

10 a. Supplement to lecture 10. Riemannian surfaces (complex curves)

The following is from the book ‘Hamilton’s Ricci flow’ by B. Chow, P. Lu and L. Ni (in preparation).

It is interesting that the Ricci flow for closed 3-manifolds with positive Ricci curvature was understood before the Ricci flow for closed surfaces with positive curvature. One of the reasons is that before Hamilton’s seminal work in 1982, closed 3-manifolds with positive Ricci curvature had not been classified, which motivated their study; whereas closed surfaces with positive curvature have long been known to be diffeomorphic to either S^2 or $\mathbb{R}P^2$ by the Gauss-Bonnet formula and the classification of surfaces. Another reason is that the contracted second Bianchi identity, which says $2 \operatorname{div}(\operatorname{Rc}) = \nabla R$, is a trivial statement when $n = 2$ since then $2 \operatorname{Rc} = Rg$; whereas when $n \geq 3$, it implies that if a metric is Einstein in the sense that $\operatorname{Rc} = \frac{1}{n}Rg$, then R is constant. The reader may think of the Ricci flow on surfaces as a **toy model for developing techniques for the Ricci flow and Kähler-Ricci flow in higher dimensions** (Ricci flow on Kähler manifolds); see [2] and [9].

Let’s first state the main result for surfaces.

Theorem 1 (Surfaces convergence theorem) *If (M^2, g_0) is a closed Riemannian surface, then there exists a unique global solution $g(t)$, $t \in [0, \infty)$, to the **normalized Ricci flow***

$$\frac{\partial}{\partial t}g = (r - R)g \tag{1}$$

with $g(0) = g_0$. As $t \rightarrow \infty$, the metrics $g(t)$ converge exponentially fast in any C^k -norm to a smooth metric g_∞ with constant curvature.

In the proof of the theorem, we may assume M^2 is orientable. The nonorientable case follows from going to the orientable double cover, lifting the solution metrics, and applying the theorem. Recall that under the normalized Ricci flow, $\operatorname{Area}(g(t))$ is independent of time. Thus, by the Gauss-Bonnet formula, the average scalar curvature $r \doteq \operatorname{Area}(g(t))^{-1} \int_{M^2} R_{g(t)} d\mu_{g(t)}$ is also constant in time. Theorem 1 was obtained by Hamilton in 1988 [9] except in the case where $M^2 \cong S^2$ or $\mathbb{R}P^2$ and $R(g_0) \not\equiv 0$. By modifying Hamilton’s techniques in the $R(g_0) > 0$ case, the first author finished the remaining case [7]. See also [1] and [10] for other proofs of the remaining case.

Exercise 2 *Show that under the unnormalized Ricci flow on surfaces, $\frac{dr}{dt} = r^2$.*

The Ricci flow on surfaces is a special case (dimension 1) of the Kähler-Ricci flow. In 1985 Huai-Dong Cao [2] proved that for closed Kähler manifolds whose first Chern class is any real multiple of the Kähler class, a solution to the normalized flow exists for all time. When $c_1 = 0$ or $c_1 < 0$, he proved the convergence to a Kähler-Einstein metric in the same Kähler class. This gave a new proof of

the Calabi Conjecture, originally proved by S.-T. Yau in 1976 [13] using elliptic methods (in particular, new a priori estimates and the continuity method). In the $c_1 > 0$ case, if the initial metric has positive bisectional curvature, then Chen and Tian [5], [6] proved the convergence to the Fubini-Study metric on $\mathbb{C}P^n$. Their proof assumes the existence of a Kähler-Einstein metric. Without having to make this assumption, Cao-Chen-Zhu [3] gave another proof of the weaker result that the curvature is uniformly bounded. In general [12], only assuming that $c_1 > 0$ and is a positive multiple of the Kähler class, Perelman [12] announced, and gave a sketch of the proof, that the scalar curvature and diameter are both uniformly bounded (this claim has been independently verified). **A fundamental open problem is to understand the limiting behavior of solutions to the Kähler-Ricci flow on manifolds with $c_1 > 0$.**

We note for future reference that if h is a fixed metric on M^2 and $g(t) = u(t)h$, then since

$$R_g = u^{-1} (R_h - \Delta_h \log u), \quad (2)$$

the Ricci flow equation $\frac{\partial}{\partial t} g = -Rg$ is equivalent to

$$\frac{\partial u}{\partial t} = \Delta_h \log u - R_h. \quad (3)$$

When h is flat we have

$$\frac{\partial u}{\partial t} = \Delta_h \log u. \quad (4)$$

As noted by Angenent in the appendix to [?], this equation is a formal limit of the **porous medium equation**

$$\frac{\partial u}{\partial \tau} = \Delta(u^m), \quad (5)$$

where $m > 0$, as $m \rightarrow 0$. More precisely, if we consider the new time variable $t \doteq m\tau$, then we may rewrite (5) as

$$\frac{\partial u}{\partial t} = \Delta_h \left(\frac{u^m - 1}{m} \right)$$

and the limit of these equations as $m \rightarrow 0$ is (4).

Exercise 3 Let h be the flat metric on \mathbb{R}^2 . Show that u is a radial solution of (4) if and only if

$$\frac{\partial u}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left(r u^{-1} \frac{\partial u}{\partial r} \right) \quad (6)$$

where $r = |x|$. Look for solutions of the form

$$u(x, t) = e^{-2at} f(re^{-at})$$

where $a \in \mathbb{R}$. Making the substitution $s = re^{-at}$ show that (6) becomes

$$-as^2 f(s) = s \frac{f'(s)}{f(s)} + b$$

for some constant b . If $b \neq 2$, then

$$f(s) = \left(cs^b + \frac{a}{2-b} s^2 \right)^{-1}$$

where $c \in \mathbb{R}$ is a solution. In terms of u this is

$$u(x, t) = e^{-2at} \left(ce^{-abt} |x|^b + \frac{a}{2-b} e^{-2at} |x|^2 \right)^{-1}$$

If $b = 2$, then we have

$$f(s) = (cs^2 + as^2 \ln s)^{-1}$$

which corresponds to the solution

$$u(x, t) = |x|^{-2} (c - a^2 t + a \ln |x|)^{-1}.$$

For which a, b, c do we obtain the cigar soliton?

Exercise 4 Given a rotationally symmetric metric on a surface

$$g = ds^2 + f(s)^2 d\theta^2,$$

change coordinates so that

$$g = u(r) (dr^2 + r^2 d\theta^2).$$

In the case of the hyperbolic cusp metric where $f(s) = e^{-s}$.

SOLUTION. If $s = h(r)$, then

$$g = [h'(r)]^2 \left(dr^2 + \frac{f(h(r))^2}{[h'(r)]^2} d\theta^2 \right).$$

Thus we impose the ODE

$$\frac{h'(r)}{f(h(r))} = \frac{1}{r}$$

so that

$$g = [h'(r)]^2 (dr^2 + r^2 d\theta^2).$$

In the case of the hyperbolic cusp metric $g = ds^2 + e^{-2s} d\theta^2$ the ODE is

$$e^{h(r)} h'(r) = \frac{1}{r}$$

to which a solution is

$$e^{h(r)} = \ln r.$$

Thus

$$h'(r) = \frac{1}{r} e^{-h(r)} = \frac{1}{r \ln r}$$

and the hyperbolic cusp metric may be written as

$$g = \frac{1}{r^2 \ln^2 r} (dr^2 + r^2 d\theta^2).$$

10.1 Curvature estimates when $\chi < 0$

We first describe the case when $\chi(M^2) < 0$, that is, when the genus of the surface (assuming it is orientable) is at least 2. By the Gauss-Bonnet formula, this is the same as $r < 0$. Here the maximum principle works well as we shall see by the following estimates. In this section we consider the *normalized Ricci flow*. The **evolution of the scalar curvature** is given by

$$\frac{\partial}{\partial t} R = \Delta R + R(R - r). \quad (7)$$

(This is a special case of the variation formula: if $\frac{\partial}{\partial s} g_{ij} = -\phi g$, then $\frac{\partial}{\partial s} R = \Delta \phi + R\phi$ when $n = 2$.) Writing this in the more transparent form

$$\begin{aligned} \frac{\partial}{\partial t} (R - r) &= \Delta (R - r) + (R - r)^2 + r(R - r) \\ &\geq \Delta (R - r) + r(R - r) \end{aligned} \quad (8)$$

and applying the maximum principle, we have the **curvature lower bound**

$$R - r \geq -Ce^{rt}, \quad (9)$$

which decays exponentially since $r < 0$.

Exercise 5 Show that under the unnormalized Ricci flow, since $\frac{\partial}{\partial t} R = \Delta R + R^2$, if $R - r \equiv 0$, then $r = 1/\tau$, where $\tau \doteq T - t$ and T is the singularity time.

To obtain a good upper estimate is more difficult since solutions to the ODE

$$\frac{d\rho}{dt} = \rho(\rho - r)$$

with $\rho(0) > 0 > r$ blow up in finite time. To remedy this we perform a clever trick which is also quite useful for the Kähler-Ricci flow. Define the **potential function** f on M by

$$\Delta f \doteq r - R. \quad (10)$$

This is possible because $\int_{M^2} (r - R) d\mu = 0$. f is determined additively up to a function of time. Choosing f appropriately, we can show that

$$\frac{\partial}{\partial t} f = \Delta f + rf. \quad (11)$$

Considering f as a unknown function (which it is not), it satisfies a linear heat equation with respect to the time-dependent metric $g(t)$. Hence we may readily apply the maximum principle. We can compute

$$\frac{\partial}{\partial t} |\nabla f|^2 = \Delta |\nabla f|^2 - 2|\nabla \nabla f|^2 + r|\nabla f|^2. \quad (12)$$

Combining this with (8), we find that the function

$$H \doteq R - r + |\nabla f|^2 \quad (13)$$

satisfies

$$\begin{aligned}\frac{\partial}{\partial t}H &= \Delta H - 2 \left| \nabla \nabla f - \frac{1}{2} \Delta f \cdot g \right|^2 + rH \\ &\leq \Delta H + rH.\end{aligned}\tag{14}$$

The good quadratic term in (12) dominates the bad quadratic term in (8). Thus by the maximum principle we have the **curvature upper bound**

$$R - r \leq R - r + |\nabla f|^2 \leq Ce^{rt}\tag{15}$$

and we have the exponentially decaying upper bound for $R - r$. Combining this with (9) we have the **exponential convergence of the scalar curvature to a constant**

$$|R - r| \leq Ce^{rt}.\tag{16}$$

The geometric motivation for considering the quantity H comes from gradient Ricci solitons.

One can also show, by applying the maximum principle, that all the derivatives of R decay exponentially. In particular, the quantity

$$\Phi \doteq |\nabla^k R|^2 - (k+1)r|\nabla^{k-1}R|^2$$

satisfies the evolution inequality

$$\frac{\partial}{\partial t}\Phi \leq \Delta\Phi + \frac{kr}{2}\Phi + Ce^{rt}$$

for some constant $C < \infty$. (See [8], Proposition 5.27 on pp. 122-3 for details.) This implies that there exist constants $C_k < \infty$ such that

$$|\nabla^k R| \leq C_k e^{rt/2}.\tag{17}$$

Define the conformal factor $v(x, t)$ by

$$g(t) \doteq e^{v(t)}g_0.$$

From this estimate and using the fact that $g(t)$ are uniformly equivalent to g_0 , we can show that for all $k \geq 1$

$$\left| \frac{\partial}{\partial t} (\nabla_{g_0}^k v(t)) \right|_{g_0} = |\nabla_{g_0}^k R_{g(t)}|_{g_0} \leq C_k e^{rt/2}$$

for some other constants $C_k < \infty$. From this we conclude that there exists a smooth function $v_\infty : M^2 \rightarrow \mathbb{R}$ such that $v(t)$ converges exponentially fast to v_∞ in each C^k norm, $k \in \mathbb{N}$. Moreover $g_\infty \doteq e^{v_\infty}g_0$ has constant curvature r since $R(g_\infty) = \lim_{t \rightarrow \infty} R(g(t))$. Thus Theorem 1 follows when $\chi(M^2) < 0$.

10.2 Surfaces with $\chi = 0$

Now we consider the Ricci flow on closed surfaces with $\chi(M^2) = 0$, that is, when M^2 is diffeomorphic to a torus or a Klein bottle. From (2) we see that to obtain a flat metric $g = e^v k$ we just need to solve the equation $\Delta_k v = R_k$. This is possible since $\int_{M^2} R_k d\mu_k = 0$. Now let h be a flat metric on M^2 . The Ricci flow equation is

$$\frac{\partial v}{\partial t} = e^{-v} \Delta_h v.$$

Immediately, from the maximum principle we see that

$$-C_1 \leq v \leq C_2,$$

where C_1 and C_2 are constants such that these inequalities hold at $t = 0$. Thus all of the **metrics $g(t)$ are uniformly equivalent**. In particular, their diameters and Sobolev constants are uniformly bounded.

Remark 6 *Note that we have the following alternate proof that the metrics are uniformly equivalent. Since the potential function satisfies $\frac{\partial}{\partial t} f = \Delta f$, we have $|f| \leq C$. From $\frac{\partial}{\partial t} f = -R$ we can directly conclude from $\frac{\partial}{\partial t} g = -Rg$ that*

$$e^{-C} g_0 \leq g(t) \leq e^C g_0. \quad (18)$$

To obtain better estimates, we first consider the **relative energy**, which is defined in general for metrics h and $g = e^v h$ (pointwise conformal to each other) on a closed surface M^2

$$\begin{aligned} E_h(g) &= \int_{M^2} \log(g/h) (R_g d\mu_g + R_h d\mu_h) \\ &= \int_{M^2} (|\nabla v|_h^2 + 2R_h v) d\mu_h. \end{aligned} \quad (19)$$

Here we integrated by parts and used (2) with $u = e^v$ to get the second equality. This energy is also the same as the determinant of the laplacian defined using the zeta function regularization:

$$E_h(g) = -48\pi \left(\log \frac{\det \Delta_g}{\det \Delta_h} - \log \frac{\text{Area}(g)}{\text{Area}(h)} \right).$$

(Compare with [4].) A calculation shows that under the normalized Ricci flow, we have the **surface energy monotonicity formula**:

$$\begin{aligned} \frac{d}{dt} E_h(g(t)) &= 2 \int_{M^2} (-\Delta_h v + R_h) (r - R_{g(t)}) d\mu_h \\ &= -2 \int_{M^2} (R_{g(t)} - r)^2 d\mu_{g(t)}, \end{aligned} \quad (20)$$

where we integrated by parts and used (2).

In particular, if $\chi(M^2) = 0$ and h is a flat metric, then $E_h(g) = \int_{M^2} |\nabla v|_h^2 d\mu_h$ and

$$\frac{d}{dt} E_h(g(t)) = -2 \int_{M^2} R_{g(t)}^2 d\mu_{g(t)} = -2 \int_{M^2} e^{-v} (\Delta_h v)^2 d\mu_h. \quad (21)$$

Since

$$E_h(g(t)) \leq C \int_{M^2} (\Delta_h v)^2 d\mu_h,$$

where $C = 1/\lambda_1(\Delta_h)$, and $v \leq C_2$, there exists a constant $c > 0$ such that

$$\frac{d}{dt} E_h(g(t)) \leq -c E_h(g(t)).$$

We conclude that

$$\int_{M^2} |\nabla v|_h^2 d\mu_h \leq C e^{-ct} \quad (22)$$

for some $C < \infty$. From (21) and (22) we deduce that for any $t_* < \infty$

$$C e^{-ct_*} \geq \int_{t_*}^{\infty} \int_{M^2} e^{-v} (\Delta_h v)^2 d\mu_h dt = \int_{t_*}^{\infty} \int_{M^2} R_{g(t)}^2 d\mu_{g(t)} dt.$$

for some constants $C < \infty$. From this we can deduce that (see [9], section 5 for details)

$$\int_{M^2} R_{g(t)}^2 d\mu_{g(t)} \leq C e^{-ct},$$

which implies

$$\int_{M^2} |\nabla_h \nabla v|_h^2 d\mu_h = \int_{M^2} (\Delta_h v)^2 d\mu_h \leq C e^{-ct}.$$

From this one can show that v tends to a constant ($\log(\int e^v d\mu_h / \int d\mu_h)$) exponentially fast. We leave it to the reader to show that all higher derivatives of v also tend to 0 exponentially fast. This implies the exponential convergence of $g(t)$ in every C^k norm to the unique flat metric $C \cdot h$ with the same area and conformal to g_0 .

It is interesting to note that when $\chi = 0$ (i.e., $r = 0$) by using only the maximum principle one can prove the following estimates (see [8], Proposition 5.33 on p. 127 for details):

$$|\nabla^k R| \leq C_k (1+t)^{-\frac{k}{2}-1}$$

where $C_k < \infty$ for all $k \geq 0$. Since the diameters of the metrics $g(t)$ are uniformly equivalent, and $\int_{M^2} R d\mu = 0$, we conclude

$$|R(x_1, t) - R(x_2, t)| \leq \text{diam}(g(t)) \max_{M^2} |\nabla R| \leq C (1+t)^{-3/2}.$$

These estimates are sufficient to prove the convergence of $g(t)$ to a flat metric, but not the exponential convergence provided by the integral estimates above.

10.3 Convergence when $R > 0$

The third main estimate for the Ricci flow on surfaces with positive curvature follows from the following formula. Let

$$J_{ij} \doteq \nabla_i \nabla_j f - \frac{1}{2} \Delta f \cdot g_{ij}.$$

Then under the normalized Ricci flow

$$\boxed{\frac{\partial}{\partial t} |J|^2 = \Delta |J|^2 - 2 |\nabla J|^2 - 2R |J|^2.} \quad (23)$$

(See section 9 of [9] or [8], Corollary 5.35 on p. 130.) This motivates us to try to obtain a uniform positive lower bound for R . For now we have

Lemma 7 ($R \geq c_0 > 0$ implies exponential decay of measure of difference from gradient soliton)
If $R \geq c_0 > 0$, then

$$|J| \leq C_0 e^{-c_0 t}. \quad (24)$$

The combination of the entropy and Harnack estimates (before (24)) enables us to prove, using standard facts in geometry such as the volume comparison theorem and Klingenberg's injectivity radius estimate, that under the normalized Ricci flow the scalar curvatures of $g(t)$ are uniformly bounded from above for all time. In particular, since $x \log x \geq -1/e$ for all $x > 0$, we have by the entropy estimate that for any subset $D \subset M^2$

$$\int_D \log R R d\mu \leq \int_{M^2} \log R R d\mu + \int_{M^2-D} \frac{1}{e} d\mu \leq C. \quad (25)$$

Given any $t_0 \in [0, \infty)$, let $x_0 \in M^2$ be such that $R(x_0, t_0) = R_{\max}(t_0)$. Now Klingenberg's injectivity radius estimate says, since $(M^2, g(t_0))$ is a closed surface with positive curvature, we have

$$\text{inj}(x_0, t_0) \geq \pi / \sqrt{R_{\max}(t_0) / 2}$$

so that $B \doteq B_{g(t_0)}(x_0, 1/\sqrt{R_{\max}(t_0)})$ is embedded. By the volume comparison theorem, since $R(x, t_0) \leq R_{\max}(t_0)$, we have the area of B is bounded from below by the area of the ball of radius $1/\sqrt{R_{\max}(t_0)}$ in the 2-sphere of constant scalar curvature $R_{\max}(t_0)$, so that

$$\text{Area}_{g(t_0)}(B) \geq \varepsilon R_{\max}(t_0)^{-1},$$

where $\varepsilon > 0$ is a universal constant. Now either using the Harnack inequality or the Bernstein-Bando-Shi estimate, we have $R(x, t_0 + \varepsilon R_{\max}(t_0)^{-1}) \geq \varepsilon R_{\max}(t_0)$ in $B(x_0, \varepsilon/\sqrt{R_{\max}(t_0)})$ for a universal constant $\varepsilon > 0$ sufficiently small. Choose this constant ε small enough so that we also have

$$\text{Area}_{g(t_0 + \varepsilon R_{\max}(t_0)^{-1})}(B) \geq \frac{1}{2} \text{Area}_{g(t_0)}(B).$$

Hence inequality (25) with $D = B$ at time $t_0 + \varepsilon R_{\max}(t_0)^{-1}$ implies

$$\log(\varepsilon R_{\max}(t_0)) \varepsilon R_{\max}(t_0) \varepsilon R_{\max}(t_0)^{-1} \leq 2C$$

and we conclude that for all $t_0 \in [0, \infty)$

$$R_{\max}(t_0) \leq C$$

for some $C < \infty$ independent of t_0 . Since the area of $g(t)$ is constant and we now have a uniform injectivity radius estimate for all time (by Klingenberg and the uniform curvature bound), we conclude that the diameter is uniformly bounded. To see this take a geodesic with length equal to the diameter of M^2 and consider a chain of balls of radius 1; the diameter bound follows since each ball has area bounded below by a constant. Given the diameter bound, we can now apply the Harnack inequality again to show that the scalar curvatures are uniformly bounded from below by a positive constant: $R \geq c_0$ for some constant $c_0 > 0$. The consequent estimate (24) for J , plus higher derivative estimates (see [9]):

$$|\nabla^k J| \leq C_k e^{-c_k t},$$

implies the following convergence result. Consider the normalized **Ricci flow modified by diffeomorphisms** (or **modified Ricci flow**):

$$\frac{\partial}{\partial t} g_{ij} = (r - R) g_{ij} - (\mathcal{L}_{\nabla f} g)_{ij} = -2J_{ij}. \quad (26)$$

Since the solutions to (26) differ from the solutions to the normalized Ricci flow by the pull back by diffeomorphisms, we also have under (26)

$$|J| \leq C_0 e^{-c_0 t}.$$

and for each $k \in \mathbb{N}$

$$|\nabla^k J| \leq C_k e^{-c_k t},$$

for some $C_k < \infty$ and $c_k > 0$ depending only on g_0 . Hence, we have

Proposition 8 (Convergence of modified RF to Ricci soliton) *If (M^2, g_0) is a closed Riemannian surface with $R(g_0) > 0$, then there exists a unique solution $g(t)$ of (26) with $g(0) = g_0$ which converges exponentially in any C^k -norm to a smooth shrinking gradient Ricci soliton metric g_∞ as $t \rightarrow \infty$.*

Proof. $J_{ij} = 0$ if and only if $R_{ij} + \nabla_i \nabla_j f - \frac{r}{2} g_{ij} = 0$. ■

It turns out that this gradient Ricci soliton metric has constant curvature. An easy way to see this is to use the **Kazdan-Warner identity**. Since $\chi(M^2) > 0$, by going to the universal cover if necessary, we may assume $M^2 \cong S^2$.

Proposition 9 (Kazdan-Warner, [11]) *If X is a conformal Killing vector field, then*

$$\int_{S^2} \langle \nabla R, X \rangle d\mu = \int_{S^2} R \operatorname{div} X d\mu = 0.$$

Corollary 10 (Ricci solitons on S^2 have constant curvature) *If (S^2, g_∞) is a Ricci soliton metric, then g_∞ is a metric of constant curvature.*

Proof. Tracing the Ricci soliton equation $(r - R)g_{ij} = \nabla_i X_j + \nabla_j X_i$ yields

$$r - R = \operatorname{div} X,$$

and hence

$$-\int_{S^2} (R - r)^2 d\mu = \int_{S^2} R \operatorname{div} X d\mu = 0$$

by the Kazdan–Warner identity. Hence $R \equiv r$, and the corollary is proved. ■

Exercise 11 *Let $\{\varphi_t\}$ be a 1-parameter family of conformal diffeomorphisms which generate the vector field X . Show that*

$$\left. \frac{d}{dt} \right|_{t=0} [E(\varphi_t^* g, g)] = -2 \int_{M^2} \langle \nabla R, X \rangle d\mu.$$

Hint: first show that $\left. \frac{\partial}{\partial t} \right|_{t=0} \varphi_t^ g = \operatorname{div}(X)g$.*

Since the solution to the modified Ricci flow (26) converges exponentially fast in each C^k norm to a constant curvature metric, we conclude that

$$|\nabla^k R| \leq C_k e^{-c_k t} \tag{27}$$

for some $C_k < \infty$ and $c_k > 0$ depending only on g_0 . These same estimates (27) also hold for the normalized Ricci flow since the solutions to that equation differs from the solutions to (26) by the pull back by diffeomorphisms, which do not affect the estimates. Hence we have proven Theorem 1 when $\chi(M^2) > 0$ and $R(g(0)) > 0$.

This result may be generalized to the case where the curvature of the initial metric on S^2 does not have positive curvature. This result was obtained in [7] by modifying the techniques of Hamilton. In particular, analogues of Hamilton’s entropy and Harnack estimates hold for solutions with curvature changing sign. For a proof using **Aleksandrov reflection** see Bartz-Struwe-Ye [1]. In Hamilton [10] a proof of convergence on S^2 was obtained by an isoperimetric estimate. There is also a proof of the main curvature estimate using **Perelman’s no local collapsing theorem** together with any of the known monotonicity formulas along the lines of Cao-Chen-Zhu [3].

There is one drawback of the proof of Corollary 10. It uses the Kazdan-Warner identity, whose proof in turn uses the uniformization theorem. Below we give another proof of the corollary which does not depend on the uniformization theorem.

References

- [1] Bartz, J.; Struwe, Michael.; Ye, Rugang. *A new approach to the Ricci flow on S^2* . Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **21** (1994), no. 3, 475–482.

- [2] Cao, Huai-Dong. *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*. Invent. Math. **81** (1985), no. 2, 359–372.
- [3] Cao, Huai-Dong; Chen, Bing-Long; Zhu, Xi-Ping. *Ricci flow on compact Kähler manifolds of positive bisectional curvature*. C. R. Math. Acad. Sci. Paris **337** (2003), no. 12, 781–784.
- [4] Chang, Sun-Yung Alice. *The Moser-Trudinger inequality and applications to some problems in conformal geometry*. Nonlinear partial differential equations in differential geometry (Park City, UT, 1992), 65–125, IAS/Park City Math. Ser., **2**, Amer. Math. Soc., Providence, RI, 1996.
- [5] Chen, Xiuxiong; Tian, Gang. *Ricci flow on Kähler-Einstein surfaces*. Invent. Math. **147** (2002), no. 3, 487–544.
- [6] Chen, Xiuxiong; Tian, Gang. *Ricci flow on Kähler-Einstein manifold*. Duke Math. J. To appear.
- [7] Chow, Bennett. *The Ricci flow on the 2-sphere*. J. Differential Geom. **33** (1991), no. 2, 325–334.
- [8] Chow, Bennett; Knopf, Dan. *The Ricci flow: An introduction*, Mathematical Surveys and Monographs, AMS, Providence, RI, 2004.
- [9] Hamilton, Richard S. *The Ricci flow on surfaces*. Mathematics and general relativity (Santa Cruz, CA, 1986), 237–262, Contemp. Math., **71**, Amer. Math. Soc., Providence, RI, 1988.
- [10] Hamilton, Richard S. *An isoperimetric estimate for the Ricci flow on the two-sphere*. Modern methods in complex analysis (Princeton, NJ, 1992), 191–200, Ann. of Math. Stud., **137**, Princeton Univ. Press, Princeton, NJ, 1995.
- [11] Kazdan, Jerry L.; Warner, Frank W. *Curvature functions for compact 2-manifolds*. Ann. of Math. (2) **99** (1974), 14–47.
- [12] Perelman, Grisha. Informal announcement about results on the Kähler-Ricci flow (2003).
- [13] Yau, Shing-Tung. *Calabi’s conjecture and some new results in algebraic geometry*. Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 5, 1798–1799.
- [14] Yau, Shing-Tung. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*. Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.v