Nearly Integrable Systems / Hamiltonian and Arnold's Diffusion.

Lecture 24. Minimal Orbits of high dimensional Hamiltonian Systems

Recall the set up

$L: TM \times \mathbb{R} \to \mathbb{R}$ C^2 $L(x,\dot{x},t) = L(x,\dot{x},t+1)$ periodic in time.

- positive definite $\frac{\partial^2 L}{\partial v_i \partial v_j}(x,v,t) > 0$
- superlinear growth $\frac{L(x,v,t)}{\|v\|} \to +\infty$ as $\|v\| \to +\infty$.

$\|\cdot\|$ metric on M

- completeness All solutions of Euler-Lagrange (EL) equation defined on all $\mathbb{R}$ (no blow up in finite time)

Let $M = \mathbb{T}^2$ $\mathbb{R}^2 = \hat{\mathbb{T}}^2$ – universal cover

Bi-infinite sequences $\{z : \mathbb{Z} \to \mathbb{R}^2\} = (\mathbb{R}^2)^{\mathbb{Z}}$ – space w product topology.

$z = \{z_i = (x_i, y_i)\}_{i \in \mathbb{Z}}$

$h : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$,

$$h(z_1, z_2) = \inf_{\gamma} \int_0^1 L(\gamma(t), \dot{\gamma}(t), t)\, dt$$

infimum over all absolut. continuous curves $\gamma(0) = z_1$, $\gamma(1) = z_2$

Extend to arbitrary finite segment

$(z_j \dots z_k)$ $j<k$ by

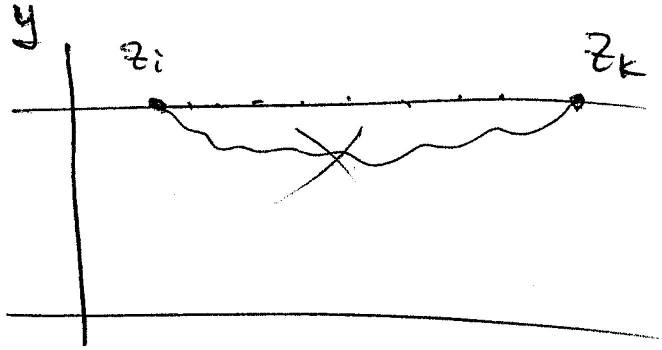
$$h(z_j \dots z_k) = \sum_{i=j}^{k-1} h(z_i\, z_{i+1})$$

$z \in (\mathbb{R}^2)^{\mathbb{Z}}$ is action-minimizing wrt h if each finite segment is action-minimizing.

Assume conditions (Ha–Hc) are satisfied following Xia

(Ha) "Invariance Condition" | $\{z_i = (x_i, y_i)\}_{i=j}^{k}$ action-minimizing

$y_j = y_k = 0$, then $y_i = 0$ $i = j \dots k$



We shall use $y=0$ being equation of cylinder

(Hb) "Uniqueness condition" | There is a positive $\eta > 0$ st the only action-minimizing recurrent traj $i \in \mathbb{Z}$ are either

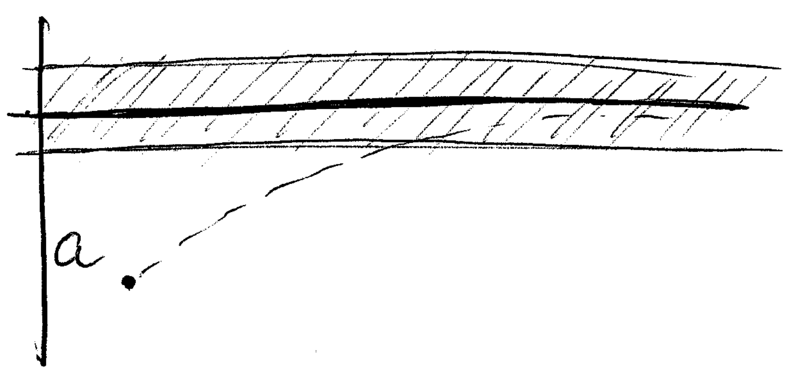
w constraints $-\eta \le y_i \le 1+\eta$

$y_i \equiv 0$ or $y_i \equiv 1$

$i \in \mathbb{Z}$

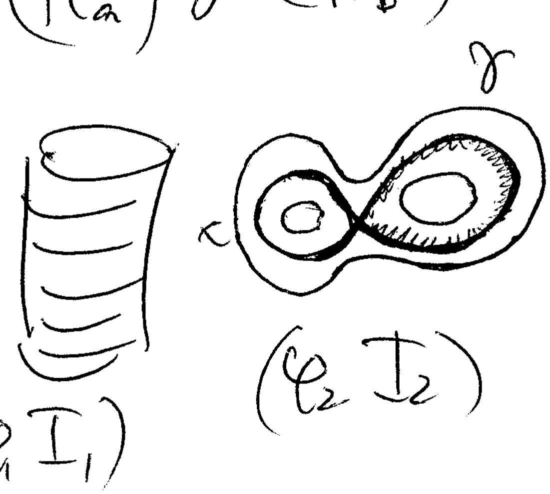
(H_c) "Limiting Condition" | There is $\delta_0 > 0$ s.t. for any $0<\delta<\delta_0$ any ~~pt~~ $a=(a_1,a_2)\in\mathbb{R}^2$ with $0<a_2<1$ and any sequence $b^k=(b_1^k, b_2^k)\in\mathbb{R}^2$ with $|b_2^k-1|\le\delta$ and $|b_1^k/k|\le B$ for some B and all $k\in\mathbb{Z}_+$ there is an integer $N\in\mathbb{Z}_+$ independent of k st for any $k\in\mathbb{Z}_+$ action-minimizing segment $z_0^k \dots z_k^k$ w $z_0^k=a$, $z_k^k=b^k$ has at most N pts outside $\mathbb{R}\times(1-\delta, 1+\delta)$

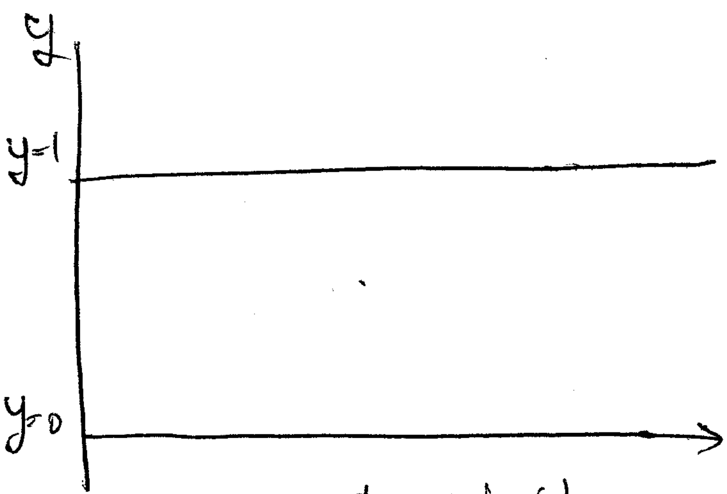
Similar condition for $|b_2^k|\le\delta$ instead of $|b_2^k-1|\le\delta$



(H_c) might follow from (H_a) & (H_b) in Arnold's example

$$H=\frac{1}{2}I_1^2+\left(\frac{1}{2}I_2^2+(\cos\varphi_2-1)\right)$$





x-component coincides w φ_1-component on the Cylinder.

Let y be a coordinate on the homoclinic loop γ in $(\varphi_2 I_2)$ plane.
Lift y to the universal cover so one lap around γ increases y from i to $i+1$.

Introduce restricted minimizers.

By (H_a) if $y_j = y_k = 0$, then $y_i = 0$ $i = j \dots k$
or all minimizers in $M^0 = \{y=0\}$ are minimizers in $\mathbb{R}^2$

Thus define $h_1(x_1, x_2) = h((x_1, 0), (x_2, 0))$.
Then h_1 satisfies hypothesis of generating functions for EAPT maps.

h_1 - defines restricted minimizers to M^0.

Similarly by periodicity in y
h_1 - defines restricted minimizers to $M^k = M^0 + (0, k)$

By analogy w twist maps

Fix $\omega \in \mathbb{R}$ (H_a) implies on $\{y \equiv 0\}$ h_1-minimizers correspond to Aubry-Mather minimizers.

Denote $\Sigma_\omega^k \subset \mathcal{M}^k$ h_1-minimizers w rot. number ω and $y_j \equiv k \in \mathbb{Z}$ for all $j \in \mathbb{Z}$

Pick any $\underset{\mathbb{R}^{\mathbb{Z}}}{\underset{\uparrow}{x^\omega}} \in \Sigma_\omega$, i.e. h_1-minimizer w rot. number ω.

Let $z^{a,\omega} = \{z_i^{a,\omega}\}_{i\in\mathbb{Z}} \in (\mathbb{R}^2)^{\mathbb{Z}}$ - configur. st

(1) $z_0^{a,\omega} = a$ and both $\{z_i^{a,\omega}\}_{i\in\mathbb{Z}_+}$ and $\{z_i^{a,\omega}\}_{i\in\mathbb{Z}_-}$ - action-minimizing

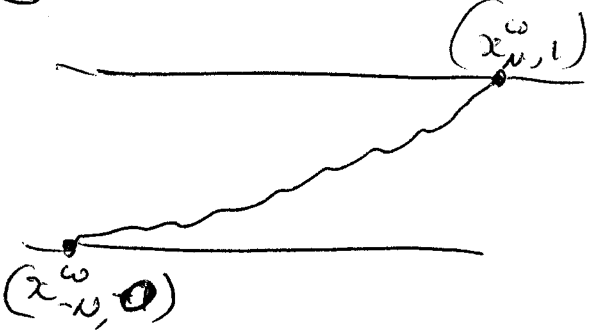
(2) ω-limit of $z^{a,\omega}$ is in $\Sigma_\omega^1 \subset \mathcal{M}^1$

(3) α-limit of $z^{a,\omega}$ is in $\Sigma_\omega^0 \subset \mathcal{M}^0$.

Lemma! $z^{a,\omega}$, as above, exists

Proof: $z^{a,\omega}$ is constructed a limit of minimizers $z^{a,\omega,N} = \{z_i^{a,\omega,N}\}_{i=-N}^{N}$. given by boundary conditions

$z_{-N}^{a,\omega,N} = (x_{-N}^\omega, 0)$, $z_N^{a,\omega,N} = (x_N^\omega, 1)$



(H_c) implies that for each N

$\{z_i^{a,\omega,N}\}_{i=0}^{N}$ has at most finitely many elements away $\mathbb{R} \times (1-\delta, 1+\delta)$

Superlinear growth implies that $\{z_i^{a,\omega,N}\}_{N\geq i}$ for fixed i and any $N \geq i$ is bounded.

Indeed, $h(z_1, z_2) \to +\infty$ as $|z_1 - z_2| \to +\infty$.

By superlinear growth for any C ~~if $|v| > B_C$, then~~ $L(x,v,t) > C|v| + D$ for some D and all x, v

So $\int_0^1 L(\gamma(t), \dot\gamma(t), t)\,dt = \int_{|\dot\gamma| < L/2} \ldots + \int_{|\dot\gamma| \geq \frac{L}{2}} \geq C \cdot \frac{L}{2} + 2D \to +\infty$ as $L \to +\infty$.

$\gamma(i) = z_{i+1}$

$|z_1 - z_2| = L$

Since $\{z_i^{a,\omega,N}\}_{N\geq i}$ is bounded in N for each i, there is a limit $z^{a,\omega,\infty} = z^\infty = \{z_i^\infty\}_{i\in\mathbb{Z}}$.

(H_c) implies at most finitely many of $\{z_i^\infty\}_{i\in\mathbb{Z}_+}$ are away $\mathbb{R}\times(1-\delta, 1+\delta)$ for any δ. Thus, ω-limit is in $\{y \equiv 1\}$. By construction ω-limit has to have rotation number ω in x-direction and, therefore, belongs Σ_ω by minimization property.

Lemma 2. Let $\omega \in \mathbb{R}$ and Σ_ω^0 and Σ_ω' are Aubry-Mather sets in $\{y=0\}$ and $\{y=1\}$ respectively. Then there is a minimizer whose α-limit set is in Σ_ω^0 and ω-limit set is in Σ_ω'.

Proof: Let $x^\omega \in \Sigma_\omega$ and $x_0^\omega \in [0,1)$.
Let $z^{\omega,N} = \{z_i^{\omega,N}\}_{i=-N}^{N}$ - minimizer subject to boundary conditions $z_{-N}^{\omega,N} = (x_{-N}^\omega, 0)$ and $z_N^{\omega,N} = (x_N^\omega, 1)$.

Noncompactness of minimizers in higher dimensional case:

Notice that the set of minimizers in Lemma 2 is not compact. Indeed, in the definition of $z^{\omega,N}$ replace $z_N^{\omega,N} = (x_N^\omega, 1)$ by $z_N^{\omega,N} = (x_N^\omega + k, 1)$ for some $k \in \mathbb{Z}_+$. The limit $z^{\infty,k}$ exists and as $k \to +\infty$ is unbounded in k.
To pick "good" minimizers additional minimization procedure is required.