The Fundamental Lemma for Unitary Groups

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Introduction

The Fundamental Lemma is a set of combinatorial identities which have been formulated by Langlands-Shelstad.

It is a key tool in proving many cases of Langlands functoriality.

Some earlier works:
- $\text{SL}(2)$ by Labesse-Langlands,
- $\text{SL}(n)$ by Waldspurger,
- $\text{Sp}(4)$ by Hales and Weissauer,
- $\text{U}(3)$ by Kottwitz and Rogawsky,
General statement

\[ \Delta(\gamma, \delta) \circ G^\kappa_\gamma(1_K) = \text{SO}_\delta^H(1_{KH}). \]

Main entries:
- \( F \) non archimedean local field
  \((\lfloor F : \mathbb{Q}_p \rfloor < +\infty \text{ or } \lfloor F : \mathbb{F}_p((t)) \rfloor < +\infty),\)
- \( G \) reductive group over \( F \),
- \( H \) endoscopic group of \( G \),
- \( \delta \in H \) elliptic \( G \)-regular semisimple,
- \( \text{O}_{\gamma}^{G,\kappa}(1_K) \) \( \kappa \)-orbital integral,
- \( \text{SO}_\delta^H(1_{KH}) \) stable orbital integral,
- \( \Delta(\gamma, \delta) \) transfer factor.

\( \chi \) rep of a rep class induced by \( \delta \)
Orbital integrals for full linear groups

- \((E_i)_{i \in I}\) finite family of finite separable extensions of \(F\),
- \(\gamma_i \in E_i^\times\) such that \(F[\gamma_i] = E_i\),
- \(E = \bigoplus_{i \in I} E_i\),
- \(\gamma = (\gamma_i)_{i \in I} \in T = E^\times \subset G = \text{Aut}_F(E)\),
- \(P_i(t) \in F[t]\) minimal polynomial of \(\gamma_i\).

Assume \(P_i(t) \neq P_j(t) \ \forall i \neq j\), so that \(T\) is the centralizer of \(\gamma\) in \(G\).

- \(1_K\) characteristic function of \(K = \text{Aut}_{O_F}(O_E) \subset G\)

\(\Rightarrow\) orbital integral

\[ O^G_{\gamma}(1_K) = \int_{T \backslash G} 1_K(g^{-1}\gamma g) \frac{dg}{dt}. \]

\(v \in \mathcal{L}(\nu, dg) = v \in \mathcal{L}(O^x_E, dt) = 1 \ 3 \)
Orbital integrals as counting lattices

- $O_F$-lattices $L$ in $E \Leftrightarrow \text{rank } n$ (free)
  $O_F$-submodules $L$ of $E$,
- $[L : O_E] = \lgth \left( \frac{L}{L \cap O_E} \right) - \lgth \left( \frac{O_E}{L \cap O_E} \right)$,
- $\mathcal{L} = \{ O_F$-lattices $L \subset E \mid \gamma L = L$ and $[L : O_E] = 0 \}$,
- $\omega_{E_i}$ uniformizer of $E_i$,
- $\Lambda = \{ \lambda \in \mathbb{Z}^I \mid \sum_{i \in I} \lambda_i = 0 \}$ acts freely
  on $\mathcal{L}$ by $\lambda \cdot L = (\omega_{E_i}^{-\lambda_i})_{i \in I} L$.

**Lemma** We have

$$O^G_\gamma(1_K) = |\mathcal{L}/\Lambda|.$$
Unitary groups

- $F'/F$ quadratic unramified extension,
- $(E_i)_{i \in I}$ finite family of finite separable extensions of $F$,
- $c_i \in E_i^\times$, $c = (c_i)_{i \in I}$,

Assume $E_i$ disjoint of $F'$.

- $E'_i = E_i F'$,
- $\text{Gal}(E'_i/E_i) = \text{Gal}(F'/F) = \{1, \tau\}$,
- $\Phi_c(x, y) = \sum_{i \in I} \text{tr}_{E'_i/F'}(c_i \tau(x_i) y_i)$ non degenerate Hermitian form on the $F'$-vector space $E' = \bigoplus_{i \in I} E'_i$.

$\Rightarrow$ unitary group

$$G = \mathbb{U}(\Phi_c) \subset \text{Aut}_{F'}(E').$$
Orbital integrals for unitary groups

- $E_i^{1} = \{ x_i \in E_i^{\times} \mid \tau(x_i)x_i = 1 \}$,
- $\gamma_i \in E_i^{1}$ such that $E_i' = F'[\gamma_i]$,
- $\gamma = (\gamma_i)_{i \in I} \in T = \Pi_{i \in I} E_i^{1} \subset U(\Phi_c)$,
- $P_i(t) \in F'[t]$ minimal polynomial of $\gamma_i$.

Assume:

- $P_i(t) \neq P_j(t) \quad \forall i \neq j$,
- $\text{disc}(c) = \sum_{i \in I} \nu_F(\text{Nr}_{E_i/F}(c_i))$ even.

$\Rightarrow$ orbital integral

$$O_\gamma^G(1_K) = \int_{T \backslash G} 1_K(g^{-1}\gamma g) \frac{dg}{dt}$$

which is the number $O_\gamma^c$ of $O_{F'}$-lattices $L \subset E'$ such that:

- $L$ is self-dual with respect to $\Phi_c$,
- $\gamma L = L$. 


Stable conjugacy

Two elements of the unitary group $G = U(\Phi_c) \subset \text{Aut}_{F'}(E')$ are stably conjugate if they are conjugate in $\text{Aut}_{F'}(E')$.

If $|I| \geq 2$ the stable conjugacy class of $\gamma$ is not equal to its conjugacy class.

Equivalently:
- for each $c' \in E^\times$ we have $\gamma \in U(\Phi_{c'})$,
- $\text{disc}(c') \equiv \text{disc}(c) \equiv 0$ modulo 2 ⇒
  $\exists \iota_{c,c'} : U(\Phi_{c'}) \xrightarrow{\sim} U(\Phi_c) = G$,
- but $\iota_{c,c'}(\gamma)$ is not necessarily conjugate to $\gamma$ in $G = U(\Phi_c)$. 
\( \kappa \)-orbital integrals

The set of conjugacy classes inside the stable conjugacy class of \( \gamma \) in \( G = \text{U}(\Phi_c) \) is isomorphic to

\[
\{ c' \in E^X \mid \text{disc}(c') \text{ even} \}/\text{Nr}_{E'/E}E'^{\times}
\]

\[
\cong \{ \overline{\lambda} \in (\mathbb{Z}/2\mathbb{Z})^I \mid \sum_{i \in I} \overline{\lambda}_i = 0 \} = \overline{\Lambda}
\]

- \( \mathcal{O}_{\gamma}^{c'} = \mathcal{O}_{\gamma}^{\overline{\lambda}} \) only depends on the class \( \overline{\lambda} \) of \( c' \) in \( \overline{\Lambda} \).

\( \kappa : \overline{\Lambda} \to \{ \pm 1 \} \) character \( \Rightarrow \) \( \kappa \)-orbital integral

\[
\mathcal{O}_{\gamma}^{G,\kappa}(1_K) = \sum_{\overline{\lambda} \in \overline{\Lambda}} \kappa(\overline{\lambda}) \mathcal{O}_{\gamma}^{\overline{\lambda}}.
\]

\( \kappa = 1 \Rightarrow \) stable orbital integral

\[
\mathcal{S}\mathcal{O}_{\gamma}^{G}(1_K) = \mathcal{O}_{\gamma}^{G,1}(1_K).
\]
Endoscopic groups

- $\kappa : \bar{\Lambda} \to \{\pm 1\} \Leftrightarrow$ partition $I = I_1 \cup I_2$,
- $c_{I_\alpha} = (c_i)_{i \in I_\alpha} \in \prod_{i \in I_\alpha} E_i^\times$,
- replacing $I$ by $I_\alpha \Rightarrow$
  \[ \gamma_{I_\alpha} = (\gamma_i)_{i \in I_\alpha} \in G_\alpha = U(\Phi_{c_{I_\alpha}}), \]
- $\delta = (\gamma_{I_1}, \gamma_{I_2}) \in T \subset H$,
- $H = G_1 \times G_2$ endoscopic group determined by $\kappa$.
\Rightarrow stable orbital integral

\[ \text{SO}_H^H(1_{K^H}) = \text{SO}_{\gamma_{I_1}}^{G_1}(1_{K^{G_1}}) \times \text{SO}_{\gamma_{I_2}}^{G_2}(1_{K^{G_2}}). \]
Fundamental Lemma for unitary groups

- \( q \) = number of elements of the residue field of \( F \),
- \( r_{ij} = v_{F^i} (\text{Res}(P_i(t), P_j(t))) \forall i \neq j \in I \),
- \( r = \sum_{i \in I_1,j \in I_2} r_{ij} \).

**THEOREM** If \( p > n = [E : F] \) we have

\[
\Delta(\delta, \gamma) O^{G, \kappa}_{\gamma}(1_K) = SO^H_\delta(1_K^H)
\]

where \( \Delta(\delta, \gamma) = (-q)^{-r} \).
Reduction (Hales, Waldspurger)

- Waldspurger: algorithm computing orbital integrals which only depends on the residue field of $F$.
- Hales and Waldspurger: the Fundamental Lemma follows from its Lie algebra version.

Lie algebra version of the Fundamental Lemma:
- $\gamma_i \in E_i^\times \leadsto \gamma_i \in E_i$,
- $\tau(\gamma_i)\gamma_i = 1 \leadsto \tau(\gamma_i) + \gamma_i = 0$,
- $\gamma L = L \leadsto \gamma L \subset L$. 
Affine Springer fibers for full linear groups

From now on:
- $F = k((\mathcal{O}_F))$, $k = \mathbb{F}_q$,
- Fundamental Lemma for Lie algebra.

$\mathcal{L} = \{L \subseteq E \mid \gamma L \subseteq L \text{ (and } [L : \mathcal{O}_E] = 0)\}$ is the set of rational points of an algebraic variety, the so-called affine Springer fiber $S^{(0)}_{\gamma}$, over $k$.

Grothendieck fixed point formula
⇒ orbital integral = trace of $\text{Frob}_q$
  on $\ell$-adic cohomology $H^\bullet(S^{(0)}_{\gamma})$.
⇒ Fundamental Lemma follows from existence of a "Gysin" isomorphism.
"Gysin" isomorphism for full linear groups

- $I = I_1 \cup I_2 \Rightarrow E = E_{I_1} \oplus E_{I_2}$,
- $\gamma = (\gamma_i)_{i \in I} \Rightarrow \gamma_{I_\alpha} = (\gamma_i)_{i \in I_\alpha} \Rightarrow$ affine Springer fiber $S_{\gamma_{I_\alpha}}$,
- closed embedding $S_{\gamma_{I_1}} \times S_{\gamma_{I_2}} \hookrightarrow S_{\gamma}$,
  $(L_1 \subset E_{I_1}, L_2 \subset E_{I_2}) \mapsto L_1 \oplus L_2 \subset E$,
- $r = \dim(S_{\gamma}) - \dim(S_{\gamma_{I_1}} \times S_{\gamma_{I_2}})$.

Question: Is there a canonical isomorphism

$$H^\bullet(S_{\gamma}) \cong H^{\bullet-2r}(S_{\gamma_{I_1}} \times S_{\gamma_{I_2}})(-r)?$$

Problem: $S_{\gamma}$ is highly singular.
Goresky-Kottwitz-MacPherson strategy

Note: \( t \cdot L = (t \oplus 1)L \subset E_{I_1} \oplus E_{I_2} = E \)
\( \Rightarrow \mathbb{G}_{m,k} \) acts on \( S_\gamma \) and the fixed point set is \( S_{\gamma I_1} \times S_{\gamma I_2} \).

Atiyah-Borel-Segal localization in equivariant \( \ell \)-adic cohomology
\( \Rightarrow \mathbb{Q}_\ell[\!x\!] \)-linear map

\[ \iota : H_{\mathbb{G}_{m,k}}^\bullet (S_\gamma) \rightarrow H^\bullet (S_{\gamma I_1} \times S_{\gamma I_2})[\!x\!]. \]

\( \Rightarrow \) "Gysin" isomorphism if we can
(1) prove that \( \iota \) is injective,
(2) compute its image,
(3) recover ordinary cohomology from equivariant one.
Purity Conjecture of Goresky-Kottwitz-MacPherson

Points (1) and (3) follow easily from:

**CONJECTURE** $H^m(S_\gamma)$ is pure of weight $m$ for all $m$.

But, any direct approach to point (2) and to the Purity Conjecture seems to be very hard.

Our approach: to try to deform the complicated affine Springer fibers into simpler ones.
Examples of affine Springer fibers

Assume: $p > 2$.

(i) $E_1 = E_2 = F$, $\gamma_1 = \omega_F$, $\gamma_2 = -\omega_F$, $S_\gamma^0$ is an infinite chain of projective lines

(ii) $E = F[x]/(x^2 - \omega_F)$, $\gamma = \omega_F^3$, $S_\gamma^0$ is a single projective line
Problem with affine Springer fibers

The affine Springer fibers don’t behave well in families:
- natural to expect that example (i) degenerates to example (ii),
- but no algebraic family whose general fiber is a chain of projective lines and whose special fiber is a single projective line.

Solution: to replace affine Springer fibers by compactified Jacobians.
Altman-Kleiman compactified Jacobians

\[ A_\gamma = \mathcal{O}_F[\gamma] \subset \mathcal{O}_E, \quad P(t) = \prod_{i \in I} P_i(t) \]
\[ \Rightarrow A_\gamma = k[[\varpi_F, t]]/(P(t)) \]

**Lemma** \exists projective geometrically irreducible curve \( C_\gamma \) over \( k \) and \( c_\gamma \in C_\gamma(k) \) such that \( g(C_\gamma) = 0 \), \( C_\gamma - \{c_\gamma\} \) smooth over \( k \) and \( \hat{\mathcal{O}}_{C_\gamma, c_\gamma} = A_\gamma \).

Compactified Jacobian \( \overline{\text{Jac}}(C_\gamma) \) of \( C_\gamma \) = moduli space of degree 0 rank 1 torsion free coherent \( \mathcal{O}_{C_\gamma} \)-Modules.

Up to homeomorphisms the affine Springer fiber \( S_\gamma^0 \) is an étale Galois covering of \( \overline{\text{Jac}}(C_\gamma) \) with Galois group \( \Lambda \).
First approach

Compactified Jacobians behave well in families: deformation of a curve $\Rightarrow$ deformation of its compactified Jacobian.

By deforming $C_\gamma$ as follows

one gets:

Purity Conjecture $\Rightarrow$ Fundamental Lemma for unitary groups.
Problem with compactified Jacobians

Problem: How to get the Purity Conjecture?

In introducing $C_\gamma$ and its compactified Jacobian, we are doing too much algebraic geometry and not enough group theory.

Solution: To replace compactified Jacobians by Hitchin fibers.
Hitchin fibration for full linear groups

Fix:
- $X$ smooth, projective, geometrically connected curve over $k$,
- $D$ ample effective divisor on $X$.

Hitchin bundle $= (\mathcal{E}, \theta)$ where:
- $\mathcal{E}$ degree 0 rank $n$ vector bundle on $X$,
- $\theta : \mathcal{E} \to \mathcal{E}(D)$ twisted endomorphism.

- $\mathcal{M} = \{\text{Hitchin bundles}\}$,
- $\mathbb{A} = \bigoplus_{i=1}^{n} H^0(X, \mathcal{O}_X(iD))$,
- $f : \mathcal{M} \to \mathbb{A}, \ (\mathcal{E}, \theta) \mapsto (a_1, \ldots, a_n)$,
  where $p_a(t) = t^n + a_1 t^{n-1} + \cdots + a_n$
  is the characteristic polynomial of $\theta$.  

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Hitchin fibers as compactified Jacobians

- $\Sigma$ total space of the line bundle $\mathcal{O}_X(D)$,
- $\Sigma \to X$ is a ruled surface,
- $\forall a \in \mathbb{A}$, $Y_a = \{p_a(t) = 0\} \subset \Sigma$ is a ramified covering of degree $n$ of $X$,
- $Y_a$ is called a spectral curve,
- $\mathbb{A}^{\text{red}} = \{a \in \mathbb{A} \mid Y_a \text{ reduced}\}$ open subset in $\mathbb{A}$.

**THEOREM** (Hitchin) $\forall a \in \mathbb{A}^{\text{red}}$, the Hitchin fiber $\mathcal{M}_a = f^{-1}(a)$ is the compactified Jacobian of $Y_a$. 
What do we do next?

- Fix a unitary group scheme $G = U(n)$ over $X$ and an endoscopic unitary group scheme $H = U(n_1) \times U(n_2)$ of $G$.
- Construct a commutative diagramm of Hitchin fibrations

$$
\begin{array}{ccc}
\mathcal{M}_H & \to & \mathcal{M}_G \\
\downarrow & & \downarrow \\
\mathcal{A}_H & \to & \mathcal{A}_G
\end{array}
$$

- Use a relative version of the Atiyah-Borel-Segal localization.
- Prove a relative version of the Purity Conjecture using Deligne’s theorem.
- ...