7 The cigar soliton, the Rosenau solution, and moving frame calculations

When making local calculations of the connection and curvature, one has the choice of either using local coordinates or moving frames. In the Ricci flow, since the metric is evolving, it has been found convenient to use time-independent local coordinates $\{x^i\}_{i=1}^n$ in an open set. When the metric possesses some symmetry it is often more convenient to compute using a moving frame. Below we describe this technique, which was primarily developed first by Elie Cartan. Later S.-S. Chern, the founding director of MSRI, was a primary practicioner of this method.

Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame field in an open set $U \subset M^n$. The dual orthonormal basis (or coframe field) $\{\omega^i\}_{i=1}^n$ of T^*M^n restricted to U is defined by

$$\omega^i(e_j) = \delta_j^i$$

for all $i, j = 1, \ldots, n$. The metric equals

$$g = \sum_{i=1}^n \omega^i \otimes \omega^i.$$

One may check this formula using $g(e_j, e_k) = \delta_{jk}$. The connection 1-forms $\left\{\omega_i^j\right\}$ are the components of the Levi-Civita connection ∇

$$\nabla_X e_i \doteqdot \sum_{j=1}^n \omega_i^j(X)\, e_j,$$

for all $i, j = 1, \ldots, n$ and all vector fields X on U.

Exercise 1 *Show that*

$$\omega_i^j = -\omega_j^i.$$

A useful formula is

$$\omega_i^k(e_j) = \frac{1}{2}\left(d\omega^i(e_j, e_k) + d\omega^j(e_i, e_k) - d\omega^k(e_j, e_i)\right). \tag{1}$$

Note the similarity between this and the formula for the Christoffel symbols. The curvature 2-forms Rm_i^j on U are defined by:

$$\mathrm{Rm}(X, Y)\, e_i \doteqdot \sum_{j=1}^n \mathrm{Rm}_i^j(X, Y)\, e_j$$

so that $\mathrm{Rm}_i^j(X, Y) = \langle \mathrm{Rm}(X, Y)\, e_i, e_j \rangle$.

$$d\omega^i = \omega^j \wedge \omega_j^i \tag{2}$$

$$\mathrm{Rm}_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j. \tag{3}$$

For a surface M^2,

$$d\omega^1 = \omega^2 \wedge \omega_2^1, \quad d\omega^2 = \omega^1 \wedge \omega_1^2,$$

$$\mathrm{Rm}\,_2^1 = d\omega_2^1.$$

Hence the Gauss curvature is given by

$$K = \langle R(e_1, e_2) e_2, e_1 \rangle = \mathrm{Rm}\,_2^1 (e_1, e_2) = d\omega_2^1 (e_1, e_2).$$

Now suppose we have a rotationally symmetric surface. Here the metric takes the form

$$g = ds^2 + \phi(s)^2 d\theta^2. \tag{4}$$

Its Gauss curvature is

$$K(s, \theta) = -\frac{\phi''(s)}{\phi(s)}. \tag{5}$$

To see this, let

$$\omega^1 = ds, \quad \omega^2 = \phi(s) d\theta.$$

From (1) we see that the connection 1-form satisfies:

$$\omega_1^2 (e_j) = \frac{1}{2} \left(d\omega^j (e_1, e_2) - d\omega^2 (e_j, e_1) \right)$$

so that

$$\omega_1^2 = \phi'(s) d\theta = \frac{\phi'(s)}{\phi(s)} \omega^2. \tag{6}$$

Hence

$$\mathrm{Rm}\,_1^2 = d\omega_1^2 = \frac{\phi''(s)}{\phi(s)} \omega^1 \wedge \omega^2,$$

and (5) follows. Now suppose that the metric g given by (4) is a steady soliton flowing along a gradient vector field ∇f, where $f = f(s)$ is a radial function. Then taking the components of the steady gradient soliton equation

$$\begin{aligned} 0 &= \frac{1}{2} R\, g(e_i, e_j) + \nabla_{e_i} \nabla_{e_j} f \\ &= K \delta_{ij} + e_i (e_j (f)) - \omega_j^k (e_i) e_k (f) \end{aligned}$$

and using (6), we obtain the equations

$$f''(s) = \frac{\phi'(s)}{\phi(s)} f'(s) = \frac{\phi''(s)}{\phi(s)}.$$

We first see that

$$f'(s) = a\phi(s)$$

so that

$$\phi'(s) = \frac{a}{2} \phi(s)^2 + b.$$

Taking $\phi = \tanh s$, which satisfies

$$\phi'(s) = \operatorname{sech}^2 s = \phi(s)^2 + 1 \quad \text{and} \quad \phi''(s) = -2\operatorname{sech}^2 s \tanh s,$$

we obtain the cigar soliton is the rotationally symmetric metric defined by

$$g_\Sigma = ds^2 + \tanh^2 s\, d\theta^2. \tag{7}$$

Since $\tanh s \to 1$ as $s \to \infty$, the metric is asymptotically cylindrical. The Gauss curvature of g_Σ is

$$K_\Sigma = 2\operatorname{sech}^2 s = \frac{2}{1+r^2}.$$

Thus the curvature decays exponentially fast as $s \to \infty$. Now let

$$f(s) = -2\log(\cosh s).$$

We have

$$f'(s) = -2\tanh s = -2\phi(s) \quad \text{and} \quad f''(s) = -2\operatorname{sech}^2 s = \frac{\phi''(s)}{\phi(s)}.$$

Exercise 2 *Show that by changing the coordinates we use, we have the following forms of the cigar soliton:*

$$\begin{aligned} g_\Sigma &= \frac{dr^2 + r^2\ d\theta^2}{1+r^2} \\ &= \frac{dx^2+dy^2}{1+x^2+y^2} \\ &= \left(1-\frac{M}{\rho}\right)d\theta^2 + \left(1-\frac{M}{\rho}\right)^{-1}\frac{d\rho^2}{4\rho^2} \\ &= \left(e^{-2z}+1\right)^{-1}\left(dz^2+d\theta^2\right), \end{aligned}$$

where $s = \operatorname{arcsinh} r = \log\left(r+\sqrt{1+r^2}\right)$, $r = \sqrt{x^2+y^2}$, $\rho = M\cosh^2 s$, *and* $z = \frac{1}{2}\log\left(\frac{r}{M}-1\right)$.

It is more proper to consider the cigar as a time-dependent metric $g(t)$, $t \in (-\infty,\infty)$, which is a solution to the Ricci flow. In this regard, it perhaps easiest to use the $\{x,y\}$ coordinates. Then

$$g_\Sigma(t) = \frac{dx^2+dy^2}{e^{4t}+x^2+y^2}.$$

One may check that

$$g_\Sigma(t) = \varphi_t^* g_\Sigma(0).$$

where $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\varphi_t(x,y) = \left(e^{-2t}x, e^{-2t}y\right)$ form a 1-parameter group of conformal diffeomorphisms of $\mathbb{R}^2$. Thus, for each two times $t_1, t_2 \in (-\infty,\infty)$, $g(t_1)$ is isometric to $g(t_2)$. In this sense, the solution is stationary.

The cigar is an *eternal* solution, that is, it is defined on the time interval $(-\infty,\infty)$. We also have the standard shrinking 2-sphere which is an *anicent* solution, that is, defined on the time interval $(-\infty,0)$. In some sense, in between these solutions is the Rosenau solution. Do describe this solution, we use cylindrical coordinates. Let $\left(\mathbb{R}\times S^1(2),h\right)$ denote the flat cylinder, where $h=dz^2+d\theta^2$ and $\theta\in S^1(2)=\mathbb{R}/4\pi\mathbb{Z}$. The Rosenau solution is $g(t)=u(t)\cdot h$, where $t<0$, defined by

$$u(z,t)=\frac{\sinh(-t)}{\cosh z+\cosh t}. \tag{8}$$

Its curvature is given by

$$R[g(t)]=-\frac{\Delta_h\log u}{u}=\frac{\cosh t\cdot\cosh z+1}{\sinh(-t)(\cosh z+\cosh t)}$$

for $t<0$, and from this we may check that $g(t)$ is a solution to the Ricci flow. Note that $g(t)$ has positive curvature and is rotationally symmetric and invariant under a reflection about the "equator" $z=0$. The metrics $g(t)$ defined on $\mathbb{R}\times S^1(2)$ extend to smooth metrics, which we also call $g(t)$, on the 2-sphere S^2, which is obtained by compactifying $\mathbb{R}\times S^1(2)$ by adding two points (see [1], p. 33 for more details).

Some limits of the Rosenau solution $g(t)$ as $t\to-\infty$ are the cigar soliton. Consider

$$u(z+t,t)=(-\cosh z\coth t-\sinh z-\coth t)^{-1},$$

so that

$$\begin{aligned}\lim_{t\to-\infty}u(z+t,t)\,h(z)&=(\cosh z-\sinh z+1)^{-1}h(z)\\&=\left(e^{-z}+1\right)^{-1}h(z).\end{aligned}$$

Changing variables by $\tilde{z}=z/2$ and $\tilde{\theta}=\theta/2$, we have

$$\left(e^{-z}+1\right)^{-1}h(z)=4\left(e^{-2\tilde{z}}+1\right)^{-1}\left(d\tilde{z}^2+d\tilde{\theta}^2\right), \tag{9}$$

where $\tilde{z}\in\mathbb{R}$ and $\tilde{\theta}\in S^1(1)$. This is the cigar soliton. We may also obtain the cylinder as a backward limit: since $\lim_{t\to-\infty}u(z,t)=1$,

$$\lim_{t\to-\infty}g(z,t)=h(z)$$

for all $z\in\mathbb{R}$.

To summarize, at the two tips of the Rosenau solution, as we go back in time toward $-\infty$, the metric looks closer and closer to the cigar soliton metric. If we consider points on the equator, then as we go back in time toward $-\infty$, the metric looks closer and closer to a cylinder.

References

[1] Chow, Bennett; Knopf, Dan. *The Ricci flow: An introduction,* Mathematical Surveys and Monographs, AMS, Providence, RI, 2004.

[2] Hamilton, Richard S. *The Ricci flow on surfaces.* Mathematics and general relativity (Santa Cruz, CA, 1986), 237–262, Contemp. Math., **71**, Amer. Math. Soc., Providence, RI, 1988.

[3] Rosenau, Philip. *On fast and super-fast diffusion.* Phys. Rev. Lett. **74** (1995) 1056–1059.

[4] Singer, I. M.; Thorpe, J. A. *Lecture notes on elementary topology and geometry.* Reprint of the 1967 edition. Undergraduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1976.