11 The Li-Yau differential Harnack estimate for the heat equation

Let's start off by recalling a fundamental theorem of Yau about harmonic functions on Riemannian manifolds.

Theorem 1 (Yau 1975) *If (M^n, g) is a complete Riemannian manifold with nonnegative Ricci curvature and if $u : M^n \to \mathbb{R}$ is a superharmonic function:*

$$\Delta u \leq 0$$

and bounded from below: $u \geq -C$, then u is constant.

The proof is based on a gradient estimate. To oversimplify things, the basic idea of the proof is to bound a quantity F by obtaining an inequality of the form:

$$F^2 \leq C_1 F + C_2$$

which of course implies $F \leq C_3$. See the appendix for some more details. It is amazing how this simple idea is ubiquitous in obtaining estimates in geometric analysis and especially geometric evolution equations.

This result was a precursor for the fundamental estimate of Peter Li and Yau for solutions of the heat equation.

Theorem 2 (Li-Yau 1986) *If (M^n, g) is a complete Riemannian manifold with nonnegative Ricci curvature and if $u : M^n \times [0, \infty) \to \mathbb{R}$ is a positive solution of the heat equation: $u > 0$ and*

$$\frac{\partial u}{\partial t} = \Delta u,$$

then

$$\Delta \log u = \frac{\partial}{\partial t} \log u - |\nabla \log u|^2 \geq -\frac{n}{2t}.$$

Exercise 3 *Prove the equality*

$$\Delta \log u = \frac{\partial}{\partial t} \log u - |\nabla \log u|^2 .$$

Note that this inequality is sharp. Simply take euclidean space $\mathbb{R}^n$ and the fundamental solution

$$h_0 (x, y, t) = (4\pi t)^{-n/2} e^{-d(x,y)^2/4t}$$

where $d(x, y) = |x - y|$ is the euclidean distance. We compute

$$\log h_0 = -\frac{n}{2} \log (4\pi t) - \frac{d (x, y)^2}{4t}.$$

Isolating the distance part of the function, we define $k_0 = \frac{d(x,y)^2}{4t}$. Then

$$\Delta \log h_0 = \Delta k_0 = -\frac{n}{2t}$$

which is the equality case of the Li-Yau estimate.

Now we sketch the proof of the Li-Yau estimate. Let

$$u(x,t) = (4\pi t)^{-n/2} e^{-f(x,t)}$$

so that

$$\log u = -\frac{n}{2} \log(4\pi t) - f.$$

Define

$$Q = \Delta \log u = -\Delta f.$$

One can compute (we'll show how in the appendix)

$$\frac{\partial Q}{\partial t} = \Delta Q + 2\nabla \log u \cdot \nabla Q + 2\,\mathrm{Rc}(\nabla \log u, \nabla \log u) + 2\left|\nabla_i \nabla_j \log u\right|^2. \tag{1}$$

Note for any symmetric 2-tensor $a = a_{ij}\omega^i \otimes \omega^j$ (where $\{\omega^i\}$ is an orthonormal basis of 1-forms), we have

$$\sum_{i,j} (a_{ij})^2 \geq \frac{1}{n}\left(\sum_i a_{ii}\right)^2.$$

As a special case we have

$$|\nabla_i \nabla_j \log u|^2 \geq \frac{1}{n}(\Delta \log u)^2 = \frac{1}{n}Q^2.$$

Hence if $\mathrm{Rc} \geq 0$, then

$$\frac{\partial Q}{\partial t} \geq \Delta Q + 2\nabla \log u \cdot \nabla Q + \frac{2}{n}Q^2.$$

Now we consider the ODE obtained formally from this by

1. changing the inequality to equality
2. dropping the laplacian term
3. dropping the gradient term:

$$\frac{dq}{dt} = \frac{2}{n}q^2.$$

The worst case senario (i.e., when q is smallest) is when

$$\lim_{t\to 0} q(t) = -\infty.$$

The solution is

$$q(t) = -\frac{n}{2t}.$$

The maximum principle now tells us

$$\Delta \log u(x,t) = Q(x,t) \geq q(t) = -\frac{n}{2t}$$

and we are done!

A consequence of this estimate is a Harnack type inequality.

Corollary 4 *If* $\mathrm{Rc} \geq 0$ *and* u *is a positive solution of the heat equation, then*

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-n/2} e^{-d(x_1,x_2)^2/4(t_2-t_1)}.$$

That is,

$$f(x_2,t_2) - f(x_1,t_1) \leq \frac{d(x_1,x_2)^2}{4(t_2-t_1)}.$$

To see roughly why this should be true, recall the Li-Yau estimate is

$$\frac{\partial}{\partial t}\log u - |\nabla \log u|^2 \geq -\frac{n}{2t}.$$

We wish to integrate this inequality along geodesics $\gamma : [t_1,t_2] \to M$ with $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$. Note that if $x_2 = x_1$ we can take a constant path and the inequality

$$\frac{\partial}{\partial t}\log u \geq -\frac{n}{2t}$$

implies

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-n/2}.$$

Now when $x_2 \neq x_1$, the path γ has a nonzero derivative $\dot{\gamma}$. When computing the total derivative

$$\frac{d}{dt}\log u(\gamma(t),t)$$

this introduces a gradient term

$$\nabla \log u \cdot \dot{\gamma}$$

which is dominated by the $|\nabla \log u|^2$ term modulo a $|\dot{\gamma}|^2/4$ term which leads to the $\frac{d(x_1,x_2)^2}{4(t_2-t_1)}$ term.

Exercise 5 *Fill in the details of the proof of the corollary.*

Now we consider the fundamental solution of the heat equation $h(x, y, t)$ which satisfies

$$\lim_{t \to 0} h(\cdot, y, t) = \delta_y.$$

The corollary implies if $\mathrm{Rc} \geq 0$, then

$$h(x, y, t) \geq (4\pi t)^{-n/2} e^{-d(x,y)^2/4t}.$$

If we define k by

$$h = (4\pi t)^{-n/2} e^{-k},$$

then this says

$$k \leq \frac{d(x, y)^2}{4t}.$$

Note that the Li-Yau inequality says that

$$\Delta k \leq \frac{n}{2t}.$$

11.1 Appendix

Proof of (1). We compute

$$\begin{aligned}
\frac{\partial}{\partial t}(\Delta \log u) &= \Delta \left(\frac{\partial}{\partial t} \log u \right) = \Delta \left(\Delta \log u + |\nabla \log u|^2 \right) \\
&= \Delta (\Delta \log u) + 2\nabla \log u \cdot \Delta \nabla \log u + 2 |\nabla\nabla \log u|^2 \\
&= \Delta (\Delta \log u) + 2\nabla \log u \cdot \nabla \Delta \log u + 2 |\nabla\nabla \log u|^2 \\
&+ 2R_{ij} \nabla_i \log u \nabla_j \log u \\
&\geq \Delta (\Delta \log u) + 2\nabla \log u \cdot \nabla (\Delta \log u) + \frac{2}{n} (\Delta \log u)^2 .
\end{aligned}$$

Here we used the Ricci identity for commuting the laplacian and covariant derivative:

$$\Delta \nabla_i f = \nabla_i \Delta f + R_{ij} \nabla_j f.$$

■

Solution of Exercise. For any path $\gamma : [t_1, t_2] \to M^2$ joining points x_1 and

x_2, we have

$$\log \frac{u(x_2,t_2)}{u(x_1,t_1)} = \int_{t_1}^{t_2} \frac{d}{dt}\left[\log u\left(\gamma\left(t\right),t\right)\right] dt$$

(fundamental theorem of calculus)

$$= \int_{t_1}^{t_2} \left(\frac{\partial}{\partial t}\log u + \nabla \log u \cdot \frac{d\gamma}{dt}\right) dt$$

(chain rule)

$$\geq \int_{t_1}^{t_2} \left(\left|\nabla \log u\right|^2 - \frac{n}{2t} + \nabla \log u \cdot \frac{d\gamma}{dt}\right) dt$$

(Li-Yau estimate)

$$\geq -\frac{n}{2}\log\left(\frac{t_2}{t_1}\right) - \frac{1}{4}\int_{t_1}^{t_2} \left|\frac{d\gamma}{dt}\left(t\right)\right|^2 dt$$

(Schwarz inequality)

and the result follows from taking γ to be a constant speed minimal geodesic. Such a geodesic satisfies:

$$\left|\frac{d\gamma}{dt}\left(t\right)\right| \equiv \frac{d(x_1,x_2)}{t_2-t_1}.$$

Idea of proof of Yau's theorem. Let

$$F\left(x\right) = \left(r^2 - d\left(x,p\right)^2\right)\left|\nabla \log u\right|\left(x\right).$$

One can take its laplacian and then apply the maximum principle while using the $\mathrm{Rc} \geq 0$ hypothesis to derive the inequality:

$$F^2 - C_1 F - C_2 r^2 \leq 0,$$

where C_1 and C_2 depend only on n. This implies $F \leq C_3 r$ for some C_3 and hence

$$\left|\nabla \log u\right|\left(x\right) \leq \frac{C_3}{r - d\left(x,p\right)}$$

for all $x \in B\left(p,r\right)$. Restricting this estimate to $x \in B\left(p,r/2\right)$ and letting $r \to \infty$ implies $\left|\nabla \log u\right| = 0$ on M.

Caveat. One technical point is that the distance function is only smooth a.e.