8 Expanding soliton on $\mathbb{R}^2$, the 3-dimensional Bryant soliton, and no closed 3-dimensional shrinkers

8.1 Expanding soliton on $\mathbb{R}^2$

Now that we have seen the cigar steady soliton and the (trivial) 2-sphere shrinking soliton, we move on to study a 2-dimensional expanding soliton.

More generally we say that a solution $(M^n, g(t))$ to the Ricci flow is an **immortal solution** if it is defined on a time interval $\alpha < t < \infty$. Expanding solitons are immortal solutions. In dimension 3 there are a number of immortal solutions which are locally homogeneous (but not Ricci solitons) and whose curvatures decay like $1/t$; these have been studied by Isenberg and Jackson [7] (see lecture 10 or Chapter 1 of [4]).

In dimension 2, there are nontrivial (that is, nonconstant curvature) examples of expanding Ricci solitons (see [5]). In particular, let $g(t)$ be a rotationally symmetric solution on $\mathbb{R}^2$ of the form:

$$g(t) = t\left(F(r)^2\, dr^2 + r^2\, d\theta^2\right), \tag{1}$$

where $F : [0,\infty) \to (0,\infty)$ is a positive function to be determined by the expanding Ricci soliton condition, and where $r \in (0,\infty)$ and $\theta \in \mathbb{R}/\left(2\pi F(0)\right)$. $g(t)$ define smooth metrics on $\mathbb{R}^2$ by extending smoothly over the origin ($r \to 0$). Clearly these metrics $g(t) = tg(1)$ are homothetically expanding for $t > 0$. One can compute that the Gauss curvatures K ($1/2$ the scalar curvature R) are given by

$$K\left[g(t)\right] = \frac{1}{t}\frac{F'(r)}{rF(r)^3}. \tag{2}$$

Exercise 1 *Show that an orthonormal coframe is given by (see lecture 7 for a brief introduction to the method of moving frames)*

$$\omega^1 = \sqrt{t}F(r)\, dr \quad \omega^2 = \sqrt{t} r d\theta,$$

and one has

$$\omega_1^2 = \frac{1}{F(r)} d\theta$$
$$\Omega_1^2 = d\omega_1^2 = -\frac{1}{t}\frac{F'(r)}{rF(r)^3}\omega^1 \wedge \omega^2.$$

Let $X(t)$ be the vector fields on $\mathbb{R}^2$ defined by

$$X(t) \doteqdot \frac{r}{tF(r)}\frac{\partial}{\partial r} = \frac{1}{t}X(1). \tag{3}$$

$X(t)$ is defined on $\mathbb{R}^2 - \{0\}$; however, it extends smoothly to $\mathbb{R}^2$. Note that for a radial function $f(r)$, its gradient with respect to $g(t)$ is given by $\operatorname{grad}_{g(t)} f = \frac{1}{tF(r)^2} f'(r) \frac{\partial}{\partial r}$. Hence

$$X(t) = \operatorname{grad}_{g(t)} f \quad \text{where } f'(r) = rF(r). \tag{4}$$

We look for F such that the homothetically expanding metrics $g(t) = tg(1)$ given by (1) form a solution to the modified Ricci flow

$$\frac{\partial}{\partial t} g(t) = -R_{g(t)} g(t) + \mathcal{L}_{X(t)} g(t). \tag{5}$$

The dual 1-forms to $X(t)$ are

$$X(t)^\flat = rF(r)\, dr = \frac{r}{\sqrt{t}} \omega^1,$$

which is time-independent. Recall that, in general, from the first structure equations:

$$\left(\nabla_V \omega^j\right)(e_i) = -\omega^j \left(\nabla_V e_i\right) = -\omega_i^j (V)$$

for any vector V, so that $\nabla \omega^j = -\omega_i^j \otimes \omega^i$. Hence, using this and $\omega_2^1 = -\frac{1}{\sqrt{t} rF(r)} \omega^2$, we find that

$$\mathcal{L}_{X(t)} g(t) = 2 \operatorname{Sym}\left(\nabla X(t)^\flat\right) = \frac{2}{\sqrt{t}} \operatorname{Sym}\left(dr \otimes \omega^1 + r \nabla \omega^1\right) = \frac{2}{tF(r)} g(t).$$

Thus (5) is equivalent to the ODE

$$F'(r) = rF(r)^2 \left(1 - \frac{F(r)}{2}\right). \tag{6}$$

Now from (4) and (6)

$$f'(r) = rF(r) = \frac{F'(r)}{F(r)\left(1 - \frac{F(r)}{2}\right)}.$$

Integrating this we have

$$f(r) = -\log\left(\frac{2}{F(r)} - 1\right). \tag{7}$$

Remark 2 *The metrics $\psi(t)^* g(t)$ satisfy (unmodified) Ricci flow if the 1-parameter family of diffeomorphisms $\psi(t)$ satisfy*

$$\left.\frac{d}{dt}\right|_{t=t_0} \left(\psi(t) \circ \psi^{-1}(t_0)\right) = -\psi(t_0)^* X(t_0).$$

Solving the separable ODE (6) we obtain

$$h(r) + \log h(r) = -r^2 + C. \tag{8}$$

where $h(r) \doteqdot \frac{2}{F(r)} - 1$. Here we have made the assumption that $0 < F < 2$ so that $h > 0$. Note that by (2) and (6) we have

$$K[g(t)] = \frac{1}{t}\left(\frac{1}{F(r)} - \frac{1}{2}\right) > 0. \tag{9}$$

Recall that the **product log** (or **Lambert-W**) **function** $W : \mathbb{R}^+ \to \mathbb{R}^+$ is the inverse of the function $w(x) = xe^x$. Hence, taking the exponential of both sides of (8), we have

$$h(r)\, e^{h(r)} = h(0)\, e^{h(0)} e^{-r^2},$$

so that

$$h(r) = W\left(h(0)\, e^{h(0)-r^2}\right).$$

In terms of F, this equation says:

$$F(r) = \frac{2}{W\left(\left(\frac{2}{F(0)} - 1\right)\exp\left(\frac{2}{F(0)} - 1 - r^2\right)\right) + 1}. \tag{10}$$

The **cone angle** of $g(t)$ at infinity (which is independent of t) is

$$\text{ConeAngle} = \frac{2\pi F(0)}{\lim_{r\to\infty} F(r)} = \pi F(0).$$

Hence the range of possible cone angles at infinity is $(0, 2\pi)$ since $F(0) \in (0, 2)$. Now (9) and (10) imply

$$K[g(t)] = \frac{1}{2t} W\left(\left(\frac{2}{F(0)} - 1\right)\exp\left(\frac{2}{F(0)} - 1 - r^2\right)\right).$$

Finally we note from (7) that we have

$$f(r) = -\log h(r) = -\log W\left(h(0)\, e^{h(0)} e^{-r^2}\right).$$

Exercise 3 (Exponential curvature decay of expanding 2d Ricci soliton) *Show that each of the metrics $g(t)$ is asymptotic at infinity to a flat cone and the curvature decays exponentially as a function of the distance to the origin.* HINT*:* $\lim_{x\to 0_+} \frac{W(x)}{x} = 1$.

In higher dimensions there are expanding **Kähler–Ricci solitons** on $\mathbb{C}^n$ due to Cao [3]; see also [6].)

8.2 Bryant soliton

Let $g_{S^{n-1}}$ denote the standard metric on the unit $(n-1)$-sphere. We search for warped product Ricci solitons on $(0,\infty) \times S^{n-1}$ which extend to Ricci solitons on $\mathbb{R}^n$ by a 1-point compactification of one end. We call the compactifying point the **origin** O. In particular, consider metrics of the form

$$g = dr^2 + \phi(r)^2 g_{S^{n-1}}. \tag{11}$$

From $\mathrm{Rc}(g_{S^{n-1}}) = (n-2)\, g_{S^{n-1}}$ and a standard formula for the Ricci tensor of a **warped product metric** (see [1], Prop. 9.106), we have

$$\mathrm{Rc}(g) = -(n-1)\frac{\phi''}{\phi} dr^2 + \left((n-2)\left(1-(\phi')^2\right) - \phi\phi''\right) g_{S^{n-1}}. \tag{12}$$

The sectional curvature of a plane passing through the radial vector $\frac{\partial}{\partial r}$ is

$$K_{\mathrm{rad}} = -\frac{\phi''(r)}{\phi(r)} \tag{13}$$

and the sectional curvature of a plane P_{sph} perpendicular to $\frac{\partial}{\partial r}$ is

$$K_{\mathrm{sph}} = \frac{1-\phi'(r)^2}{\phi(r)^2}. \tag{14}$$

Exercise 4 *Show that (13) and (14) imply (12).*

The hessian of a function f is given by

$$\nabla\nabla f = f''(r) dr^2 + \phi\phi' f' g_{S^{n-1}}.$$

The Ricci soliton equation $\mathrm{Rc}(g) + \nabla\nabla f = 0$ becomes the following system of two second order ODE

$$f'' = (n-1)\frac{\phi''}{\phi}, \qquad \phi\phi' f' = -(n-2)(1-(\phi')^2) + \phi\phi''.$$

For g and f to extend smoothly over the origin, we need $\phi(0) = 0$, $\phi'(0) = 1$ and $f'(0) = 0$. It can be shown (see Chapter 1 of [**?**]) that the exists such a solution which defines a complete Ricci soliton g on $\mathbb{R}^n$ of the form (11) with positive curvature operator, called the **Bryant soliton** [2], [8]. Let K_{sph} and K_{rad} denote the sectional curvatures of planes tangent to the spheres and tangent to the radial direction, respectively. From the formulas for the sectional curvatures (13), (14), and $C^{-1}r^{1/2} \le \phi(r) \le Cr^{1/2}$, $\phi'(r) = O\left(r^{-1/2}\right)$ and $\phi''(r) = O\left(r^{-3/2}\right)$, we have

$$K_{\mathrm{rad}} = O\left(r^{-2}\right), \quad K_{\mathrm{sph}} = O\left(r^{-1}\right).$$

That is, we have the following.

Theorem 5 (Bryant soliton) *For all $n \geq 3$, there exists a unique (up to homothety) complete, steady, gradient Ricci soliton metric on $\mathbb{R}^n$ with positive curvature operator. The eigenspaces of the curvature operator consist of 2-forms that are the wedge product of two 1-forms. The corresponding planes are either tangent to the spheres, in which case the sectional curvatures decay inverse linearly in distance to the origin, or pass through the radial direction, in which case the sectional curvatures decay inverse quadratically.*

In [10], Ivey constructed Ricci solitons on doubly warped products which generalize the Bryant soliton.

8.3 No closed 3-dimensional shrinkers

From Lemma 5 of lecture 5 we have the following.

Theorem 6 *A shrinking soliton on a closed 3-manifold with positive Ricci curvature is isometric to a shrinking constant positive sectional curvature metric.*

Exercise 7 *Prove this.*

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