

3 Heat equations and maximum principles

The heat equation is the linear model for the Ricci flow. Recall that given a Riemannian manifold (M^n, g) , we say that a time-dependent function

$$u : M^n \times [0, T) \rightarrow \mathbb{R}$$

is a solution of the heat equation if

$$\frac{\partial u}{\partial t} = \Delta u.$$

The heat equation smooths out functions. For example, if $M^n = S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and

$$f(x) = \sum_{m=0}^{\infty} (a_m \cos mx + b_m \sin mx),$$

then

$$u(x, t) = \sum_{m=0}^{\infty} e^{-m^2 t} (a_m \cos mx + b_m \sin mx)$$

is a solution of the heat equation with $u(0) = f$. From this one can deduce that

$$\lim_{t \rightarrow \infty} u(x_0, t) = a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx$$

for all $x_0 \in S^1$. One can also show that

$$\left| \frac{\partial u}{\partial x}(x, t) \right| \leq C e^{-t}.$$

One manifestation of the smoothing property of the heat equation is the maximum principle. The maximum principle is simply the first and second derivative tests from calculus. If a function $f : M \rightarrow \mathbb{R}$ attains its minimum at a point $x_0 \in M$, then

$$\nabla f(x_0) = 0 \quad \text{and} \quad \Delta f(x_0) \geq 0.$$

If we have a time-dependent function $u : M^n \times [0, T) \rightarrow \mathbb{R}$ such that on some time interval $[0, t_0]$ u attains its minimum at some point x_0 at time t_0 , then

$$\frac{\partial u}{\partial t}(x_0, t_0) \leq 0$$

in addition to

$$\nabla u(x_0, t_0) = 0 \quad \text{and} \quad \Delta u(x_0, t_0) \geq 0.$$

The above elementary facts imply the following (see also [1] for example).

Theorem 1 Let $g(t)$ be a family of metrics on a closed manifold M^n and suppose $u : M^n \times [0, T] \rightarrow \mathbb{R}$ satisfies

$$\frac{\partial}{\partial t} u \geq \Delta_{g(t)} u + X_t(u),$$

where $X_t(u)$ denotes the directional derivative of u in the direction of some time-dependent vector field X_t . If $u \geq c$ at $t = 0$ for some $c \in \mathbb{R}$, then $u \geq c$ for all $t \geq 0$.

Proof. Given any $\varepsilon > 0$, define $u_\varepsilon : M \times [0, T] \rightarrow \mathbb{R}$ by

$$u_\varepsilon = u + \varepsilon(1 + t).$$

Since $u \geq c$ at $t = 0$, we have $u_\varepsilon > c$ at $t = 0$. If for some $\varepsilon > 0$, $u_\varepsilon \leq c$ somewhere in $M \times (0, T)$, then since M is closed, there exists $(x_1, t_1) \in M \times (0, T)$ such that $u_\varepsilon(x_1, t_1) = c$ and $u_\varepsilon(x, t) > c$ for all $x \in M$ and $t \in [0, t_1)$. We then have at (x_1, t_1)

$$0 \geq \frac{\partial u_\varepsilon}{\partial t} \geq \Delta_{g(t)} u_\varepsilon + \varepsilon > 0,$$

which is a contradiction. Hence $u_\varepsilon > c$ on $M \times [0, T]$ for all $\varepsilon > 0$. Taking the limit as $\varepsilon \rightarrow 0$, we obtain $u \geq c$ on $M \times [0, T]$. ■

Exercise 2 Suppose

$$\frac{\partial}{\partial t} u \geq \Delta_{g(t)} u + X_t(u) + Cu$$

whenever $u \leq 0$ for some $c \in \mathbb{R}$. Show that if $u \geq 0$ at $t = 0$ for some $c \in \mathbb{R}$, then $u \geq 0$ for all $t \geq 0$. Hint: consider the function $v(x, t) = e^{-Ct}u(x, t)$.

We can apply the above exercise to the Ricci flow. Recall that under the Ricci flow the scalar curvature evolves by

$$\frac{\partial}{\partial t} R = \Delta R + 2|\text{Rc}|^2.$$

Since $|\text{Rc}|^2 \geq \frac{1}{n}R^2$ (see lecture 11), we have

$$\frac{\partial}{\partial t} R \geq \Delta R + \frac{2}{n}R^2. \tag{1}$$

Now we apply a trick which is useful for obtaining sharp estimates. Let $\rho(t)$ be a solution of the ODE

$$\frac{d\rho}{dt} = \frac{2}{n}\rho^2.$$

This ODE is obtained by replacing \geq by $=$ and dropping the Δ term in (1). We then have

$$\frac{\partial}{\partial t} (R - \rho) \geq \Delta (R - \rho) + \frac{2}{n} (R + \rho) (R - \rho).$$

Let $[0, T)$ be the time interval of existence of the solution of Ricci flow. For any $0 < \tau < T$ we have $R + \rho \leq C$ for some constant $C < \infty$. Hence

$$\frac{\partial}{\partial t}(R - \rho) \geq \Delta(R - \rho) + \frac{2C}{n}(R - \rho).$$

whenever $R - \rho \leq 0$. Now choose ρ so that

$$\rho(0) = \min_{x \in M} R(x, 0).$$

By the exercise, since $R - \rho \geq 0$ at $t = 0$, we have $R - \rho \geq 0$ on $M \times [0, \tau]$. Since $\tau < T$ is arbitrary, it follows that $R - \rho \geq 0$ on $M \times [0, T)$. We also easily see that the solution to $\frac{d\rho}{dt} = \frac{2}{n}\rho^2$ is

$$\rho(t) = \left(\frac{1}{\rho(0)} - \frac{2}{n}t \right)^{-1}$$

unless $\rho(0) = 0$ in which case $\rho(t) \equiv 0$.

We conclude

Lemma 3

$$R(x, t) \geq \left(\frac{1}{\min_{x \in M} R(x, 0)} - \frac{2}{n}t \right)^{-1}$$

if $\min_{x \in M} R(x, 0) \neq 0$. If $\min_{x \in M} R(x, 0) = 0$, then we have $R(x, t) \geq 0$.

In particular, if $\min_{x \in M} R(x, 0) > 0$, then the maximal time interval of existence $[0, T)$ satisfies

$$T < \frac{n}{2 \min_{x \in M} R(x, 0)}.$$

Now the scalar curvature is a function. We are also interested in the Ricci curvatures and sectional curvatures, which form tensors. For this reason we would like to have a maximum principle which applies to tensors. For such a maximum principle to hold, we need a notion of positivity for the tensors we are considering. Fortunately, symmetric 2-tensors and curvature operator like tensors (we'll define this later) are bilinear forms for which one can define positivity (or nonnegativity).

In Hamilton's 1982 paper [2] a maximum principle for symmetric 2-tensors was proved. If γ and δ are symmetric 2-tensors, we write $\gamma \geq \delta$ if $\gamma - \delta$ is positive semi-definite.

Theorem 4 *Let $(M^n, g(t))$ be a solution of the Ricci flow on a closed manifold and suppose that $\alpha(t)$ is a symmetric 2-tensor such that*

$$\frac{\partial}{\partial t}\alpha \geq \Delta_{g(t)}\alpha + \beta,$$

where β is a polynomial in α using the metric $g(t)$ to contract indices. Further suppose β satisfies the **null eigenvector assumption** that if A_{ij} is a non-negative symmetric 2-tensor at a point (x, t) and if V is such that $A_{ij}V^j = 0$, then

$$\beta_{ij}(A, g)V^iV^j \geq 0.$$

If $\alpha(0) \geq 0$, then $\alpha(t) \geq 0$ for all $t \geq 0$ as long as the solution exists.

It can be shown that the evolution of the Ricci tensor on a 3-manifold is given by:

$$\frac{\partial}{\partial t}R_{ij} = \Delta R_{ij} + 3RR_{ij} - 6R_{ip}R_{jp} + \left(2|\text{Rc}|^2 - R^2\right)g_{ij}$$

Corollary 5 *If $(M^3, g(t))$ is a solution to the Ricci flow on a closed 3-manifold with nonnegative Ricci curvature at the initial time, then the Ricci curvature is nonnegative as long as the solution exists.*

Proof. The tensor

$$\beta_{ij} = 3RR_{ij} - 6R_{ip}R_{jp} + \left(2|\text{Rc}|^2 - R^2\right)g_{ij}$$

satisfies the null eigenvector assumption with respect to $\alpha_{ij} = R_{ij}$: if $\text{Rc}(V) = 0$ at some point and time, then at that point and time: $|\text{Rc}|^2 \geq \frac{1}{2}R^2$ so that

$$\beta_{ij}V^iV^j = \left(2|\text{Rc}|^2 - R^2\right)|V|^2 \geq 0.$$

■

In [3] a general maximum principle which applies to curvature operator like tensors was proved. We will discuss this in the next lecture and consider applications to estimating the curvature.

References

- [1] Hamilton, Richard S. *Harmonic maps of manifolds with boundary*. Lecture Notes in Mathematics, Vol. **471**. Springer-Verlag, Berlin-New York, 1975.
- [2] Hamilton, Richard S. *Three-manifolds with positive Ricci curvature*. J. Differential Geom. **17** (1982), no. 2, 255–306.
- [3] Hamilton, Richard S. *Four-manifolds with positive curvature operator*. J. Differential Geom. **24** (1986), no. 2, 153–179.