

Hall(Q)

Basis = $\text{Reps}(Q)$
in \mathbb{F}_9 vect spaces

$$A * B = \sum_C r_{AB}^C \cdot C, \quad A, B, C \in \text{Reps}(Q)$$

$$r_{AB}^C = \# \{ B' \subset C \mid B' \cong B, C/B' \cong A \}$$

Hall (Q)

associative, not necessarily commutative

(2)

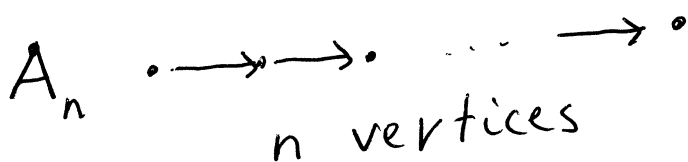
• $r_{A, B}^C = r_{A, B}^C(q)$ polynomial in q .

• $C/B \cong A \Rightarrow \dim C = \dim A + \dim B$

Q without loops



Lie algebra
 $Lie(Q)$



$(n+1) \times (n+1)$ matrices
trace = 0

Thm, $Hall(Q)|_{q=1} \cong$

Upper triangular part $Lie(Q)$

$$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & * \\ & 0 & & 0 \end{pmatrix}$$

upper triangular part

(3)

$n = 2$

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e_{12} = [e_1, e_2] = e_1 e_2 - e_2 e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

e_{12} central

$$e_1 \cdot e_{12} = e_{12} \cdot e_1$$

$$e_2 \cdot e_{12} = e_{12} \cdot e_2$$

q -analogue

$$e_1 \cdot e_{12} = q \cdot e_{12} \cdot e_1$$

$$Q \quad \bullet \longrightarrow \bullet$$

$$\mathbb{F} = \mathbb{F}_9$$

Irreps:

$$e_1: \begin{array}{ccc} \mathbb{F} & & 0 \\ \bullet & \longrightarrow & \bullet \end{array}$$

$$e_2: \begin{array}{ccc} 0 & & \mathbb{F} \\ \bullet & \longrightarrow & \bullet \end{array}$$

$$e_1 * e_2 = \begin{array}{ccc} \mathbb{F} & \sim & \mathbb{F} \\ \bullet & \longrightarrow & \bullet \end{array} + \begin{array}{ccc} \mathbb{F} & 0 & \mathbb{F} \\ \bullet & \longrightarrow & \bullet \end{array}$$

$$C/e_2 \cong e_1$$

$$e_2 * e_1 = \begin{array}{ccc} \mathbb{F} & 0 & \mathbb{F} \\ \bullet & \longrightarrow & \bullet \end{array}$$

$$C/e_1 \cong e_2$$

$$e_1 * e_2 - e_2 * e_1 = \begin{array}{ccc} \mathbb{F} & \sim & \mathbb{F} \\ \bullet & \longrightarrow & \bullet \end{array} =: e_{12}$$

$$e_{12} * e_1 = \begin{array}{ccc} \mathbb{F}^2 & \xrightarrow{f} & \mathbb{F} \\ \bullet & \longrightarrow & \bullet \end{array}$$

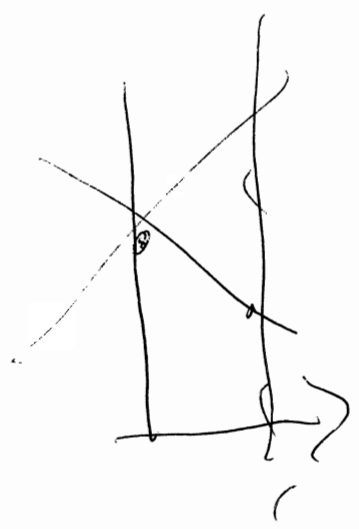
$$C/e_1 \cong e_{12}$$

surjective quotient

$$e_1 = \begin{array}{ccc} \mathbb{F} & \longrightarrow & 0 \\ \bullet & \longrightarrow & \bullet \end{array} = \begin{array}{ccc} \text{Ker } f & \longrightarrow & 0 \\ \bullet & \longrightarrow & \bullet \end{array}$$

$$e_1 * e_{12} =$$

$$\begin{array}{ccc} \mathbb{F}^2 & \xrightarrow{f} & \mathbb{F} \\ \bullet & \longrightarrow & \bullet \\ \cup & & \parallel \\ \mathbb{F} & \xrightarrow{\sim} & \mathbb{F} \\ \bullet & \longrightarrow & \bullet \end{array}$$



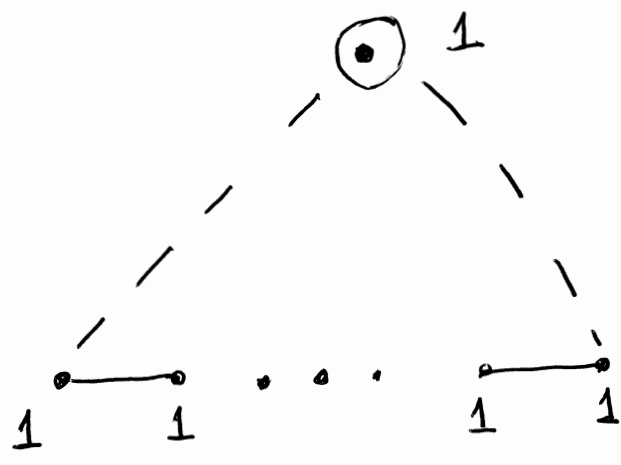
$$C/e_{12} \cong e_1$$

surjective sub rep

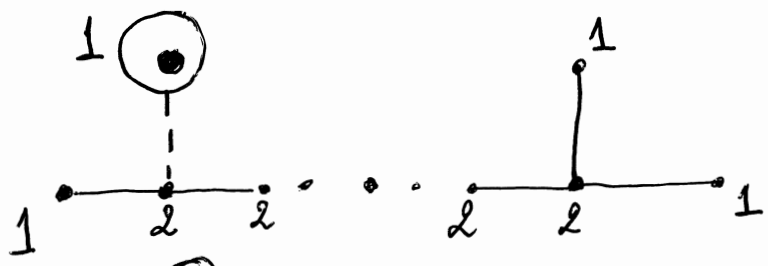
$$\# \text{ subreps} = \# \text{ lines in } \mathbb{F}^2 \neq \text{Ker } f = 9$$

$$e_1 * e_{12} = 9, e_{12} * e_1$$

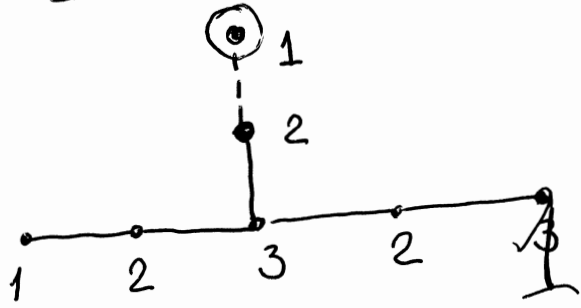
A_n



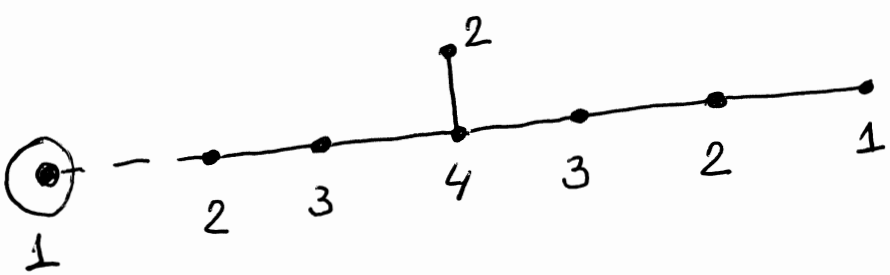
D_n



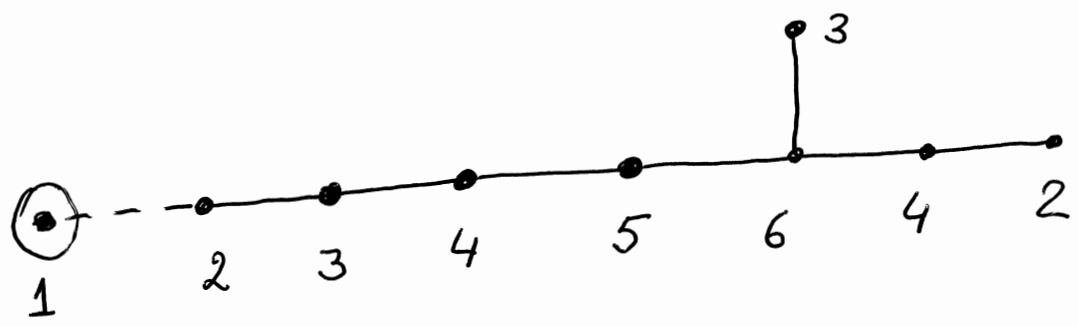
E_6



E_7



E_8



$$a_{ij} = \# \{ \text{edges between } i \text{ \& } j \}$$

Adjacency
matrix

$$A = (a_{ij})$$

Cartan matrix

$$C = \begin{pmatrix} 2 & & & 0 \\ & 2 & & \\ & & \ddots & \\ 0 & & & 2 \end{pmatrix} - A$$

$$C = (c_{ij})$$

$$c_{ij} = c_{ji}$$

$$\mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$$

$$q(x, y) = \sum_{ij} c_{ij} x_i y_j$$

Thm. connected graph is Dynkin $\Leftrightarrow q(x, x) > 0$ for any $x \neq 0$.

or extended Dynkin $\Leftrightarrow q(x, x) \geq 0$.

Lemma Let $d = (d_1, \dots, d_m) \in \mathbb{R}^m$

- $d_i > 0$ for all i .
- $q(x, d) = 0$ for all $x \in \mathbb{R}^m$

Then

$$q(x, x) \geq 0 \quad \forall x \in \mathbb{R}^m.$$

Pf:

$$q(x, x) = \sum_{i \neq j} a_{ij} \frac{d_i d_j}{4} \left(\frac{x_i}{d_i} - \frac{x_j}{d_j} \right)^2$$

$$SL_2(\mathbb{C}) = \left\{ \begin{array}{l} \text{matrices } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \det(g) = ad - bc = 1 \end{array} \right\}$$

\cup
 G finite subgroup

$$L = \mathbb{C}^2 \quad \curvearrowright \quad G$$

vertices = Irreps(G)

$$i = \dim \text{Hom}_G(V_i, L \otimes V_i)$$

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Claim For McKay graph have
 $q(x, d) = 0 \quad \forall x \in \mathbb{R}^m$

Schur lemma.

Corollary For McKay graph $q(x) \geq 0$

McKay
graphs

=

Extended
Dynkin
graphs

Example: A_{n-1}

$$\varepsilon^n = 1$$

$$G = \left\{ \begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{pmatrix}, \varepsilon = \text{primitive } n\text{-th root of } 1 \right\}$$

$k = 0, 1, \dots, n-1.$

$$G \cong \mathbb{Z}/n\mathbb{Z}$$

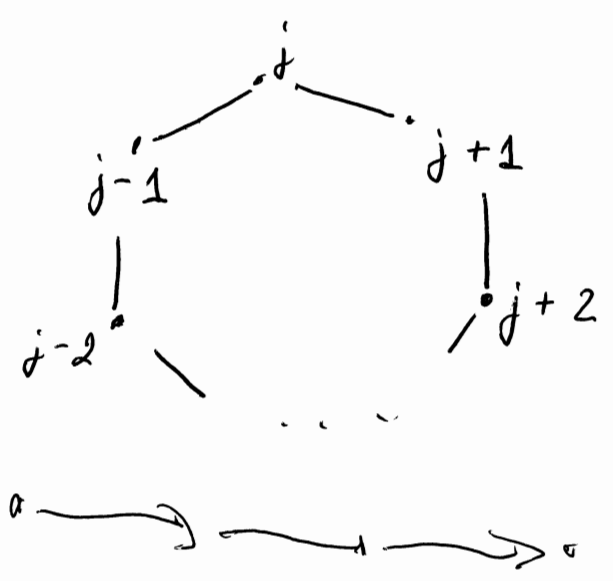
Irreps (G) = n-th roots of unity = $\{0, \dots, n-1\}$

$$V_i : \begin{pmatrix} \varepsilon & \\ & \varepsilon^{-1} \end{pmatrix} \longmapsto \text{multiplication by } \varepsilon^i$$

$$L = V_1 \oplus V_{-1} \quad (V_{-1} = V_{n-1})$$

$$L \otimes V_j = V_{j+1} \oplus V_{j-1}$$

$$\text{Hom}(V_i, V_{j+1} \oplus V_{j-1})$$



$$M = \mathbb{C}^2 / G$$

$$\dim_{\mathbb{C}} M = 2$$

$$\text{Fun}(M) = \left\{ \begin{array}{l} f \in \mathbb{C}[u, v] \\ f \text{ constant on } G\text{-orbits} \end{array} \right\}$$

Ex A_{n-1} type

$$\begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{pmatrix} : (u, v) \longmapsto (\varepsilon^k u, \varepsilon^{-k} v)$$

$$f(u, v) = f(\varepsilon u, \varepsilon^{-1} v) \quad \forall u, v \in \mathbb{C}$$

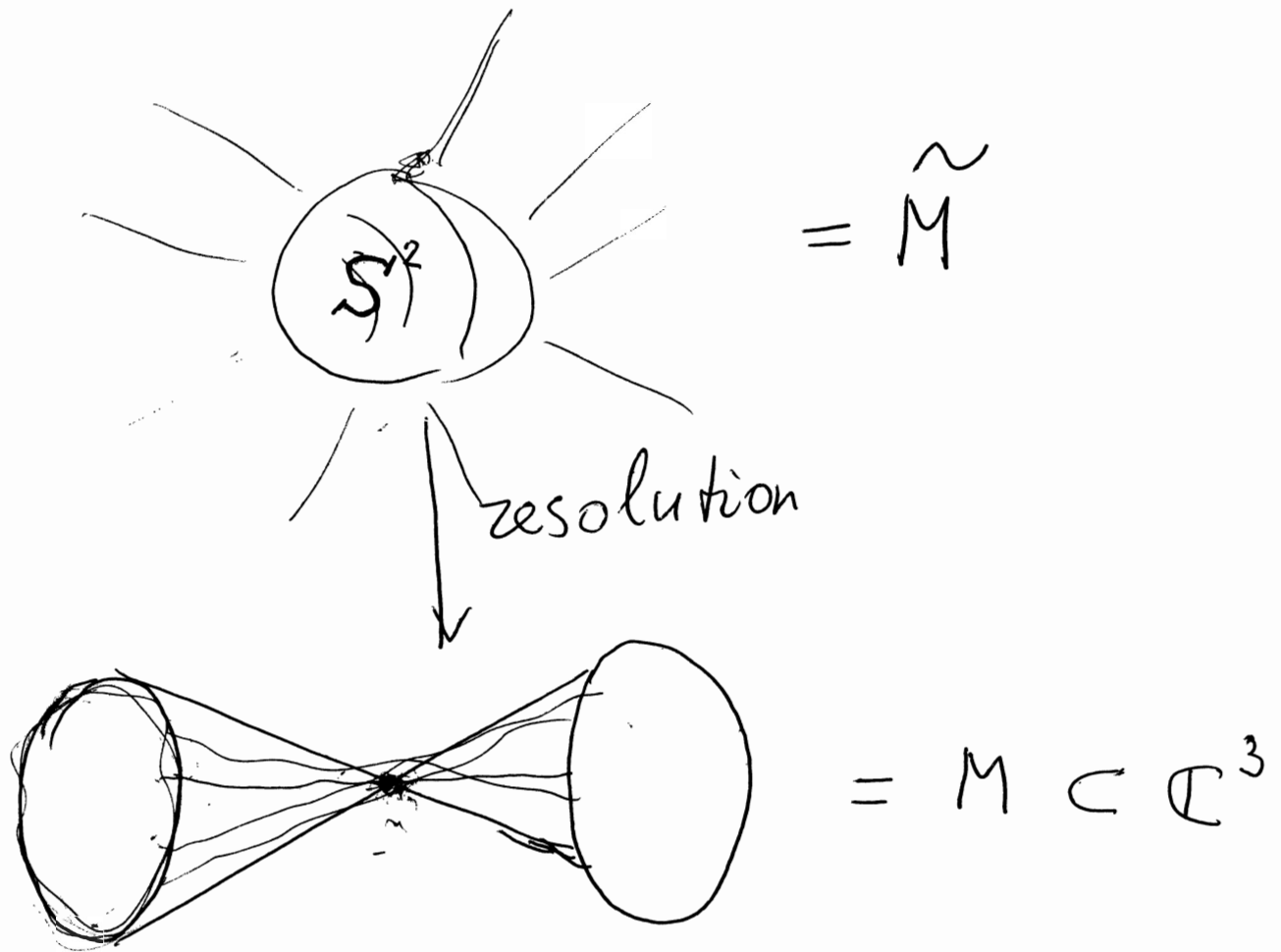
$$x := u^n, \quad y = v^n, \quad z = uv$$

$$f = f(x, y, z)$$

$$\boxed{z^n = xy}$$

$$\text{Fun}(M) = \mathbb{C}[x, y, z] / (z^n - xy)$$

$$M = \left\{ (x, y, z) \in \mathbb{C}^3 \mid z^2 = xy \right\}$$



$$n = 2 : \quad z^2 = xy$$

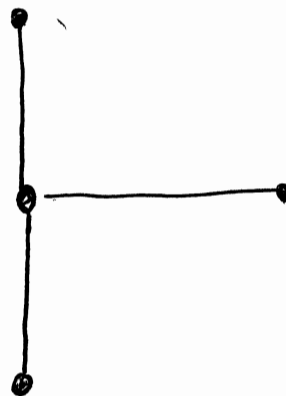
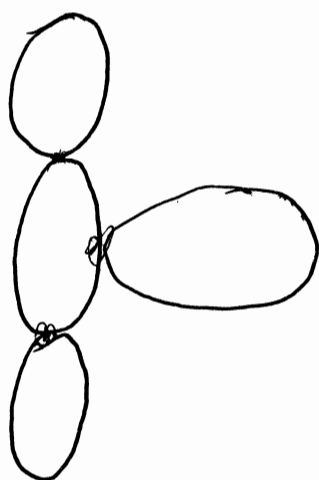
General case

$\tilde{M} \setminus \{\text{zero fiber}\} \subset \tilde{M}$ resolution

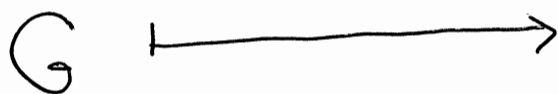
$S \downarrow$
 $M \setminus \{0\} \subset M = \mathbb{C}^2/G$

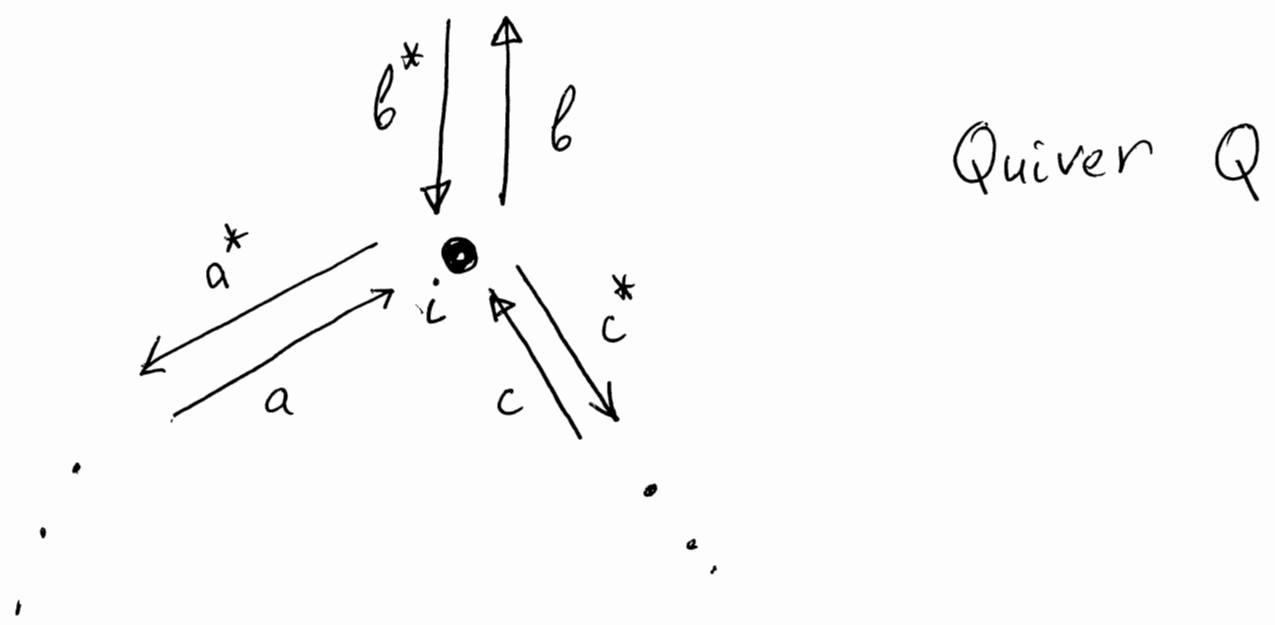


zero fiber



non extended
Dynkin
graph.



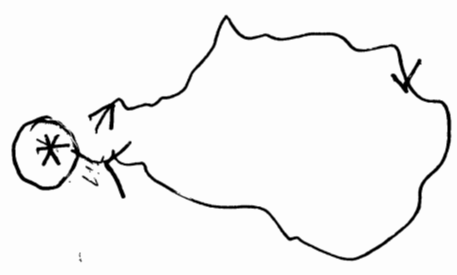


$$p_i = -aa^* + b^*b - cc^* \in \text{Path}(Q)$$

$$A = \frac{\text{Path}(Q)}{\{p_i, i \in I\}}$$

two-sided ideal

$$B = \left. \begin{array}{l} \text{paths with source and end} \\ \text{at } * = \text{extending vertex} \end{array} \right\}$$



Thm.
 $\text{Fun}(M) \cong B$