{3}C^{-1}Rg$. Thus:

(**Ricci pinching is preserved**) there exists a constant $\varepsilon > 0$ such that

$$\boxed{\mathrm{Rc} \geq \varepsilon R g. \quad (n=3)} \tag{2}$$

In particular, when M^3 is compact, we have that $\mathrm{Rc} > 0$ is preserved.

2. Given $C_0 > 0$, $C_1 \geq 1/2$, $C_2 < \infty$ and $\delta > 0$, let

$$K = \left\{ \mathbf{M} : \begin{array}{c} \lambda_3(\mathbf{M}) - \lambda_1(\mathbf{M}) - C_2(\lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}) + \lambda_3(\mathbf{M}))^{1-\delta} \leq 0 \\ \lambda_3(\mathbf{M}) \leq C_1(\lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M})) \\ \lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}) + \lambda_3(\mathbf{M}) \geq C_0 \end{array} \right\}.$$

K is a convex set since $\lambda_3 - \lambda_1 - C_2(\lambda_1 + \lambda_2 + \lambda_3)^{1-\delta}$ is a convex function for $C_2 > 0$. Observe that if $\mathbf{M} \in K$, then $\lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}) > 0$ by the last two inequalities in the definition of K. We have already seen that the inequalities $\lambda_1 + \lambda_2 + \lambda_3 \geq C_0$ and $\lambda_3 \leq C_1(\lambda_1 + \lambda_2)$ are preserved under the ODE. Since $C_0 > 0$, we can compute

$$\begin{aligned} &\frac{d}{dt}\log\left(\frac{\lambda_3 - \lambda_1}{(\lambda_1 + \lambda_2 + \lambda_3)^{1-\delta}}\right) \\ &= \delta(\lambda_1 + \lambda_3 - \lambda_2) - (1-\delta)\frac{(\lambda_1 + \lambda_2)\lambda_2 + (\lambda_2 - \lambda_1)\lambda_3 + \lambda_2^2}{\lambda_1 + \lambda_2 + \lambda_3} \\ &\leq \delta(\lambda_1 + \lambda_3 - \lambda_2) - (1-\delta)\frac{\lambda_2^2}{\lambda_1 + \lambda_2 + \lambda_3}. \end{aligned}$$

Note that

$$\frac{\lambda_2^2}{\lambda_1 + \lambda_2 + \lambda_3} \geq \frac{1}{6}\frac{(\lambda_1 + \lambda_2)\lambda_2}{\lambda_3} \geq \frac{1}{6C_1}\lambda_2$$

since $\lambda_2 + \lambda_3 \leq 2\lambda_3 \leq 2C_1(\lambda_1 + \lambda_2)$, and we also have

$$\lambda_1 + \lambda_3 - \lambda_2 \leq \lambda_3 \leq C_1(\lambda_1 + \lambda_2) \leq 2C_1\lambda_2.$$

Hence, choosing $\delta > 0$ small enough so that $\frac{\delta}{1-\delta} \leq \frac{1}{12C_1^2}$, we have

$$\frac{d}{dt}\log\left(\frac{\lambda_3 - \lambda_1}{(\lambda_1 + \lambda_2 + \lambda_3)^{1-\delta}}\right) \leq 0.$$

Since $\lambda_3 - \lambda_1 \geq \left|\mathrm{Rc} - \frac{1}{3}Rg\right|$, this implies:

(**Ricci pinching improves**) there exist constants $C < \infty$ and $\delta > 0$ such that

$$\boxed{\left|\mathrm{Rc} - \frac{1}{3}Rg\right| \leq CR^{1-\delta}. \quad (n = 3)} \tag{3}$$