The Fundamental Lemma for Unitary Groups

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Introduction

The Fundamental Lemma is a set of combinatorial identities which have been formulated by Langlands-Shelstad.

It is a key tool in proving many cases of Langlands functoriality.

Some earlier works:
- SL(2) by Labesse-Langlands,
- SL(n) by Waldspurger,
- Sp(4) by Hales and Weissauer,
- U(3) by Kottwitz and Rogawsky,
- unramified equal valuation case by Goresky-Kottwitz-MacPherson.

General statement

$$\Delta(\gamma,\delta)\,\mathrm{O}_{\gamma}^{G,\kappa}(1_K) = \mathrm{SO}_{\delta}^{H}(1_{K^H}).$$

Main entries:

- F non archimedean local field ($[F:\mathbb{Q}_p]<+\infty$ or $[F:\mathbb{F}_p((t))]<+\infty$),
- G reductive group over F,
- H endoscopic group of G,
- $\delta \in H$ elliptic G-regular semisimple,
- $\mathrm{O}_{\gamma}^{G,\kappa}(1_K)$ κ-orbital integral,
- $\mathrm{SO}_{\delta}^{H}(1_{K^H})$ stable orbital integral,
- $\Delta(\gamma,\delta)$ transfer factor.

γ repr of a conj class induced by δ

Orbital integrals for full linear groups

- $(E_i)_{i\in I}$ finite family of finite separable extensions of F,
- $\gamma_i \in E_i^\times$ such that $F[\gamma_i] = E_i$,
- $E = \oplus_{i\in I} E_i$,
- $\gamma = (\gamma_i)_{i\in I} \in T = E^\times \subset G = \mathrm{Aut}_F(E)$,
- $P_i(t) \in F[t]$ minimal polynomial of γ_i.

Assume $P_i(t) \neq P_j(t)$ $\forall i \neq j$, so that T is the centralizer of γ in G.

- 1_K characteristic function of $K = \mathrm{Aut}_{\mathcal{O}_F}(\mathcal{O}_E) \subset G$

$\mathcal{O}_E = \bigoplus_{i\in I} \mathcal{O}_{E_i}$.

$\mathcal{O}_{F,E}$ = ring of integers of F, E

$\Rightarrow$ orbital integral

$$\mathrm{O}_\gamma^G(1_K) = \int_{T\backslash G} 1_K(g^{-1}\gamma g)\frac{\mathrm{d}g}{\mathrm{d}t}.$$

$\mathrm{vol}(K, dg) = \mathrm{vol}(\mathcal{O}_E^\times, dt) = 1$

Orbital integrals as counting lattices

- $\mathcal{O}_F$-lattices L in E $\Leftrightarrow$ rank n (free) $\mathcal{O}_F$-submodules L of E,
- $[L : \mathcal{O}_E] = \mathrm{lgth}\left(\frac{L}{L\cap\mathcal{O}_E}\right) - \mathrm{lgth}\left(\frac{\mathcal{O}_E}{L\cap\mathcal{O}_E}\right)$,
- $\mathcal{L} = \{\mathcal{O}_F\text{-lattices } L \subset E \mid \gamma L = L \text{ and } [L : \mathcal{O}_E] = 0\}$,
- ϖ_{E_i} uniformizer of E_i,
- $\Lambda = \{\lambda \in \mathbb{Z}^I \mid \sum_{i\in I} \lambda_i = 0\}$ acts freely on $\mathcal{L}$ by $\lambda \cdot L = (\varpi_{E_i}^{-\lambda_i})_{i\in I} L$.

LEMMA *We have*

$$\mathrm{O}_\gamma^G(1_K) = |\mathcal{L}/\Lambda|.$$

Unitary groups

- F'/F quadratic unramified extension,
- $(E_i)_{i\in I}$ finite family of finite separable extensions of F,
- $c_i \in E_i^\times$, $c = (c_i)_{i\in I}$,

Assume E_i disjoint of F'.

- $E'_i = E_i F'$,
- $\mathrm{Gal}(E'_i/E_i) = \mathrm{Gal}(F'/F) = \{1, \tau\}$,
- $\Phi_c(x, y) = \sum_{i\in I} \mathrm{tr}_{E'_i/F'}(c_i \tau(x_i) y_i)$ non degenerate Hermitian form on the F'-vector space $E' = \oplus_{i\in I} E'_i$.

$\Rightarrow$ unitary group

$$G = \mathrm{U}(\Phi_c) \subset \mathrm{Aut}_{F'}(E').$$

Orbital integrals for unitary groups

- $E_i'^1 = \{x_i \in E_i'^\times \mid \tau(x_i)x_i = 1\}$,
- $\gamma_i \in E_i'^1$ such that $E_i' = F'[\gamma_i]$,
- $\gamma = (\gamma_i)_{i\in I} \in T = \prod_{i\in I} E_i'^1 \subset \mathrm{U}(\Phi_c)$,
- $P_i(t) \in F'[t]$ minimal polynomial of γ_i.

Assume:

- $P_i(t) \neq P_j(t)\ \forall i \neq j$,
- $\mathrm{disc}(c) = \sum_{i\in I} v_F(\mathrm{Nr}_{E_i/F}(c_i))$ *even*.

$\Rightarrow$ orbital integral

$$\mathrm{O}_\gamma^G(1_K) = \int_{T\backslash G} 1_K(g^{-1}\gamma g)\frac{\mathrm{d}g}{\mathrm{d}t}$$

which is the number O_γ^c of $\mathcal{O}_{F'}$-lattices $L \subset E'$ such that:

- L is self-dual with respect to Φ_c,
- $\gamma L = L$.

Stable conjugacy

Two elements of the unitary group $G = \mathsf{U}(\Phi_c) \subset \mathsf{Aut}_{F'}(E')$ are stably conjugate if they are conjugate in $\mathsf{Aut}_{F'}(E')$.

If $|I| \geq 2$ the stable conjugacy class of γ is not equal to its conjugacy class.

Equivalently:
- for each $c' \in E^\times$ we have $\gamma \in \mathsf{U}(\Phi_{c'})$,
- $\mathsf{disc}(c') \equiv \mathsf{disc}(c) \equiv 0$ modulo 2 $\Rightarrow$ $\exists\ \iota_{c,c'} : \mathsf{U}(\Phi_{c'}) \xrightarrow{\sim} \mathsf{U}(\Phi_c) = G$,
- but $\iota_{c,c'}(\gamma)$ is not necessarily conjugate to γ in $G = \mathsf{U}(\Phi_c)$.

κ-orbital integrals

The set of conjugacy classes inside the stable conjugacy class of γ in $G = \mathsf{U}(\Phi_c)$ is isomorphic to

$$\{c' \in E^\times \mid \mathrm{disc}(c') \text{ even}\}/\mathrm{Nr}_{E'/E}E'^\times$$
$$\cong \{\overline{\lambda} \in (\mathbb{Z}/2\mathbb{Z})^I \mid \sum_{i\in I} \overline{\lambda}_i = 0\} = \overline{\Lambda}$$

- $\mathsf{O}_\gamma^{c'} = \mathsf{O}_\gamma^{\overline{\lambda}}$ only depends on the class $\overline{\lambda}$ of c' in $\overline{\Lambda}$.

$\kappa : \overline{\Lambda} \to \{\pm 1\}$ character $\Rightarrow$ κ-orbital integral

$$\mathsf{O}_\gamma^{G,\kappa}(1_K) = \sum_{\overline{\lambda}\in\overline{\Lambda}} \kappa(\overline{\lambda})\mathsf{O}_\gamma^{\overline{\lambda}}.$$

$\kappa = 1 \Rightarrow$ stable orbital integral

$$\mathsf{SO}_\gamma^{G}(1_K) = \mathsf{O}_\gamma^{G,1}(1_K).$$

Endoscopic groups

- $\kappa : \overline{\Lambda} \to \{\pm 1\} \Leftrightarrow$ partition $I = I_1 \dot{\cup} I_2$,
- $c_{I_\alpha} = (c_i)_{i \in I_\alpha} \in \prod_{i \in I_\alpha} E_i^\times$,
- replacing I by $I_\alpha \Rightarrow$

  $\gamma_{I_\alpha} = (\gamma_i)_{i \in I_\alpha} \in G_\alpha = \mathrm{U}(\Phi_{c_{I_\alpha}})$,
- $\delta = (\gamma_{I_1}, \gamma_{I_2}) \in T \subset H$,
- $H = G_1 \times G_2$ endoscopic group

  determined by κ.

$\Rightarrow$ stable orbital integral

$$\mathrm{SO}_\delta^H(1_{K^H}) = \mathrm{SO}_{\gamma_{I_1}}^{G_1}(1_{K^{G_1}}) \times \mathrm{SO}_{\gamma_{I_2}}^{G_2}(1_{K^{G_2}}).$$

Fundamental Lemma for unitary groups

- $q =$ number of elements of the residue field of F,
- $r_{ij} = v_{F'}(\text{Res}(P_i(t), P_j(t)))$ $\forall i \neq j \in I$,
- $r = \sum_{i \in I_1, j \in I_2} r_{ij}$.

THEOREM If $p > n = [E : F]$ we have

$$\Delta(\delta, \gamma) \mathrm{O}_{\gamma}^{G,\kappa}(1_K) = \mathrm{SO}_{\delta}^{H}(1_{K^H})$$

where $\Delta(\delta, \gamma) = (-q)^{-r}$.

Reduction (Hales, Waldspurger)

• Waldspurger: algorithm computing orbital integrals which only depends on the residue field of F.

• Hales and Waldspurger: the Fundamental Lemma follows from its Lie algebra version.

Lie algebra version of the Fundamental Lemma:

- $\gamma_i \in E_i^\times \rightsquigarrow \gamma_i \in E_i$,
- $\tau(\gamma_i)\gamma_i = 1 \rightsquigarrow \tau(\gamma_i) + \gamma_i = 0$,
- $\gamma L = L \rightsquigarrow \gamma L \subset L$.

Affine Springer fibers for full linear groups

From now on:

- $F = k((\varpi_F))$, $k = \mathbb{F}_q$,
- Fundamental Lemma for Lie algebra.

$\mathcal{L} = \{L \subset E \mid \gamma L \subset L (\textit{and } [L : \mathcal{O}_E] = 0)\}$ *is the set of rational points of an algebraic variety, the so-called affine Springer fiber* $\mathcal{S}_\gamma^{(0)}$, *over* k.

Grothendieck fixed point formula

$\Rightarrow$ orbital integral = trace of Frob_q on ℓ-adic cohomology $H^\bullet(\mathcal{S}_\gamma^{0})$.

$\Rightarrow$ Fundamental Lemma follows from existence of a "Gysin" isomorphism.

"Gysin" isomorphism for full linear groups

- $I = I_1 \dot{\cup} I_2 \Rightarrow E = E_{I_1} \oplus E_{I_2}$,
- $\gamma = (\gamma_i)_{i \in I} \Rightarrow \gamma_{I_\alpha} = (\gamma_i)_{i \in I_\alpha} \Rightarrow$ affine Springer fiber $\mathcal{S}_{\gamma_{I_\alpha}}$,
- closed embedding $\mathcal{S}_{\gamma_{I_1}} \times \mathcal{S}_{\gamma_{I_2}} \hookrightarrow \mathcal{S}_\gamma$, $(L_1 \subset E_{I_1}, L_2 \subset E_{I_2}) \mapsto L_1 \oplus L_2 \subset E$,
- $r = \dim(\mathcal{S}_\gamma) - \dim(\mathcal{S}_{\gamma_{I_1}} \times \mathcal{S}_{\gamma_{I_2}})$.

Question: Is there a canonical isomorphism

$$H^\bullet(\mathcal{S}_\gamma) \cong H^{\bullet - 2r}(\mathcal{S}_{\gamma_{I_1}} \times \mathcal{S}_{\gamma_{I_2}})(-r)?$$

Problem: $\mathcal{S}_\gamma$ is highly singular.

Goresky-Kottwitz-MacPherson strategy

Note: $t \cdot L = (t \oplus 1)L \subset E_{I_1} \oplus E_{I_2} = E$

$\Rightarrow \mathbb{G}_{\mathrm{m},k}$ acts on $\mathcal{S}_\gamma$ and the fixed point set is $\mathcal{S}_{\gamma_{I_1}} \times \mathcal{S}_{\gamma_{I_2}}$.

Atiyah-Borel-Segal localization in equivariant ℓ-adic cohomology

$\Rightarrow \mathbb{Q}_\ell[x]$-linear map

$$\iota : H^\bullet_{\mathbb{G}_{\mathrm{m},k}}(\mathcal{S}_\gamma) \to H^\bullet(\mathcal{S}_{\gamma_{I_1}} \times \mathcal{S}_{\gamma_{I_2}})[x].$$

$\Rightarrow$ "Gysin" isomorphism if we can

(1) prove that ι is injective,

(2) compute its image,

(3) recover ordinary cohomology from equivariant one.

Purity Conjecture of Goresky-Kottwitz-MacPherson

Points (1) and (3) follow easily from:

CONJECTURE $H^m(\mathcal{S}_\gamma)$ *is pure of weight* m *for all* m.

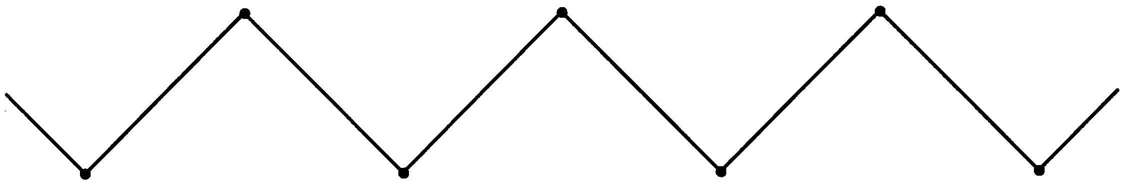
But, any direct approach to point (2) and to the Purity Conjecture seems to be very hard.

Our approach: to try to deform the complicated affine Springer fibers into simpler ones.

Examples of affine Springer fibers

Assume: $p > 2$.

(i) $E_1 = E_2 = F$, $\gamma_1 = \varpi_F$, $\gamma_2 = -\varpi_F$, $\mathcal{S}_\gamma^0$ is an infinite chain of projective lines



(ii) $E = F[x]/(x^2 - \varpi_F)$, $\gamma = \varpi_F^3$, $\mathcal{S}_\gamma^0$ is a single projective line

Problem with affine Springer fibers

The affine Springer fibers don't behave well in families:

- natural to expect that example (i) degenerates to example (ii),
- but no algebraic family whose general fiber is a chain of projective lines and whose special fiber is a single projective line.

Solution: to replace affine Springer fibers by compactified Jacobians.

Altman-Kleiman compactified Jacobians

$A_\gamma = \mathcal{O}_F[\gamma] \subset \mathcal{O}_E$, $P(t) = \prod_{i\in I} P_i(t)$
$\Rightarrow A_\gamma = k[[\varpi_F, t]]/(P(t))$

LEMMA $\exists$ *projective geometrically irreducible curve* C_γ *over* k *and* $c_\gamma \in C_\gamma(k)$ *such that* $g(C_\gamma) = 0$, $C_\gamma - \{c_\gamma\}$ *smooth over* k *and* $\widehat{\mathcal{O}}_{C_\gamma, c_\gamma} = A_\gamma$.

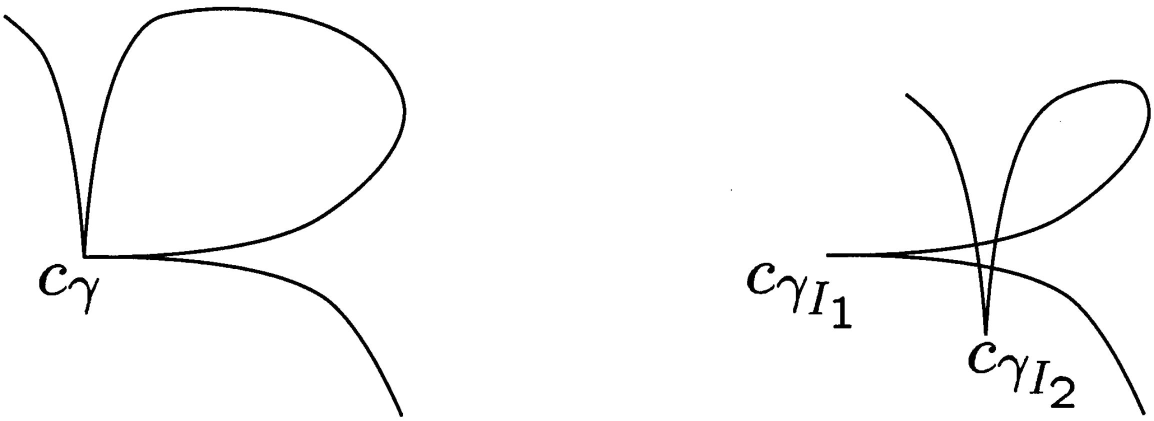
Compactified Jacobian $\overline{\mathrm{Jac}}(C_\gamma)$ of C_γ = moduli space of degree 0 rank 1 torsion free coherent $\mathcal{O}_{C_\gamma}$-Modules.

Up to homeomorphisms the affine Springer fiber $\mathcal{S}_\gamma^0$ is an étale Galois covering of $\overline{\mathrm{Jac}}(C_\gamma)$ with Galois group Λ.

First approach

Compactified Jacobians behave well in families: deformation of a curve $\Rightarrow$ deformation of its compactified Jacobian.

By deforming C_γ as follows



one gets:

Purity Conjecture $\Rightarrow$ Fundamental Lemma for unitary groups.

Problem with compactified Jacobians

Problem: How to get the Purity Conjecture?

In introducing C_γ and its compactified Jacobian, we are doing too much algebraic geometry and not enough group theory.

Solution: To replace compactified Jacobians by Hitchin fibers.

Hitchin fibration for full linear groups

Fix:

- X smooth, projective, geometrically connected curve over k,
- D ample effective divisor on X.

Hitchin bundle $= (\mathcal{E}, \theta)$ where:

- $\mathcal{E}$ degree 0 rank n vector bundle on X,
- $\theta : \mathcal{E} \to \mathcal{E}(D)$ twisted endomorphism.

- $\mathcal{M} = \{\text{Hitchin bundles}\}$,
- $\mathbb{A} = \oplus_{i=1}^{n} H^0(X, \mathcal{O}_X(iD))$,
- $f : \mathcal{M} \to \mathbb{A}$, $(\mathcal{E}, \theta) \mapsto (a_1, \ldots, a_n)$, where $p_a(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ is the characteric polynomial of θ.

Hitchin fibers as compactified Jacobians

- Σ total space of the line bundle $\mathcal{O}_X(D)$,
- $\Sigma \to X$ is a ruled surface,
- $\forall a \in \mathbb{A}$, $Y_a = \{p_a(t) = 0\} \subset \Sigma$ is a ramified covering of degree n of X,
- Y_a is called a spectral curve,
- $\mathbb{A}^{\text{red}} = \{a \in \mathbb{A} \mid Y_a \text{ reduced}\}$ open subset in $\mathbb{A}$.

THEOREM (Hitchin) $\forall a \in \mathbb{A}^{red}$, *the Hitchin fiber* $\mathcal{M}_a = f^{-1}(a)$ *is the compactified Jacobian of* Y_a.

What do we do next?

- Fix a unitary group scheme $G = \mathrm{U}(n)$ over X and an endoscopic unitary group scheme $H = \mathrm{U}(n_1) \times \mathrm{U}(n_2)$ of G.
- Construct a commutative diagramm of Hitchin fibrations

$$\begin{array}{ccc} \mathcal{M}_H & \to & \mathcal{M}_G \\ \downarrow & & \downarrow \\ \mathbb{A}_H & \to & \mathbb{A}_G \end{array}$$

- Use a relative version of the Atiyah-Borel-Segal localization.
- Prove a relative version of the Purity Conjecture using Deligne's theorem.
- ...