The following supplement to week's one lectures is excerpted from the book [1]. There are several references to find this (or related) material such as in the books of Thierry Aubin, Isaac Chavel, David Gilbarg and Neil Trudinger, Frank Warner, etc.

The **divergence** of a $(p,0)$-tensor is defined as

$$\operatorname{div}(\alpha)_{i_1\cdots i_{p-1}} \doteqdot g^{jk}\nabla_j \alpha_{k i_1\cdots i_{p-1}} = \nabla_j \alpha_{j i_1 \cdots i_{p-1}}. \tag{1}$$

In particular if X is a 1-form, then

$$\operatorname{div}(X) = g^{ij}\nabla_i X_j.$$

A basic tool is integration by parts. Recall that Stokes' theorem says that

Theorem 1 *If α is an $(n-1)$-form on a compact differentiable manifold M^n with (possibly empty) boundary ∂M, then*

$$\int_M d\alpha = \int_{\partial M} \alpha.$$

The **divergence theorem** says

Theorem 2 *Let (M,g) be a compact Riemannian manifold. If X is a 1-form, then*

$$\int_{M^n} \operatorname{div}(X)\, d\mu = \int_{\partial M^n} \langle X, \nu \rangle\, d\sigma. \tag{2}$$

*Here ν is the unit outward normal, $d\mu$ denotes the **volume form** of g (see (**??**) for its formula in local coordinates), and $d\sigma \doteqdot \iota_\nu(d\mu)$ is the volume form of the boundary ∂M^n with respect to the induced metric.*

Proof. Define the $(n-1)$-form α by

$$\alpha = \iota_X(d\mu).$$

Using $d^2 = 0$ we compute

$$d\alpha = d \circ \iota_X(d\mu) = (d\circ\iota_X + \iota_X \circ d)(d\mu) = \mathcal{L}_X(d\mu) = \operatorname{div}(X) d\mu,$$

where to obtain the last equality, we may compute in an orthonormal frame $e_1, \ldots, e_n$:

$$\begin{aligned}\mathcal{L}_X(d\mu)(e_1,\ldots,e_n) &= \sum_{i=1}^n d\mu(e_1,\ldots,\nabla_{e_i}X,\ldots,e_n)\\ &= \operatorname{div}(X) d\mu(e_1,\ldots,e_n).\end{aligned}$$

Now Stokes' theorem implies

$$\int_M \operatorname{div}(X) d\mu = \int_M d\alpha = \int_{\partial M}\alpha = \int_{\partial M}\iota_X(d\mu) = \int_{\partial M} X(\nu) d\sigma,$$

and the theorem is proved. ■

Exercise 3 *Derive the following consequences of the divergence theorem.*

1. *On a closed manifold,* $\int_{M^n} \Delta u d\mu = 0$.
2. (Green) *On a compact manifold,*
$$\int_{M^n} (u\Delta v - v\Delta u)\, d\mu = \int_{\partial M^n} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) d\sigma.$$
In particular, on a closed manifold
$$\int_{M^n} u\Delta v d\mu = \int_{M^n} v\Delta u d\mu.$$
3. *Show that if* f *is a function and* X *is a 1-form, then*
$$\int_{M^n} f \operatorname{div}(X)\, d\mu = -\int_{M^n} \langle \nabla f, X\rangle\, d\mu + \int_{\partial M^n} f\langle X, \nu\rangle\, d\sigma.$$

Corollary 4 *Let* (M^n, g) *be a closed Riemannian manifold. If* α *is an* (r,s)*-tensor and* β *is an* $(r-1,s)$*-tensor, then*
$$\int_M \langle \alpha, \nabla\beta\rangle\, dV = \int_M \langle \operatorname{div}(\alpha), \beta\rangle\, dV.$$

Proof. Let $X_j = \alpha_{ji_2\cdots i_r}^{k_1\cdots k_s}\beta_{i_2\cdots i_r}^{k_1\cdots k_s}$. We compute that
$$\operatorname{div} X = \langle \operatorname{div}(\alpha), \beta\rangle + \langle \alpha, \nabla\beta\rangle\ ,$$
and the result follows from the divergence theorem. ■

Exercise 5 *Show that on a closed manifold*
$$\int_{M^n} |\nabla\nabla f|^2\, d\mu + \int_{M^n} \operatorname{Rc}(\nabla f, \nabla f)\, d\mu = \int_{M^n} (\Delta f)^2\, d\mu. \tag{3}$$
Since $|\nabla\nabla f|^2 \geq \frac{1}{n}(\Delta f)^2$, *this implies*
$$\int_{M^n} \operatorname{Rc}(\nabla f, \nabla f)\, d\mu \leq \frac{n-1}{n}\int_{M^n} (\Delta f)^2\, d\mu. \tag{4}$$

Exercise 6 (Lichnerowicz) *Suppose* f *is an eigenfunction of the laplacian with eigenvalue* λ*:*
$$\Delta f + \lambda f = 0.$$
Use (4) to show that if $\operatorname{Rc} \geq (n-1)Kg$, *where* $K > 0$ *is a constant, then*
$$\lambda \geq nK.$$
Equality is obtained by linear functions on the sphere of radius $1/\sqrt{K}$.

References

[1] Chow, Bennett; Lu, Peng; Ni, Lei. *Hamilton's Ricci Flow.* In preparation.