5 Convergence of Ricci flow for closed 3-manifolds with positive Ricci curvature

In this lecture we sketch the proof of Theorem 4 of lecture 4:

Theorem 1 *If* (M^3, g_0) *is a closed Riemannian* 3*-manifold with* $\mathrm{Rc}\,(g_0) > 0$*, then there exists a unique solution* $g\,(t)$ *of the normalized Ricci flow with* $g\,(0) = g_0$ *for all* $t \geq 0$*. As* $t \to \infty$*, the metrics* $g(t)$ *converge exponentially fast in every* C^m*-norm to a* C^∞ *metric* g_∞ *with constant positive sectional curvature.*

The idea of the proof is to first obtain pointwise estimates for the curvature via estimates very specific to the Ricci flow equation and then to estimate the higher derivatives of the curvature using estimates which hold for rather general geometric evolution equations. Our tool is the maximum principle for systems, Theorem 6 of lecture 4:

Theorem 2 *Let* $g\,(t)\,,\ t \in [0, T)$*, be a solution to the Ricci flow on a closed manifold* M^n*. Let* $K \subset E$ *be a subset which is invariant under parallel translation and whose intersection* $K_x \doteqdot K \cap E_x$ *with each fiber is closed and convex. Suppose the* ODE

$$\begin{aligned}\frac{d\lambda_1}{dt} &= \lambda_1^2 + \lambda_2\lambda_3\\ \frac{d\lambda_2}{dt} &= \lambda_2^2 + \lambda_1\lambda_3\\ \frac{d\lambda_3}{dt} &= \lambda_3^2 + \lambda_1\lambda_2.\end{aligned} \tag{1}$$

has the property that for any $\mathbf{M}\,(0) \in K$*, we have* $\mathbf{M}\,(t) \in K$ *for all* $t \in [0, T)$*. If* $\mathrm{Rm}\,(0) \in K$*, then* $\mathrm{Rm}\,(t) \in K$ *for all* $t \in [0, T)$*.*

We construct closed, fiberwise convex sets K, invariant under parallel translation, which are preserved by the ODE. Each such set corresponds to an a priori estimate for the curvature Rm and each successive estimate gives more control on Rm.

We have the following estimates for Rm and corresponding sets $K \subset E$ which are invariant under parallel translation and such that for each $x \in M$, K_x is closed, convex and preserved by the ODE (1). Recall $E = \wedge^2 M^n \otimes_S \wedge^2 M^n$ and Rm is a section of E.

Lemma 3 *If* $\mathrm{Rc} \geq 0$ *at* $t = 0$*, then* $\mathrm{Rc} \geq 0$ *for* $t > 0$*.*

Here we let

$$K = \{\mathbf{M} \in E : \lambda_1\,(\mathbf{M}) + \lambda_2\,(\mathbf{M}) \geq 0\}$$

and note that $\mathrm{Rm} \in K$ if and only if $\mathrm{Rc} \geq 0$ since the smallest eigenvalue of Rc is $\lambda_1\,(\mathrm{Rm}) + \lambda_2\,(\mathrm{Rm})$. The function $\lambda_1 + \lambda_2$ is concave:

$$(\lambda_1 + \lambda_2)\,(\mathbf{M}) = \min\{\mathbf{M}\,(V_1, V_1) + \mathbf{M}\,(V_2, V_2) : \{V_1, V_2\} \text{ orthonormal}\}\,.$$

We compute

$$\frac{d}{dt}(\lambda_1+\lambda_2)=\lambda_1^2+\lambda_2^2+(\lambda_1+\lambda_2)\lambda_3\geq 0$$

whenever $\lambda_1+\lambda_2\geq 0$. So we see $\lambda_1+\lambda_2\geq 0$ is preserved under the ODE, that is, K is preserved under the ODE. The strong maximum principle actually tells us

$$\text{Rc}>0 \text{ is preserved under the Ricci flow.}$$

Lemma 4 *Assuming* Rc >0 *at* $t=0$, *there exists a constant* $C\geq 1/2$ *such that*

$$\lambda_3(\text{Rm})\leq C(\lambda_1(\text{Rm})+\lambda_2(\text{Rm})) \tag{2}$$

at $t=0$. *This estimate is preserved under the Ricci flow.*

To see this let

$$K=\{\mathbf{M}:\lambda_3(\mathbf{M})\leq C(\lambda_1(\mathbf{M})+\lambda_2(\mathbf{M}))\}$$

so that (2) holds if and only if Rm $\in K$. Since

$$\lambda_3(\mathbf{M})=\max_{|V|=1}\mathbf{M}(V,V),$$

λ_3 is convex. Since $\lambda_1+\lambda_2$ is concave, we have K_x is convex for all $x\in M$. To see each K_x is preserved by the ODE, we calculate:

$$\frac{d}{dt}[\lambda_3-C(\lambda_1+\lambda_2)]=\lambda_3(\lambda_3-C(\lambda_1+\lambda_2))-C\left(\lambda_1^2-\frac{1}{C}\lambda_1\lambda_2+\lambda_2^2\right).$$

In particular, if $\lambda_3-C(\lambda_1+\lambda_2)=0$ and $C\geq 1/2$, then

$$\frac{d}{dt}[\lambda_3-C(\lambda_1+\lambda_2)]\leq 0.$$

Since Rm $(g(0))\subset K$, by the maximum principle for systems, Rm $(g(t))\subset K$ and inequality (2) is true for all $t\geq 0$. Note (2) implies

$$\text{Rc}\geq C^{-1}\lambda_3(\text{Rm})\,g\geq \varepsilon Rg.$$

where $\varepsilon=\frac{1}{3}C^{-1}$. In particular, when M^3 is compact, we have that Rc >0 is preserved.

Lemma 5 *There exists* $\delta, C>0$ *such that the eigenvalues of* Rm *satisfy*

$$\frac{\lambda_3-\lambda_1}{\lambda_1+\lambda_2+\lambda_3}\leq C(\lambda_1+\lambda_2+\lambda_3)^{-\delta} \tag{3}$$

for all points and times.

Given $C_0 > 0$, $C_1 \geq 1/2$, $C_2 < \infty$ and $\delta > 0$, let

$$K = \left\{ \mathbf{M} : \begin{array}{c} \lambda_3(\mathbf{M}) - \lambda_1(\mathbf{M}) - C_2\left(\lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}) + \lambda_3(\mathbf{M})\right)^{1-\delta} \leq 0 \\ \lambda_3(\mathbf{M}) \leq C_1\left(\lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M})\right) \\ \lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}) + \lambda_3(\mathbf{M}) \geq C_0 \end{array} \right\}.$$

K is a convex set since $\lambda_3 - \lambda_1 - C_2\left(\lambda_1 + \lambda_2 + \lambda_3\right)^{1-\delta}$ is a convex function for $C_2 > 0$. If $\mathbf{M} \in K$, then $\lambda_1(\mathbf{M}) + \lambda_2(\mathbf{M}) > 0$ by the last two inequalities in the definition of K. The inequalities $\lambda_1 + \lambda_2 + \lambda_3 \geq C_0$ and $\lambda_3 \leq C_1(\lambda_1 + \lambda_2)$ are preserved under the ODE. Since $C_0 > 0$, we can compute

$$\begin{aligned} & \frac{d}{dt} \log\left(\frac{\lambda_3 - \lambda_1}{\left(\lambda_1 + \lambda_2 + \lambda_3\right)^{1-\delta}} \right) \\ & = \delta\left(\lambda_1 + \lambda_3 - \lambda_2\right) - (1-\delta) \frac{\left(\lambda_1 + \lambda_2\right)\lambda_2 + \left(\lambda_2 - \lambda_1\right)\lambda_3 + \lambda_2^2}{\lambda_1 + \lambda_2 + \lambda_3} \\ & \leq \delta\left(\lambda_1 + \lambda_3 - \lambda_2\right) - (1-\delta) \frac{\lambda_2^2}{\lambda_1 + \lambda_2 + \lambda_3}. \end{aligned}$$

Note that

$$\frac{\lambda_2^2}{\lambda_1 + \lambda_2 + \lambda_3} \geq \frac{1}{3} \frac{\left(\lambda_1 + \lambda_2\right)\lambda_2}{\lambda_2 + \lambda_3} \geq \frac{1}{6C_1}\lambda_2$$

since $\lambda_2 + \lambda_3 \leq 2\lambda_3 \leq 2C_1\left(\lambda_1 + \lambda_2\right)$, and we also have

$$\lambda_1 + \lambda_3 - \lambda_2 \leq \lambda_3 \leq C_1\left(\lambda_1 + \lambda_2\right) \leq 2C_1\lambda_2.$$

Hence, choosing $\delta > 0$ small enough so that $\frac{\delta}{1-\delta} \leq \frac{1}{12C_1^2}$, we have

$$\frac{d}{dt} \log\left(\frac{\lambda_3 - \lambda_1}{\left(\lambda_1 + \lambda_2 + \lambda_3\right)^{1-\delta}} \right) \leq 0.$$

Now given this $\delta > 0$, we choose C such that (3) holds at $t = 0$. By the above calculation the ODE (1) preserves the set K and the lemma follows from the maximum principle for systems.

Exercise 6 *Show that* $\lambda_3 - \lambda_1 \geq \left|\operatorname{Rc} - \frac{1}{3}Rg\right|$ *and hence this implies there exist constants* $C < \infty$ *and* $\delta > 0$ *such that*

$$\left|\operatorname{Rm} - \frac{1}{3}R \operatorname{Id}_{\wedge^2}\right|^2 = \left|\operatorname{Rc} - \frac{1}{3}Rg\right| \leq CR^{1-\delta} \tag{4}$$

Next we have the following gradient estimate for the scalar curvature.

Proposition 7 *Let* $\left(M^3, g(0)\right)$ *be a closed 3-manifold with positive Ricci curvature. For any* $\varepsilon > 0$, *there exists* $C(\varepsilon)$ *depending only on* ε *and* $g(0)$ *such that*

$$|\nabla R|^2(x,t) \leq \varepsilon R(x,t)^3 + C(\varepsilon)$$

as long as the solution exists.

The proof is based on the maximum principle and we refer the reader to [2] or [1]. One shows that there exist constants $\beta_0 > 0$ and $\delta > 0$ depending only on $g(0)$ such that for all $\beta \in [0, \beta_0]$

$$W \doteqdot \frac{|\nabla R|^2}{R} + \frac{37}{2}\left(8\sqrt{3}+1\right)\left(|\mathrm{Rc}|^2 - \frac{1}{3}R^2\right) - \beta R^{2-\delta}$$

satisfies

$$\frac{\partial}{\partial t} W \leq \Delta W + C,$$

where C depends only on β and $g(0)$. The maximum principle tells us

$$W(t) \leq \max W(0) + C_1 t \leq C_2.$$

Using the gradient of scalar curvature estimate and Myers' theorem, one can show:

Lemma 8 (Global scalar curvature pinching) *We have*

$$\lim_{t \to T} \frac{R_{\max}(t)}{R_{\min}(t)} = 1. \tag{5}$$

In fact, there exist constants $C < \infty$ and $\gamma > 0$ depending only on $g(0)$ such that

$$\frac{R_{\min}(t)}{R_{\max}(t)} \geq 1 - C R_{\max}(t)^{-\gamma} \tag{6}$$

for all $t \in [0, T)$.

Remark 9 *Since $\lim_{t \to T} R_{\max}(t) = \infty$, (6) implies (5) and*

$$\lim_{t \to T} R_{\min}(t) = \infty.$$

We then deduce:

Lemma 10 (Global sectional curvature pinching) *For every $\varepsilon \in (0,1)$, there exists $\tau < T$ such that for all $t \in [\tau, T)$ the sectional curvatures of $g(t)$ are positive and*

$$\min_{x \in M^3} \lambda_1(\mathrm{Rm})(x,t) \geq (1-\varepsilon) \max_{x \in M^3} \lambda_3(\mathrm{Rm})(x,t).$$

Proof. By (4), there exists $C < \infty$ and $\delta > 0$ such that

$$\frac{\lambda_1(\mathrm{Rm})}{\lambda_3(\mathrm{Rm})}(x,t) \geq 1 - C\frac{R^{1-\delta}}{\lambda_3(\mathrm{Rm})}(x,t) \geq 1 - 3C R_{\min}(t)^{-\delta}$$

for all $x \in M^3$ and $t \in [0, T)$. ■

Now we go back to the normalized flow

$$\frac{\partial}{\partial t}\tilde{g}_{ij} = -2\tilde{R}_{ij} + \frac{2}{3}\tilde{r}\tilde{g}_{ij}$$

which is equivalent to the original Ricci flow by rescaling time and space.

Each of the following estimates represents in some way the fact that under the normalized Ricci flow the metrics converge to constant curvature exponentially fast.

Lemma 11 (Estimates for the normalized RF) *If* $\left(M^3, g(0)\right)$ *is a closed 3-manifold with positive Ricci curvature, then under the normalized Ricci flow we have the following estimates. Let* $[0, \tilde{T})$ *denote the maximal time interval of existence of the normalized Ricci flow. There exist constants* $C < \infty$ *and* $\delta > 0$ *such that*

1. $$\lim_{\tilde{t} \to \tilde{T}} \frac{\tilde{R}_{\max}\left(\tilde{t}\right)}{\tilde{R}_{\min}\left(\tilde{t}\right)} = 1$$

2. $$\widetilde{\mathrm{Rc}} \geq \delta \tilde{R} \tilde{g}$$

3. $$\tilde{R}_{\max}\left(\tilde{t}\right) \leq C$$

4. $$\tilde{T} = \infty$$

5. $$\lim_{\tilde{t} \to \infty} \left(\max_{\tilde{x} \in M^3} \frac{\left| \widetilde{\mathrm{Rc}} - \frac{1}{3} \tilde{R} \tilde{g} \right|^2}{\tilde{R}^2} \left(\tilde{x}, \tilde{t}\right) \right) = 0$$

6. $$\tilde{R}_{\min}\left(\tilde{t}\right) \geq \frac{1}{C}$$

   and hence $\operatorname{diam}\left(\tilde{g}(t)\right) \leq C$

7. $$\left| \widetilde{\mathrm{Rc}} - \frac{1}{3} \tilde{R} \tilde{g} \right| \leq C e^{-\delta \tilde{t}}$$

8. $$\tilde{R}_{\max}\left(\tilde{t}\right) - \tilde{R}_{\min}\left(\tilde{t}\right) \leq C e^{-\delta \tilde{t}}$$

9. $$\left| \widetilde{\mathrm{Rc}} - \frac{1}{3} \tilde{r} \tilde{g} \right| \leq C e^{-\delta \tilde{t}} \tag{7}$$

10. $$\left| \tilde{\nabla}^k \widetilde{\mathrm{Rc}} \right| \leq C e^{-\delta \tilde{t}} \tag{8}$$

    for all $k \in \mathbb{N}$.

From the above lemma and the fact that we can estimate the derivatives of the metrics in terms of the estimates for the derivatives of the Ricci tensor, one can complete the proof of Theorem 1. In particular, by (7) there exists a constant $C < \infty$ such that

$$\frac{1}{C}\tilde{g}\left(0\right) \leq \tilde{g}\left(\tilde{t}\right) \leq C\tilde{g}\left(0\right)$$

for all $\tilde{t} \in [0, \infty)$, and the metrics $\tilde{g}\left(\tilde{t}\right)$ converge uniformly on compact sets to a continuous metric $\tilde{g}\left(\infty\right)$ as $\tilde{t} \rightarrow \infty$. The estimates (8) imply the exponential convergence in each C^k-norm of $\tilde{g}\left(\tilde{t}\right)$ to $\tilde{g}\left(\infty\right)$.

References

[1] Chow, Bennett; Knopf, Dan. *The Ricci flow: An introduction,* Mathematical Surveys and Monographs, AMS, Providence, RI, 2004.

[2] Hamilton, Richard S. *Three-manifolds with positive Ricci curvature.* J. Differential Geom. **17** (1982), no. 2, 255–306.