Lecture 17. Aubry-Mather theory and Peierl's Barrier.

Continuation of Lecture 7.

To construct minimal "rational" configuration we need two steps.

Step 1. Consider variational problem

Finite piece $\theta_0 = \theta,\ \theta_q = \theta + p$

Find $\min\limits_{\theta_0 \dots \theta_{q-1}} \sum\limits_{n=0}^{q-1} h(\theta_n, \theta_{n+1})$.

Exercise 1. Prove that minimum should occur in a compact region (condition at $+\infty$ for h lecture 3 lemma 2). Then use the fact $\sum\limits_{n=0}^{q-1} h(\theta_n, \theta_{n+1})$ is continuous.

Let $\theta_0^*, \dots, \theta_q^* = \theta_0^* + p$ realize minimum.

Step 2 <u>Periodic extension of finite piece and minimality</u>

Consider periodic continuation

$\theta^*_{k+nq} = \theta^*_k + np$ for any $0 \le k < q$, $n \in \mathbb{Z}$.

Lemma 1. $\{\theta^*_m\}_{m \in \mathbb{Z}}$ is minimal.

Proof: Suppose it is not minimal. Then for some $j < k \in \mathbb{Z}$ we have $\{\theta'_m\}_{m=j}^{k-1}$ such that $\theta'_j = \theta^*_j$, $\theta'_k = \theta^*_k$, and

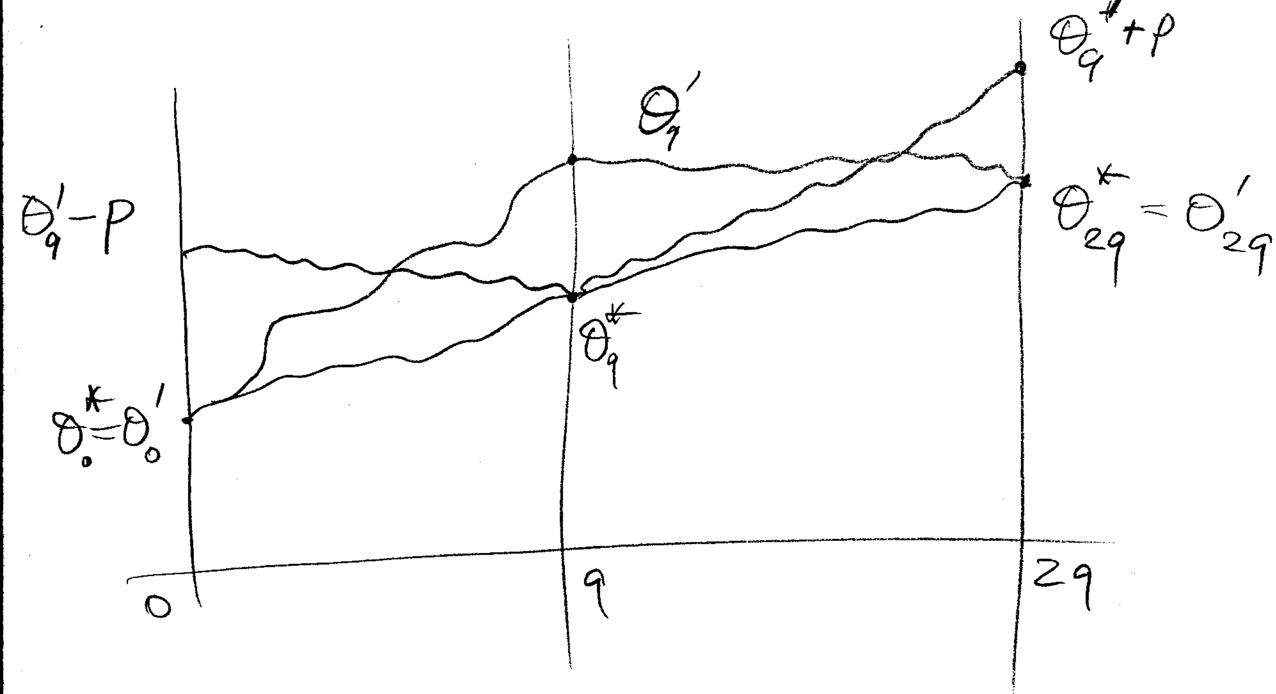
$$\sum_{n=j}^{k-1} h(\theta'_n, \theta'_{n+1}) < \sum_{n=j}^{k-1} h(\theta^*_n, \theta^*_{n+1})$$

By periodicity assume $0 \le j < q$ and $(s-1)q \le k < sq$.

For simplicity we put $s=2$.

For general s the proof is the same

Extend θ' to the whole interval $[0, 2q]$ by $\theta'_n = \theta^*_n$ outside $[j, k]$.



Translate $[0,q]$ part to $[q,2q]$
and $[q,2q]$ part to $[0,q]$

$$\tilde{\theta}_n = \theta'_{n+q} - p \qquad 0 \le n \le q$$

$$\tilde{\theta}_n = \theta'_{n-q} + p \qquad q \le n \le 2q$$

Translation does not change action.

If $\{\theta_n'\}_{n=0}^{2q}$ were minimal between any configuration with $\theta_{2q}' = \theta_0' + 2p$, then $\{\tilde{\theta}_n'\}_{n=0}^{2q}$ is minimal too.

Their Aubry graphs intersect twice.
Contradiction. This proves Lemma 1.

Exercise 2. Extend the proof above to the case s (= number of periods) > 2.

II <u>Construction of minimal configurations w "irrational" rotation and fundamental property of planar minimizers</u>

Exercise 3. Limit of minimizers is minimizer

$\{\theta_n^s\}_{n\in\mathbb{Z}} \to \{\theta_n\}_{n\in\mathbb{Z}}$

Proof by contradiction.

Let ω be irrational, $p_s/q_s \to \omega$.

Let $\{\theta_n^s\}_{n\in\mathbb{Z}}$ $\theta_{k+mq_s} = \theta_k + m p_s$ be periodic minimal $\theta_0^s \in [0,1]$.

Then there is a converging subsequence.

Exercise 4, Using Fundamental lemma prove $\{\theta_n\}_{n\in\mathbb{Z}}$ in the limit has rotation number ω.

Fundamental Theorem. Every minimizer $\{\theta_n\}_n$ has rotation number.

Proof is left as Exercise 5

Hint: Define $\omega = \min\{ p/q : T_{p,q}\theta$ has Aubry graph above the one of $\theta\}$

Similarly

$\omega = \max\{ p/q : T_{p,q}\theta$ —‖— below the one of $\theta\}$

III Peierl's Barrier (rational case)

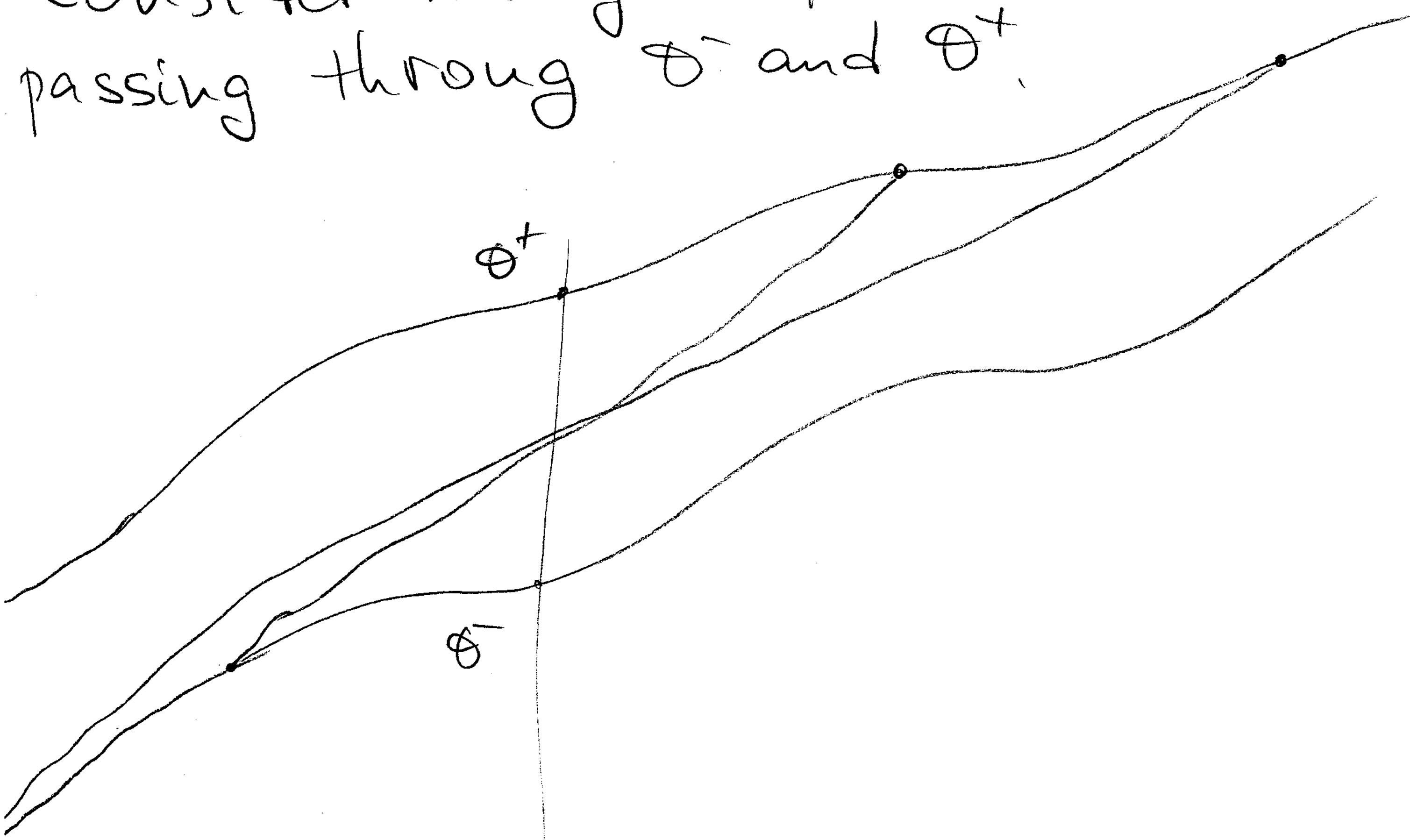
Let $\Sigma_{p/q}$ be the set of minimal periodic points. Suppose for simplicity $\Sigma_{p/q}$ – one periodic orbit.

By Graph Theorem $\pi \Sigma_{p/q}$ has exactly q points $\theta_0 \dots \theta_{q-1} \in S^1$.

Pick two neighbors, denote them by θ^- and θ^+ So no $\theta_k \in (\theta^-, \theta^+)$

Construction of homoclinic orbit (or class A geodesic, or $(p/q)^{\pm}$ minimizer.

Consider Aubry Graphs $G(\theta^-)$ and $G(\theta^+)$ passing throug θ^- and θ^+.



Consider sequence of finite minimizers

$$\min_{\substack{\theta_{-n}=\theta^-_{-n} \\ \theta_n=\theta^+_n}} \sum_{m=-n}^{n-1} h(\theta_m, \theta_{m+1})$$

Denote minimizing one $\{\theta^n_m\}_{m=-n}^{n}$.

Lemma 2A limit $\{\theta^*_m\}_{m\in\mathbb{Z}}$ exists and tends to θ^+ at $+\infty$, to θ^- at $-\infty$.