2 The Ricci flow equation and associated equations

Hamilton's **Ricci flow equation** is

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij}.$$

It has been called the heat equation for metrics. As such we expect it to smooth out metrics and make them better. Let's first take a look at a couple of simple examples.

Consider the unit n-sphere S^n with the standard metric g_{S^n}. If $g_0 = r_0^2 g_{S^n}$ for some $r_0 > 0$ (r_0 is the radius), then

$$g(t) \doteqdot \left(r_0^2 - 2(n-1)t\right) g_{S^n} \tag{1}$$

is a solution to the Ricci flow with $g(0) = g_0$ defined on the maximal time interval $(-\infty, T)$, where $T \doteqdot r_0^2/2(n-1)$. This explicit solution exhibits a few basic properties of Ricci flow. Positive Ricci curvature causes the metric to shrink. Singularities often form in finite time. Constant curvature metrics evolve homothetically (without changing their shape).

One nice thing about the Ricci tensor is its scale-invariance. For any constant $a > 0$ we have $\mathrm{Rc}(ag) = \mathrm{Rc}(g)$.

Exercise 1 *Let g_0 be an Einstein metric:*

$$\mathrm{Rc}(g_0) \equiv c g_0$$

for some $c \in \mathbb{R}$. Show that $g(t) = (1 - 2ct) g_0$ is a solution to the Ricci flow. Note that $g(t)$ is homothetic to the initial metric g_0 and shrinks, is stationary, or expands depending on whether c is positive, zero, or negative, respectively.

Under the Ricci flow $\frac{\partial}{\partial t}g_{ij} = -2R_{ij}$, we have

$$\boxed{\frac{\partial}{\partial t}\Gamma_{ij}^k = -g^{k\ell}\left(\nabla_i R_{j\ell} + \nabla_j R_{i\ell} - \nabla_\ell R_{ij}\right).} \tag{2}$$

Equation (2) follows from substituting in $v_{ij} = -2R_{ij}$ in the variation formula for the Christoffel symbols given in the first lecture. If you had difficulty verifying that variation formula, either see [3] or [2], or use the following hint: given any $p \in M^n$, compute in local coordinates where $\Gamma_{ij}^k(p) = 0$. Note that in such coordinates $\nabla_i v_{j\ell}(p) = \frac{\partial}{\partial x^i} v_{j\ell}(p)$.

Let Δ denote the laplacian acting on functions, which in local coordinates is given by:

$$\Delta \doteqdot g^{ij}\nabla_i\nabla_j = g^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial}{\partial x^k}\right). \tag{3}$$

The laplacian is one of the most fundamental partial differential operators. The heat equation is

$$\frac{\partial u}{\partial t} = \Delta u.$$

Exercise 2 *If $(M^n, g(t))$ is a solution to the Ricci flow, show that*

$$\frac{\partial}{\partial t}\left(\Delta_{g(t)}\right) = 2R_{ij} \cdot \nabla_i \nabla_j,$$

where $\Delta_{g(t)}$ is the laplacian acting on functions with respect to $g(t)$. Hint: use the contracted second Bianchi identity and evolution equation for the Christoffel symbols.

We first state the evolution equation for the scalar curvature:

$$\boxed{\frac{\partial}{\partial t} R = \Delta R + 2\left|\mathrm{Rc}\right|^2.} \tag{4}$$

This also follows directly form the variation formula for the scalar curvature given in the first lecture. Note that $\operatorname{div}(\mathrm{Rc}) = \frac{1}{2}\nabla R$ by the contracted second Bianchi identity so that $\operatorname{div}(\operatorname{div}(\mathrm{Rc})) = \frac{1}{2}\Delta R$. *Thus we see that the scalar curvature satisfies a heat type equation.*

Of course curvature is not the only geometric quantity around. Of fundamental importance is volume. The evolution of the volume form $d\mu$ is given by

$$\frac{\partial}{\partial t} d\mu = -R d\mu. \tag{5}$$

This follows from the variation formula stated in the first lecture. The volume $\mathrm{Vol}(g) \doteqdot \int_{M^n} d\mu$ evolves by

$$\frac{d}{dt}\mathrm{Vol}(g(t)) = -\int_M R d\mu. \tag{6}$$

Since the volume is not constant and often we would like to prevent the solution from shrinking to a point or expanding to infinity, we consider the normalized Ricci flow:

$$\frac{\partial}{\partial t}\hat{g}_{ij} = -2\hat{R}_{ij} + \frac{2}{n}\hat{r}\hat{g}_{ij} \tag{7}$$

where $\hat{r} = \mathrm{Vol}(\hat{g})^{-1} \cdot \int_{M^n} \hat{R}\, d\hat{\mu}$ is the average scalar curvature. We then have

$$\frac{d}{dt}\mathrm{Vol}(\hat{g}(t)) = 0. \tag{8}$$

Now Einstein metrics are the fixed points of the Ricci flow.

Next we examine the evolution equation for the Ricci tensor under the Ricci flow. In higher dimensions the formula involves the whole Riemann curvature tensor. We can avoid this and get a formula just in terms of the Ricci tensor if we restrict ourselves to dimension 3, in which case we have

$$\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} + 3RR_{ij} - 6R_{ip}R_{jp} + \left(2\left|\mathrm{Rc}\right|^2 - R^2\right) g_{ij}. \tag{9}$$

Just like the evolution of the scalar curvature we get a heat equation with a quadratic nonlinearity.

This evolution equation take a little effort to obtain. We first describe some nice related formulas. Given a variation $\frac{\partial}{\partial s} g_{ij} = v_{ij}$, we have

$$\frac{\partial}{\partial s} R_{ij} = -\frac{1}{2}\left(\Delta_L v_{ij} + \nabla_i \nabla_j V - \nabla_i (\operatorname{div} v)_j - \nabla_j (\operatorname{div} v)_i\right) \tag{10}$$

where Δ_L denotes the Lichnerowicz laplacian, which is defined on symmetric 2-tensors by

$$\Delta_L v_{ij} \doteqdot \Delta v_{ij} + 2R_{kij\ell} v_{k\ell} - R_{ik} v_{jk} - R_{jk} v_{ik}.$$

We leave it to the motivated reader to verify this formula (see [2] or for example). Since

$$\nabla_i \nabla_j R - \nabla_i (\operatorname{div} \mathrm{Rc})_j - \nabla_j (\operatorname{div} \mathrm{Rc})_i = 0$$

by the contracted second Bianchi identity, we have that under the Ricci flow

$$\frac{\partial}{\partial t} R_{ij} = (\Delta_L \mathrm{Rc})_{ij} = \Delta R_{ij} + 2R_{kij\ell} R_{k\ell} - R_{ik} R_{jk} - R_{jk} R_{ik}.$$

This formula holds in all dimensions. When $n = 3$, we can use the formula:

$$R_{kij\ell} = R_{k\ell} g_{ij} + R_{ij} g_{k\ell} - R_{kj} g_{i\ell} - R_{i\ell} g_{kj} - \frac{1}{2} R (g_{k\ell} g_{ij} - g_{kj} g_{i\ell})$$

to obtain (9).

Finally we state the evolution equation for the Riemann curvature tensor. See [3] or [2] for the derivation of this equation:

$$\begin{aligned} \frac{\partial}{\partial t} R_{ijk\ell} = {} & \Delta R_{ijk\ell} + 2 (B_{ijk\ell} - B_{ij\ell k} + B_{ikj\ell} - B_{i\ell jk}) \\ & - (R_{ip} R_{pjk\ell} + R_{jp} R_{ipk\ell} + R_{kp} R_{ijp\ell} + R_{\ell p} R_{ijkp}), \end{aligned} \tag{11}$$

where

$$B_{ijk\ell} \doteqdot -g^{pr} g^{qs} R_{ipjq} R_{kr\ell s} = -R_{pijq} R_{q\ell kp}. \tag{12}$$

Although the formula looks complicated, the derivation is rather straightforward. In later lectures we shall try to illuminate this equation.

Since in this lecture we have introduced the laplacian and the laplacian is fundamental in what is known as the Bochner technique (in fact much of the Ricci flow is in essense the Bochner technique), we end this lecture with some exercises.

Exercise 3 *Show that for any function f*

$$\Delta \nabla_i f = \nabla_i \Delta f + R_{ij} \nabla_j f. \tag{13}$$

Exercise 4 *Show that for any smooth function f*

$$\Delta |\nabla f|^2 = 2 |\nabla_i \nabla_j f|^2 + 2R_{ij} \nabla_i f \nabla_j f + 2\nabla_i f \nabla_i (\Delta f). \tag{14}$$

Conclude from this that if $\mathrm{Rc} \geq 0$, $\Delta f \equiv 0$ *and* $|\nabla f| \equiv 1$, *then* ∇f *is parallel, i.e.,* $\nabla\nabla f \equiv 0$, *and* $\mathrm{Rc}(\nabla f, \nabla f) \equiv 0$.

References

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[2] Chow, Bennett; Knopf, Dan. *The Ricci flow: An introduction,* Mathematical Surveys and Monographs, AMS, Providence, RI, 2004.

[3] Hamilton, Richard S. *Three-manifolds with positive Ricci curvature.* J. Differential Geom. **17** (1982), no. 2, 255–306.