

## 9 Basic formulas associated to the gradient Ricci soliton equation

A Ricci soliton is a solution of the Ricci flow where the metric  $g(t)$  changes only by a time-dependent scale factor and the pull-back by diffeomorphisms. That is, for any  $t_1$  and  $t_2$  in the time interval of existence, there exists a positive constant  $c = c(t_1, t_2)$  and a diffeomorphism  $\varphi = \varphi(t_1, t_2)$  such that

$$g(t_2) = c\varphi^*g(t_1).$$

Fixing a time  $t_0$  we have for any time  $t$ ,

$$g(t) = c(t)\varphi(t)^*g(t_0).$$

A gradient Ricci soliton is a Ricci soliton where the vector fields  $X(t)$  generated by the 1-parameter family of diffeomorphisms  $\varphi(t)$  are the gradients of functions  $f(t)$ . The single time version of this definition is as follows.

**Definition 1** A quadruple  $(M^n, g_0, f_0, \varepsilon)$ , where  $(M^n, g_0)$  is a Riemannian manifold,  $f_0 : M^n \rightarrow \mathbb{R}$  is a function and  $\varepsilon \in \mathbb{R}$ , is a **gradient Ricci soliton** if

$$\text{Rc}(g_0) + \nabla^{g_0}\nabla^{g_0}f_0 + \frac{\varepsilon}{2}g_0 = 0. \quad (1)$$

We say that  $g_0$  is **expanding**, **shrinking**, or **steady**, if  $\varepsilon > 0$ ,  $\varepsilon < 0$ , or  $\varepsilon = 0$ , respectively. The gradient soliton is called **complete** if  $(M^n, g_0)$  is complete and the vector field  $\text{grad}_{g_0}f_0$  is complete.

The following result gives a canonical form for the associated time-dependent version of a gradient Ricci soliton (see [1]).

**Theorem 2 (Gradient Ricci solitons)** *If  $(M^n, g_0, f_0, \varepsilon)$  is a complete gradient Ricci soliton, then there exists a solution  $g(t)$  of the Ricci flow with  $g(0) = g_0$ , diffeomorphisms  $\varphi(t)$  with  $\varphi(0) = \text{id}_{M^n}$ , functions  $f(t)$  with  $f(0) = f_0$  defined for all  $t$  with*

$$\tau(t) \doteq \varepsilon t + 1 > 0, \quad (2)$$

such that

1. 
$$\frac{\partial}{\partial t}\varphi(t)(x) = \frac{1}{\tau(t)}(\text{grad}_{g_0}f_0)(\varphi(t)(x)).$$

2. 
$$g(t) = \tau(t)\varphi(t)^*g_0, \quad (3)$$

3. 
$$f(t) = f_0 \circ \varphi(t) = \varphi(t)^*(f_0), \quad (4)$$

Moreover,

$$\text{Rc}(g(t)) + \nabla^{g(t)} \nabla^{g(t)} f(t) + \frac{\varepsilon}{2\tau} g(t) = 0 \quad (5)$$

and

$$\frac{\partial f}{\partial t}(t) = \left| \text{grad}_{g(t)} f(t) \right|_{g(t)}^2. \quad (6)$$

**Proof.** Define  $\tau(t) = \varepsilon t + 1$ . Since the vector field  $\text{grad}_{g_0} f_0$  is complete, there exists a 1-parameter family of diffeomorphisms  $\varphi(t) : M^n \rightarrow M^n$  generated by the vector fields  $\frac{1}{\tau(t)} \text{grad}_{g_0} f_0$  defined for all  $t$  such that  $\tau(t) > 0$ . Then define  $f(t) = f_0 \circ \varphi(t)$  and  $g(t) = \tau(t) \varphi(t)^* g_0$ . We compute

$$\frac{\partial}{\partial t} \Big|_{t=t_0} g(t) = \frac{\varepsilon}{\tau(t_0)} g(t_0) + \tau(t_0) \frac{\partial}{\partial t} \Big|_{t=t_0} (\varphi(t)^* g_0).$$

We have

$$\begin{aligned} \tau(t_0) \frac{\partial}{\partial t} \Big|_{t=t_0} (\varphi(t)^* g_0) &= \tau(t_0) \mathcal{L}_{(\varphi(t_0)^{-1})_* \frac{\partial}{\partial t} \Big|_{t=t_0} \varphi(t)} \varphi(t_0)^* g_0 \\ &= \mathcal{L}_{\text{grad}_{g(t_0)} f(t_0)} g(t_0) \end{aligned}$$

which holds since

$$\frac{\partial}{\partial t} \Big|_{t=t_0} \varphi(t) = \frac{1}{\tau(t_0)} \text{grad}_{g_0} f_0 = \varphi(t_0)_* (\text{grad}_{g(t_0)} f(t_0)).$$

Hence (evaluating at  $t$  instead of  $t_0$ )

$$\frac{\partial}{\partial t} g(t) = \frac{\varepsilon}{\tau(t)} g(t) + \mathcal{L}_{\text{grad}_{g(t)} f(t)} g(t).$$

We find that

$$\begin{aligned} -2 \text{Rc}(g(t)) &= \varphi(t)^* (-2 \text{Rc}(g_0)) = \varphi(t)^* (\varepsilon g_0 + \mathcal{L}_{\text{grad}_{g_0} f_0} g_0) \\ &= \frac{\varepsilon}{\tau(t)} g(t) + \mathcal{L}_{\text{grad}_{g(t)} f(t)} g(t). \end{aligned}$$

Hence

$$\frac{\partial}{\partial t} g(t) = \frac{\varepsilon}{\tau} g(t) + \mathcal{L}_{\text{grad}_{g(t)} f(t)} g(t) = -2 \text{Rc}(g(t)). \quad (7)$$

Finally we calculate

$$\begin{aligned} \frac{\partial f}{\partial t}(x, t) &= \left( \frac{\partial}{\partial t} \varphi(t) \right) (f_0)(x) = \frac{1}{\tau(t)} |\text{grad}_{g_0} f_0|^2 (\varphi(t)(x)) \\ &= \left| \text{grad}_{g(t)} f(t) \right|_{g(t)}^2 (x). \end{aligned}$$

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We now derive some equations which are consequences of the gradient Ricci soliton equation. Taking the trace of (5) we have

$$R + \Delta f + \frac{n\varepsilon}{2\tau} = 0. \quad (8)$$

Subtracting (8) from the RHS of (6), we get

$$\frac{\partial f}{\partial t} = -\Delta f - R + |\nabla f|^2 - \frac{n\varepsilon}{2\tau}. \quad (9)$$

Next we take the divergence of equation (5). By the contracted second Bianchi identity  $\operatorname{div}(\operatorname{Rc}) = \frac{1}{2}\nabla R$  and commuting derivatives, we obtain:

$$\begin{aligned} 0 &= \nabla_j (R_{ij} + \nabla_i \nabla_j f + \lambda g_{ij}) = \frac{1}{2} \nabla_i R + \nabla_i \Delta f - R_{jijk} \nabla_k f \\ &= -\frac{1}{2} \nabla_i R + R_{ik} \nabla_k f. \end{aligned} \quad (10)$$

Substituting (5) into (10), we get

$$0 = \nabla_i R + \nabla_i |\nabla f|^2 + \frac{\varepsilon}{\tau} \nabla_i f.$$

Hence

$$R + |\nabla f|^2 + \frac{\varepsilon}{\tau} f \stackrel{\text{G}}{=} C(t) \quad (11)$$

is constant in the space variables.

By taking twice of (8) and subtracting (11), we find that

$$V_\varepsilon \doteq \tau \left( R + 2\Delta f - |\nabla f|^2 \right) - \varepsilon (f - n) \quad (12)$$

is equal to  $-\tau C(t)$ , which is a function only of  $t$ .

One can also show the following (see [1]).

**Lemma 3** *If  $(M^n, g(t), f(t))$  is a gradient Ricci soliton and if  $f$  has a critical point in space, then  $V_\varepsilon(x, t)$  is a constant independent of  $t$ :*

$$V_\varepsilon \stackrel{\text{G}}{=} C. \quad (13)$$

The above considerations are related to Perelman's entropy monotonicity which are discussed in the lectures by Kleiner/Lott/Tian.

## References

- [1] Chow, Bennett; Lu, Peng; Ni, Lei. *Hamilton's Ricci Flow*. In preparation.