1 Connections, curvatures, and variation formulas

The reader is assumed to have a basic familiarity with Riemannian geometry. For her or his convenience we review some of the basic facts. We refer the reader to the book by Cheeger and Ebin [2] for Riemannian and comparison geometry and the author's book [3] with Dan Knopf for an introduction to Ricci flow. The reader may find [1] as a convenient reference for some papers on Ricci flow including [6] on singularity formation. If the reader is having difficulty filling in some details in these lectures, we refer her or him to [3] and also the forthcoming book by Peng Lu, Lei Ni and the author [4].

Let M^n be an n-dimensional differentiable manifold and g be a Riemannian metric. The Levi-Civita covariant derivative $\nabla_X : C^\infty(TM) \to C^\infty(TM)$ is the unique linear map such that

$$\nabla_{X+cY} = \nabla_X + c\nabla_Y$$
$$\nabla_X (Y + fZ) = \nabla_X Y + (Xf) Z + f\nabla_X Z$$

$$X (g (Y, Z)) = g (\nabla_X Y, Z) + g (Y, \nabla_X Z) \tag{1}$$
$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{2}$$

for any vector fields X, Y, Z, constant c, and function f. *The covariant derivative tells us how to differentiate vector fields.*

Exercise 1 ([2]) *Show that*

$$2g (\nabla_X Y, Z) = X (g (Y, Z)) + Y (g (X, Z)) - Z (g (X, Y)) + g ([X, Y], Z) - g ([X, Z], Y) - g ([Y, Z], X). \tag{3}$$

Let $\{x^i\}_{i=1}^n$ be a local coordinate system defined in an open set U in M^n. The Christoffel symbols are defined in U by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \doteqdot \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

From now on we shall use the Einstein summation convention, where we sum over repeated indices and omit the summation sign $\sum$. By (3) and $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$ ([5])

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell} \left(\frac{\partial}{\partial x^i} g_{j\ell} + \frac{\partial}{\partial x^j} g_{i\ell} - \frac{\partial}{\partial x^\ell} g_{ij} \right). \tag{4}$$

Many calculations in Ricci flow are carried out in local coordinates.

Using the covariant derivative we may define the Riemann curvature $(3,1)$-tensor Rm by

$$\operatorname{Rm}(X, Y) Z \doteqdot \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The Riemann curvature tensor measures how noncommutative covariant differentiation is; its expresses how curved the manifold is.

Exercise 2

$$\text{Rm}\,(fX, Y)\,Z = \text{Rm}\,(X, fY)\,Z = \text{Rm}\,(X, Y)\,(fZ) = f\,\text{Rm}\,(X, Y)\,Z. \tag{5}$$

Since Rm *is linear over* C^∞ *functions, it is indeed a tensor.*

The components R_{ijk}^{ℓ} of Rm are defined by

$$\text{Rm}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} \doteqdot R_{ijk}^{\ell}\frac{\partial}{\partial x^\ell}$$

and we also define $R_{ijk\ell} \doteqdot g_{\ell m} R_{ijk}^{m}$ as the components of the Riemann $(4,0)$-tensor:

$$R_{ijk\ell} = \left\langle \text{Rm}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\right\rangle.$$

Exercise 3 *Show that*

$$\boxed{R_{ijk}^{\ell} = \partial_i \Gamma_{jk}^{\ell} - \partial_j \Gamma_{ik}^{\ell} + \Gamma_{jk}^{p}\Gamma_{ip}^{\ell} - \Gamma_{ik}^{p}\Gamma_{jp}^{\ell}} \tag{6}$$

Exercise 4 *Show the following basic symmetries of the Riemann curvature tensor:*

$$R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij}.$$

The Ricci tensor Rc is the trace

$$\text{Rc}\,(Y, Z) \doteqdot \text{trace}\,(X \mapsto \text{Rm}\,(X, Y)\,Z)\,.$$

Its components $R_{jk} \doteqdot \text{Rc}\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right)$ are given by

$$R_{jk} = \sum_{i=1}^{n} R_{ijk}^{i}.$$

The scalar curvature is the trace of the Ricci tensor:

$$R = g^{ij} R_{ij}$$

where $g^{ij} \doteqdot \left(g^{-1}\right)_{ij}$ is the inverse matrix.

By the product rule, covariant differentiation is defined on tensors. The first and second Bianchi identities are:

$$R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0 \tag{7}$$

$$\nabla_i R_{jk\ell m} + \nabla_j R_{ki\ell m} + \nabla_k R_{ij\ell m} = 0. \tag{8}$$

Exercise 5 *Prove the (twice) contracted second Bianchi identity*

$$\boxed{2\nabla_q R_{pq} = \nabla_p R.} \tag{9}$$

The Lie derivative of α with respect to X is defined by

$$\mathcal{L}_X \alpha \doteqdot \lim_{t \to 0} \frac{1}{t} \left(\alpha - (\varphi_t)_* \alpha \right). \tag{10}$$

The Lie derivative is related to the diffeomorphism invariance of the tensor α. We have the following fact: if f is a function, then

$$\boxed{\left(\mathcal{L}_{\mathrm{grad}_g f} g \right)_{ij} = 2 \nabla_i \nabla_j f.} \tag{11}$$

The Ricci flow equation is

$$\frac{\partial}{\partial t} g_{ij} = -2 R_{ij}.$$

In Ricci flow we want to see how various geometric quantities evolve given a solution to the Ricci flow. For this reason we compute the variation formulas for the Christoffel symbols and curvature tensors. The variation of the Christoffel symbols is given as follows. If $g(s)$ is a one-parameter family of metrics with

$$\frac{\partial}{\partial s} g_{ij} = v_{ij},$$

then

$$\frac{\partial}{\partial s} \Gamma_{ij}^k = \frac{1}{2} g^{k\ell} \left(\nabla_i v_{j\ell} + \nabla_j v_{i\ell} - \nabla_\ell v_{ij} \right). \tag{12}$$

From this we calculate the variation of the Ricci tensor:

$$\frac{\partial}{\partial s} R_{ij} = \nabla_p \left(\frac{\partial}{\partial s} \Gamma_{ij}^p \right) - \nabla_i \left(\frac{\partial}{\partial s} \Gamma_{pj}^p \right). \tag{13}$$

and the variation of scalar curvature:

$$\boxed{\frac{\partial}{\partial s} R = -\Delta V + \operatorname{div}(\operatorname{div} v) - \langle v, \mathrm{Rc} \rangle ,} \tag{14}$$

where $V = g^{ij} v_{ij} = \operatorname{trace}(v)$ is the trace of v.

The volume form is given in a positively oriented local coordinate system $\{x^i\}$ by

$$d\mu = \sqrt{\det g_{ij}} dx^1 \wedge \cdots \wedge dx^n \tag{15}$$

(we assume that M is oriented). If $\frac{\partial}{\partial s} g_{ij} = v_{ij}$, then

$$\frac{\partial}{\partial s} d\mu = \frac{1}{2} V d\mu. \tag{16}$$

The Einstein-Hilbert functional is

$$E(g) \doteqdot \int_M R d\mu.$$

The above formulas imply that if $\frac{\partial}{\partial s} g_{ij} = v_{ij}$, then

$$\begin{aligned}\frac{d}{ds} E &= \int_M \left(-\Delta V + \nabla_p \nabla_q v_{pq} - \langle v, \mathrm{Rc} \rangle + \frac{1}{2} R V \right) d\mu \\ &= \int_M \left\langle v, \frac{1}{2} R g - \mathrm{Rc} \right\rangle d\mu.\end{aligned}$$

Thus we see the critical points E satisfy Einstein's equation $\frac{1}{2} R g - \mathrm{Rc} = 0$. The gradient flow of E is given by

$$\frac{\partial}{\partial t} g_{ij} = 2 \left(\nabla E \left(g \right) \right)_{ij} = R g_{ij} - 2 R_{ij}. \tag{17}$$

This is almost the Ricci flow.

References

[1] Cao, Huai-Dong; Chow, Bennett; Chu, Sun-Chin, Yau, Shing-Tung, editors. *Collected papers on Ricci flow.* Internat. Press, Somerville, MA, 2003.

[2] Cheeger, Jeff; Ebin, David G. *Comparison theorems in Riemannian geometry.* North-Holland Mathematical Library, Vol. 9. North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1975.

[3] Chow, Bennett; Knopf, Dan. *The Ricci flow: An introduction,* Mathematical Surveys and Monographs, AMS, Providence, RI, 2004.

[4] Chow, Bennett; Lu, Peng; Ni, Lei. *Hamilton's Ricci flow.* Preliminary version of book to be published by Science Press, China.

[5] Eisenhart, Luther Pfahler. *Riemannian geometry.* Eighth printing. Princeton Landmarks in Mathematics. Princeton Paperbacks. Princeton University Press, Princeton, NJ, 1997. x+306 pp.

[6] Hamilton, Richard S. *The formation of singularities in the Ricci flow.* Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Internat. Press, Cambridge, MA, 1995.