

## 12 Hamilton's trace Harnack estimate for the Ricci flow on surfaces and its consequences

In our last lecture we saw the Li-Yau estimate for the heat equation. It is somewhat surprising that there is a similar estimate for the Ricci flow. This similarity is especially striking when the dimension of the manifold is equal to 2. For this reason we first consider the case of a surface.

Recall that on a surface, the Ricci flow is

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} = -Rg_{ij}.$$

Thus it is the same as the Yamabe flow  $\frac{\partial}{\partial t} g_{ij} = -Rg_{ij}$  and for this reason the metrics stay in a conformal class, that is, there exists a function  $u(t) : M^2 \rightarrow \mathbb{R}$  such that  $g(t) = e^{u(t)}g(0)$ . This is not true in general when  $n \geq 3$ . Another good thing about the Ricci flow on surfaces is that the evolution equation for the scalar curvature (when  $n \geq 2$ )

$$\frac{\partial}{\partial t} R = \Delta R + 2|\text{Rc}|^2$$

becomes (when  $n = 2$ )

$$\frac{\partial}{\partial t} R = \Delta R + R^2.$$

Such a heat type equation is called semi-linear. The nonlinearity arises from the  $R^2$  term on the right hand side.

So what is the analogue of the Li-Yau estimate for the Ricci flow on surfaces? One should think of the scalar curvature  $R$  as the solution. Thus we assume  $R > 0$ . Under this assumption Hamilton 1988 proved the following. Let

$$Q = \Delta \log R + R = \frac{\partial}{\partial t} \log R - |\nabla \log R|^2. \tag{1}$$

This is very analogous to the Li-Yau quantity for the heat equation. The only changes are that we have replace  $R$  by  $u$  and we have an extra  $+R$  term which comes from the extra  $+R^2$  term in the evolution equation for the scalar curvature as compared to the heat equation.

**Exercise 1** *Prove the second equality.*

**Theorem 2**

$$Q(x, t) \geq -\frac{1}{t} \tag{2}$$

for all  $x \in M^2$  and  $t > 0$ .

**Proof.** A calculation similar to the previous lecture shows that

$$\frac{\partial}{\partial t} Q \geq \Delta Q + 2 \langle \nabla \log R, \nabla Q \rangle + Q^2. \tag{3}$$

So the theorem follows from the maximum principle. ■

An immediate consequence of this is:

**Corollary 3** For each fixed point  $x \in M^2$ , the function

$$t \mapsto tR(x, t)$$

is nondecreasing.

**Proof.** We have  $\frac{\partial}{\partial t} \log R + \frac{1}{t} \geq 0$ . ■

**Exercise 4** Show that by integrating (2) over graphs  $(\gamma(t), t)$  of paths  $\gamma : [t_1, t_2] \rightarrow M^2$  joining points  $x_1$  and  $x_2$ , we get

$$\frac{R(x_2, t_2)}{R(x_1, t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-1} \exp \left\{ -\frac{1}{4} \inf_{\gamma} \int_{t_1}^{t_2} \left| \frac{d\gamma}{dt}(t) \right|_{g(t)}^2 dt \right\}.$$

Since  $R > 0$ , we have  $g(t) \leq g(t_1)$  for all  $t \geq t_1$ . Hence:

**Corollary 5**

$$\frac{R(x_2, t_2)}{R(x_1, t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-1} \exp \left\{ -\frac{d(x_1, x_2, t_1)^2}{4(t_2 - t_1)} \right\}$$

where  $d(x_1, x_2, t_1)$  denotes the distance from  $x$  to  $y$  with respect to the metric  $g(t)$ .

Note that if we let

$$L(\gamma) = \frac{1}{4} \int_{t_1}^{t_2} \left| \frac{d\gamma}{dt}(t) \right|_{g(t)}^2 dt$$

and

$$\ell(x_1, t_1; x_2, t_2) = \inf_{\substack{\gamma(t_1)=x_1 \\ \gamma(t_2)=x_2}} L(\gamma),$$

then we may write the Harnack inequality as:

$$\frac{R(x_2, t_2)}{R(x_1, t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-1} e^{-\ell(x_1, t_1; x_2, t_2)}.$$

**Exercise 6** Prove the following generalization of Hamilton's inequality. If  $(M^2, g(t))$  is a solution to the Ricci flow on a closed surface with  $R > 0$ , and if  $u$  is a positive solution to

$$\frac{\partial u}{\partial t} = \Delta u + Ru,$$

then

$$\frac{\partial}{\partial t} \log u - |\nabla \log u|^2 + \frac{1}{t} = \Delta \log u + R + \frac{1}{t} \geq 0. \quad (4)$$

Hint: Let  $Q = \Delta \log u + R$  and compute  $\frac{\partial Q}{\partial t}$ . See the proof below for more details.

Here's an important application of Hamilton's inequality.

**Theorem 7** *If  $(M^2, g(t))$ ,  $t \in (-\infty, \infty)$ , is a complete solution to the Ricci flow with positive curvature and such that  $\sup_{M^2 \times (-\infty, \infty)} R$  is attained at some point in space and time, then  $(M^2, g(t))$  is a gradient Ricci soliton. By the classification theorem, it must be the cigar soliton.*

**Proof.** One can compute that

$$\frac{\partial}{\partial t} Q = \Delta Q + 2 \langle \nabla \log R, \nabla Q \rangle + 2 \left| \nabla \nabla \log R + \frac{1}{2} Rg \right|^2. \quad (5)$$

Hence

$$\frac{\partial}{\partial t} Q \geq \Delta P + 2 \langle \nabla \log R, \nabla Q \rangle + Q^2$$

where we applied the inequality  $|a_{ij}|^2 \geq \frac{1}{n} (\text{tr } a)^2$  to

$$a = \nabla \nabla \log R + \frac{1}{2} Rg$$

with  $n = 2$ . Now we used our (strong) assumption that the solution exists on all of  $(-\infty, \infty)$ . In particular, for any  $\alpha \in \mathbb{R}$ , the solution exists on the interval  $(\alpha, \infty)$ . Since the solution to the ODE

$$\frac{dq}{dt} = q^2$$

with  $\lim_{t \rightarrow \alpha} q(t) = \infty$  is  $q(t) = -\frac{1}{t-\alpha}$ , the maximum principle says

$$Q = \Delta \log R + R \geq -\frac{1}{t-\alpha}$$

for all  $t > \alpha$ . Hence, on all of  $M^2 \times (-\infty, \infty)$ , by taking  $\alpha \rightarrow -\infty$ , we get

$$\Delta \log R + R \geq 0.$$

By our hypothesis, there's a point  $(x_0, t_0) \in M^2 \times (-\infty, \infty)$  such that

$$R(x_0, t_0) = \sup_{M^2 \times (-\infty, \infty)} R.$$

At  $(x_0, t_0)$  we have

$$\frac{\partial R}{\partial t} = 0 \quad \text{and} \quad |\nabla R| = 0,$$

and hence  $Q(x_0, t_0) = 0$ . Since  $Q \geq 0$ , applying the strong maximum principle to (5) we see that

$$\Delta \log R + R = Q \equiv 0.$$

Plugging this back into (5), we get

$$\nabla \nabla \log R + \frac{1}{2} Rg \equiv 0$$

on  $M^2 \times (-\infty, \infty)$ . This says that  $g(t)$  is a gradient Ricci soliton flowing along  $\nabla \log R$ :

$$\frac{\partial}{\partial t} g = -Rg = 2\nabla \nabla \log R.$$

■

Finally, Hamilton has generalized his inequality to a “matrix” inequality.

**Theorem 8**

$$\nabla_i \nabla_j \log R + \frac{1}{2} R g_{ij} \geq -\frac{1}{2t} g_{ij}. \quad (6)$$

We do not give the details of the proof except to note that the basic computation is as follows. Let

$$Q_{ij} \doteq \nabla_i \nabla_j \log R + \frac{1}{2} R g_{ij}$$

so that its trace is  $g^{ij} Q_{ij} = Q$ . Then  $Q_{ij}$  satisfies the evolution equation:

$$\frac{\partial}{\partial t} Q_{ij} = \Delta Q_{ij} + 2\nabla_k Q_{ij} \nabla_k L + 2Q_{ik} Q_{jk} - 3RQ_{ij} + RQg_{ij}.$$

The idea is that the positive quadratic term  $+2Q_{ik} Q_{jk}$  pushes  $Q_{ij}$  up as time evolves.