6 Gradient Ricci solitons, related monotonicity on surfaces and the Kazdan-Warner identity

A solution $(M^n, g(t))$, $t \in (-\infty, 0)$, is said to be a shrinking gradient Ricci soliton if there exists a time-dependent function $f(t)$ such that

$$R_{ij} + \nabla_i \nabla_j f + \frac{1}{2t} g_{ij} = 0 \tag{1}$$

and

$$\frac{\partial f}{\partial t}(t) = |\nabla f|^2_{g(t)}. \tag{2}$$

Note that (1) is equivalent to

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} = (L_{\nabla f} g)_{ij} - \frac{1}{|t|} g_{ij}$$

since $t < 0$ and the Lie derivative of the metric is given by

$$(L_{\nabla f} g)_{ij} = 2\nabla_i \nabla_j f.$$

(More generally, if X is a vector field, then $(L_X g)_{ij} = \nabla_i X_j^* + \nabla_j X_i^*$, where X^* is the 1-form dual to X.)

Given a single metric g and function f with

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{2} g_{ij} = 0,$$

it is a theorem that we may extend g to a shrinking gradient Ricci soliton $(M^n, g(t), f(t))$ with

$$\mathrm{Rc}(g(t)) + \nabla^{g(t)} \nabla^{g(t)} f(t) + \frac{1}{2t} g(t) = 0$$

such that $g(-1) = g$ and (1) and (2) hold. Moreover, if $\varphi(t) : M^n \to M^n$ is the 1-parameter family of diffeomorphisms such that

$$\frac{\partial}{\partial t} \varphi(t)(x) = -\frac{1}{t} \left(\mathrm{grad}_{g(-1)} f(-1) \right) (\varphi(t)(x)),$$

then

1. $$g(t) = -t\varphi(t)^* g(-1), \tag{3}$$

2. $$f(t) = f(-1) \circ \varphi(t). \tag{4}$$

So a shrinking gradient Ricci soliton simply shrinks homothetically up to isometry. Examples of shrinking gradient Ricci solitons are the Einstein solutions with positive scalar curvature, where f may be taken to be 0 and

$$R_{ij} = \frac{1}{2\,|t|} g_{ij}$$

for $t < 0$. For example, a shrinking sphere. Now

Theorem 1 (Hamilton and Ivey) *In dimensions 2 and 3 any shrinking gradient Ricci soliton on a closed manifold is Einstein.*

Remark 2 *In dimension 4 there is an example due to Koiso [5] of a non-Einstein shrinking gradient Ricci soliton on a closed manifold.*

In the following we state some facts and give reasons for these facts without going into very much detail. The reader may refer to [2] for an exposition of these facts.

1. First, one sees from the maximum principle that since a shrinking gradient Ricci soliton on a closed manifold is an ancient solution with bounded curvature, it has nonnegative sectional curvature.

2. Next one shows that:

   (a) In dimension 2, the solution must have positive curvature.

   (b) In dimension 3, the solution is either

      i. a quotient of the product of a 2-dimensional shrinking gradient Ricci soliton with $\mathbb{R}$, or

      ii. has positive sectional curvature.

3. (a) The only 2-dimensional shrinking gradient Ricci soliton is the round 2-sphere (or its $\mathbb{Z}_2$ quotient $\mathbb{R}P^2$).

   (b) Any 3-dimensional shrinking gradient Ricci soliton with positive sectional curvature on a closed manifold has constant sectional curvature and hence is a quotient of S^3.

An easy way to see statement 3a above is to use the Kazdan-Warner identity. Since $\chi\left(M^2\right) > 0$, by going to the universal cover if necessary, we may assume $M^2 \cong S^2$.

Proposition 3 ([4]) *If X is a conformal vector field, then*

$$\int_{S^2} \langle \nabla R, X \rangle \, d\mu = \int_{S^2} R \operatorname{div} X \, d\mu = 0.$$

Let $\left(M^2, g(t)\right)$ be a shrinking gradient Ricci soliton on a closed surface. Tracing (1) yields

$$\Delta f = r - R.$$

Since ∇f is a conformal vector field,

$$-\int_{S^2} (R-r)^2 \, d\mu = \int_{S^2} R \operatorname{div} X \, d\mu = 0$$

by the Kazdan–Warner identity. Hence $R \equiv r$.

Now how are gradient Ricci solitons related to monotonicity formulas? You will see in the lectures by Kleiner, Lott and Tian how Perelman acheives this with his entropy and reduced volume functionals. Here we will consider Hamilton's entropy functional for surfaces with positive curvature.

Hamilton's entropy N is defined for a metric of strictly positive curvature on a closed surface M^2 by

$$N(g) \doteqdot \int_{M^2} \log(RA) \, R d\mu,$$

where $A = \int_{M^2} d\mu$ is the area of g and $d\mu$ is the area (volume) form.

Theorem 4 *If $\left(M^2, g(t)\right)$ is a solution of the Ricci flow on a closed surface with $R_{g(0)} > 0$, then*

$$\frac{dN}{dt} = -\int_{M^2} \frac{|\nabla R|^2}{R} d\mu + \int_{M^2} (R-r)^2 \, d\mu \tag{5}$$

$$= -\int_{M^2} |\nabla \log R - \nabla f|^2 \, R d\mu - 2 \int_{M^2} \left| \nabla\nabla f - \frac{1}{2} \Delta f \cdot g \right|^2 d\mu \leq 0 \tag{6}$$

where f is defined by $\Delta f = r - R$ and $r = A^{-1} \int_M R d\mu$ is the average scalar curvature.

Remark 5 *In general, if h is a smooth function on a closed Riemannian manifold (M^n, g) with $\int_M h d\mu = 0$, then there exists a function f such that $\Delta f = h$.*

Remark 6 *(5)-(6) implies the Poincaré type inequality:*

$$\int_{M^2} (R-r)^2 \, d\mu \leq \int_{M^2} \frac{|\nabla R|^2}{R} d\mu. \tag{7}$$

Proof. Formula (5) follows from the evolution equations for the scalar curvature and volume form. We leave its proof as an exercise. (6) flows from an integration by parts. For any function f on a Riemannian manifold M^n

$$\int_{M^n} (\Delta f)^2 \, d\mu = \int_{M^n} \left(|\nabla_i \nabla_j f|^2 + R_{ij} \nabla_i f \nabla_j f \right) d\mu. \tag{8}$$

When $n=2$, we have $R_{ij}\nabla_i f\nabla_j f=\frac{1}{2}R\left|\nabla f\right|^2$. We leave it an exercise to show that (6) follows from the above and:

$$\int_{M^2}\nabla R\cdot\nabla f d\mu=\int_{M^2}(R-r)^2\,d\mu.$$

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Exercise 7 *Prove (8).*

Solution (sketch).

$$\int_{M^n}\nabla_i\nabla_j f\nabla_i\nabla_j f d\mu=-\int_{M^n}\nabla_j f\Delta\nabla_j f d\mu=-\int_{M^n}\left(\nabla_j f\nabla_j\Delta f+R_{ij}\nabla_i f\nabla_j f\right)d\mu$$

and

$$-\int_{M^n}\nabla_j f\nabla_j\Delta f d\mu=\int_{M^n}(\Delta f)^2\,d\mu.$$

Note that another way to write the 2-tensor appearing on the RHS of (6) is:

$$\nabla_i\nabla_j f-\frac{1}{2}\Delta f\cdot g_{ij}=\nabla_i\nabla_j f+R_{ij}-\frac{r}{2}g_{ij}.$$

If $T<\infty$ is the singularity time so that $r=1/\tau$, where $\tau=T-t$, we may rewrite (6) as:

$$\begin{aligned}\frac{dN}{dt}&=-2\int_{M^2}\left|R_{ij}+\nabla_i\nabla_j f-\frac{1}{2\tau}g_{ij}\right|^2 d\mu \\ &\quad-\int_{M^2}\frac{\left|\operatorname{div}\left(\mathrm{Rc}+\nabla\nabla f-\frac{1}{2\tau}g\right)\right|^2}{R}\,d\mu.\end{aligned} \tag{9}$$

References

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