Tilings[*]

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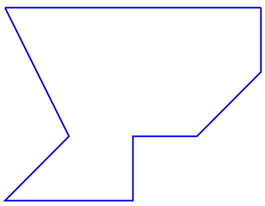
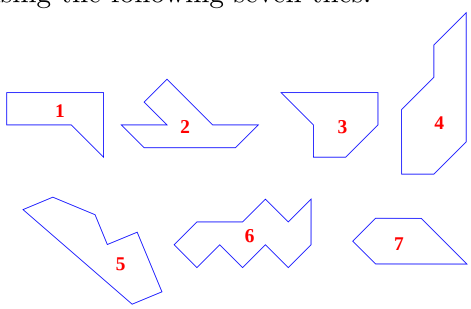
1 Introduction.

Consider the following puzzle. The goal is to cover the region

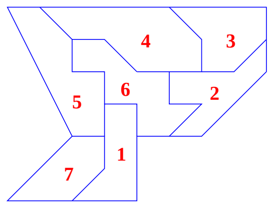


using the following seven tiles.



The region must be covered entirely without any overlap. It is allowed to shift and rotate the seven pieces in any way, but each piece must be used exactly once.

One could start by observing that some of the pieces fit nicely in certain parts of the region. However, the solution can really only be found through trial and error.



For that reason, even though this is an amusing puzzle, it is not very intriguing mathematically.

This is, in any case, an example of a tiling problem. A tiling problem asks us to cover a given region using a given set of tiles, completely and without any overlap. Such a covering is called a tiling. Of course, we will focus our attention on specific regions and tiles which give rise to interesting mathematical problems.

Given a region and a set of tiles, there are many different questions we can ask. Some of the questions that we will address are the following:

- Is there a tiling?
- How many tilings are there?
- About how many tilings are there?
- Is a tiling easy to find?
- Is it easy to prove that a tiling does not exist?
- Is it easy to convince someone that a tiling does not exist?
- What does a "typical" tiling look like?

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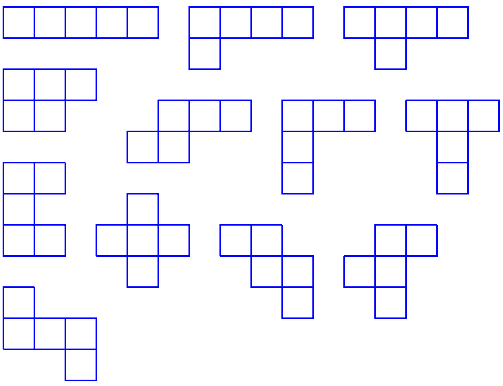
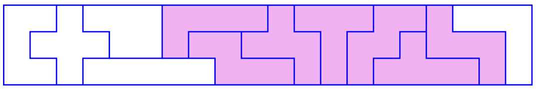
- Are there relations among the different tilings?
- Is it possible to find a tiling with special properties, such as symmetry?

2 Is there a tiling?

From looking at the set of tiles and the region we wish to cover, it is not always clear whether such a task is even possible. The puzzle of Section 1 is such a situation. Let us consider a similar puzzle, where the set of tiles is more interesting mathematically.

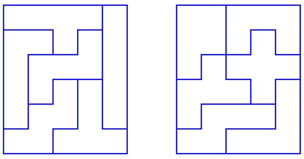


A *pentomino* is a collection of five unit squares arranged with coincident sides. Pentominoes can be flipped or rotated freely. The figure shows the twelve different pentominoes. Since their total area is 60, we can ask, for example: Is it possible to tile a 3×20 rectangle using each one of them exactly once?



This puzzle can be solved in at least two ways. One solution is shown above. A different solution is obtained if we rotate the shaded block by 180°. In fact, after spending some time trying to find a tiling, one discovers that these (and their rotations and reflections) are the only two possible solutions.

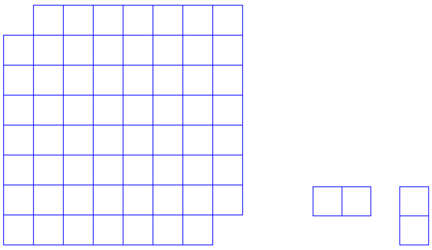
One could also ask whether it is possible to tile two 6×5 rectangles using each pentomino exactly once. There is a unique way to do it, shown below. The problem is made more interesting (and difficult) by the uniqueness of the solution.



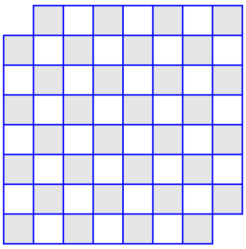
Knowing that, one can guess that there are several tilings of a 6×10 rectangle using the twelve pentominoes. However, one might not predict just how many there are. An exhaustive computer search has found that there are 2339 such tilings.

These questions make nice puzzles, but are not the kind of interesting mathematical problem that we are looking for. To illustrate what we mean by this, let us consider a problem which is superficially somewhat similar, but which is much more amenable to mathematical reasoning.

Suppose we remove two opposite corners of an 8×8 chessboard, and we ask: Is it possible to tile the resulting figure with 31 dominoes?

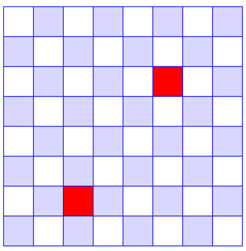


Our chessboard would not be a chessboard if its cells were not colored black and white alternatingly. As it turns out, this coloring is crucial in answering the question at hand.

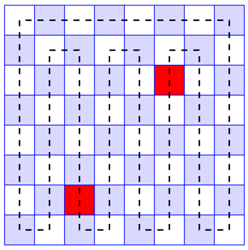


Notice that, regardless of where it is placed, a domino will cover one black and one white square of the board. Therefore, 31 dominoes will cover 31 black squares and 31 white squares. However, the board has 32 black squares and 30 white squares in all, so a tiling does *not* exist. This is an example of a *coloring argument*; such arguments are very common in showing that certain tilings are impossible.

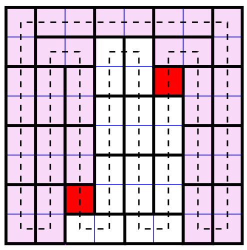
A natural variation of this problem is to now remove one black square and one white square from the chessboard. Now the resulting board has the same number of black squares and white squares; is it possible to tile it with dominoes?



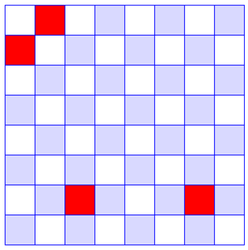
Let us show that the answer is *yes*, regardless of which black square and which white square we remove. Consider any closed path that covers all the cells of the chessboard, like the one shown below.



Now start traversing the path, beginning with the point immediately after the black hole of the chessboard. Cover the first and second cell of the path with a domino; they are white and black, respectively. Then cover the third and fourth cells with a domino; they are also white and black, respectively. Continue in this way, until the path reaches the second hole of the chessboard. Fortunately, this second hole is white, so there is no gap between the last domino placed and this hole. We can therefore skip this hole, and continue covering the path with successive dominoes. When the path returns to the first hole, there is again no gap between the last domino placed and the hole. Therefore, the board is entirely tiled with dominoes. This procedure is illustrated below.

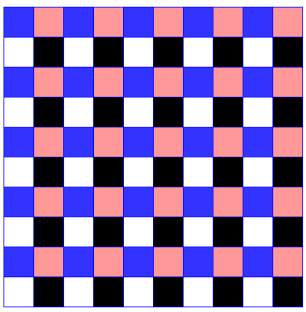


What happens if we remove *two* black squares and *two* white squares? If we remove the four squares closest to a corner of the board, a tiling with dominoes obviously exists. On the other hand, in the example below, a domino tiling does not exist, since there is no way for a domino to cover the upper left square.

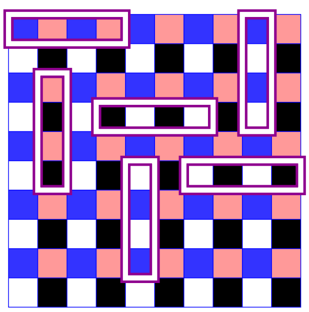


This question is clearly more subtle than the previous one. The problem of describing which subsets of the chessboard can be tiled by dominoes leads to some very nice mathematics. We will say more about this topic in Section 5.

Let us now consider a more difficult example of a coloring argument, to show that a 10×10 board *cannot* be tiled with 1×4 rectangles.



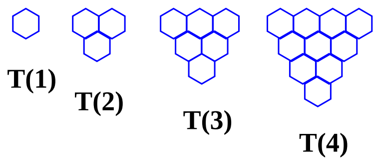
Giving the board a chessboard coloring gives us no information about the existence of a tiling. Instead, let us use four colors, as shown above. Any 1×4 tile that we place on this board will cover an *even* number (possibly zero) of squares of each color.



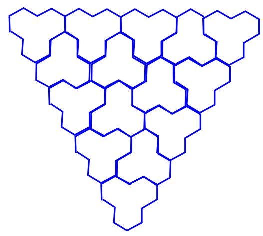
Therefore, if we had a tiling of the board, the total number of squares of each color would be even. But there are 25 squares of each color, so a tiling is impossible.

With these examples in mind, we can invent many similar situations where a certain coloring of the board makes a tiling impossible. Let us now discuss a tiling problem which cannot be solved using such a coloring argument.

Consider the region $T(n)$ consisting of a triangular array of $n(n+1)/2$ unit regular hexagons.



Call $T(2)$ a *tribone*. We wish to know the values of n for which $T(n)$ be tiled by tribones. For example, $T(9)$ can be tiled as follows.



Since each tribone covers 3 hexagons, $n(n+1)/2$ must be a multiple of 3 for $T(n)$ to be tileable. However, this does not explain why regions such as $T(3)$ and $T(5)$ cannot be tiled.

Conway [20] showed that the triangular array $T(n)$ can be tiled by tribones if and only if $n = 12k, 12k+2, 12k+9$ or $12k+11$ for some $k \geq 0$. The smallest values of n for which $T(n)$ can be tiled are 0, 2, 9, 11, 12, 14, 21, 23, 24, 26, 33, and 35. Conway's proof uses a certain nonabelian group which detects information about the tiling that no coloring can detect, while coloring arguments can always be rephrased in terms of *abelian* groups. In fact, it is possible to prove that no coloring argument can establish Conway's result [15].

3 Counting tilings, exactly.

Once we know that a certain tiling problem can be solved, we can go further and ask: How many solutions are there?

As we saw earlier, there are 2339 ways (up to symmetry) to tile a 6×10 rectangle using each one of the 12 pentominoes exactly once. It is perhaps interesting that this number is so large, but the exact answer is not so interesting, especially since it was found by a computer search.

The first significant result on tiling enumeration was obtained independently in 1961 by Fisher and Temperley [7] and by Kasteleyn [12]. They found that the number of tilings of a $2m \times 2n$ rectangle with $2mn$ dominoes is equal to

$$4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

Here Π denotes *product* and π denotes $180°$, so the number above is given by 4^{mn} times a product of sums of two squares of cosines, such as

$$\cos \frac{2\pi}{5} = \cos 72° = 0.3090169938\ldots.$$

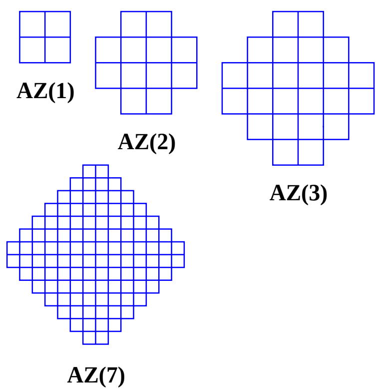
This is a remarkable formula! The numbers we are multiplying are not integers; in most cases, they are not even rational numbers. When we multiply these numbers we miraculously obtain an integer, and this integer is exactly the number of domino tilings of the $2m \times 2n$ rectangle.

For example, for $m = 2$ and $n = 3$, we get:

$$\begin{aligned} & 4^6(\cos^2 36° + \cos^2 25.71\ldots°) \times \\ & (\cos^2 36° + \cos^2 51.43\ldots°) \times \\ & (\cos^2 36° + \cos^2 77.14\ldots°) \times \\ & (\cos^2 72° + \cos^2 25.71\ldots°) \times \\ & (\cos^2 72° + \cos^2 51.43\ldots°) \times \\ & (\cos^2 72° + \cos^2 77.14\ldots°) \\ = \; & 4^6(1.4662\ldots)(1.0432\ldots)(0.7040\ldots) \times \\ & (0.9072\ldots)(0.4842\ldots)(0.1450\ldots) \\ = \; & 281. \end{aligned}$$

Skeptical readers with a lot of time to spare are invited to find all domino tilings of a 4×6 rectangle and check that there are, indeed, exactly 281 of them.

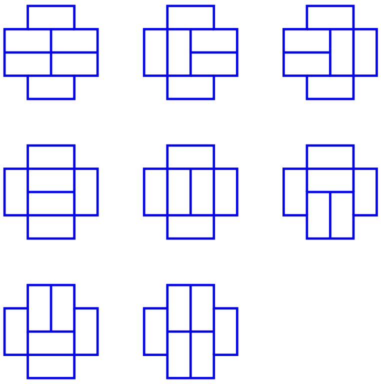
Let us say a couple of words about the proofs of this result. Kasteleyn expressed the answer in terms of a certain Pfaffian, and reduced its computation to the evaluation of a related determinant. Fisher and Temperley gave a different proof using the transfer matrix method, a technique often used in statistical mechanics and enumerative combinatorics.



There is a different family of regions for which the number of domino tilings is surprisingly simple. The Aztec diamond AZ(n)

is obtained by stacking successive centered rows of length $2, 4, \ldots, 2n, 2n, \ldots, 4, 2$, as shown above.

The Aztec diamond of order 2, AZ(2), has the following eight tilings:



Elkies, Kuperberg, Larsen and Propp [6] showed that the number of domino tilings of AZ(n) is $2^{n(n+1)/2}$. The following table shows the number of tilings of AZ(n) for the first few values of n.

1	2	3	4	5	6
2	8	64	1024	32768	2097152

Since $2^{(n+1)(n+2)/2} \,/\, 2^{n(n+1)/2} = 2^{n+1}$, one could try to associate 2^{n+1} domino tilings of the Aztec diamond of order $n+1$ to each domino tiling of the Aztec diamond of order n, so that each tiling of order $n+1$ occurs exactly once. This is one of the four original proofs found in [6]; there are now around 12 proofs of this result. None of these proofs is quite as simple as the answer $2^{n(n+1)/2}$ might suggest.

4 Counting tilings, approximately.

Sometimes we are interested in estimating the number of tilings of a certain region. In some cases, we will want to do this because we are not able to find an exact formula. In other cases, somewhat paradoxically, we might prefer an approximate formula over an exact formula. A good example is the number of tilings of a rectangle. We have an exact formula for this number, but this formula does not give us any indication of how large this number is.

For instance, since Aztec diamonds are "skewed" squares, we might wonder: How do the number of domino tilings of an Aztec diamond and a square of about the same size compare? After experimenting a bit with these shapes, one notices that placing a domino on the boundary of an Aztec diamond almost always forces the position of several other dominoes. This almost never happens in the square. This might lead us to guess that the square should have more tilings than the Aztec diamond.

To try to make this idea precise, let us make a definition. If a region with N squares has T tilings, we will say that it has $\sqrt[N]{T}$ *degrees of freedom per square*. The motivation, loosely speaking, is the following: If each square could decide independently how it would like to be covered, and it had $\sqrt[N]{T}$ possibilities to choose from, then the total number of choices would be T.

The Aztec diamond AZ(n) consists of $N = 2n(n+1)$ squares, and it has $T = 2^{n(n+1)/2}$ tilings. Therefore, the number of degrees of freedom per square in AZ(n) is:

$$\sqrt[N]{T} = \sqrt[4]{2} = 1.189207115\ldots$$

For the $2n \times 2n$ square, the exact formula for the number of tilings is somewhat unsatisfactory, because it does not give us any indication of how large this number is. Fortunately, as Kasteleyn, Fisher and Temperley observed, one can use their formula to show that the number of domino tilings of a $2n \times 2n$ square is approximately C^{4n^2}, where

$$\begin{aligned} C &= e^{G/\pi} \\ &= 1.338515152\ldots. \end{aligned}$$

Here G denotes the *Catalan constant*, which is defined as follows:

$$\begin{aligned} G &= 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots \\ &= 0.9159655941\ldots. \end{aligned}$$

Thus our intuition was correct. The square board is "easier" to tile than the Aztec diamond, in the sense that it has approximately $1.3385\ldots$ degrees of freedom per square, while the Aztec diamond has $1.1892\ldots$.

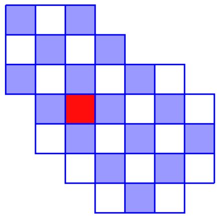
5 Demonstrating that a tiling does not exist.

As we saw in Section 2, there are many tiling problems where a tiling exists, but finding it is a difficult task. However, once we have found it, it is very easy to demonstrate its existence to someone: We can simply show them the tiling!

Can we say something similar in the case that a tiling does not exist? As we also saw in Section 2, it can be difficult to show that a tiling does not exist. Is it true, however, that if a tiling does not exist, then there is an easy way of demonstrating that to someone?

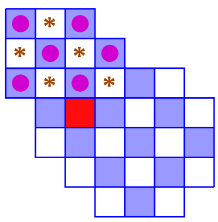
In a precise sense, the answer to this question is almost certainly *no* in general, even for tilings of regions using 1×3 rectangles [1]. Surprisingly, though, the answer is *yes* for domino tilings!

Before stating the result in its full generality, let us illustrate it with an example. Consider the following region, consisting of 16 black squares and 16 white squares. (The dark shaded cell is a hole in the region.)



One can use a case by case analysis to become convinced that this region cannot be tiled with dominoes. Knowing this, can we find an easier, faster way to convince someone that this is the case?

One way of doing it is the following. Consider the six black squares marked with a $\bullet$. They are adjacent to a total of five white squares, which are marked with a $*$. We would need six different tiles to cover the six marked black squares, and each one of these tiles would have to cover one of the five marked white squares. This makes a tiling impossible.



Philip Hall [10] showed that in *any* region which cannot be tiled with dominoes, one can find such a demonstration of impossibility. More precisely, one can find k cells of one color which have fewer than k neighbors.

Therefore, to demonstrate to someone that tiling the region is impossible, we can simply show them those k cells and their neighbors!

Hall's statement is more general than this, and is commonly known as the *marriage theorem*. The name comes from thinking of the black cells as men and the white cells as women. These men and women are not very adventurous: They are only willing to marry one of their neighbors. We are the matchmakers; we are trying to find an arrangement in which everyone can be happily married. The marriage theorem tells us exactly when such an arrangement exists.

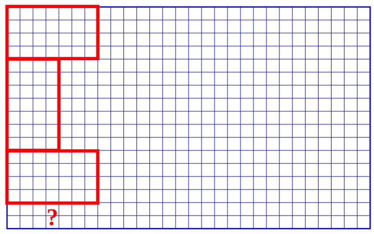
6 Tiling rectangles with rectangles.

One of the most natural tiling situations is that of tiling a rectangle with smaller rectangles. We now present three beautiful results of this form.

The first question we wish to explore is: When can an $m \times n$ rectangle be tiled with $a \times b$ rectangles (in any orientation)? Let us start this discussion with some motivating examples.

Can a 7×10 rectangle be tiled with 2×3 rectangles? This is clearly impossible, because each 2×3 rectangle contains 6 squares, while the number of squares in a 7×10 rectangle is 70, which is not a multiple of 6. For a tiling to be possible, the number of cells of the large rectangle must be divisible by the number of cells of the small rectangle. Is this condition enough?

Let us try to tile a 17×28 rectangle with 4×7 rectangles. The argument of the previous paragraph does not apply here; it only tells us that the number of tiles needed is 17. Let us try to cover the leftmost column first.



Our first attempt failed. After covering the first 4 cells of the column with the first tile, the following 7 cells with the second tile, and the following 4 cells with the third tile, there is no room for a fourth tile to cover the remaining two cells. In fact, if we manage to cover the 17 cells of the first column with 4×7 tiles, we will have written 17 as a sum of 4s and 7s. But it is easy to check that this cannot be done, so a tiling does not exist. We have found a second reason for a tiling not to exist: It may be impossible to cover the first row or column, because either m or n cannot be written as a sum of as and bs.

Is it then possible to tile a 10×15 rectangle using 1×6 rectangles? 150 is in fact a multiple of 6, and both 10 and 15 can be written as a sum of 1s and 6s. However, this tiling problem is still impossible!

The full answer to our question was given by de Bruijn and Klarner [4, 13]. They proved that an $m \times n$ rectangle can be tiled with $a \times b$ rectangles if and only if:

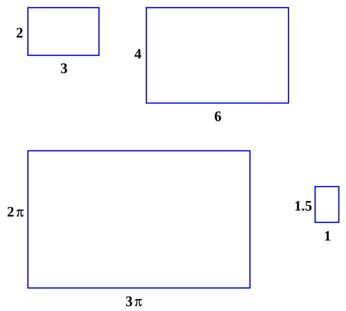
- mn is divisible by ab,
- the first row and column can be covered; *i.e.*, both m and n can be written as sums of as and bs, and
- either m or n is divisible by a, and either m or n is divisible by b.

Since neither 10 nor 15 are divisible by 6, the 10×15 rectangle *cannot* be tiled with

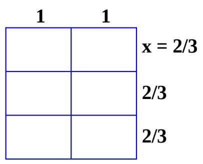
1×6 rectangles. There are now many proofs of de Bruijn and Klarner's theorem. A particularly elegant one uses properties of the complex roots of unity [4, 13]. For an interesting variant with fourteen (!) proofs, see [21].

The second problem we wish to discuss is the following. Let $x > 0$, such as $x = \sqrt{2}$. Can a square be tiled with finitely many rectangles *similar* to a $1 \times x$ rectangle (in any orientation)? In other words, can a square be tiled with finitely many rectangles, all of the form $a \times ax$ (where a may vary)?

For example, for $x = 2/3$, some of the tiles we can use are the following:

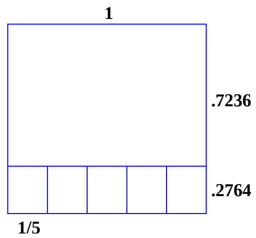


They have the same shape, but different sizes. In this case, however, we only need one size, because we can tile a 2×2 square with six $1 \times 2/3$ squares.



For reasons which will become clear later, we point out that $x = 2/3$ satisfies the equation $3x - 2 = 0$. Notice also that a similar construction will work for any positive rational number $x = p/q$.

Let us try to construct a tiling of a square with similar rectangles of at least two different sizes. There is a tiling approximately given by the picture below. The rectangles are similar because $0.7236\ldots/1 = 0.2/0.2764\ldots$.



How did we find this configuration? Suppose that we want to form a square by putting five copies of a rectangle in a row, and then stacking on top of them a larger rectangle of the same shape on its side, as shown. Assume that we know the square has side length 1, but we do not know the dimensions of the rectangles. Let the dimensions of the large rectangle be $1 \times x$. Then the height of each small rectangle is equal to $1-x$. Since the small rectangles are similar to the large one, their width is $x(1-x)$. Sitting together in the tiling, their total width is $5x(1-x)$, which should be equal to 1.

Therefore, the picture above is a solution to our problem if x satisfies the equation $5x(1-x) = 1$, which we rewrite as $5x^2 - 5x + 1 = 0$. One value of x which satisfies this equation is

$$x = \frac{5+\sqrt{5}}{10} = 0.7236067977\ldots,$$

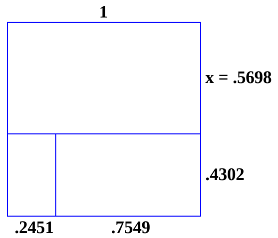
giving rise to the tiling illustrated above.

But recall that any quadratic polynomial has two roots; the other one is

$$x = \frac{5-\sqrt{5}}{10} = 0.2763932023\ldots,$$

and it gives rise to a different tiling which also satisfies the conditions of the problem.

It may be unexpected that our tiling problem has a solution for these two somewhat complicated values of x. In fact, the situation can get much more intricate. Let us find a tiling using three similar rectangles of different sizes.



Say that the largest rectangle has dimensions $1 \times x$. Imitating the previous argument, we find that x satisfies the equation

$$x^3 - x^2 + 2x - 1 = 0.$$

One value of x which satisfies this equation is

$$x = 0.5698402910\ldots.$$

For this value of x, the tiling problem can be solved as above. The polynomial above has degree three, so it has two other solutions. They are approximately $0.215 + 1.307\sqrt{-1}$ and $0.215 - 1.307\sqrt{-1}$. These two complex numbers do not give us real solutions to the tiling problem.

In the general situation, Laczkovich and Szekeres [14] gave the following amazing answer to this problem. A square can be tiled with finitely many rectangles similar to a $1 \times x$ rectangle if and only if:

- x is the root of a polynomial with integer coefficients, and
- for the polynomial of least degree satisfied by x, any root $a + b\sqrt{-1}$ satisfies $a > 0$.

It is very surprising that these complex roots, which seem completely unrelated to the tiling problem, actually play a fundamental role in it. In the example above, a solution for a $1 \times 0.5698\ldots$ rectangle is only possible because $0.215\ldots$ is a positive number. Let us further illustrate this result with some examples.

The value $x = \sqrt{2}$ does satisfy a polynomial equation with integer coefficients, namely $x^2 - 2 = 0$. However, the other root of the equation is $-\sqrt{2} < 0$. Thus a square *cannot* be tiled with finitely many rectangles similar to a $1 \times \sqrt{2}$ rectangle.

On the other hand, $x = \sqrt{2} + \frac{17}{12}$ satisfies the quadratic equation $144x^2 - 408x + 1 = 0$, whose other root is $-\sqrt{2} + \frac{17}{12} = 0.002453\cdots > 0$. Therefore, a square *can* be tiled with finitely many rectangles similar to a $1 \times (\sqrt{2} + \frac{17}{12})$ rectangle. How would we actually do it?

Similarly, $x = \sqrt[3]{2}$ satisfies the equation $x^3 - 2 = 0$. The other two roots of this equation are $-\frac{\sqrt[3]{2}}{2} \pm \frac{\sqrt[3]{2}\sqrt{3}}{2}\sqrt{-1}$. Since $-\frac{\sqrt[3]{2}}{2} < 0$, a square *cannot* be tiled with finitely many rectangles similar to a $1 \times \sqrt[3]{2}$ rectangle.

Finally, let r/s be a rational number and let $x = \frac{r}{s} + \sqrt[3]{2}$. One can check that this is still a root of a cubic polynomial, whose other two roots are:

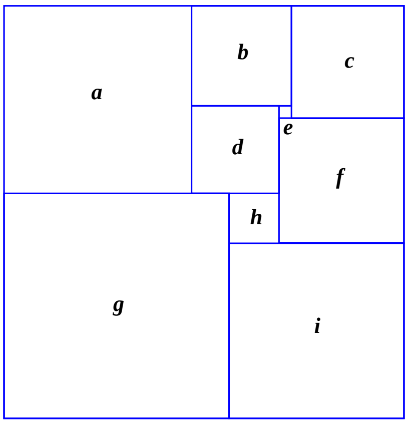
$$\left(\frac{r}{s} - \frac{\sqrt[3]{2}}{2}\right) \pm \frac{\sqrt[3]{2}\sqrt{3}}{2}\sqrt{-1}.$$

It follows that a square can be tiled with finitely many rectangles similar to a $1 \times (\frac{r}{s} + \sqrt[3]{2})$ rectangle if and only if

$$\frac{r}{s} > \frac{\sqrt[3]{2}}{2}.$$

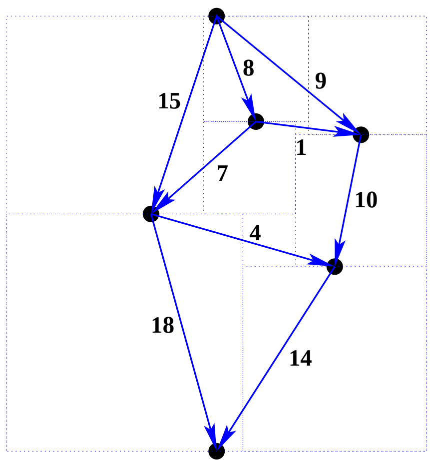
As a nice puzzle, the reader can pick his or her favorite fraction larger than $\sqrt[3]{2}/2$, and tile a square with rectangles similar to a $1 \times (\frac{r}{s} + \sqrt[3]{2})$ rectangle.

The third problem we wish to discuss is motivated by the following remarkable tiling of a rectangle into nine squares, all of which have different sizes. (We will soon see what the sizes of the squares and the rectangle are.) Such tilings are now known as *perfect tilings*.



To find perfect tilings of rectangles, we can use the approach of the previous problem. We start by proposing a tentative layout of the squares, such as the pattern shown above, without knowing what sizes they have. We denote the side length of each square by a variable. For each horizontal line inside the rectangle, we write the following equation: The total length of the squares sitting on the line is equal to the total length of the squares hanging from the line. For example, we have the "horizontal equations" $a + d = g + h$ and $b = d + e$. Similarly, we get a "vertical equation" for each vertical line inside the rectangle, such as $a = b + d$ or $d + h = e + f$. Finally, we write the equations that say that the top and bottom sides of the rectangle are equal, and the left and right sides of the rectangle are equal. In this case, they are $a+b+c = g+i$ and $a+g = c+f+i$. It then remains to hope that the resulting system of linear equations has a solution; and furthermore, one where the values of the variables are positive and distinct. For the layout proposed above, the system has a unique solution up to scaling: $(a, b, c, d, e, f, g, h, i) = (15, 8, 9, 7, 1, 10, 18, 4, 14)$. The large rectangle has dimensions 32×33.

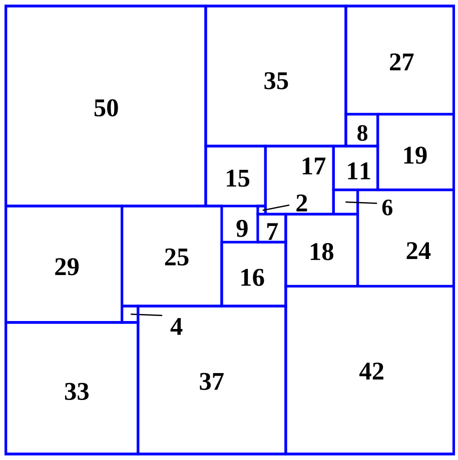
Amazingly, the resulting system of linear equations *always* has a unique solution up to scaling, for *any* proposed layout of squares. (Unfortunately, the resulting "side lengths" are usually not positive and distinct.) In 1936, Brooks, Smith, Stone, and Tutte [2] gave a beautiful explanation of this result. They constructed a directed graph whose vertices are the horizontal lines found in the rectangle. There is one edge for each small square, which goes from its top horizontal line to its bottom horizontal line. The diagram below shows the resulting graph for our perfect tiling of the 32×33 rectangle.



We can think of this graph as an electrical network of unit resistors, where the current flowing through each wire is equal to the length of the corresponding square in the tiling. The "horizontal equations" for the side lengths of the squares are equivalent to the equations for conservation of current in this network, and the "vertical equations" are equivalent to Ohm's law. Knowing this, our statement is essentially equivalent to Kirchhoff's theorem: The flow in each wire is determined uniquely, once we know the potential difference between some two vertices (*i.e.*, up to scaling).

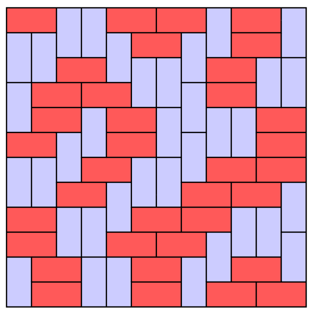
Brooks, Smith, Stone, and Tutte were especially interested in studying perfect tilings of squares. This also has a nice interpretation in terms of the network. To find tilings of squares, we would need an additional linear equation, stating that the vertical and horizontal side lengths of the rectangle are equal. In the language of the electrical network, this is equivalent to saying that the network has total resistance 1.

While this correspondence between tilings and networks is very nice conceptually, it does not necessarily make it easy to construct perfect tilings of squares, or even rectangles. In fact, after developing this theory, Stone spent some time trying to prove that a perfect tiling of a square was impossible. Roland Sprague finally constructed one in 1939, tiling a square of side length 4205 with 55 squares. Since then, much effort and computer hours have been spent trying to find better constructions. Duijvestijn and his computer [5] showed that the smallest possible number of squares in a perfect tiling of a square is 21; the only such tiling is shown below.

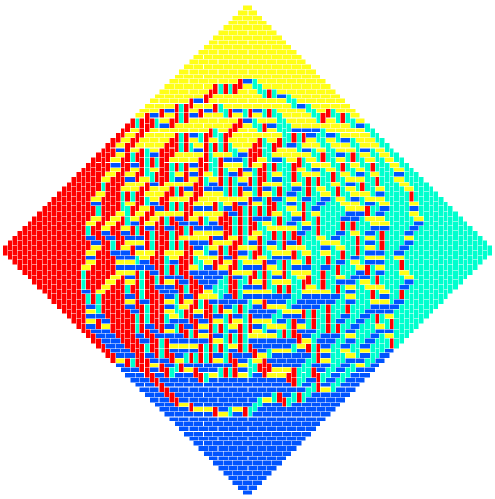


7 What does a typical tiling look like?

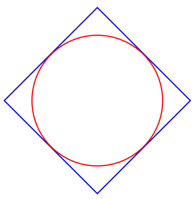
Suppose that we draw each possible solution to a tiling problem on a sheet of paper, put these sheets of paper in a bag, and pick one of them at random. Can we predict what we will see?



The random domino tiling of a 12×12 square shown above, with horizonal dominoes shaded darkly and vertical dominoes shaded lightly, exhibits no obvious structure. Compare this with a random tiling of the Aztec diamond of order 50. Here there are two shades of horizontal dominoes and two shades of vertical dominoes, assigned according to a certain rule not relevant here. These pictures were created by Jim Propp's Tilings Research Group.



This very nice picture suggests that something interesting can be said about random tilings. The tiling is clearly very regular at the corners, and gets more chaotic as we move away from the edges. There is a well defined *region of regularity*, and we can predict its shape. Jockusch, Propp and Shor [11] showed that for very large n, and for "most" domino tilings of the Aztec diamond AZ(n), the region of regularity "approaches" the outside of a circle tangent to the four limiting sides. Sophisticated probability theory is needed to make the terms "most" and "approaches" precise, but the intuitive meaning should be clear.

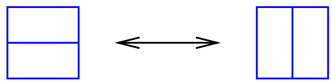


This result is known as the *Arctic circle theorem*. The tangent circle is the Arctic circle; the tiling is "frozen" outside of it. Many similar phenomena have since been observed and (in some cases) proved for other tiling problems.

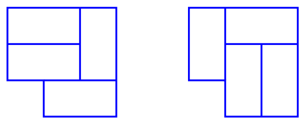
8 Relations among tilings

When we study the set of all tilings of a region, it is often useful to be able to "navigate" this set in a nice way. Suppose we have one solution to a tiling problem, and we want to find another one. Instead of starting over, it is probably easier to find a second solution by making small changes to the first one. We could then try to obtain a third solution from the second one, then a fourth solution, and so on.

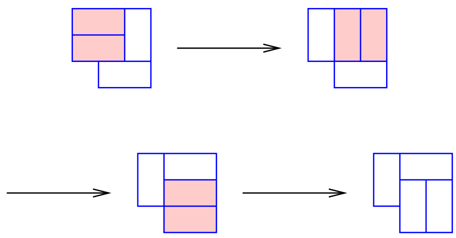
In the case of domino tilings, there is a very easy way to do this. A *flip* in a domino tiling consists of reversing the orientation of two dominoes forming a 2×2 square.



This may seem like a trivial transformation to get from one tiling to another. However, it is surprisingly powerful. Consider the two following tilings of a region.

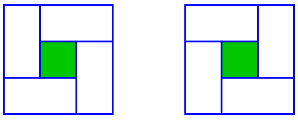


Although they look very different from each other, one can in fact reach one from the other by successively flipping 2×2 blocks.



Thurston [20] showed that this is a general phenomenon. For *any* region R with no holes, *any* domino tiling of R can be reached from *any* other by a sequence of flips.

This domino flipping theorem has numerous applications in the study of domino tilings. We point out that the theorem can be false for regions with holes, as shown by the two tilings of a 3×3 square with a hole in the middle. There is a version due to Propp [18] of the domino flipping theorem for regions with holes, but we will not discuss it here.



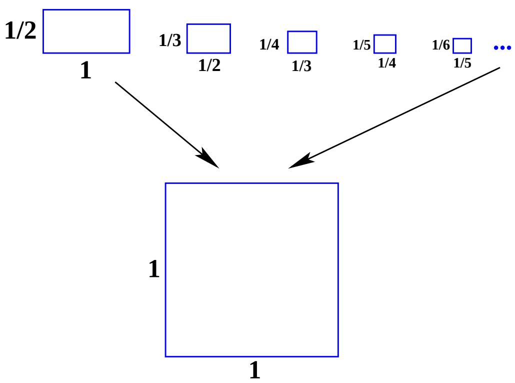
9 Confronting infinity.

We now discuss some tiling questions which involve arbitrary large regions or arbitrarily small tiles.

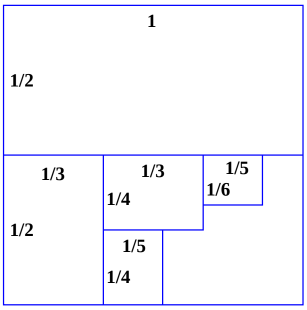
The first question is motivated by the following identity:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots = 1.$$

Consider infinitely many rectangular tiles of dimensions $1 \times \frac{1}{2}, \frac{1}{2} \times \frac{1}{3}, \frac{1}{3} \times \frac{1}{4}, \ldots$. These tiles get smaller and smaller, and the above equation shows that their total area is exactly equal to 1. Can we tile a unit square using each one of these tiles exactly once?



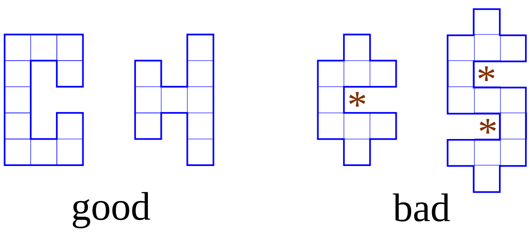
This seems to be quite a difficult problem. An initial attempt shows how to fit the first five pieces nicely. However, it is difficult to imagine how we can fit all of the pieces into the square, without leaving any gaps.



To this day, no one has been able to find a tiling, or prove that it does not exist. Paulhus [16] has come very close; he found a way to fit all these rectangles into a square of side length 1.000000001. Of course Paulhus's packing is not a tiling as we have defined the term, since there is leftover area.

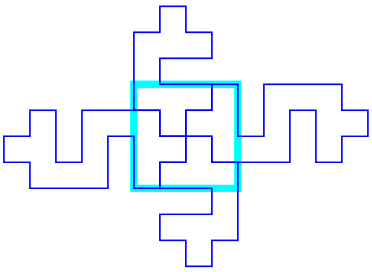
Let us now discuss a seemingly simple problem about tilings that makes it necessary to consider indeterminately large regions. Recall that a polyomino is a collection of unit squares arranged with coincident sides.

Let us call a collection of polyominoes "good" if it is possible to tile the whole plane using the collection as tiles, and "bad" otherwise. A good and a bad collection of polyominoes are shown below.



It is easy to see why it is impossible to tile the whole plane with the bad collection shown above. Once we lay down a tile, the square(s) marked with an asterisk cannot be covered by any other tile.

However, we can still ask: How large of a square region can we cover with a tiling? After a few tries, we will find that it is possible to cover a 4×4 square.



It is impossible, however to cover a 5×5 square. Any attempt to cover the central cell of the square with a tile will force one of the asterisks of that tile to land inside the square as well.

In general, the question of whether a given collection of polyominoes can cover a given square is a tremendously difficult one. A deep result from mathematical logic states that there does not exist an algorithm to decide the answer to this question.[1]

An unexpected consequence of this deep fact is the following. Consider all the bad collections of polyominoes which have a total of n unit cells. Let $L(n)$ be the side length of the largest square which can be covered with one of them. The bad collection of our example, which has a total of 22 unit squares, shows that $L(22) \geq 4$.

One might expect $L(22)$ to be reasonably small. Given a bad collection of tiles with a total of 22 squares, imagine that we start laying down tiles to fit together nicely and cover as large a square as possible. Since the collection is bad, at some point we will inevitably form a hole which we cannot cover. It seems plausible to assume that this will happen fairly soon, since our tiles are quite small.

Surprisingly, however, the numbers $L(n)$ are incredibly large! If $f(n)$ is *any* function that can be computed on a computer, even with infinite memory, then $L(n) > f(n)$ for all large enough n. Notice that computers can compute functions which grow *very* quickly, such as

$$f(n) = n^n, \quad f(n) = n^{n^n}, \quad \text{or}$$

$$f(n) = n^{n^{\cdot^{\cdot^{\cdot^{n}}}}} \quad \text{(a tower of length } n), \quad \ldots .$$

In fact, all of these functions are *tiny* in comparison with certain other computable functions. In turn, every computable function is *tiny* in comparison with $L(n)$.

We can give a more concrete consequence of this result. There exists a collection of polyominoes with a modest number of unit squares[2], probably no more than 100, with the following property: It is impossible to tile the whole plane with this collection; however, it is possible to completely cover Australia[3], with a tiling.

A very important type of problem is concerned with tilings of infinite (unbounded) regions, in particular, tilings of the entire plane. This is a vast subject (the 700-page book [9] by Grünbaum and Shephard is devoted primarily to this topic), but lack of space prevents us from saying more than a few words.

[1] A related question is the following: Given a polyomino P, does there exist a rectangle which can be *tiled* using copies of P? Despite many statements to the contrary in the literature, it is not known whether there exists an algorithm to decide this.

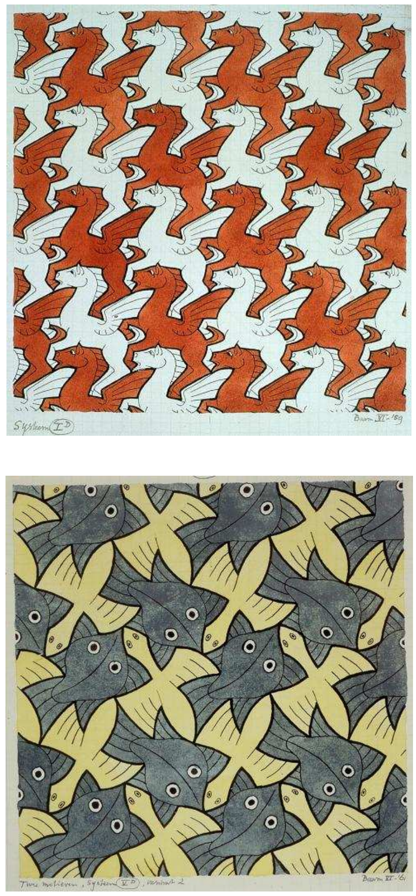
[2] Say "unit squares" have a side length of 1 cm.

[3] which is very large and very flat

A famous result in mathematical crystallography states that there are 17 essentially different tiling patterns of the plane that have symmetries in two independent directions [9, Sec. 6.2]. These symmetry types are called *plane crystallographic groups.* The Alhambra palace in Granada, Spain, dating to the 13th and 14th century, is especially renowned for its depiction of many of these tiling patterns. We give two samples below.

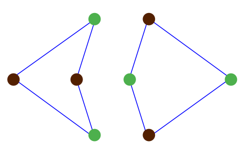
Owen Jones, The Grammar of Ornament, views 90 and 93. ©1998 Octavo and the Rochester Institute of Technology. Used with permission. Imaged by Octavo, www.octavo.com.

Another well-known source of plane tiling patterns is the drawings, lithographs, and engravings of the Dutch graphic artist Maurits Cornelis Escher (1898–1972). Again we give two samples below.

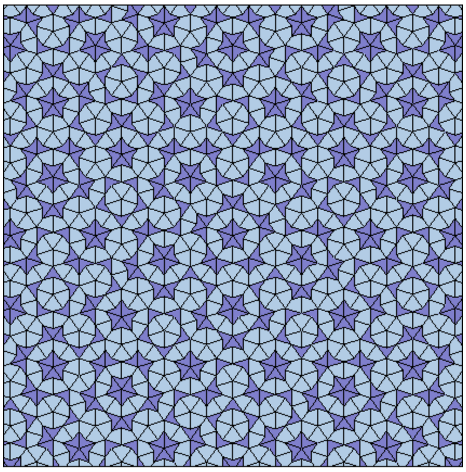


M.C. Escher's Symmetry Drawings E105 and E110. ©2004 The M.C. Escher Company, Baarn, Holland. All rights reserved.

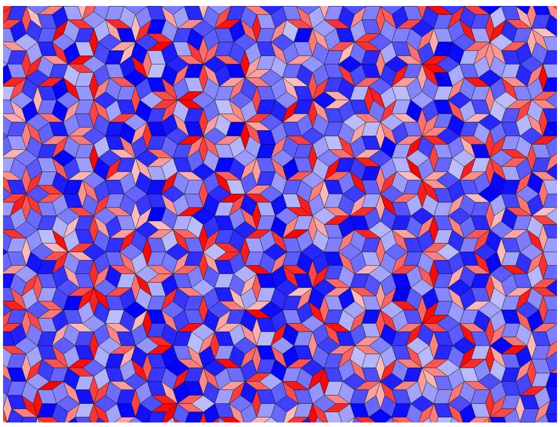
In the opposite direction to plane tilings with lots of symmetry are tilings with *no* symmetry. The most interesting are those discovered by Sir Roger Penrose. Dart and kite tilings are the best known example: We wish to tile the plane using the tiles shown below, with the rule that tiles can only be joined at vertices which have the same color.



The coloring of the tiles makes it impossible to cover the plane by repeating a small pattern in a regular way, as was done in the four previous tilings. However, there are infinitely many different dart and kite tilings of the plane [8, 17]. Below is a sketch of such a tiling, created by Franz Gähler and available at `www.itap.physik.uni-stuttgart.de/~gaehler`; it has many pleasing features, but does not follow any obvious pattern.



These *Penrose tilings* have many remarkable properties; for instance, any Penrose tiling of the plane contains infinitely many copies of *any* finite region which one can form using the tiles.



Our last example is another kind of Penrose tiling, which is obtained by gluing two kinds of rhombi, following a similar rule. This figure was created by Russell Towle in Dutch Flat, CA, with a *Mathematica* notebook available at `library.wolfram.com/infocenter/MathSource/1197/`.

We leave the reader to investigate further the fascinating subject of tilings of the plane.

References

[1] D. Beauquier, M. Nivat, E. Rémila and M. Robson. Tiling figures of the plane with two bars. *Comput. Geom.* **5** (1995), 1-25.

The authors consider the problem of tiling a region with horizontal $n \times 1$ and vertical $1 \times m$ rectangles. Their main result is that, for $n \geq 2$ and $m > 2$, deciding whether such a tiling exists is an NP-complete question. They also study several specializations of this problem.

[2] R. Brooks, C. Smith, A. Stone and W. Tutte. The dissection of rectangles into squares. *Duke Math. J.* **7** (1940), 312-340.

To each perfect tilings of a rectangle, the authors associate a certain graph and a flow of electric current through it. They show how the properties of the tiling are reflected in the electrical network. They use this point of view to prove several results about perfect tilings, and to provide new methods for constructing them.

[3] J. Conway and J. Lagarias. Tiling with polyominoes and combinatorial group theory. *J. Combin. Theory Ser. A* **53** (1990), 183-208.

Conway and Lagarias study the existence of a tiling of a region in a regular lattice in $\mathbb{R}^2$ using a finite set of tiles. By studying the way in which the boundaries of the tiles fit together to give the boundary of the region, they give a necessary condition for a tiling to exist, using the language of combinatorial group theory.

[4] N. de Bruijn. Filling boxes with bricks. *Amer. Math. Monthly* **76** (1969), 37-40.

The author studies the problem of tiling an n-dimensional box of integer dimensions $A_1 \times \cdots \times A_n$ with bricks of integer dimensions $a_1 \times \cdots \times a_n$. For a tiling to exist, de Bruijn proves that every a_i must have a multiple among $A_1, \ldots, A_n$.
The box is called a *multiple* of the brick if it can be tiled in the trivial way. It is shown that, if $a_1|a_2, a_2|a_3, \ldots, a_{n-1}|a_n$, then the brick can only tile boxes which are multiples of it. The converse is also shown to be true.

[5] A. Duijvestijn. Simple perfect squared square of lowest order. *J. Combin. Theory Ser. B* **25** (1978), 240-243.

The unique perfect tiling of a square using the minimum possible number of squares, 21, is exhibited.

[6] N. Elkies, G. Kuperberg, M. Larsen and J. Propp. Alternating sign matrices and domino tilings I, II. *J. Algebraic Combin.* **1** (1992), 111-132, 219-234.

It is shown that the Aztec diamond of order n has $2^{n(n+1)/2}$ domino tilings. Four proofs are given, exploiting the connections of this object with alternating-sign matrices, monotone triangles, and the representation theory of $GL(n)$. The relation with Lieb's square-ice model is also explained.

[7] M. Fisher and H. Temperley. Dimer problem in statistical mechanics – an exact result. *Philos. Mag.* **6** (1961), 1061-1063.

A formula for the number of domino tilings of a rectangle is given in the language of statistical mechanics.

[8] M. Gardner. Extraordinary nonperiodic tiling that enriches the theory of tiles. *Scientific American* **236** (1977), 110-121.

This expository article discusses the quest for finding a set of tiles which can tile the plane, but cannot do so periodically. After some historical background, Gardner focuses on the properties of the best known example: Penrose's kite and dart tilings.

[9] B. Grünbaum and G. Shephard. *Tilings and patterns.* W. H. Freeman and Company, New York, 1987.

This book provides an extensive account of various aspects of tilings, with an emphasis on tilings of the plane with a finite set of tiles. For example, the authors carry out the task of classifying several types of tiling patterns in the plane. Other topics discussed include perfect tilings of rectangles, and aperiodic tilings of the plane.

[10] P. Hall. On representatives of subsets. *J. London Math. Soc.* **10** (1935), 26-30.

Given m subsets $T_1, \ldots, T_m$ of a set S, Hall defines a *complete system of distinct representatives* to be a set of m distinct elements $a_1, \ldots, a_m$ of S such that $a_i \in T_i$ for each i. He proves that such a system exists if and only if, for each $k = 1, \ldots, m$, the union of any k of the sets contains at least k elements.

[11] W. Jockusch, J. Propp and P. Shor. Random domino tilings and the Arctic circle theorem, preprint, 1995, `arXiv:math.CO/9801068`.

In a domino tiling of an Aztec diamond, the diamond is partitioned into five regions: four outer regions near the corners where the tiles are neatly lined up, and one central region where they do not follow a predictable pattern. The authors prove the Arctic circle theorem: In a random tiling of a large Aztec diamond, the central region is extremely close to a perfect circle inscribed in the diamond.

[12] P. Kasteleyn. The statistics of dimers on a lattice I. The number of dimer arrangements on a quadratic lattice. *Phys.* **27** (1961), 1209-1225.

Kasteleyn proves exact and asymptotic formulas for the number of domino tilings of a rectangle, with edges or with periodic boundary conditions. He then discusses the relationship between this problem and the Ising model of statistical mechanics.

[13] D. Klarner. Packing a rectangle with congruent n-ominoes. *J. Combin. Theory* **7** (1969), 107-115.

Klarner investigates the problem of tiling a rectangle using an odd number of copies of a single polyomino. He also characterizes the rectangles which can be tiled with copies of an $a \times b$ rectangle, and the rectangles which can be tiled with copies of a certain octomino.

[14] M. Laczkovich and G. Szekeres. Tilings of the square with similar rectangles. *Discrete Comput. Geom.* **13** (1995), 569-572.

The authors show that a square can be tiled with similar copies of the $1 \times u$ rectangle if and only if u is a root of a polynomial with integer coefficients, all of whose roots have positive real part.

[15] I. Pak. Tile invariants: new horizons. *Theoret. Comput. Sci.* **303** (2003), 303-331.

Given a finite set of tiles T, the group of invariants $G(T)$ consists of the linear relations that must hold between the number of tiles of each type in tilings of the same region. This paper surveys what is known about $G(T)$. These invariants are shown to be much stronger than classical coloring arguments.

[16] M. Paulhus. An algorithm for packing squares. *J. Combin. Theory Ser. A* **82** (1998), 147-157.

Paulhus presents an algorithm for packing an infinite set of increasingly small rectangles with total area A into a rectangle of area very slightly larger than A. He applies his algorithm to three known problems of this sort, obtaining extremely tight packings.

[17] R. Penrose. Pentaplexity. *Math. Intelligencer* **2** (1979), 32-37.

The author describes his discovery of kites and darts: two kinds of quadrilaterals which can tile the plane, but cannot do so periodically. He briefly surveys some of the properties of these tilings.

[18] J. Propp. Lattice structure for orientations of graphs, preprint, 1994.

It is shown that the set of orientations of a graph which have the same flow-differences around all circuits can be given the structure of a distributive lattice. This generalizes similar constructions for alternating sign matrices and matchings.

[19] S. Stein and S. Szabó. *Algebra and tiling. Homomorphisms in the service of geometry.* Mathematical Association of America, Washington, DC, 1994.

This book discusses the solution of several tiling problems using tools from modern algebra. Two sample problems are the following: A square cannot be tiled with $30^\circ - 60^\circ - 90^\circ$ triangles, and a square of odd integer area cannot be tiled with triangles of unit area.

[20] W. Thurston. Conway's tiling groups. *Amer. Math. Monthly* **97** (1990), 757-773.

The author presents a technique of Conway for studying tiling problems. Sometimes it is possible to label the edges of the tiles with elements of a group, so that a region can be tiled if and only if the product (in order) of the labels on its boundary is the identity element. The idea of a height function which lifts tilings to a three-dimensional picture is also presented. These techniques are applied to tilings with dominoes, lozenges, and tribones.

[21] S. Wagon. Fourteen proofs of a result about tiling a rectangle. *Amer. Math. Monthly* **94** (1987), 601-617.

Wagon gives fourteen proofs of the following theorem: If a rectangle can be tiled by rectangles, each of which has at least one integral side, then the tiled rectangle has at least one integral side.

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