13 More gradient Ricci solitons, Hamilton's matrix Harnack estimate for the Ricci flow, and eternal solutions

Recall that Hamilton's matrix Harnack estimate in dimension two says the following. If $\left(M^2, g\left(t\right)\right)$ is a complete solution of the Ricci flow with bounded and positive curvature, then

$$\nabla_i \nabla_j \log R + \frac{1}{2} R g_{ij} \geq -\frac{1}{2t} g_{ij}. \tag{1}$$

We apply some simple algebraic manipulations to put this inequality in a form more suggestive of the higher dimensional generalization. In particular (1) is equivalent to the following inequality: for any 1-form V

$$\nabla_i \nabla_j R + \frac{1}{2} R^2 g_{ij} + \frac{1}{t} g_{ij} + \nabla_i R V_j + \nabla_j R V_i + R V_i V_j \geq 0. \tag{2}$$

We see this equivalence by noting that dividing by R and completing the square, we have

$$\begin{aligned} 0 \leq & \frac{1}{R} \nabla_i \nabla_j R + \frac{1}{2} \left(R + \frac{1}{t} \right) g_{ij} - \frac{1}{R^2} \nabla_i R \nabla_j R \\ & + \left(V_i + \frac{1}{R} \nabla_i R \right) \left(V_j + \frac{1}{R} \nabla_j R \right). \end{aligned}$$

Hence the minimizing choice of V is $-\frac{1}{R} \nabla R$ and we obtain that (1) and (2) are equivalent.

Recall that a gradient expanding Ricci soliton satisfies

$$-\frac{1}{2t} g_{ij} = R_{ij} + \nabla_i \nabla_j f \tag{3}$$

and when $n = 2$, $R_{ij} = \frac{1}{2} R g_{ij}$ so that the equation (obtained by taking the divergence of the above equation)

$$0 = \nabla R - 2 \operatorname{Rc} \left(\nabla f \right)$$

implies

$$\nabla \log R = \nabla f$$

provided $R > 0$. Hence

$$\nabla_i \nabla_j \log R + \frac{1}{2} R g_{ij} = -\frac{1}{2t} g_{ij}.$$

Thus, on a gradient expanding Ricci soliton with positive curvature on a surface, we have equality in the matrix Harnack estimate (1).

Now let's see what the generalization of this is to higher dimensions, taking the point of view of inequality (2). First note that very roughly speaking, (2) takes the form

$$\text{curvature}\,(V,V)+\text{derivative}\,(\text{curvature})\,V+\text{second derivative}\,(\text{curvature})\geq 0.$$

When $n=2$ we have

$$R_{ijk\ell}=\frac{1}{2}R\left(g_{i\ell}g_{jk}-g_{ik}g_{j\ell}\right)$$

so that when we consider the curvature, we only see the scalar curvature (twice the Gauss curvature). In higher dimensions, the Riemann curvature is naturally an operator on two forms. When $n\geq 3$ the Harnack quantity is an extension of the Riemann curvature operator. Without further ado, it is the following (we'll explain what this means in a moment). Given a 1-form W and a 2-form U, let

$$H\left(W,U\right)=M_{ij}W_iW_j+2P_{pij}U_{pi}W_j+R_{pijq}U_{pi}U_{qj}$$

where

$$P_{kij}=\nabla_kR_{ij}-\nabla_iR_{kj}$$

and

$$M_{ij}=\triangle R_{ij}-\frac{1}{2}\nabla_i\nabla_jR+2R_{kij\ell}R_{k\ell}-R_{ip}R_{pj}+\frac{1}{2t}R_{ij}$$

Roughly, this looks like

$$\text{curv}\,(U,U)+\text{deriv}\,(\text{curv})\,(U,W)+\text{2nd deriv}\,(\text{curv})\,(W,W)\geq 0.$$

More precisely,

Exercise 1

$$P_{kij}=\nabla_\ell R_{kij\ell}$$

and

$$M_{ij}=\nabla_kP_{kij}+R_{kij\ell}R_{k\ell}+\frac{1}{2t}R_{ij}.$$

When we take a covariant derivative of a tensor and use the metric to contract a pair of indices, we get the divergence. P is the divergence of Rm. *M is almost the divergence of P.*

Hamilton's matrix Harnack inequality says the following.

Theorem 2 ([1]) *If $(M^n,g(t))$ is a complete solution to the Ricci flow with bounded nonnegative curvature operator, then for any 1-form W and a 2-form U*

$$H\left(W,U\right)=M_{ij}W_iW_j+2P_{pij}U_{pi}W_j+R_{pijq}U_{pi}U_{qj}\geq 0. \tag{4}$$

Exercise 3 *Show that when $n=2$ (4) is equivalent to (2).*

How did Hamilton find such a complicated inequality? By considering the equations satisfied by gradient expanding Ricci solitons. In particular, differentiating equation (3) yields:

$$\nabla_i R_{jk} - \nabla_j R_{ik} = -\nabla_i \nabla_j \nabla_k f + \nabla_j \nabla_i \nabla_k f = R_{ijk\ell} \nabla_\ell f$$

so that

$$P_{ijk} - R_{ijk\ell} \nabla_\ell f = 0. \tag{5}$$

Next we take a divergence of (5):

$$\begin{aligned} 0 &= \nabla_i P_{ijk} - R_{ijk\ell} \nabla_i \nabla_\ell f - \nabla_i R_{ijk\ell} \nabla_\ell f \\ &= M_{jk} - R_{ijk\ell} R_{i\ell} - \frac{1}{2t} R_{jk} - R_{ijk\ell} \nabla_i \nabla_\ell f + \nabla_i R_{k\ell ji} \nabla_\ell f \\ &= M_{jk} + P_{k\ell j} \nabla_\ell f. \end{aligned} \tag{6}$$

Combining (5) and (6) into a quadratic, we get:

$$M_{jk} + 2P_{k\ell j} \nabla_\ell f - R_{k\ell ji} \nabla_i f \nabla_\ell f = 0$$

so that for any 1-form W we have

$$M_{jk} W_j W_k - 2P_{k\ell j} W_k \nabla_\ell f W_j + R_{k\ell ij} W_k \nabla_\ell f \nabla_i f W_j = 0. \tag{7}$$

Using the symmetries of the curvature tensor, if we take

$$U_{ij} = \frac{1}{2} \left(\nabla_i f W_j - \nabla_j f W_i \right),$$

then (7) implies

$$H(U, W) = M_{jk} W_j W_k + 2P_{k\ell j} U_{k\ell} W_j + R_{k\ell ij} U_{k\ell} U_{ji} = 0.$$

That is, for an expanding Ricci soliton flowing along the gradient vector field ∇f,

$$H(\nabla f \wedge W, W) = 0 \tag{8}$$

for any 1-form W.

A more digestible consequence of the matrix Harnack inequality is the so-called trace inequality.

Corollary 4 *Under the same hypotheses as the theorem,*

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2\nabla_i R V^i + 2R_{ij} V^i V^i \geq 0 \tag{9}$$

for any vector field V. Taking $V = 0$, we have

$$\frac{\partial}{\partial t}(tR) \geq 0.$$

Proof. We leave this as an exercise with the hint to take $\{\omega^a\}_{a=1}^n$ to be an orthonormal coframe and apply (4) to $U = \omega^a \wedge V$ and $W = \omega^a$ in

$$\sum_{a=1}^{n} H(\omega^a \wedge V, \omega^a) \geq 0.$$

■

Corollary 5 *If in addition to the above hypotheses, the solution is ancient, that is,* $g(t)$ *is defined for* $t \in (-\infty, 0]$, *then*

$$\frac{\partial R}{\partial t} \geq 0.$$

Proof. Exercise. ■

An important application of the matrix Harnack inequality is the following characterization of steady gradient Ricci solitons.

Theorem 6 ([2]) *If* $(M^n, g(t))$, $t \in (-\infty, \omega)$, *is a complete solution to the Ricci flow with nonnegative curvature operator and such that* $\sup_{M\times(-\infty,\omega)} R$ *is attained at some point in space and time, then* $(M^n, g(t))$ *is a steady gradient Ricci soliton.*

When $n = 2$ we saw a proof of this in the previous lecture.

References

[1] Hamilton, Richard S. *The Harnack estimate for the Ricci flow.* J. Differential Geom. **37** (1993), no. 1, 225–243.

[2] Hamilton, Richard S. *Eternal solutions to the Ricci flow.* J. Diff. Geom. **38** (1993) 1-11.