HISTORY AND BACKGROUND

In the latter part of the nineteenth century, the French mathematician Henri Poincaré was studying the problem of whether the solar system is stable. Do the planets and asteroids in the solar system continue in regular orbits for all time, or will some of them be ejected into the far reaches of the galaxy or, alternatively, crash into the sun? In this work he was led to topology, a still new kind of mathematics related to geometry, and to the study of shapes (compact manifolds) of all dimensions.

The simplest such shape was the circle, or distorted versions of it such as the ellipse or something much wilder: lay a piece of string on the table, tie one end to the other to make a loop, and then move it around at random, making sure that the string does not touch itself. The next simplest shape is the two-sphere, which we find in nature as the idealized skin of an orange, the surface of a baseball, or the surface of the earth, and which we find in Greek geometry and philosophy as the "perfect shape." Again, there are distorted versions of the shape, such as the surface of an egg, as well as still wilder objects. Both the circle and the two-sphere can be described in words or in equations as the set of points at a fixed distance from a given point (the center). Thus it makes sense to talk about the three-sphere, the four-sphere, etc. These shapes are hard to visualize, since they naturally are contained in four-dimensional space, five-dimensional space, and so on, whereas we live in three-dimensional space. Nonetheless, with mathematical training, shapes in higher-dimensional spaces can be studied just as well as shapes in dimensions two and three.

In topology, two shapes are considered the same if the points of one correspond to the points of another in a continuous way. Thus the circle, the ellipse, and the wild piece of string are considered the same. This is much like what happens in the geometry of Euclid. Suppose that one shape can be moved, without changing lengths or angles, onto another shape. Then the two shapes are considered the same (think of congruent triangles). A round, perfect two-sphere, like the surface of a ping-pong ball, is topologically the same as the surface of an egg.

In 1904 Poincaré asked whether a three-dimensional shape that satisfies the "simple connectivity test" is the same, topologically, as the ordinary round three-sphere. The round three-sphere is the set of points equidistant from a given point in four-dimensional space. His test is something that can be performed by an imaginary being who lives inside the three-dimensional shape and cannot see it from "outside." The test is that every loop in the shape can be drawn back to the point of departure without leaving the shape. This can be done for the two-sphere and the three-sphere. But it cannot be done for the surface of a doughnut, where a loop may get stuck around the hole in the doughnut.

The question raised became known as the Poincaré conjecture. Over the years, many outstanding mathematicians tried to solve it—Poincaré himself, Whitehead, Bing, Papakirioukopolos, Stallings, and others. While their efforts frequently led to the creation of significant new mathematics, each time a flaw was found in the proof. In 1961 came astonishing news. Stephen Smale, then of the University of California at Berkeley (now at the City University of Hong Kong) proved that the analogue of the Poincaré conjecture was true for spheres of five or more dimensions. The higher-dimensional version of the conjecture required a more stringent version of Poincaré's test; it asks whether a so-called homotopy sphere is a true sphere. Smale's theorem was an achievement of extraordinary proportions. It did not, however, answer Poincaré's original question. The search for an answer became all the more alluring.

Smale's theorem suggested that the theory of spheres of dimensions three and four was unlike the theory of spheres in higher dimension. This notion was confirmed a decade later, when Michael Freedman, then at the University of California, San Diego, now of Microsoft Research Station Q, announced a proof of the Poincaré conjecture in dimension four. His work used techniques quite different from those of Smale. Freedman also gave a classification, or kind of species list, of all simply connected four-dimensional manifolds.

Both Smale (in 1966) and Freedman (in 1986) received Fields medals for their work.

There remained the original conjecture of Poincaré in dimension three. It seemed to be the most difficult of all, as the continuing series of failed efforts, both to prove and to disprove it, showed. In the meantime, however, there came three developments that would play crucial roles in Perelman's solution of the conjecture.

Geometrization

The first of these developments was William Thurston's geometrization conjecture. It laid out a program for understanding all three-dimensional shapes in a coherent way, much as had been done for two-dimensional shapes in the latter half of the nineteenth century. According to Thurston, three-dimensional shapes could be broken down into pieces governed by one of eight geometries, somewhat as a molecule can be broken into its constituent, much simpler atoms. This is the origin of the name, "geometrization conjecture."

A remarkable feature of the geometrization conjecture was that it implied the Poincaré conjecture as a special case. Such a bold assertion was accordingly thought to be far, far out of reach—perhaps a subject of research for the twenty-second century. Nonetheless, in an imaginative *tour de force* that drew on many fields of mathematics, Thurston was able to prove the geometrization conjecture for a wide class of shapes (Haken manifolds) that have a sufficient degree of complexity. While these methods did not apply to the three-sphere, Thurston's work shed new light on the central role of Poincaré's conjecture and placed it in a far broader mathematical context.

Limits of spaces

The second current of ideas did not appear to have a connection with the Poincaré conjecture until much later. While technical in nature, the work, in which the names of Cheeger and Perelman figure prominently, has to do with how one can take limits of geometric shapes, just as we learned to take limits in beginning calculus class. Think of Zeno and his paradox: you walk half the distance from where you are standing to the wall of your living room. Then you walk half the remaining distance. And so on. With each step you get closer to the wall. The wall is your "limiting position," but you never reach it in a finite number of steps. Now imagine a shape changing with time. With each "step" it changes shape, but can nonetheless be a "nice" shape at each step— smooth, as the mathematicians say. For the limiting shape the situation is different. It may be nice and smooth, or it may have special points that are different from all the others, that is, singular points, or "singularities." Imagine a Y-shaped piece of tubing that is collapsing: as time increases, the diameter of the tube gets smaller and smaller. Imagine further that one second after the tube begins its collapse, the diameter has gone to zero. Now the shape is different: it is a Y shape of infinitely thin wire. The point where the arms of the Y meet is different from all the others. It is the singular point of this shape. The kinds of shapes that can occur as limits are called Aleksandrov spaces, named after the Russian mathematician A. D. Aleksandrov who initiated and developed their theory.

Differential equations

The third development concerns differential equations. These equations involve rates of change in the unknown quantities of the equation, e.g., the rate of change of the position of an apple as it falls from a tree towards the earth's center. Differential equations are expressed in the language of calculus, which Isaac Newton invented in the 1680s in order to explain how material bodies (apples, the moon, and so on) move under the influence of an external force. Nowadays physicists use differential equations to study a great range of phenomena: the motion of galaxies and the stars within them, the flow of air and water, the propagation of sound and light, the conduction of heat, and even the creation, interaction, and annihilation of elementary particles such as electrons, protons, and quarks.

In our story, conduction of heat and change of temperature play a special role. This kind of physics was first treated mathematically by Joseph Fourier in his 1822 book, *Théorie Analytique de la Chaleur.* The differential equation that governs change of temperature is called the heat equation. It has the remarkable property that as time increases, irregularities in the distribution of temperature decrease.

Differential equations apply to geometric and topological problems as well as to physical ones. But one studies not the rate at which temperature changes, but rather the rate of change in some geometric quantity as it relates to other quantities such as curvature. A piece of paper lying on the table has curvature zero. A sphere has positive curvature. The curvature is a large number for a small sphere, but is a small number for a large sphere such as the surface of the earth. Indeed, the curvature of the earth is so small that its surface has sometimes mistakenly been thought to be flat. For an example of negative curvature, think of a point on the bell of a trumpet. In some directions the metal bends away from your eye; in others it bends towards it.

An early landmark in the application of differential equations to geometric problems was the 1963 paper of J. Eells and J. Sampson. The authors introduced the "harmonic map equation," a kind of nonlinear version of Fourier's heat equation. It proved to be a powerful tool for the solution of geometric and topological problems. There are now many important nonlinear heat equations—the equations for mean curvature flow, scalar curvature flow, and Ricci flow.

Also notable is the Yang-Mills equation, which came into mathematics from the physics of quantum fields. In 1983 this equation was used to establish very strong restrictions on the topology of four-dimensional shapes on which it was possible to do calculus [D]. These results helped renew hopes of obtaining other strong geometric results from analytic arguments—that is, from calculus and differential equations. Optimism for such applications had been tempered to some extent by the examples of René Thom (on cycles not representable by smooth submanifolds) and Milnor (on diffeomorphisms of the six-sphere).

Ricci flow

The differential equation that was to play a key role in solving the Poincaré conjecture is the Ricci flow equation. It was discovered two times, independently. In physics, the equation originated with the thesis of Friedan [F, 1985], although it was perhaps implicit in the work of Honerkamp [Ho, 1972]. In mathematics it originated with the 1982 paper of Richard Hamilton [Ha1]. The physicists were working on the renormalization group of quantum field theory, while Hamilton was interested in geometric applications of the Ricci flow equation itself. Hamilton, now at Columbia University, was then at Cornell University.

On the left-hand side of the Ricci flow equation is a quantity that expresses how the geometry changes with time—the derivative of the metric tensor, as the mathematicians like to say. On the right-hand side is the Ricci tensor, a measure of the extent to which the shape is curved. The Ricci tensor, based on Riemann's theory of geometry (1854), also appears in Einstein's equations for general relativity (1915). Those equations govern the interaction of matter, energy, curvature of space, and the motion of material bodies.

The Ricci flow equation is the analogue, in the geometric context, of Fourier's heat equation. The idea, *grosso modo*, for its application to geometry is that, just as Fourier's heat equation disperses temperature, the Ricci flow equation disperses curvature. Thus, even if a shape was irregular and distorted, Ricci flow would gradually remove these anomalies, resulting in a very regular shape whose topological nature was evident. Indeed, in 1982 Hamilton showed that for positively curved, simply connected shapes of dimension three (compact three-manifolds) the Ricci flow transforms the shape into one that is ever more like the round three-sphere. In the long run, it becomes almost indistinguishable from this perfect, ideal shape. When the curvature is not strictly positive, however, solutions of the Ricci flow equation behave in a much more complicated way. This is because the equation is nonlinear. While parts of the shape may evolve towards a smoother, more regular state, other parts might develop singularities. This richer behavior posed serious difficulties. But it also held promise: it was conceivable that the formation of singularities could reveal Thurston's decomposition of a shape into its constituent geometric atoms.

Richard Hamilton

Hamilton was the driving force in developing the theory of Ricci flow in mathematics, both conceptually and technically. Among his many notable results is his 1999 paper [Ha2], which showed that in a Ricci flow, the curvature is pushed towards the positive near a singularity. In that paper Hamilton also made use of the collapsing theory [C-G] mentioned earlier. Another result [Ha3], which played a crucial role in Perelman's proof, was the Hamilton Harnack inequality, which generalized to positive Ricci flows a result of Peter Li and Shing-Tung Yau for positive solutions of Fourier's heat equation.

Hamilton had established the Ricci flow equation as a tool with the potential to resolve both conjectures as well as other geometric problems. Nevertheless, serious obstacles barred the way to a proof of the Poincaré conjecture. Notable among these obstacles was lack of an adequate understanding of the formation of singularities in Ricci flow, akin to the formation of black holes in the evolution of the cosmos. Indeed, it was not at all clear how or if formation of singularities could be understood. Despite the new front opened by Hamilton, and despite continued work by others using traditional topological tools for either a proof or a disproof, progress on the conjectures came to a standstill.

Such was the state of affairs in 2000, when John Milnor wrote an article describing the Poincaré conjecture and the many attempts to solve it. At that writing, it was not clear whether the conjecture was true or false, and it was not clear which method might decide the issue. Analytic methods (differential equations) were mentioned in a later version (2004). See [M1] and [M2].

Perelman announces a solution of the Poincaré conjecture

It was thus a huge surprise when Grigoriy Perelman announced, in a series of preprints posted on ArXiv.org in 2002 and 2003, a solution not only of the Poincaré conjecture, but also of Thurston's geometrization conjecture [P1, P2, P3].

The core of Perelman's method of proof is the theory of Ricci flow. To its applications in topology he brought not only great technical virtuosity, but also new ideas. One was to combine collapsing theory in Riemannian geometry with Ricci flow to give an understanding of the parts of the shape that were collapsing onto a lower-dimensional space. Another was the introduction of a new quantity, the entropy, which instead of measuring disorder at the atomic level, as in the classical theory of heat exchange, measures disorder in the global geometry of the space. Perelman's entropy, like the thermodynamic entropy, is increasing in time: there is no turning back. Using his entropy function and a related local version (the L-length functional), Perelman was able to understand the nature of the singularities that formed under Ricci flow. There were just a few kinds, and one could write down simple models of their formation. This was a breakthrough of first importance.

Once the simple models of singularities were understood, it was clear how to cut out the parts of the shape near them as to continue the Ricci flow past the times at which they would otherwise form. With these results in hand, Perelman showed that the formation times of the singularities could not run into Zeno's wall: imagine a singularity that occurs after one second, then after half a second more, then after a quarter of a second more, and so on. If this were to occur, the "wall," which one would reach two seconds after departure, would correspond to a time at which the mathematics of Ricci flow would cease to hold. The proof would be unattainable. But with this new mathematics in hand, attainable it was.

The posting of Perelman's preprints and his subsequent talks at MIT, SUNY–Stony Brook, Princeton, and the University of Pennsylvania set off a worldwide effort to understand and verify his groundbreaking work. In the US, Bruce Kleiner and John Lott wrote a set of detailed notes on Perelman's work. These were posted online as the verification effort proceeded. A final version was posted to ArXiv.org in May 2006, and the refereed article appeared in *Geometry and Topology* in 2008. This was the first time that work on a problem of such importance was facilitated via a public website. John Morgan and Gang Tian wrote a book-long exposition of Perelman's proof, posted on ArXiv.org in July of 2006, and published by the American Mathematical Society in CMI's monograph series (August 2007). These expositions, those by other teams, and, importantly, the multi-year scrutiny of the mathematical community, provided the needed verification. Perelman had solved the Poincaré conjecture. After a century's wait, it was settled!

Among other articles that appeared following Perelman's work is a paper in the *Asian Journal of Mathematics*, posted on ArXiv.org in June of 2006 by the American-Chinese team, Huai-Dong Cao (Lehigh University) and Xi-Ping Zhu (Zhongshan University). Another is a paper by the European group of Bessières, Besson, Boileau, Maillot, and Porti, posted on ArXiv.org in June of 2007. It was accepted for publication by *Inventiones Mathematicae* in October of 2009. It gives an alternative approach to the last step in Perelman's proof of the geometrization conjecture.

Perelman's proof of the Poincaré and geometrization conjectures is a major mathematical advance. His ideas and methods have already found new applications in analysis and geometry; surely the future will bring many more.

— JC, March 18, 2010 (corrections, 3/19/2010)

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