10 b. Supplement to Lecture 10. Ricci flow of left-invariant metrics on 3-dimensional unimodular Lie groups

Proof of Lemma 2, part 2. By the definition of the Riemann curvature tensor

$$\begin{aligned}\langle \mathrm{Rm}(X,Y)Z,W\rangle &= \langle \nabla_X\nabla_Y Z,W\rangle - \langle \nabla_Y\nabla_X Z,W\rangle - \left\langle \nabla_{[X,Y]}Z,W\right\rangle \\ &= X\left\langle \nabla_Y Z,W\right\rangle - \langle \nabla_Y Z,\nabla_X W\rangle \\ &- Y\left\langle \nabla_X Z,W\right\rangle + \langle \nabla_X Z,\nabla_Y W\rangle - \left\langle \nabla_{[X,Y]}Z,W\right\rangle.\end{aligned}$$

The formula for the Riemann curvature tensor now follows from this and the fact that $\langle \nabla_X Z,W\rangle$ is a constant function. ■

10.1 Details for the curvature calculations

The definition $[f_i,f_j]=c_{ij}^k f_k$ implies

$$[e_i,e_j]=\frac{\lambda_k c_{ij}^k}{(\lambda_i\lambda_j\lambda_k)^{1/2}}e_k$$

where $\lambda_1=A$, $\lambda_2=B$ and $\lambda_3=C$. By the formula for the Levi-Civita connection (1) in the Lecture 10 notes, the components of the Levi-Civita connection are

$$\begin{aligned}\langle \nabla_{e_i}e_j,e_k\rangle &= \frac{1}{2}\left(\langle [e_i,e_j],e_k\rangle - \langle [e_i,e_k],e_j\rangle - \langle [e_j,e_k],e_i\rangle\right) \\ &= \frac{1}{2\left(\lambda_i\lambda_j\lambda_k\right)^{1/2}}\left(\lambda_k c_{ij}^k - \lambda_j c_{ik}^j - \lambda_i c_{jk}^i\right).\end{aligned}$$

Substituting this into the formula for Rm (2) in the Lecture 10 notes and since $\nabla_{e_j}e_j=0$, we have

$$\begin{aligned}\langle \mathrm{Rm}(e_i,e_j)e_j,e_i\rangle &= \left\langle \nabla_{e_i}e_j,\nabla_{e_j}e_i\right\rangle - \left\langle \nabla_{e_j}e_j,\nabla_{e_i}e_i\right\rangle - \left\langle \nabla_{[e_i,e_j]}e_j,e_i\right\rangle \\ &= \frac{1}{4\lambda_i\lambda_j\lambda_k}\left(\lambda_k c_{ij}^k - \lambda_j c_{ik}^j - \lambda_i c_{jk}^i\right)\left(\lambda_k c_{ji}^k - \lambda_i c_{jk}^i - \lambda_j c_{ik}^j\right) \\ &- \frac{1}{2\lambda_i\lambda_j\lambda_k}\lambda_k c_{ij}^k\left(\lambda_i c_{kj}^i - \lambda_j c_{ki}^j - \lambda_k c_{ji}^k\right) \\ &= \frac{1}{4\lambda_i\lambda_j\lambda_k}\left(\left(\lambda_i c_{jk}^i - \lambda_j c_{ki}^j\right)^2 - \left(\lambda_k c_{ij}^k\right)^2\right) \\ &+ \frac{2}{4\lambda_i\lambda_j\lambda_k}\lambda_k c_{ij}^k\left(\lambda_i c_{jk}^i + \lambda_j c_{ki}^j - \lambda_k c_{ij}^k\right)\end{aligned}$$

where $\{i,j,k\}$ is a permutation of $\{1,2,3\}$ and we have used the antisymmetry of c_{ij}^k in i and j. Thus the sectional curvatures $K\left(e_i\wedge e_j\right)=\langle \mathrm{Rm}(e_i,e_j)e_j,e_i\rangle$

are given by:

$$K(e_2 \wedge e_3) = \frac{(\mu B - \nu C)^2}{4ABC} + \lambda \frac{2\mu B + 2\nu C - 3\lambda A}{4BC}$$
$$K(e_3 \wedge e_1) = \frac{(\nu C - \lambda A)^2}{4ABC} + \mu \frac{2\nu C + 2\lambda A - 3\mu B}{4AC}$$
$$K(e_1 \wedge e_2) = \frac{(\lambda A - \mu B)^2}{4ABC} + \nu \frac{2\lambda A + 2\mu B - 3\nu C}{4AB}$$

and the $\langle \mathrm{Rm}(e_k, e_i) e_j, e_k \rangle = 0$ for any $i \neq j$ and k. (Exercise 5 in the Lecture 10 notes is a special case of the above.) From this we can easily derive that the Ricci tensor is diagonal and given by:

$$\mathrm{Rc}(e_1, e_1) = \frac{(\lambda A)^2 - (\mu B - \nu C)^2}{2ABC} \tag{1}$$
$$\mathrm{Rc}(e_2, e_2) = \frac{(\mu B)^2 - (\nu C - \lambda A)^2}{2ABC} \tag{2}$$
$$\mathrm{Rc}(e_1, e_1) = \frac{(\nu C)^2 - (\lambda A - \mu B)^2}{2ABC}. \tag{3}$$

10.2 Details for the Ricci flow equation form for left-invariant metrics on 3-dimensional unimodular Lie groups

Hence the Ricci flow equation is equivalent to the following system:

$$\frac{dA}{dt} = \frac{(\mu B - \nu C)^2 - (\lambda A)^2}{BC} \tag{4}$$
$$\frac{dB}{dt} = \frac{(\nu C - \lambda A)^2 - (\mu B)^2}{AC} \tag{5}$$
$$\frac{dC}{dt} = \frac{(\lambda A - \mu B)^2 - (\nu C)^2}{AB}. \tag{6}$$

We note that the normalized Ricci flow is:

$$\frac{dA}{dt} = \frac{-4(\lambda A)^2 + 2(\mu B)^2 + 2(\nu C)^2 - 4\mu B \cdot \nu C + 2\nu C \cdot \lambda A + 2\lambda A \cdot \mu B}{3BC} \tag{7}$$
$$\frac{dB}{dt} = \frac{2(\lambda A)^2 - 4(\mu B)^2 + 2(\nu C)^2 + 2\mu B \cdot \nu C - 4\nu C \cdot \lambda A + 2\lambda A \cdot \mu B}{3BC} \tag{8}$$
$$\frac{dC}{dt} = \frac{2(\lambda A)^2 + 2(\mu B)^2 - 4(\nu C)^2 + 2\mu B \cdot \nu C + 2\nu C \cdot \lambda A - 4\lambda A \cdot \mu B}{3BC} \tag{9}$$

since by summing up (1)-(3) we see that the scalar curvature is

$$R = \frac{-(\lambda A)^2 - (\mu B)^2 - (\nu C)^2 + 2\mu B \cdot \nu C + 2\nu C \cdot \lambda A + 2\lambda A \cdot \mu B}{2ABC}.$$

Taking into account $(ABC)(t) \equiv 8/3$, equations (3)-(5) in the Lecture 10 notes is a special case of (7)-(9).