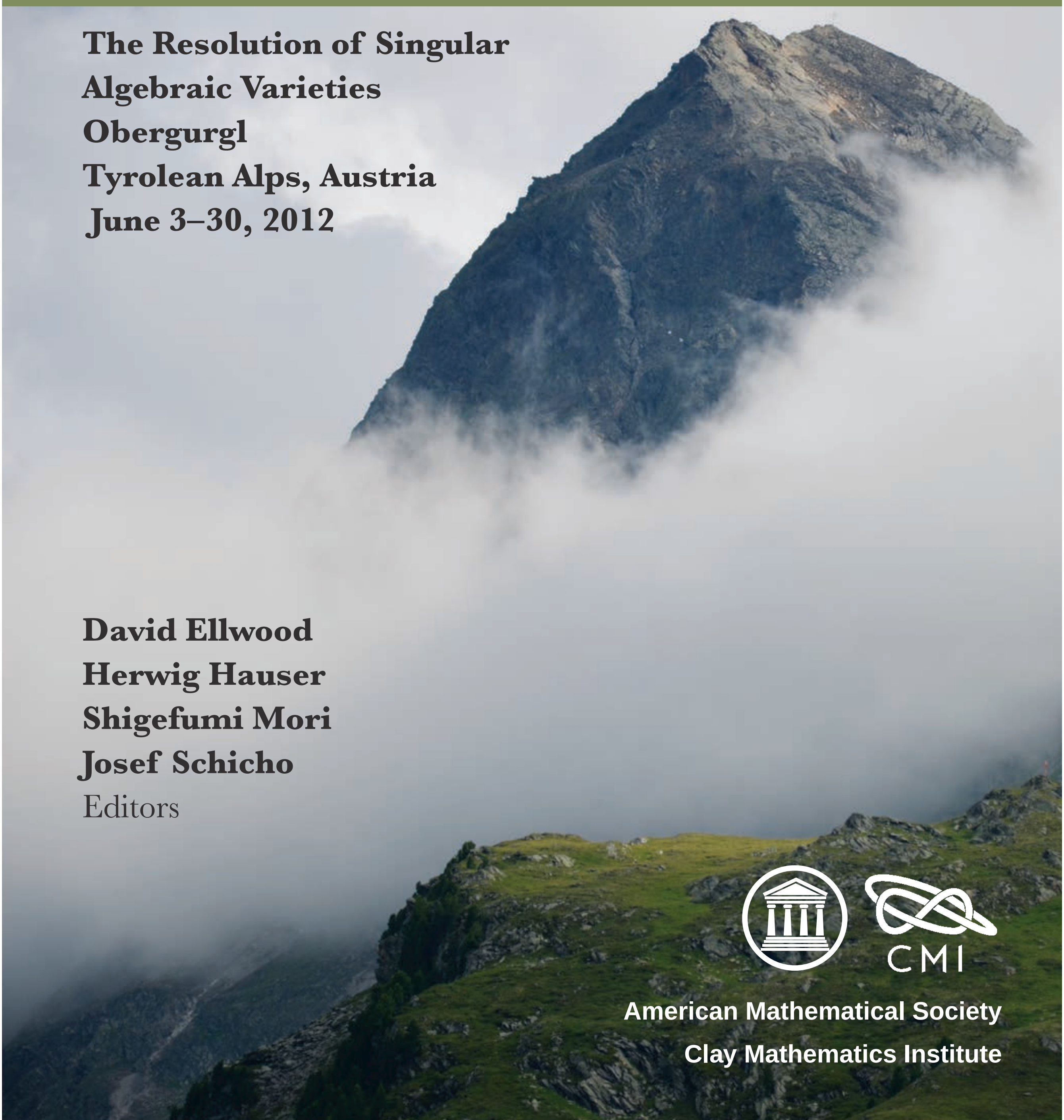
Clay Mathematics Proceedings
Volume 20

# The Resolution of Singular Algebraic Varieties

Clay Mathematics Institute Summer School 2012



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Obergurgl
Tyrolean Alps, Austria
June 3–30, 2012

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2010 *Mathematics Subject Classification.* Primary 14-01, 14-06, 14Bxx, 14Exx, 13-01, 13-06, 13Hxx, 32Bxx, 32Sxx, 58Kxx.

Cover photo of Obergurgl, Austria is by Alexander Zainzinger.

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**Library of Congress Cataloging-in-Publication Data**

Clay Mathematics Institute Summer School (2012 : Obergurgl Center)

The resolution of singular algebraic varieties: Clay Mathematics Institute Summer School, the resolution of singular algebraic varieties, June 3–30, 2012, Obergurgl Center, Tyrolean Alps, Austria / David Ellwood, Herwig Hauser, Shigefumi Mori, Josef Schicho, editors.

pages cm. — (Clay mathematics proceedings ; volume 20)

Includes bibliographical references and index.

ISBN 978-0-8218-8982-4 (alk. paper)

1. Algebraic varieties—Congresses. 2. Commutative algebra—Congresses. I. Ellwood, David, 1966– II. Hauser, H. (Herwig), 1956– III. Mori, Shigefumi. IV. Schicho, Josef, 1964– V. Title.

QA564.C583 2012
516.3′5—dc23

2014031965

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A tribute to Shreeram Abhyankar and Heisuke Hironaka

# Contents

# Preface

A remote place in the Tyrolean Alps at 2000 meters of altitude, a venue with perfect working facilities and a clear cut goal: to excite young mathematicians in the resolution of singularities of algebraic varieties by offering them a four week program of classes and problem sessions—this was the scene of the 12th Clay Summer School at Obergurgl, Austria.

It was previewed from the outset that such a school should go beyond mere mathematical education: it should represent a decisive step in the career of participants by teaching how to grasp and incorporate the main features of a complicated theory, to evaluate and combine the many ideas involved in its proofs, and to develop an overall picture of what mathematics can be good for with respect to intellectual, cultural and personal development. This scientific intention was to be matched by social effects: the communication with colleagues and teachers, group work, mutual respect and stimulation, the dialectic of modesty versus ambition.

The topic: Resolution of singularities consists in constructing for a given algebraic variety $X$ an algebraic manifold $\widetilde{X}$ together with a surjective map $\pi : \widetilde{X} \longrightarrow X$. This map gives a parametrization of the singular variety by a smooth variety. Algebraically, this means to find, for a given system of polynomial equations, a systematic transformation of the polynomials by means of blowups which transform the system into one that satisfies the assumption of the implicit function theorem, so that certain variables can be expressed as functions of the remaining variables. This transformation allows one to interpret the solution set of the given system as the projection of a graph to the singular variety.

The existence of resolutions is instrumental in many circumstances since it allows one to deduce properties of the variety from properties of the parametrizing manifold. Applications abound.

The pioneer in this problem was Oscar Zariski. He introduced abstract algebraic ideas and techniques to the field, and proved many important cases, both in small dimensions and, for more restrictive assertions, in arbitrary dimension. His perspective was mostly based on varieties defined over fields of characteristic zero. He recommended to his student Shreeram Abhyankar to abandon, after several vain attempts, the difficult positive characteristic case of surfaces. As a matter of protest and stubbornness, Abhyankar intensified his efforts and succeeded in his thesis to settle this case[1].

At that time, another of Zariski's many students, Heisuke Hironaka, was a friend of Abhyankar, and together they were discussing this subject at the end of the fifties. It seems that these conversations produced the key idea for the characteristic

[1] Shreeram Abhyankar, Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$, Ann. Math. (2) 63 (1956), 491–526.

zero case of arbitrary dimension, the so called notion of hypersurfaces of maximal contact. In a technically enormously challenging though rather elementary tour de force Hironaka established this result[2].

It took some thirty years to really understand this proof and to put it on a transparent logical fundament: the systematic use of a local resolution invariant which measures the complexity of a singularity. It serves two purposes: Firstly, the stratum where it attains its maximum (i.e., where the worst singularities occur) is smooth and can be taken as the center of the next blowup. Secondly, under this blowup, the invariant drops at each point above points of the center. In this way, induction can be applied, and there will exist (a possibly very long) sequence of blowups which makes the invariant eventually drop to zero. At that time, it is shown that the variety has become smooth.

The last years have seen further simplifications of the proof and strengthening of the result. However, the case of positive characteristic in arbitrary dimension still resists. It is one of the main challenges of modern algebraic geometry.

The participants: Around eighty students were selected for the school from about 250 applicants. They were recommended to acquaint themselves with the basic prerequisites in advance of the school to ensure a common framework for the presentation of the courses. They were also advised that the four weeks of the school would be an intensive experience, demanding a strong personal dedication to learning the material of the classes, solving the daily problems and exercises, as well as sharing their insights and doubts with the other participants. Special emphasis was laid on the respectful contact among peers which is essential for the success of such a four week event.

The lecturers: Herwig Hauser developed the various contexts in which resolution may appear (algebraic, analytic, local, global, formal, . . . ), exposed the main concepts and techniques (singularities, order functions, transversality, blowups, exceptional divisors, transforms of ideals and varieties, hypersurfaces of maximal contact, resolution invariants), and presented the logical and technical structure of the characteristic zero proof. His article is reminiscent of the engaging style of his lectures at the school, with a vast number of exercises and examples. It gives a concise yet comprehensive overview of resolution techniques including much of the required basics from commutative algebra and/or algebraic geometry. Blowups (in various forms) and transforms of subvarieties are introduced in lectures 4–6. Lecture 7 describes the precise statements of resolution of singularities. Lecture 8 introduces the order invariant of singularities which is used in one of the main inductions in the proof of resolution. This invariant is then refined via the use of a hypersurface of maximal contact and coefficient ideals which are discussed in lectures 9 and 10 respectively. Lecture 11 gives a detailed outline of the proof of strong resolution of a variety in characteristic zero. Finally lecture 12 concerns positive characteristic: it discusses problems that arise and gives references to topics of recent progress.

Orlando Villamayor concentrated on the commutative algebra side of resolution: local multiplicity and Hilbert-Samuel functions, integral ring extensions, finite morphisms, Rees algebras, actions of symmetry groups and a replacement of hypersurfaces of maximal contact by means of generic projections. This gives a new proof of resolution in characteristic zero and has very good chances to be applicable in

[2]Heisuke Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero. Parts I. & II., Ann. Math. 79 (1964): 109–203, 205–326.

arbitrary characteristic. His comprehensive set of notes have been compiled jointly with Ana Bravo. The final results provide a natural smooth stratification of the locus of maximal multiplicity (or maximal Hilbert function) on a variety X (valid in all characteristics) and use this stratification to give a strategy for reducing the maximal value of multiplicity (or Hilbert function) by blowing up smooth centers in characteristic zero. The motivating ideas come from Hironaka's idealistic exponents (of which the Rees algebra gives a generalization) and the classical local interpretation of multiplicity as the degree of a (formal) realization of the singularity as a finite extension over the germ of a regular local ring.

Josef Schicho presented an axiomatic approach to the resolution of singularities in the form of a parlor game. Instead of the various constructions of the classical proof one distills their essential properties and uses them to define a game between two players. The existence of a winning strategy is then a purely combinatorial problem which can be solved by logical arguments. The second part consists in showing that there do exist constructions fulfilling the many axioms or moves of the game. This works up to now only in zero characteristic. Only elementary algebra is needed. In principle, the game also prepares a prospective approach for the case of positive characteristic. Schicho's lecture notes describe the precise rules of the game, the reduction of the resolution problem to the game, and the winning strategy. As an aside, several other games with a mathematical flavor were discussed, and FlipIt was one of them. Together with Jaap Top (Schicho learned about FlipIt from his webpage), a formulation of the game in terms of linear algebra over the field with two elements is provided.

The last week of the school culminated in a series of mini-courses in which invited experts reported on their latest research on resolution, several of whom also contributed to this volume.

Steven Cutkovsky's paper gives an overview of the resolution problem in positive characteristic, enriched with some significant computations on a range of important topics of current interest. The first part of the paper provides the central results, illustrated by an exposition of the main ideas of resolution of surfaces. The second part deals with the problem of making an algebraic map monomial, in appropriate local coordinates, by sequences of blowups. The application by the author and Piltant to a general form of local ramification for valuations is explained. Some results on monomialization in positive characteristic and on global monomialization in characteristic zero conclude the paper.

The article by Santiago Encinas provides an informative survey about resolution of toric varieties, with an emphasis on the resolution algorithm of Blanco and the author. The procedure uses the binomial equations defining the toric variety as an embedded subvariety in an affine space. The article contains an introduction to toric varieties as well as several worked examples.

Anne Frühbis-Krüger gives an introduction to the computational applications of desingularization. In the first part, she details the data that is actually computed by the resolution algorithm. This is a priori not clear because theoretically the result arises by compositions of blowups and is not naturally embedded in an affine or projective space; rather it comes as a union of affine charts. The applications described in the second part include the dual graph of an isolated surface singularity, the log canonical threshold, the topological Zeta function, and the Bernstein-Sato polynomial.

Hiraku Kawanoue gives an overview of his Idealistic Filtration Program, a program for resolution of singularities. The presentation gives a clear overview of the main ideas, and it contains several examples that illustrate the general strategy through clever examples both in characteristic zero and in characteristic p. A new result concerning the monomial case of the Idealistic Filtration Program using the radical saturation is also included.

Takehiko Yasuda presents his geometric approach to the resolution of singularities (higher Semple-Nash blowups, considered mainly in characteristic zero, and F-blowups, defined only in positive characteristic). In this method a series of blowups is constructed directly from the given variety, using not only first-order data, but also higher order ones. Each blowup is the parameter space of some geometric object on the given variety. The final section provides some open problems associated with this approach.

The venue: The Obergurgl Center is owned by the University of Innsbruck. It serves as a conference and research venue for up to 120 participants. Lecture halls, seminar rooms, library and meeting lounges are of the highest standards. They are complemented by cosy rooms in a traditional alpine style, various leisure facilities, excellent food and a very attentive personnel. Above all, the natural surroundings are spectacular, culminating in an inspiring skyline of alpine peaks.

Our thanks go to the former staff of the Clay Mathematics Institute, particularly Julie Feskoe, Vida Salahi and Lina Chen, as well as all the personnel of the Obergurgl Center. We are also grateful for the support of the University of Innsbruck, the Obergurgl tourist office, and the referees of this volume, who participated in an unusually detailed refereeing process.

David Alexandre Ellwood, Herwig Hauser, Josef Schicho, Shigefumi Mori
Boston, Vienna, Linz, Kyoto
August 2014

Clay Mathematics Proceedings
Volume **20**, 2014

# Blowups and Resolution

Herwig Hauser

*To the memory of Sheeram Abhyankar with great respect*

Contents



This manuscript originates from a series of lectures the author[1] gave at the Clay Summer School on Resolution of Singularities at Obergurgl in the Tyrolean Alps in June 2012. A hundred young and ambitious students gathered for four weeks to hear and learn about resolution of singularities. Their interest and dedication became essential for the success of the school.

The reader of this article is ideally an algebraist or geometer having a rudimentary acquaintance with the main results and techniques in resolution of singularities. The purpose is to provide quick and concrete information about specific topics in the field. As such, the article is modelled like a dictionary and not particularly suited to be read from the beginning to the end (except for those who like to read dictionaries). To facilitate the understanding of selected portions of the text

2010 *Mathematics Subject Classification.* Primary: 14B05, 14E15, 14J17, 13H05, 32S45.

*Key words and phrases.* Blowups, resolution, singularities, algebraic varieties.

[1]Supported in part by the Austrian Science Fund FWF within the projects P-21461 and P-25652.

without reading the whole earlier material, a certain repetition of definitions and assertions has been accepted.

Background information on the historic development and the motivation behind the various constructions can be found in the cited literature, especially in [Obe00, Hau03, Hau10a, FH10, Cut04, Kol07, Lip75]. Complete proofs of several more technical results appear in [EH02].

All statements are formulated for algebraic varieties and morphism between them. They are mostly also valid, with the appropriate modifications, for schemes. In certain cases, the respective statements for schemes are indicated separately.

Each chapter concludes with a broad selection of examples, ranging from computational exercises to suggestions for additional material which could not be covered in the text. Some more challenging problems are marked with a superscript $^+$. The examples should be especially useful for people planning to give a graduate course on the resolution of singularities. Occasionally the examples repeat or specialize statements which have appeared in the text and which are worth to be done personally before looking at the given proof. In the appendix, hints and answers to a selection of examples marked by a superscript $^\triangleright$ are given.

Several results appear without proof, due to lack of time and energy of the author. Precise references are given whenever possible. The various survey articles contain complementary bibliography.

The Clay Mathematics Institute chose resolution of singularities as the topic of the 2012 summer school. It has been a particular pleasure to cooperate in this endeavour with its research director David Ellwood, whose enthusiasm and interpretation of the school largely coincided with the approach of the organizers, thus creating a wonderful working atmosphere. His sensitiveness of how to plan and realize the event has been exceptional.

The CMI director Jim Carlson and the CMI secretary Julie Feskoe supported very efficiently the preparation and realization of the school.

Xudong Zheng provided a preliminary write-up of the lectures, Stefan Perlega and Eleonore Faber completed several missing details in preliminary drafts of the manuscript. Faber also produced the two visualizations. The discussion of the examples in the appendix was worked out by Perlega and Valerie Roitner. Anonymous referees helped substantially with their remarks and criticism to eliminate deficiencies of the exposition. Barbara Beeton from the AMS took care of the TeX-layout. All this was very helpful.

## 1. Lecture I: A First Example of Resolution

Let $X$ be the zeroset in affine three space $\mathbb{A}^3$ of the polynomial

$$f = 27x^2y^3z^2 + (x^2 + y^3 - z^2)^3$$

over a field $\mathbb{K}$ of characteristic different from 2 and 3. This is an algebraic surface, called *Camelia*, with possibly singular points and curves, and certain symmetries. For instance, the origin 0 is singular on $X$, and $X$ is symmetric with respect to the automorphisms of $\mathbb{A}^3$ given by replacing $x$ by $-x$ or $z$ by $-z$, and also by replacing $y$ by $-y$ while interchanging $x$ with $z$. Sending $x$, $y$ and $z$ to $t^3x$ , $t^2y$ and $t^3z$ for $t \in \mathbb{K}$ also preserves $X$. See figure 1 for a plot of the real points of $X$. The intersections of $X$ with the three coordinate hyperplanes of $\mathbb{A}^3$ are given as the zerosets of the equations

$$x = (y^3 - z^2)^3 = 0,$$
$$y = (x^2 - z^2)^3 = 0,$$
$$z = (x^2 + y^3)^3 = 0.$$

These intersections are plane curves: two perpendicular cusps lying in the $xy$- and $yz$-plane, respectively the union of the two diagonals in the $xz$-plane. The singular locus $\mathrm{Sing}(X)$ of $X$ is given as the zeroset of the partial derivatives of $f$ inside $X$. This yields for $\mathrm{Sing}(X)$ the additional equations

$$x \cdot [9y^3z^2 + (x^2 + y^3 - z^2)^2] = 0,$$
$$y^2 \cdot [9x^2z^2 + (x^2 + y^3 - z^2)^2] = 0,$$
$$z \cdot [9x^2y^3 - (x^2 + y^3 - z^2)^2] = 0.$$

Combining these equations with $f = 0$, it results that the singular locus of $X$ has six irreducible components, defined respectively by

$$x = y^3 - z^2 = 0,$$
$$z = x^2 + y^3 = 0,$$
$$y = x + z = 0,$$
$$y = x - z = 0,$$
$$x^2 - y^3 = x + \sqrt{-1} \cdot z = 0,$$
$$x^2 - y^3 = x - \sqrt{-1} \cdot z = 0.$$

The first four components of $\mathrm{Sing}(X)$ coincide with the four curves given by the three coordinate hyperplane sections of $X$. The last two components are plane cusps in the hyperplanes given by $x \pm \sqrt{-1} \cdot z = 0$. At points $a \neq 0$ on the first two singular components of $\mathrm{Sing}(X)$, the intersections of $X$ with a plane through $a$ and transversal to the component are again cuspidal curves.

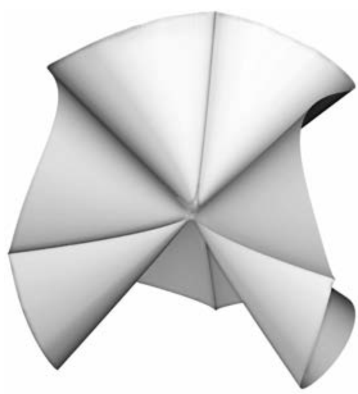


FIGURE 1. The surface *Camelia*: $27x^2y^3z^2 + (x^2 + y^3 - z^2)^3 = 0$.

Consider now the surface $Y$ in $\mathbb{A}^4$ which is given as the cartesian product $C \times C$ of the plane cusp $C : x^2 - y^3 = 0$ in $\mathbb{A}^2$ with itself. It is defined by the equations $x^2 - y^3 = z^2 - w^3 = 0$. The singular locus $\mathrm{Sing}(Y)$ is the union of the two cusps $C_1 = C \times 0$ and $C_2 = 0 \times C$ defined by $x^2 - y^3 = z = w = 0$, respectively $x = y = z^2 - w^3 = 0$. The surface $Y$ admits the parametrization

$$\gamma : \mathbb{A}^2 \to \mathbb{A}^4, (s, t) \mapsto (s^3, s^2, t^3, t^2).$$

The image of $\gamma$ is $Y$. The composition of $\gamma$ with the linear projection

$$\pi : \mathbb{A}^4 \to \mathbb{A}^3, (x, y, z, w) \mapsto (x, -y + w, z)$$

yields the map

$$\delta = \pi \circ \gamma : \mathbb{A}^2 \to \mathbb{A}^3, (s, t) \mapsto (s^3, -s^2 + t^2, t^3).$$

Replacing in the polynomial $f$ of $X$ the variables $x$, $y$, $z$ by $s^3$, $-s^2 + t^2$, $t^3$ gives 0. This shows that the image of $\delta$ lies in $X$. As $X$ is irreducible of dimension 2 and $\delta$ has rank 2 outside 0 the image of $\delta$ is dense in (and actually equal to) $X$. Therefore the image of $Y$ under $\pi$ is dense in $X$: This interprets $X$ as a contraction of $Y$ by means of the projection $\pi$ from $\mathbb{A}^4$ to $\mathbb{A}^3$. The two surfaces $X$ and $Y$ are not isomorphic because, for instance, their singular loci have a different number of components. The simple geometry of $Y$ as a cartesian product of two plane curves is scrambled up when projecting it down to $X$.

The point blowup of $Y$ in the origin produces a surface $Y_1$ whose singular locus has two components. They map to the two components $C_1$ and $C_2$ of $\mathrm{Sing}(Y)$ and are regular and transversal to each other. The blowup $X_1$ of $X$ at 0 will still be the image of $Y_1$ under a suitable projection. The four singular components of $\mathrm{Sing}(X)$ will become regular in $X_1$ and will either meet pairwise transversally or not at all. The two regular components of $\mathrm{Sing}(X)$ will remain regular in $X_1$ but will no longer meet each other, cf. figure 2.

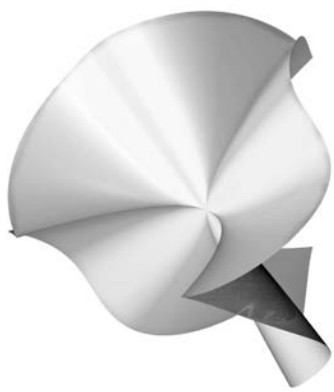


FIGURE 2. The surface $X_1$ obtained from *Camelia* by a point blowup.

The point blowup of $Y_1$ in the intersection point of the two curves of $\mathrm{Sing}(Y_1)$ separates the two curves and yields a surface $Y_2$ whose singular locus consists of two disjoint regular curves. Blowing up these separately yields a regular surface $Y_3$ and thus resolves the singularities of $Y$. The resolution of the singularities of $X$ is more complicated, see the examples below.

## *Examples*

EXAMPLE 1.1. ▹ Show that the surface $X$ defined in $\mathbb{A}^3$ by $27x^2y^3z^2 + (x^2 + y^3 - z^2)^3 = 0$ is the image of the cartesian product $Y$ of the cusp $C : x^3 - y^2 = 0$ with itself under the projection from $\mathbb{A}^4$ to $\mathbb{A}^3$ given by $(x, y, z, w) \mapsto (x, -y+w, z)$.

EXAMPLE 1.2. ▹ Find additional symmetries of $X$ aside from those mentioned in the text.

Example 1.3. $^{\triangleright}$ Produce a real visualization of the surface obtained from *Camelia* by replacing in the equation $z$ by $\sqrt{-1} \cdot z$. Determine the components of the singular locus.

Example 1.4. $^{\triangleright}$ Consider at the point $a = (0, 1, 1)$ of $X$ the plane $P : 2y + 3z = 5$ through $a$. It is transversal to the component of the singular locus of $X$ passing through $a$ (i.e., this component is regular at $a$ and its tangent line at $a$ does not lie in $P$). Determine the singularity of $X \cap P$ at $a$. The normal vector $(0, 2, 3)$ to $P$ is the tangent vector at $t = 1$ of the parametrization $(0, t^2, t^3)$ of the component of $\mathrm{Sing}(X)$ defined by $x = y^3 - z^2$.

Example 1.5. The point $a$ on $X$ with coordinates $(1, 1, \sqrt{-1})$ is a singular point of $X$, and a non-singular point of the curve of $\mathrm{Sing}(X)$ passing through it. A plane transversal to $\mathrm{Sing}(X)$ at $a$ is given e.g. by

$$P : 3x + 2y + 3\sqrt{-1} \cdot z = 9.$$

Determine the singularity of $X \cap P$ at $a$. The normal vector $(3, 2, 3\sqrt{-1})$ to $P$ is the tangent vector at $s = 1$ of the parametrization $(s^3, s^2, \sqrt{-1} \cdot s^3)$ of the curve of $\mathrm{Sing}(X)$ through $a$ defined by $x^2 - y^3 = x - \sqrt{-1} \cdot z = 0$.

Example 1.6. $^{\triangleright}$ Blow up $X$ and $Y$ in the origin, describe exactly the geometry of the transforms $X_1$ and $Y_1$ and produce visualizations of $X$ and $X_1$ over $\mathbb{R}$ in all coordinate charts. For $Y_1$ the equations are in the $x$-chart $1 - xy^3 = z^3x - w^2 = 0$ and in the $y$-chart $x^2 - y = z^3y - w^2 = 0$.

Example 1.7. Blow up $X_1$ and $Y_1$ along their singular loci. Compute a resolution of $X_1$ and compare it with the resolution of $Y_1$.

## 2. Lecture II: Varieties and Schemes

The following summary of basic concepts of algebraic geometry shall clarify the terminology used in later sections. Detailed definitions are available in [Mum99, Sha94, Har77, EH00, Liu02, Kem11, GW10, Gro61, ZS75, Nag75, Mat89, AM69].

### *Varieties*

Definition 2.1. Write $\mathbb{A}^n = \mathbb{A}^n_{\mathbb{K}}$ for the *affine $n$-space* over some field $\mathbb{K}$. The points of $\mathbb{A}^n$ are identified with $n$-tuples $a = (a_1, \dots, a_n)$ of elements $a_i$ of $\mathbb{K}$. The space $\mathbb{A}^n$ is equipped with the *Zariski topology*: the closed sets are the *algebraic subsets* of $\mathbb{A}^n$, i.e., the zerosets $V(I) = \{a \in \mathbb{A}^n,\ f(a) = 0 \text{ for all } f \in I\}$ of ideals $I$ of $\mathbb{K}[x_1, \dots, x_n]$.

To $\mathbb{A}^n$ one associates its *coordinate ring*, given as the polynomial ring $\mathbb{K}[\mathbb{A}^n] = \mathbb{K}[x_1, \dots, x_n]$ in $n$ variables over $\mathbb{K}$.

Definition 2.2. A *polynomial map* $f : \mathbb{A}^n \to \mathbb{A}^m$ between affine spaces is given by a vector $f = (f_1, \dots, f_m)$ of polynomials $f_i = f_i(x) \in \mathbb{K}[x_1, \dots, x_n]$. It induces a $\mathbb{K}$-algebra homomorphism $f^* : \mathbb{K}[y_1, \dots, y_m] \to \mathbb{K}[x_1, \dots, x_n]$ sending $y_i$ to $f_i$ and a polynomial $h = h(y)$ to $h \circ f = h(f(x))$. A *rational map* $f : \mathbb{A}^n \to \mathbb{A}^m$ is given by a vector $f = (f_1, \dots, f_m)$ of elements $f_i \in \mathbb{K}(x_1, \dots, x_n) = \mathrm{Quot}(\mathbb{K}[x_1, \dots, x_n])$ in the quotient field of $\mathbb{K}[x_1, \dots, x_n]$. Albeit the terminology, a rational map need not define a set-theoretic map on whole affine space $\mathbb{A}^n$. It does this only on the open subset $U$ which is the complement of the union of the zerosets of the denominators of the elements $f_i$. The induced map $f_{|U} : U \to \mathbb{A}^m$ is then called a *regular map* on $U$.

Definition 2.3. An *affine (algebraic) variety* $X$ is a subset of an affine space $\mathbb{A}^n$ which is defined as the zeroset $V(I)$ of a radical ideal $I$ of $\mathbb{K}[x_1, \ldots, x_n]$ and equipped with the topology induced by the Zariski topology of $\mathbb{A}^n$. It is thus a closed subset of $\mathbb{A}^n$. In this text, the ideal $I$ need not be prime, hence $X$ is not required to be irreducible. The ideal $I_X$ of $\mathbb{K}[x_1, \ldots, x_n]$ of all polynomials $f$ vanishing on $X$ is the largest ideal such that $X = V(I_X)$. If the field $\mathbb{K}$ is algebraically closed and $X = V(I)$ is defined by the radical ideal $I$, the ideal $I_X$ coincides with $I$ by Hilbert's Nullstellensatz.

The *(affine) coordinate ring of* $X$ is the factor ring $\mathbb{K}[X] = \mathbb{K}[x_1, \ldots, x_n]/I_X$. As $I_X$ is radical, $\mathbb{K}[X]$ is reduced, i.e., has no nilpotent elements. If $X$ is irreducible, $I$ is a prime ideal and $\mathbb{K}[X]$ is an integral domain. To each point $a = (a_1, \ldots, a_n)$ of $X$ one associates the maximal ideal $m_{X,a}$ of $\mathbb{K}[X]$ generated by the residue classes of the polynomials $x_1 - a_1, \ldots, x_n - a_n$. If the field $\mathbb{K}$ is algebraically closed, this defines, by Hilbert's Nullstellensatz, a bijection of the points of $X$ and the maximal ideals of $\mathbb{K}[X]$.

The *function field* of an irreducible variety $X$ is the quotient field $\mathbb{K}(X) = \mathrm{Quot}(\mathbb{K}[X])$.

The *local ring* of an affine variety $X$ at a point $a$ is the localization $\mathcal{O}_{X,a} = \mathbb{K}[X]_{m_{X,a}}$ of $\mathbb{K}[X]$ at the maximal ideal $m_{X,a}$ of $\mathbb{K}[X]$ associated to $a$ in $X$. It is isomorphic to the factor ring $\mathcal{O}_{\mathbb{A}^n,a}/I_a$ of the local ring $\mathcal{O}_{\mathbb{A}^n,a}$ of $\mathbb{A}^n$ at $a$ by the ideal $I_a$ generated by the ideal $I$ defining $X$ in $\mathcal{O}_{\mathbb{A}^n,a}$. If $Z \subset X$ is an irreducible subvariety defined by the prime ideal $p_Z$, the local ring $\mathcal{O}_{X,Z}$ of $\mathcal{O}_X$ along $Z$ is defined as the localization $\mathbb{K}[X]_{p_Z}$. The local ring $\mathcal{O}_{X,a}$ gives rise to the *germ* $(X, a)$ of $X$ at $a$, see below. By abuse of notation, the (unique) maximal ideal $m_{X,a} \cdot \mathcal{O}_{X,a}$ of $\mathcal{O}_{X,a}$ is also denoted by $m_{X,a}$. The factor ring $\kappa_a = \mathcal{O}_{X,a}/m_{X,a}$ is called the *residue field* of $X$ at $a$.

Definition 2.4. A *principal open subset* of an affine variety $X$ is the complement in $X$ of the zeroset $V(g)$ of a single non-zero divisor $g$ of $\mathbb{K}[X]$. Principal open subsets are thus dense in $X$. They form a basis of the Zariski-topology. A *quasi-affine variety* is an open subset of an affine variety. A *(closed) subvariety* of an affine variety $X$ is a subset $Y$ of $X$ which is defined as the zeroset $Y = V(J)$ of an ideal $J$ of $\mathbb{K}[X]$. It is thus a closed subset of $X$.

Definition 2.5. Write $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{K}}$ for the *projective* $n$*-space* over some field $\mathbb{K}$. The points of $\mathbb{P}^n$ are identified with equivalence classes of $(n+1)$-tuples $a = (a_0, \ldots, a_n)$ of elements $a_i$ of $\mathbb{K}$, where $a \sim b$ if $a = \lambda \cdot b$ for some non-zero $\lambda \in \mathbb{K}$. Points are given by their *projective coordinates* $(a_0 : \ldots : a_n)$, with $a_i \in \mathbb{K}$ and not all zero. Projective space $\mathbb{P}^n$ is equipped with the Zariski topology whose closed sets are the algebraic subsets of $\mathbb{P}^n$, i.e., the zerosets $V(I) = \{a \in \mathbb{P}^n, f(a_0, \ldots, a_n) = 0$ for all $f \in I\}$ of homogeneous ideals $I$ of $\mathbb{K}[x_0, \ldots, x_n]$ different from the ideal generated by $x_0, \ldots, x_n$. As $I$ is generated by homogeneous polynomials, the definition of $V(I)$ does not depend on the choice of the affine representatives $(a_0, \ldots, a_n) \in \mathbb{A}^{n+1}$ of the points $a$. Projective space is covered by the affine open subsets $U_i \simeq \mathbb{A}^n$ formed by the points whose $i$-th projective coordinate does not vanish $(i = 0, \ldots, n)$.

The *homogeneous coordinate ring* of $\mathbb{P}^n$ is the graded polynomial ring $\mathbb{K}[\mathbb{P}^n] = \mathbb{K}[x_0, \ldots, x_n]$ in $n+1$ variables over $\mathbb{K}$, the grading being given by the degree.

Definition 2.6. A *projective algebraic variety* $X$ is a subset of projective space $\mathbb{P}^n$ defined as the zeroset $V(I)$ of a homogeneous radical ideal $I$ of $\mathbb{K}[x_0, \ldots, x_n]$

and equipped with the topology induced by the Zariski topology of $\mathbb{P}^n$. It is thus a closed algebraic subset of $\mathbb{P}^n$. The ideal $I_X$ of $\mathbb{K}[x_0, \ldots, x_n]$ of all homogeneous polynomials $f$ which vanish at all (affine representatives of) points of $X$ is the largest ideal such that $X = V(I_X)$. If the field $\mathbb{K}$ is algebraically closed and $X = V(I)$ is defined by the radical ideal $I$, the ideal $I_X$ coincides with $I$ by Hilbert's Nullstellensatz.

The *homogeneous coordinate ring of* $X$ is the graded factor ring $\mathbb{K}[X] = \mathbb{K}[x_0, \ldots, x_n]/I_X$ equipped with the grading given by degree.

Definition 2.7. A *principal open subset* of a projective variety $X$ is the complement of the zeroset $V(g)$ of a single homogeneous non-zero divisor $g$ of $\mathbb{K}[X]$. A *quasi-projective variety* is an open subset of a projective variety. A *(closed) subvariety* of a projective variety $X$ is a subset $Y$ of $X$ which is defined as the zeroset $Y = V(J)$ of a homogeneous ideal $J$ of $\mathbb{K}[X]$. It is thus a closed algebraic subset of $X$.

Remark 2.8. Abstract algebraic varieties are obtained by gluing affine algebraic varieties along principal open subsets, cf. [Mum99] I, §3, §4, [Sha94] V, §3. This allows to develop the category of algebraic varieties with the usual constructions therein. All subsequent definitions could be formulated for abstract algebraic varieties, but will only be developed in the affine or quasi-affine case to keep things simple.

Definition 2.9. Let $X$ and $Y$ be two affine or quasi-affine algebraic varieties $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$. A *regular map* or *(regular) morphism* from $X$ to $Y$ is a map $f : X \to \mathbb{A}^m$ sending $X$ into $Y$ with components rational functions $f_i \in \mathbb{K}(x_1, \ldots, x_n)$ whose denominators do not vanish on $X$. If $X$ and $Y$ are affine varieties, a morphism induces a $\mathbb{K}$-algebra homomorphism $f^* : \mathbb{K}[Y] \to \mathbb{K}[X]$ between the coordinate rings, which, in turn, determines $f$. Over algebraically closed fields, a morphism between affine varieties is the restriction to $X$ of a polynomial map $f : \mathbb{A}^n \to \mathbb{A}^m$ sending $X$ into $Y$, i.e., such that $f^*(I_Y) \subset I_X$, [Har77], Chap. I, Thm. 3.2, p. 17.

Definition 2.10. A *rational map* $f : X \to Y$ between affine varieties is a morphism $f : U \to Y$ on some dense open subset $U$ of $X$. One then says that $f$ is *defined* on $U$. Albeit the terminology, it need not induce a set-theoretic map on whole $X$. For irreducible varieties, a rational map is given by a $\mathbb{K}$-algebra homomorphism $\alpha_f : \mathbb{K}(Y) \to \mathbb{K}(X)$ of the function fields. A *birational map* $f : X \to Y$ is a regular map $f : U \to Y$ on some dense open subset $U$ of $X$ such that $V = f(U)$ is open in $Y$ and such that $f_{|U} : U \to V$ is a regular isomorphism, i.e., admits an inverse morphism. In this case $U$ and $V$ are called *biregularly isomorphic*, and $X$ and $Y$ are *birationally isomorphic*. For irreducible varieties, a birational map is given by an isomorphism $\alpha_f : \mathbb{K}(Y) \to \mathbb{K}(X)$ of the function fields. A *birational morphism* $f : X \to Y$ is a birational map which is defined on whole $X$, i.e., a morphism $f : X \to Y$ which admits a rational inverse map defined on a dense open subset $V$ of $Y$.

Definition 2.11. A morphism $f : X \to Y$ between algebraic varieties is called *separated* if the diagonal $\Delta \subset X \times_Y X$ is closed in the fibre product $X \times_Y X = \{(a, b) \in X \times X,\ f(a) = f(b)\}$. A morphism $f : X \to Y$ between varieties is *proper* if it is separated and universally closed, i.e., if for any variety $Z$ and morphism

$Z \to Y$ the induced morphism $g : X \times_Y Z \to Z$ is closed. Closed immersions and compositions of proper morphisms are proper.

DEFINITION 2.12. The *germ of a variety* $X$ *at a point* $a$, denoted by $(X, a)$, is the equivalence class of open neighbourhoods $U$ of $a$ in $X$ where two neighbourhoods of $a$ are said to be equivalent if they coincide on a (possibly smaller) neighbourhood of $a$. To a germ $(X, a)$ one associates the local ring $\mathcal{O}_{X,a}$ of $X$ at $a$, i.e., the localization $\mathcal{O}_{X,a} = \mathbb{K}[X]_{m_{X,a}}$ of the coordinate ring $\mathbb{K}[X]$ of $X$ at the maximal ideal $m_{X,a}$ of $\mathbb{K}[X]$ defining $a$ in $X$.

DEFINITION 2.13. Let $X$ and $Y$ be two varieties, and let $a$ and $b$ be points of $X$ and $Y$. The *germ of a morphism* $f : (X, a) \to (Y, b)$ at $a$ is the equivalence class of a morphism $\widetilde{f} : U \to Y$ defined on an open neighbourhood $U$ of $a$ in $X$ and sending $a$ to $b$; here, two morphisms defined on neighbourhoods of $a$ in $X$ are said to be equivalent if they coincide on a (possibly smaller) neighbourhood of $a$. The morphism $\widetilde{f} : U \to Y$ is called a *representative* of $f$ on $U$. Equivalently, the germ of a morphism is given by a local $\mathbb{K}$-algebra homomorphism $\alpha_f = f^* : \mathcal{O}_{Y,b} \to \mathcal{O}_{X,a}$.

DEFINITION 2.14. The *tangent space* $\mathrm{T}_a X$ to a variety $X$ at a point $a$ is defined as the $\mathbb{K}$-vector space

$$\mathrm{T}_a X = (m_{X,a}/m_{X,a}^2)^* = \mathrm{Hom}(m_{\mathrm{X,a}}/m_{\mathrm{X,a}}^2, \mathbb{K})$$

with $m_{X,a} \subset \mathbb{K}[X]$ the maximal ideal of $a$. Equivalently, one may take the maximal ideal of the local ring $\mathcal{O}_{X,a}$. The tangent space of $X$ at $a$ thus only depends on the germ of $X$ at $a$. The *tangent map* $\mathrm{T}_a f : \mathrm{T}_a X \to \mathrm{T}_b Y$ of a morphism $f : X \to Y$ sending a point $a \in X$ to $b \in Y$ or of a germ $f : (X, a) \to (Y, b)$ is defined as the linear map induced naturally by $f^* : \mathcal{O}_{Y,b} \to \mathcal{O}_{X,a}$.

DEFINITION 2.15. The *embedding dimension* $\mathrm{embdim_a X}$ of a variety $X$ at a point $a$ is defined as the $\mathbb{K}$-dimension of $\mathrm{T}_a X$.

## *Schemes*

DEFINITION 2.16. An *affine scheme* $X$ is a commutative ring $R$ with 1, called the *coordinate ring* or *ring of global sections* of $X$. The set $\mathrm{Spec}(R)$, also denoted by $X$ and called the *spectrum* of $R$, is defined as the set of prime ideals of $R$. Here, $R$ does not count as a prime ideal, but 0 does if it is prime, i.e., if $R$ is an integral domain. A *point* of $X$ is an element of $\mathrm{Spec}(R)$. In this way, $R$ is the underlying algebraic structure of an affine scheme, whereas $X = \mathrm{Spec}(R)$ is the associated geometric object. To be more precise, one would have to define $X$ as the pair consisting of the coordinate ring $R$ and the spectrum $\mathrm{Spec}(R)$.

The spectrum is equipped with the *Zariski topology*: the closed sets $V(I)$ are formed by the prime ideals containing a given ideal $I$ of $R$. It also comes with a sheaf of rings $\mathcal{O}_X$, the *structure sheaf of* $X$, whose stalks at points of $X$ are the localizations of $R$ at the respective prime ideals [Mum99, Har77, Sha94]. For affine schemes, the sheaf $\mathcal{O}_X$ is completely determined by the ring $R$. In particular, to define affine schemes it is not mandatory to introduce sheaves or locally ringed spaces.

An affine scheme is *of finite type over some field* $\mathbb{K}$ if its coordinate ring $R$ is a finitely generated $\mathbb{K}$-algebra, i.e., a factor ring $\mathbb{K}[x_1, \ldots, x_n]/I$ of a polynomial ring by some ideal $I$. If $I$ is radical, the scheme is called *reduced.* Over algebraically closed fields, affine varieties can be identified with reduced affine schemes of finite type over $\mathbb{K}$ [Har77], Chap. II, Prop. 2.6, p. 78.

Definition 2.17. As a scheme, *affine n-space* over some field $\mathbb{K}$ is defined as the scheme $\mathbb{A}^n = \mathbb{A}^n_{\mathbb{K}}$ given by the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ over $\mathbb{K}$ in $n$ variables. Its underlying topological space $\mathbb{A}^n = \mathrm{Spec}(\mathbb{K}[x_1, \ldots, x_n])$ consists of the prime ideals of $\mathbb{K}[x_1, \ldots, x_n]$. The points of $\mathbb{A}^n$ when considered as a scheme therefore correspond to the irreducible subvarieties of $\mathbb{A}^n$ when considered as an affine variety. A point is *closed* if the respective prime ideal is a maximal ideal. Over algebraically closed fields, the closed points of $\mathbb{A}^n$ as a scheme correspond, by Hilbert's Nullstellensatz, to the points of $\mathbb{A}^n$ as a variety.

Definition 2.18. Let $X$ be an affine scheme of coordinate ring $R$. A *closed subscheme* $Y$ of $X$ is a factor ring $S = R/I$ of $R$ by some ideal $I$ of $R$, together with the canonical homomorphism $R \to R/I$. The set $Y = \mathrm{Spec}(R/I)$ of prime ideals of $R/I$ can be identified with the closed subset $V(I)$ of $X = \mathrm{Spec}(R)$ of prime ideals of $R$ containing $I$. The homomorphism $R \to R/I$ induces an injective continuous map $Y \to X$ between the underlying topological spaces. In this way, the Zariski topology of $Y$ as a scheme coincides with the topology induced by the Zariski topology of $X$.

A *point* $a$ of $X$ is a prime ideal $I$ of $R$, considered as an element of $\mathrm{Spec}(R)$. To a point one may associate a closed subscheme of $X$ via the spectrum of the factor ring $R/I$. The point $a$ is called *closed* if $I$ is a maximal ideal of $R$.

A *principal open subset* of $X$ is an affine scheme $U$ defined by the ring of fractions $R_g$ of $R$ with respect to the multiplicatively closed set $\{1, g, g^2, \ldots\}$ for some non-zero divisor $g \in R$. The natural ring homomorphism $R \to R_g$ sending $h$ to $h/1$ interprets $U = \mathrm{Spec}(R_g)$ as an open subset of $X = \mathrm{Spec}(R)$. Principal open subsets form a basis of the Zariski topology of $X$. Arbitary open subsets of $X$ need not admit an interpretation as affine schemes.

Remark 2.19. Abstract schemes are obtained by gluing affine schemes along principal open subsets [Mum99] II, §1, §2, [Har77] II.2, [Sha94] V, §3. Here, two principal open subsets will be identified or patched together if their respective coordinate rings are isomorphic. This works out properly because the passage from a ring $R$ to its ring of fractions $R_g$ satisfies two key algebraic properties: An element $h$ of $R$ is determined by its images in $R_g$, for all $g$, and, given elements $h_g$ in the rings $R_g$ whose images in $R_{gg'}$ coincide for all $g$ and $g'$, there exists an element $h$ in $R$ with image $h_g$ in $R_g$, for all $g$. This allows in particular to equip arbitrary open subsets of (affine) schemes $X$ with a natural structure of a scheme, which will then be called *open subscheme* of $X$.

The gluing of affine schemes along principal open sets can also be formulated on the sheaf-theoretic level, even though, again, this is not mandatory. For local considerations, one can mostly restrict to the case of affine schemes.

Definition 2.20. A *morphism* $f : X \to Y$ between affine schemes $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ is a (unitary) ring homomorphism $\alpha = \alpha_f : S \to R$. It induces a continuous map $\mathrm{Spec}(R) \to \mathrm{Spec}(S)$ between the underlying topological spaces by sending a prime ideal $I$ of $R$ to the prime ideal $\alpha^{-1}(I)$ of $S$. Morphisms between arbitary schemes are defined by choosing coverings by affine schemes and defining the morphism locally subject to the obvious conditions on the overlaps of the patches. A *rational map* $f : X \to Y$ between schemes is a morphism $f : U \to Y$

defined on a dense open subscheme $U$ of $X$. It need not be defined on whole $X$. A rational map is *birational* if the morphism $f : U \to Y$ admits on a dense open subscheme $V$ of $Y$ an inverse morphism $V \to U$, i.e., if $f_{|U} : U \to V$ is an isomorphism. A *birational morphism* $f : X \to Y$ is a morphism $f : X \to Y$ which is also a birational map, i.e., induces an isomorphism $U \to V$ of dense open subschemes. In contrast to birational maps, a birational morphism is defined on whole $X$, while its inverse map is only defined on a dense open subset of $Y$.

DEFINITION 2.21. Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring. The set $X = \mathrm{Proj}(R)$ of homogeneous prime ideals of $R$ not containing the *irrelevant ideal* $M = \oplus_{i\geq 1}^{\infty} R_i$ is equipped with a topology, the *Zariski topology*, and with a sheaf of rings $\mathcal{O}_X$, the *structure sheaf* of $X$ [Mum99, Har77, Sha94]. It thus becomes a scheme. Typically, $R$ is generated as an $R_0$-algebra by the homogeneous elements $g \in R_1$ of degree 1. An open covering of $X$ is then given by the affine schemes $X_g = \mathrm{Spec}(R_g^\circ)$, where $R_g^\circ$ denotes, for any non-zero divisor $g \in R_1$, the subring of elements of degree 0 in the ring of fractions $R_g$.

REMARK 2.22. A graded ring homomorphism $S \to R$ induces a morphism of schemes $\mathrm{Proj}(R) \to \mathrm{Proj}(S)$.

DEFINITION 2.23. Equip the polynomial ring $\mathbb{K}[x_0, \ldots, x_n]$ with the natural grading given by the degree. The scheme $\mathrm{Proj}(\mathbb{K}[x_0, \ldots, x_n])$ is called $n$-dimensional *projective space* over $\mathbb{K}$, denoted by $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{K}}$. It is is covered by the affine open subschemes $U_i \simeq \mathbb{A}^n$ which are defined through the principal open sets associated to the rings of fractions $\mathbb{K}[x_0, \ldots, x_n]_{x_i}$ $(i = 0, \ldots, n)$.

DEFINITION 2.24. A morphism $f : X \to Y$ is called *projective* if it factors, for some $k$, into a closed embedding $X \hookrightarrow Y \times \mathbb{P}^k$ followed by the projection $Y \times \mathbb{P}^k \to Y$ onto the first factor [Har77] II.4, p. 103.

## *Formal germs*

DEFINITION 2.25. Let $R$ be a ring and $I$ an ideal of $R$. The powers $I^k$ of $I$ induce natural homomorphisms $R/I^{k+1} \to R/I^k$. The *$I$-adic completion* of $R$ is the inverse limit $\widehat{R} = \varprojlim R/I^k$, together with the canonical homomorphism $R \to \widehat{R}$. If $R$ is a local ring with maximal ideal $m$, the $m$-adic completion $\widehat{R}$ is called the *completion* of $R$. For an $R$-module $M$, one defines the *$I$-adic completion* $\widehat{M}$ of $M$ as the inverse limit $\widehat{M} = \varprojlim M/I^k \cdot M$.

LEMMA 2.26. Let $R$ be a noetherian ring with prime ideal $I$ and $I$-adic completion $\widehat{R}$.

(1) If $J$ is another ideal of $R$ with $(I \cap J)$-adic completion $\widehat{J}$, then $\widehat{J} = J \cdot \widehat{R}$ and $\widehat{R/J} \simeq \widehat{R}/\widehat{J}$.
(2) If $J_1$, $J_2$ are two ideals of $R$ and $J = J_1 \cdot J_2$, then $\widehat{J} = \widehat{J_1} \cdot \widehat{J_2}$.
(3) If $R$ is a local ring with maximal ideal $m$, the completion $\widehat{R}$ is a noetherian local ring with maximal ideal $\widehat{m} = m \cdot \widehat{R}$, and $\widehat{m} \cap R = m$.
(4) If $J$ is an arbitrary ideal of a local ring $R$, then $\widehat{J} \cap R = \bigcap_{i\geq 0}(J + m^i)$.
(5) Passing to the $I$-adic completion defines an exact functor on finitely generated $R$-modules.
(6) $\widehat{R}$ is a faithfully flat $R$-algebra.

(7) If $M$ a finitely generated $R$-module, then $\widehat{M} = M \otimes_R \widehat{R}$.

PROOF. (1) By [ZS75] VIII, Thm. 6, Cor. 2, p. 258, it suffices to verify that $J$ is closed in the $(I \cap J)$-adic topology. The closure $\overline{J}$ of $J$ equals $\overline{J} = \bigcap_{i \geq 0}(J + (I \cap J)^i) = J$ by [ZS75] VIII, Lemma 1, p. 253.

(2) By definition, $\widehat{J} = J_1 \cdot J_2 \cdot \widehat{R} = J_1 \cdot \widehat{R} \cdot J_2 \cdot \widehat{R} = \widehat{J}_1 \cdot \widehat{J}_2$.

(3), (4) The ring $\widehat{R}$ is noetherian and local by [AM69] Prop. 10.26, p. 113, and Prop. 10.16, p. 109. The ideal $\widehat{I} \cap R$ equals the closure $\overline{I}$ of $I$ in the $m$-adic topology. Thus, $\widehat{I} \cap R = \bigcap_{i \geq 0}(I + m^i)$ and $\widehat{m} \cap R = m$.

(5) – (7) [AM69] Prop. 10.12, p. 108, Prop. 10.14, p. 109, Prop. 10.14, p. 109, and [Mat89], Thm. 8.14, p. 62. □

DEFINITION 2.27. Let $X$ be a variety and let $a$ be a point of $X$. The local ring $\mathcal{O}_{X,a}$ of $X$ at $a$ is equipped with the $m_{X,a}$-adic topology whose basis of neighbourhoods of 0 is given by the powers $m_{X,a}^k$ of the maximal ideal $m_{X,a}$ of $\mathcal{O}_{X,a}$. The induced completion $\widehat{\mathcal{O}}_{X,a}$ of $\mathcal{O}_{X,a}$ is called the *complete local ring of* $X$ *at* $a$ [Nag75] II, [ZS75] VIII, §1, §2. The scheme defined by $\widehat{\mathcal{O}}_{X,a}$ is called the *formal neighbourhood* or *formal germ* of $X$ at $a$, denoted by $(\widehat{X}, a)$. If $X = \mathbb{A}^n_{\mathbb{K}}$ is the affine $n$-space over $\mathbb{K}$, the complete local ring $\widehat{\mathcal{O}}_{\mathbb{A}^n,a}$ is isomorphic as a local $\mathbb{K}$-algebra to the formal power series ring $\mathbb{K}[[x_1, \ldots, x_n]]$ in $n$ variables over $\mathbb{K}$. The natural ring homomorphism $\mathcal{O}_{X,a} \to \widehat{\mathcal{O}}_{X,a}$ defines a morphism $(\widehat{X}, a) \to (X, a)$ in the category of schemes from the formal neighbourhood to the germ of $X$ at $a$. A *formal subvariety* $(\widehat{Y}, a)$ of $(\widehat{X}, a)$, also called a *formal local subvariety* of $X$ at $a$, is (the scheme defined by) a factor ring $\widehat{\mathcal{O}}_{X,a}/I$ for an ideal $I$ of $\widehat{\mathcal{O}}_{X,a}$.

As $m_{X,a}/m_{X,a}^2 \simeq \widehat{m}_{X,a}/\widehat{m}_{X,a}^2$, one defines the tangent space $\mathrm{T}_a(\widehat{X}, a)$ of the formal germ as the tangent space $\mathrm{T}_a(X)$ of the variety at $a$.

A map $f : (\widehat{X}, a) \to (\widehat{Y}, b)$ between two formal germs is defined as a local algebra homomorphism $\alpha_f : \widehat{\mathcal{O}}_{Y,b} \to \widehat{\mathcal{O}}_{X,a}$. It is also called a *formal map* between $X$ and $Y$ at $a$. It induces in a natural way a linear map, the *tangent map*, $\mathrm{T}_a f : \mathrm{T}_a X \to \mathrm{T}_b Y$ between the tangent spaces.

Two varieties $X$ and $Y$ are *formally isomorphic* at points $a \in X$, respectively $b \in Y$, if the complete local rings $\widehat{\mathcal{O}}_{X,a}$ and $\widehat{\mathcal{O}}_{Y,b}$ are isomorphic. If $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^n$ are subvarieties of the same affine space $\mathbb{A}^n$ over $\mathbb{K}$, and $a = b = 0$, this is equivalent to saying that there is a local algebra automorphism of $\mathcal{O}_{\mathbb{A}^n,0} \simeq \mathbb{K}[[x_1, \ldots, x_n]]$ sending the completed ideals $\widehat{I}$ and $\widehat{J}$ of $X$ and $Y$ onto each other. Such an automorphism is also called a *formal coordinate change* of $\mathbb{A}^n$ at 0. It is given by a vector of $n$ formal power series without constant term whose Jacobian matrix of partial derivatives is invertible when evaluated at 0.

REMARK 2.28. The inverse function theorem does not hold for algebraic varieties and regular maps between them, but it holds in the category of formal germs and formal maps: A formal map $f : (\widehat{X}, a) \to (\widehat{Y}, b)$ is an isomorphism if and only if its tangent map $\mathrm{T}_a f : \mathrm{T}_a X \to \mathrm{T}_b Y$ is a linear isomorphism.

DEFINITION 2.29. A morphism $f : X \to Y$ between varieties is called *étale* if for all $a \in X$ the induced maps of formal germs $f : (\widehat{X}, a) \to (\widehat{Y}, f(a))$ are isomorphisms, or, equivalently, if all tangent maps $\mathrm{T}_a f$ of $f$ are isomorphisms, [Har77], Chap. III, Ex. 10.3 and 10.4, p. 275.

Definition 2.30. A morphism $f : X \to Y$ between varieties is called *smooth* if for all $a \in X$ the induced maps of formal germs $f : (\widehat{X}, a) \to (\widehat{Y}, f(a))$ are submersions, i.e., if the tangent maps $\mathrm{T}_a f$ of $f$ are surjective, [Har77], Chap. III, Prop. 10.4, p. 270.

Remark 2.31. The category of formal germs and formal maps admits the usual concepts and constructions as e.g. the decomposition of a variety in irreducible components, intersections of germs, inverse images, or fibre products. Similarly, when working of $\mathbb{R}$, $\mathbb{C}$ or any complete valued field $\mathbb{K}$, one can develop, based on rings of convergent power series, the category of analytic varieties and analytic spaces, respectively of their germs, and analytic maps between them [dJP00].

## *Examples*

Example 2.32. Compare the algebraic varieties $X$ satisfying $(X, a) \simeq (\mathbb{A}^d, 0)$ for all points $a \in X$ (where $\simeq$ stands for biregularly isomorphic germs) with those where the isomorphism is just formal, $(\widehat{X}, a) \simeq (\widehat{\mathbb{A}}^d, 0)$. Varieties with the first property are called *plain*, cf. Def. 3.11.

Example 2.33. It can be shown that any complex regular (Def. 3.4) and rational surface (i.e., a surface which is birationally isomorphic to the affine plane $\mathbb{A}^2$ over $\mathbb{C}$) is plain, [BHSV08] Prop. 3.2. Show directly that $X$ defined in $\mathbb{A}^3$ by $x - (x^2 + z^2)y = 0$ is plain by exhibiting a local isomorphism (i.e., isomorphism of germs) of $X$ at 0 with $\mathbb{A}^2$ at 0. Is there an algorithm to construct such a local isomorphism for any complex regular and rational surface?

Example 2.34.$^\triangleright$ Let $X$ be an algebraic variety and $a \in X$ be a point. What is the difference between the concept of regular system of parameters (Def. 3.2) in $\mathcal{O}_{X,a}$ and $\widehat{\mathcal{O}}_{X,a}$?

Example 2.35. The formal neighbourhood $(\widehat{\mathbb{A}}^n, a)$ of affine space $\mathbb{A}^n$ at a point $a = (a_1, \ldots, a_n)$ is given by the formal power series ring $\widehat{\mathcal{O}}_{\mathbb{A}^n,a} \simeq \mathbb{K}[[x_1 - a_1, \ldots, x_n - a_n]]$. If $X \subset \mathbb{A}^n$ is an affine algebraic variety defined by the ideal $I$ of $\mathbb{K}[x_1, \ldots, x_n]$, the formal neighbourhood $(\widehat{X}, a)$ is given by the factor ring $\widehat{\mathcal{O}}_{X,a} = \widehat{\mathcal{O}}_{\mathbb{A}^n,a}/\widehat{I} \simeq \mathbb{K}[[x_1 - a_1, \ldots, x_n - a_n]]/\widehat{I}$, where $\widehat{I} = I \cdot \widehat{\mathcal{O}}_{\mathbb{A}^n,a}$ denotes the extension of $I$ to $\widehat{\mathcal{O}}_{\mathbb{A}^n,a}$.

Example 2.36. The map $f : \mathbb{A}^1 \to \mathbb{A}^2$ given by $t \mapsto (t^2, t^3)$ is a regular morphism which induces a birational isomorphism onto the curve $X$ in $\mathbb{A}^2$ defined by $x^3 = y^2$. The inverse $(x, y) \mapsto y/x$ is a rational map on $X$ and regular on $X \setminus \{0\}$.

Example 2.37. The map $f : \mathbb{A}^1 \to \mathbb{A}^2$ given by $t \mapsto (t^2 - 1, t(t^2 - 1))$ is a regular morphism which induces a birational isomorphism onto the curve $X$ in $\mathbb{A}^2$ defined by $x^2 + x^3 = y^2$. The inverse $(x, y) \mapsto y/x$ is a rational map on $X$ and regular on $X \setminus \{0\}$. The germ $(X, 0)$ of $X$ at 0 is not isomorphic to the germ $(Y, 0)$ of the union $Y$ of the two diagonals in $\mathbb{A}^2$ defined by $x^2 = y^2$. The formal germs $(\widehat{X}, 0)$ and $(\widehat{Y}, 0)$ are isomorphic via the map $(x, y) \mapsto (x\sqrt{1 + x}, y)$.

Example 2.38.$^\triangleright$ The map $f : \mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (xy, y)$ is a birational morphism with inverse the rational map $(x, y) \mapsto (\frac{x}{y}, y)$. The inverse defines a regular morphism outside the $x$-axis.

EXAMPLE 2.39. The maps $\varphi_{ij} : \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ given by

$$(x_0, \dots, x_n) \mapsto (\frac{x_0}{x_j}, \dots, \frac{x_{i-1}}{x_j}, 1, \frac{x_{i+1}}{x_j}, \dots, \frac{x_n}{x_j}).$$

are birational maps for each $i, j = 0, \dots, n$. They are the transition maps between the affine charts $U_j = \mathbb{P}^n \setminus V(x_j) \simeq \mathbb{A}^n$ of projective space $\mathbb{P}^n$.

EXAMPLE 2.40. $^\triangleright$ The maps $\varphi_{ij} : \mathbb{A}^n \to \mathbb{A}^n$ given by

$$(x_1, \dots, x_n) \mapsto \Big(\frac{x_1}{x_j}, \dots, \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_{i+1}}{x_j}, \dots, \frac{x_{j-1}}{x_j}, x_i x_j, \frac{x_{j+1}}{x_j}, \dots, \frac{x_n}{x_j}\Big)$$

are birational maps for $i, j = 1, \dots, n$. They are the transition maps between the affine charts of the blowup $\widetilde{\mathbb{A}}^n \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ of $\mathbb{A}^n$ at the origin (cf. Lecture IV) .

EXAMPLE 2.41. $^\triangleright$ Assume that the characteristic of the ground field is different from 2. The elliptic curve $X$ defined in $\mathbb{A}^2$ by $y^2 = x^3 - x$ is formally isomorphic at each point $a$ of $X$ to $(\widehat{\mathbb{A}}^1, 0)$, whereas the germs $(X, a)$ are not biregularly isomorphic to $(\mathbb{A}^1, 0)$.

EXAMPLE 2.42. $^\triangleright$ The projection $(x, y) \mapsto x$ from the hyperbola $X$ defined in $\mathbb{A}^2$ by $xy = 1$ to the $x$-axis $\mathbb{A}^1 = \mathbb{A}^1 \times \{0\} \subset \mathbb{A}^2$ has open image $\mathbb{A}^1 \setminus \{0\}$. In particular, it is not proper.

EXAMPLE 2.43. $^\triangleright$ The curves $X$ and $Y$ defined in $\mathbb{A}^2$ by $x^3 = y^2$, respectively $x^5 = y^2$ are not formally isomorphic to each other at 0, whereas the curve $Z$ defined in $\mathbb{A}^2$ by $x^3 + x^5 = y^2$ is formally isomorphic to $X$ at 0.

## 3. Lecture III: Singularities

Let $X$ be an affine algebraic variety defined over a field $\mathbb{K}$. Analog concepts to the ones given below can be defined for abstract varieties and schemes.

DEFINITION 3.1. A noetherian local ring $R$ with maximal ideal $m$ is called a *regular ring* if $m$ can be generated by $n$ elements, where $n$ is the Krull dimension of $R$. The number $n$ is then given as the vector space dimension of $m/m^2$ over the residue field $R/m$. A noetherian local ring $R$ is regular if and only if its completion $\widehat{R}$ is regular, [AM69] p. 123.

DEFINITION 3.2. Let $R$ be a noetherian regular local ring with maximal ideal $m$. A minimal set of generators $x_1, \dots, x_n$ of $m$ is called a *regular system of parameters* or *local coordinate system* of $R$.

REMARK 3.3. A regular system of parameters of a local ring $R$ is also a regular system of parameters of its completion $\widehat{R}$, but not conversely, cf. Ex. 3.27.

DEFINITION 3.4. A point $a$ of $X$ is a *regular* or *non-singular point* of $X$ if the local ring $\mathcal{O}_{X,a}$ of $X$ at $a$ is a regular ring. Equivalently, the $m_{X,a}$-adic completion $\widehat{\mathcal{O}}_{X,a}$ of $\mathcal{O}_{X,a}$ is isomorphic to the completion $\widehat{\mathcal{O}}_{\mathbb{A}^d,0} \simeq \mathbb{K}[[x_1, \dots, x_d]]$ of the local ring of some affine space $\mathbb{A}^d$ at 0. Otherwise $a$ is called a *singular point* or a *singularity* of $X$. The set of singular points of $X$ is denoted by $\mathrm{Sing}(X)$, its complement by $\mathrm{Reg}(X)$. The variety $X$ is *regular* or *non-singular* if all its points are regular. Over perfect fields regular varieties are the same as *smooth* varieties, [Cut04, Liu02].

Proposition 3.5. A subvariety $X$ of a regular variety $W$ is regular at $a$ if and only if there are local coordinates $x_1, \ldots, x_n$ of $W$ at $a$ so that $X$ can be defined locally at $a$ by $x_1 = \cdots = x_k = 0$ where $k$ is the codimension of $X$ in $W$ at $a$.

Proof. [dJP00], Cor. 4.3.20, p. 155. □

Remark 3.6. Affine and projective space are regular at each of their points. The germ of a variety at a regular point need not be biregularly isomorphic to the germ of an affine space at 0. However, it is formally isomorphic, i.e., after passage to the formal germ, cf. Ex. 3.28.

Proposition 3.7. Assume that the field $\mathbb{K}$ is perfect. Let $X$ be a hypersurface defined in a regular variety $W$ by the square-free equation $f = 0$. The point $a \in X$ is singular if and only if all partial derivatives $\partial_{x_1} f, \ldots, \partial_{x_n} f$ of vanish at $a$.

Proof. [Zar47], Thm. 7, [Har77] I.5, [dJP00] 4.3. □

Definition 3.8. The characterization of singularities by the vanishing of the partial derivatives as in Prop. 3.7 is known as the *Jacobian criterion for smoothness*.

Remark 3.9. A similar statement holds for irreducible varieties which are not hypersurfaces, using instead of the partial derivatives the $k \times k$-minors of the Jacobian matrix of a system of equations of $X$ in $W$, with $k$ the codimension of $X$ in $W$ at $a$ [dJP00], [Cut04] pp. 7-8, [Liu02] pp. 128 and 142. The choice of the system of defining polynomials of the variety is significant: The ideal generated by them has to be radical, otherwise the concept of singularity has to be developed scheme-theoretically, allowing non-reduced schemes. Over non-perfect fields, the criterion of the proposition does not hold [Zar47].

Corollary 3.10. The singular locus $\mathrm{Sing}(X)$ of $X$ is closed in $X$.

Definition 3.11. A point $a$ of $X$ is a *plain point* of $X$ if the local ring $\mathcal{O}_{X,a}$ is isomorphic to the local ring $\mathcal{O}_{\mathbb{A}^d,0} \simeq \mathbb{K}[x_1, \ldots, x_d]_{(x_1,\ldots,x_d)}$ of some affine space $\mathbb{A}^d$ at 0 (with $d$ the dimension of $X$ at $a$). Equivalently, there exists an open neighbourhood $U$ of $a$ in $X$ which is biregularly isomorphic to an open subset $V$ of some affine space $\mathbb{A}^d$. The variety $X$ is *plain* if all its points are plain.

Theorem 3.12 (Bodnár-Hauser-Schicho-Villamayor). The blowup of a plain variety defined over an infinite field along a regular center is again plain.

Proof. [BHSV08], Thm. 4.3. □

Definition 3.13. An irreducible variety $X$ is *rational* if it has a dense open subset $U$ which is biregularly isomorphic to a dense open subset $V$ of some affine space $\mathbb{A}^d$. Equivalently, the function field $\mathbb{K}(X) = \mathrm{Quot}(\mathbb{K}[X])$ is isomorphic to the field of rational functions $K(x_1, \ldots, x_d)$ of $\mathbb{A}^d$.

Remark 3.14. A plain complex variety is smooth and rational. The converse is true for curves and surfaces, and unknown in arbitrary dimension [BHSV08].

Definition 3.15. A point $a$ is a *normal crossings point* of $X$ if the formal neighbourhood $(\widehat{X}, a)$ is isomorphic to the formal neighbourhood $(\widehat{Y}, 0)$ of a union $Y$ of coordinate subspaces of $\mathbb{A}^n$ at 0. The point $a$ is a *simple normal crossings point* of $X$ if it is a normal crossings point and all components of $X$ passing through $a$ are regular at $a$. The variety $X$ has *normal crossings*, respectively *simple normal crossings*, if the property holds at all of its points.

REMARK 3.16. In the case of schemes, a normal crossings scheme may be non-reduced in which case the components of the scheme $Y$ are equipped with multiplicities. Equivalently, $Y$ is defined locally at 0 in $\mathbb{A}^n$ up to a formal isomorphism by a monomial ideal of $\mathbb{K}[x_1, \ldots, x_n]$.

PROPOSITION 3.17. A point $a$ is a normal crossings point of a subvariety $X$ of a regular ambient variety $W$ if and only if there exists a regular system of parameters $x_1, \ldots, x_n$ of $\mathcal{O}_{W,a}$ so that the germ $(X, a)$ is defined in $(W, a)$ by a radical monomial ideal in $x_1, \ldots, x_n$. This is equivalent to saying that each component of $(X, a)$ is defined by a subset of the coordinates.

REMARK 3.18. In the case of schemes, the monomial ideal need not be radical.

DEFINITION 3.19. Two subvarieties $X$ and $Y$ of a regular ambient variety $W$ meet *transversally* at a point $a$ of $W$ if they are regular at $a$ and if the union $X \cup Y$ has normal crossings at $a$.

REMARK 3.20. The definition differs from the corresponding notion in differential geometry, where it is required that the tangent spaces of the two varieties at intersection points sum up to the tangent space of the ambient variety at the respective point. In the present text, an inclusion $Y \subset X$ of two regular varieties is considered as being transversal, and also any two coordinate subspaces in $\mathbb{A}^n$ meet transversally. In the case of schemes, the union $X \cup Y$ has to be defined by the product of the defining ideals, not their intersection.

DEFINITION 3.21. A variety $X$ is a *cartesian product*, if there exist positive dimensional varieties $Y$ and $Z$ such that $X$ is biregularly isomorphic to $Y \times Z$. Analogous definitions hold for germs $(X, a)$ and formal germs $(\widehat{X}, a)$ (the cartesian product of formal germs has to be taken in the category of complete local rings).

DEFINITION 3.22. A variety $X$ is *(formally) a cylinder over a subvariety $Z \subset X$* at a point $a$ of $Z$ if the formal neighborhood $(\widehat{X}, a)$ is isomorphic to a cartesian product $(\widehat{Y}, a) \times (\widehat{Z}, a)$ for some (positive-dimensional) subvariety $Y$ of $X$ which is regular at $a$. One also says that $X$ is *trivial* or a *cylinder along $Y$ at $a$ with transversal section $Z$*.

DEFINITION 3.23. Let $X$ and $F$ be varieties, and let 0 be a distinguished point on $F$. The set $S$ of points $a$ of $X$ where the formal neighborhood $(\widehat{X}, a)$ is isomorphic to $(\widehat{F}, 0)$ is called the *triviality locus of $X$ of singularity type* $(\widehat{F}, 0)$.

THEOREM 3.24 (Ephraim, Hauser-Müller). For any complex variety $F$ and point 0 on $F$, the triviality locus $S$ of $X$ of singularity type $(\widehat{F}, 0)$ is locally closed and regular in $X$, and $X$ is a cylinder along $S$. Any subvariety $Z$ of $X$ such that $(\widehat{X}, a) \simeq (\widehat{S}, a) \times (\widehat{Z}, a)$ is unique up to formal isomorphism at $a$.

PROOF. [Eph78], Thm. 2.1, [HM89], Thm. 1. □

COROLLARY 3.25. The singular locus $\mathrm{Sing}(X)$ and the non-normal crossings locus of a variety $X$ are closed subvarieties.

PROOF. [Mum99], III.4, Prop. 3, p. 170, [Bod04]. □

THEOREM 3.26 (Hauser-Müller). Let $X$, $Y$ and $Z$ be complex varieties with points $a$, $b$ and $c$ on them, respectively. The formal germs of $X \times Z$ at $(a, c)$ and $Y \times Z$ at $(b, c)$ are isomorphic if and only if $(\widehat{X}, a)$ and $(\widehat{Y}, b)$ are isomorphic.

Proof. [HM90], Thm. 1. □

*Examples*

Example 3.27.▷ The element $x\sqrt{1+x}$ is a regular parameter of the completion $\mathbb{K}[[x]]$ of the local ring $K[x]_{(x)}$ which does not stem from a regular parameter of $K[x]_{(x)}$.

Example 3.28.▷ The elements $y^2 - x^3 - x$ and $y$ form a regular parameter system of $\mathcal{O}_{\mathbb{A}^2,0}$ but the zeroset $X$ of $y^2 = x^3 - x$ is not locally at 0 isomorphic to $\mathbb{A}^1$. However, $(\widehat{X}, 0)$ is formally isomorphic to $(\widehat{\mathbb{A}}^1, 0)$.

Example 3.29. The set of plain points of a variety $X$ is Zariski open (cf. Def. 3.11).

Example 3.30. Let $X$ be the plane cubic curve defined by $x^3 + x^2 - y^2 = 0$ in $\mathbb{A}^2$. The origin $0 \in X$ is a normal crossings point, but $X$ is not locally at 0 biregularly isomorphic to the union of the two diagonals of $\mathbb{A}^2$ defined by $x = \pm y$.

Example 3.31.▷ Let $X$ be the complex surface in $\mathbb{A}^3_{\mathbb{C}}$ defined by $x^2 - y^2 z = 0$, the *Whitney umbrella* or *pinch point singularity.* The singular locus is the $z$-axis. The origin is not a normal crossings point of $X$. For $a \neq 0$ a point on the $z$-axis, $a$ is a normal crossings point but not a simple normal crossings point of $X$. The formal neighbourhoods of $X$ at points $a \neq 0$ on the $z$-axis are isomorphic to each other, since they are isomorphic to the formal neighbourhood at 0 of the union of two transversal planes in $\mathbb{A}^3$.

Example 3.32. Prove that the non-normal crossings locus of a variety is closed. Try to find equations for it [Bod04].

Example 3.33.+ Find a local invariant that measures reasonably the distance of a point $a$ of $X$ from being a normal crossings point.

Example 3.34.▷ Do the varieties defined by the following equations have normal crossings, respectively simple normal crossings, at the origin? Vary the ground field.

(a) $x^2 + y^2 = 0$,
(b) $x^2 - y^2 = 0$,
(c) $x^2 + y^2 + z^2 = 0$,
(d) $x^2 + y^2 + z^2 + w^2 = 0$,
(e) $xy(x - y) = 0$,
(f) $xy(x^2 - y) = 0$,
(g) $(x - y)z(z - x) = 0$.

Example 3.35.▷ Visualize the zeroset of $(x - y^2)(x - z)z = 0$ in $\mathbb{A}^3_{\mathbb{R}}$.

Example 3.36.+ Formulate and then prove the theorem of local analytic triviality in positive characteristic, cf. Thm. 1 of [HM89].

Example 3.37. Find a coordinate free description of normal crossings singularities. Find an algorithm that tests for normal crossings [Fab11, Fab12].

Example 3.38.▷ The singular locus of a cartesian product $X \times Y$ is the union of $\mathrm{Sing}(X) \times Y$ and $X \times \mathrm{Sing}(Y)$.

Example 3.39.$^\triangleright$ Call a finite union $X = \bigcup X_i$ of regular varieties *mikado* if all possible intersections of the components $X_i$ are (scheme-theoretically) regular. The intersections are defined by the sums of the ideals, not taking their radical. Find the simplest example of a variety which is not mikado but for which all pairwise intersections are non-singular.

Example 3.40. Interpret the family given by taking the germs of a variety $X$ at varying points $a \in X$ as the germs of the fibers of a morphism of varieties, equipped with a section.

## 4. Lecture IV: Blowups

Blowups, also known as monoidal transformations, can be introduced in several ways. The respective equivalences will be proven in the second half of this section. All varietes are reduced but not necessarily irreducible, and subvarieties are closed if not mentioned differently. Schemes will be noetherian but not necessarily of finite type over a field. To ease the exposition they are often assumed to be affine, i.e., of the form $X = \mathrm{Spec}(R)$ for some ring $R$. Points of varieties are closed, points of schemes can also be non-closed. The coordinate ring of an affine variety $X$ is denoted by $\mathbb{K}[X]$ and the structure sheaf of a scheme $X$ by $\mathcal{O}_X$, with local rings $\mathcal{O}_{X,a}$ at points $a \in X$.

References providing additional material on blowups are, among many others, [Hir64], Chap. III, and [EH00], Chap. IV.2.

Definition 4.1. A subvariety $Z$ of a variety $X$ is called a *hypersurface in $X$* if the codimension of $Z$ in $X$ at any point $a$ of $Z$ is 1,

$$\dim_a Z = \dim_a X - 1.$$

Remark 4.2. In the case where $X$ is non-singular and irreducible, a hypersurface is locally defined at any point $a$ by a single non-trivial equation, i.e., an equation given by a non-zero and non-invertible element $h$ of $\mathcal{O}_{X,a}$. This need not be the case for singular varieties, see Ex. 4.21. Hypersurfaces are a particular case of *effective Weil divisors* [Har77] Chap. II, Rmk. 6.17.1, p. 145.

Definition 4.3. A subvariety $Z$ of an irreducible variety $X$ is called a *Cartier divisor* in $X$ at a point $a \in Z$ if $Z$ can be defined locally at $a$ by a single equation $h = 0$ for some non-zero element $h \in \mathcal{O}_{X,a}$. If $X$ is not assumed to be irreducible, $h$ is required to be a non-zero divisor of $\mathcal{O}_{X,a}$. This excludes the possibility that $Z$ is a component or a union of components of $X$. The subvariety $Z$ is called a *Cartier divisor* in $X$ if it is a Cartier divisor at each of its points. The empty subvariety is considered as a Cartier divisor. A (non-empty) Cartier divisor is a hypersurface in $X$, but not conversely, and its complement $X \setminus Z$ is dense in $X$. Cartier divisors are, in a certain sense, the largest closed and properly contained subvarieties of $X$. If $Z$ is Cartier in $X$, the ideal $I$ defining $Z$ in $X$ is called *locally principal* [Har77] Chap. II, Prop. 6.13, p. 144, [EH00] III.2.5, p. 117.

Definition 4.4. (Blowup via universal property) Let $Z$ be a (closed) subvariety of a variety $X$. A variety $\widetilde{X}$ together with a morphism $\pi : \widetilde{X} \to X$ is called a *blowup of $X$ with center $Z$ of $X$*, or a *blowup of $X$ along $Z$*, if the inverse image $E = \pi^{-1}(Z)$ of $Z$ is a Cartier divisor in $\widetilde{X}$ and $\pi$ is universal with respect to this property: For any morphism $\tau : X' \to X$ such that $\tau^{-1}(Z)$ is a Cartier divisor in $X'$, there exists a unique morphism $\sigma : X' \to \widetilde{X}$ so that $\tau$ factors through $\sigma$, say $\tau = \pi \circ \sigma$,

$$\begin{array}{ccc} X' & \overset{\exists_1 \sigma}{\dashrightarrow} & \widetilde{X} \\ & \searrow^{\tau} & \downarrow \pi \\ & & X \end{array}$$

The morphism $\pi$ is also called the *blowup map*. The subvariety $E$ of $\widetilde{X}$ is a Cartier divisor, in particular a hypersurface, and called the *exceptional divisor* or *exceptional locus* of the blowup. One says that $\pi$ *contracts* $E$ *to* $Z$.

Remark 4.5. By the universal property, a blowup of $X$ along $Z$, if it exists, is unique up to unique isomorphism. It is therefore called *the* blowup of $X$ along $Z$. If $Z$ is already a Cartier divisor in $X$, then $\widetilde{X} = X$ and $\pi$ is the identity by the universal property. In particular, this is the case when $X$ is non-singular and $Z$ is a hypersurface in $X$.

Definition 4.6. The *Rees algebra* of an ideal $I$ of a commutative ring $R$ is the graded $R$-algebra

$$\mathrm{Rees}(I) = \bigoplus_{i=0}^{\infty} I^i = \bigoplus_{i=0}^{\infty} I^i \cdot t^i \subset R[t],$$

where $I^i$ denotes the $i$-fold power of $I$, with $I^0$ set equal to $R$. The variable $t$ is given degree 1 so that the elements of $I^i \cdot t^i$ have degree $i$. Write $\widetilde{R}$ for Rees$(I)$ when $I$ is clear from the context. The Rees algebra of $I$ is generated by elements of degree 1, and $R$ embeds naturally into $\widetilde{R}$ by sending an element $g$ of $R$ to the degree 0 element $g \cdot t^0$. If $I$ is finitely generated by elements $g_1, \ldots, g_k \in R$, then $\widetilde{R} = R[g_1 t, \ldots, g_k t]$. The Rees algebras of the zero-ideal $I = 0$ and of the whole ring $I = R$ equal $R$, respectively $R[t]$. If $I$ is a principal ideal generated by a non-zero divisor of $R$, the schemes Proj$(\widetilde{R})$ and Spec$(R)$ are isomorphic. The Rees algebras of an ideal $I$ and its $k$-th power $I^k$ are isomorphic as graded $R$-algebras, for any $k \geq 1$.

Definition 4.7 (Blowup via Rees algebra). Let $X = \mathrm{Spec}(R)$ be an affine scheme and let $Z = \mathrm{Spec}(R/I)$ be a closed subscheme of $X$ defined by an ideal $I$ of $R$. Denote by $\widetilde{R} = \mathrm{Rees}(I)$ the Rees algebra of $I$ over $R$, equipped with the induced grading. The *blowup of* $X$ *along* $Z$ is the scheme $\widetilde{X} = \mathrm{Proj}(\widetilde{R})$ together with the morphism $\pi : \widetilde{X} \to X$ given by the natural ring homomorphism $R \to \widetilde{R}$. The subscheme $E = \pi^{-1}(Z)$ of $\widetilde{X}$ is called the *exceptional divisor* of the blowup.

Definition 4.8 (Blowup via secants). Let $W = \mathbb{A}^n$ be affine space over $\mathbb{K}$ (taken as a variety) and let $p$ be a fixed point of $\mathbb{A}^n$. Equip $\mathbb{A}^n$ with a vector space structure by identifying it with its tangent space $\mathrm{T}_p\mathbb{A}^n$. For a point $a \in \mathbb{A}^n$ different from $p$, denote by $g(a)$ the secant line in $\mathbb{A}^n$ through $p$ and $a$, considered as an element of projective space $\mathbb{P}^{n-1} = \mathbb{P}(\mathrm{T}_p\mathbb{A}^n)$. The morphism

$$\gamma : \mathbb{A}^n \setminus \{p\} \to \mathbb{P}^{n-1}, a \mapsto g(a),$$

is well defined. The Zariski closure $\widetilde{X}$ of the graph $\Gamma$ of $\gamma$ inside $\mathbb{A}^n \times \mathbb{P}^{n-1}$ together with the restriction $\pi : \widetilde{X} \to X$ of the projection map $\mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$ is the *point blowup of* $\mathbb{A}^n$ *with center* $p$.

Definition 4.9 (Blowup via closure of graph). Let $X$ be an affine variety with coordinate ring $\mathbb{K}[X]$ and let $Z = \mathrm{V}(I)$ be a subvariety of $X$ defined by an ideal $I$ of $\mathbb{K}[X]$ generated by elements $g_1, \dots, g_k$. The morphism

$$\gamma : X \setminus Z \to \mathbb{P}^{k-1}, a \mapsto (g_1(a) : \cdots : g_k(a)),$$

is well defined. The Zariski closure $\widetilde{X}$ of the graph $\Gamma$ of $\gamma$ inside $X \times \mathbb{P}^{k-1}$ together with the restriction $\pi : \widetilde{X} \to X$ of the projection map $X \times \mathbb{P}^{k-1} \to X$ is the *blowup of $X$ along $Z$*. It does not depend, up to isomorphism over $X$, on the choice of the generators $g_i$ of $I$.

Definition 4.10 (Blowup via equations). Let $X$ be an affine variety with coordinate ring $\mathbb{K}[X]$ and let $Z = \mathrm{V}(I)$ be a subvariety of $X$ defined by an ideal $I$ of $\mathbb{K}[X]$ generated by elements $g_1, \dots, g_k$. Assume that $g_1, \dots, g_k$ form a regular sequence in $\mathbb{K}[X]$. Let $(u_1 : \dots : u_k)$ be projective coordinates on $\mathbb{P}^{k-1}$. The subvariety $\widetilde{X}$ of $X \times \mathbb{P}^{k-1}$ defined by the equations

$$u_i \cdot g_j - u_j \cdot g_i = 0, \qquad i, j = 1, \dots, k,$$

together with the restriction $\pi : \widetilde{X} \to X$ of the projection map $X \times \mathbb{P}^{k-1} \to X$ is the *blowup of $X$ along $Z$*. It does not depend, up to isomorphism over $X$, on the choice of the generators $g_i$ of $I$.

Remark 4.11. This is a special case of the preceding definition of blowup as the closure of a graph. If $g_1, \dots, g_k$ do not form a regular sequence, the subvariety $\widetilde{X}$ of $X \times \mathbb{P}^{k-1}$ may require more equations, see Ex. 4.43.

Definition 4.12 (Blowup via affine charts). Let $W = \mathbb{A}^n$ be affine space over $\mathbb{K}$ with chosen coordinates $x_1, \dots, x_n$. Let $Z \subset W$ be a coordinate subspace defined by equations $x_j = 0$ for $j$ in some subset $J \subset \{1, \dots n\}$. Set $U_j = \mathbb{A}^n$ for $j \in J$, and glue two affine charts $U_j$ and $U_\ell$ via the transition maps

$$\begin{aligned} x_i &\mapsto x_i/x_j, && \text{if } i \in J \setminus \{j, \ell\}, \\ x_j &\mapsto 1/x_\ell, \\ x_\ell &\mapsto x_j x_\ell, \\ x_i &\mapsto x_i, && \text{if } i \notin J. \end{aligned}$$

This yields a variety $\widetilde{W}$. Define a morphism $\pi : \widetilde{W} \to W$ by the chart expressions $\pi_j : \mathbb{A}^n \to \mathbb{A}^n$ for $j \in J$ as follows,

$$\begin{aligned} x_i &\mapsto x_i, && \text{if } i \notin J \setminus \{j\}, \\ x_i &\mapsto x_i x_j, && \text{if } i \in J \setminus \{j\}. \end{aligned}$$

The variety $\widetilde{W}$ together with the morphism $\pi : \widetilde{W} \to W$ is the *blowup of $W = \mathbb{A}^n$ along $Z$*. The map $\pi_j : \mathbb{A}^n \to \mathbb{A}^n$ is called the *$j$-th affine chart* of the blowup map, or the *$x_j$-chart*. The maps $\pi_j$ depend on the choice of coordinates in $\mathbb{A}^n$, whereas the blowup map $\pi : \widetilde{W} \to W$ only depends on $W$ and $Z$.

Definition 4.13 (Blowup via ring extensions). Let $I$ be a non-zero ideal in a noetherian integral domain $R$, generated by non-zero elements $g_1, \dots, g_k$ of $R$. The *blowup of $R$ in $I$* is given by the ring extensions

$$R \hookrightarrow R_j = R\left[\frac{g_1}{g_j}, \dots, \frac{g_k}{g_j}\right], \qquad j = 1, \dots, k,$$

inside the rings $R_{g_j} = R[\frac{1}{g_j}]$, where the rings $R_j$ are glued pairwise via the natural inclusions $R_{g_j}, R_{g_\ell} \subset R_{g_j g_\ell}$, for $j, \ell = 1, \dots, k$. The blowup does not depend, up to isomorphism over $X$, on the choice of the generators $g_i$ of $I$.

REMARK 4.14. The seven concepts of blowup given in the preceding definitions are all essentially equivalent. It will be convenient to prove the equivalence in the language of schemes, though the substance of the proofs is inherent to varieties.

DEFINITION 4.15 (Local blowup via germs). Let $\pi : X' \to X$ be the blowup of $X$ along $Z$, and let $a'$ be a point of $X'$ mapping to the point $a \in X$. The *local blowup of $X$ along $Z$ at $a'$* is the morphism of germs $\pi : (X', a') \to (X, a)$. It is given by the dual homomorphism of local rings $\pi^* : \mathcal{O}_{X,a} \to \mathcal{O}_{X',a'}$. This terminology is also used for the completions of the local rings giving rise to a morphism of formal neighborhoods $\pi : (\widehat{X}', a') \to (\widehat{X}, a)$ with dual homomorphism $\pi^* : \widehat{\mathcal{O}}_{X,a} \to \widehat{\mathcal{O}}_{X',a'}$.

DEFINITION 4.16 (Local blowup via localizations of rings). Let $R = (R, m)$ be a local ring and let $I$ be an ideal of $R$ with Rees algebra $\widetilde{R} = \bigoplus_{i=0}^\infty I^i$. Any non-zero element $g$ of $I$ defines a homogeneous element of degree 1 in $\widetilde{R}$, also denoted by $g$. The ring of quotients $\widetilde{R}_g$ inherits from $\widetilde{R}$ the structure of a graded ring. The set of degree 0 elements of $\widetilde{R}_g$ forms a ring, denoted by $R[Ig^{-1}]$, and consists of fractions $f/g^\ell$ with $f \in I^\ell$ and $\ell \in \mathbb{N}$. The localization $R[Ig^{-1}]_p$ of $R[Ig^{-1}]$ at any prime ideal $p$ of $R[Ig^{-1}]$ containing the maximal ideal $m$ of $R$ together with the natural ring homomorphism $\alpha : R \to R[Ig^{-1}]_p$ is called the *local blowup* of $R$ with center $I$ associated to $g$ and $p$.

REMARK 4.17. The same definition can be made for non-local rings $R$, giving rise to a local blowup $R_q \to R_q[Ig^{-1}]_p$ with respect to the localization $R_q$ of $R$ at the prime ideal $q = p \cap R$ of $R$.

THEOREM 4.18. Let $X = \operatorname{Spec}(R)$ be an affine scheme and $Z$ a closed subscheme defined by an ideal $I$ of $R$. The blowup $\pi : \widetilde{X} \to X$ of $X$ along $Z$ when defined as $\operatorname{Proj}(\widetilde{R})$ of the Rees algebra $\widetilde{R}$ of $I$ satisfies the universal property of blowups.

PROOF. (a) It suffices to prove the universal property on the affine charts of an open covering of $\operatorname{Proj}(\widetilde{R})$, since on overlaps the local patches will agree by their uniqueness. This in turn reduces the proof to the local situation.

(b) Let $\beta : R \to S$ be a homomorphism of local rings such that $\beta(I) \cdot S$ is a principal ideal of $S$ generated by a non-zero divisor, for some ideal $I$ of $R$. It suffices to show that there is a unique homomorphism of local rings $\gamma : R' \to S$ such that $R'$ equals a localization $R' = R[Ig^{-1}]_p$ for some non-zero element $g$ of $I$ and a prime ideal $p$ of $R[Ig^{-1}]$ containing the maximal ideal of $R$, and such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & R' \\ \downarrow{\scriptstyle\beta} & \swarrow{\scriptstyle\gamma} & \\ S & & \end{array}$$

commutes, where $\alpha : R \to R'$ denotes the local blowup of $R$ with center $I$ specified by the choice of $g$ and $p$. The proof of the local statement goes in two steps.

(c) There exists an element $f \in I$ such that $\beta(f)$ generates $\beta(I) \cdot S$: Let $h \in S$ be a non-zero divisor generating $\beta(I) \cdot S$. Write $h = \sum_{i=1}^n \beta(f_i) s_i$ with elements

$f_1, \dots, f_n \in I$ and $s_1, \dots, s_n \in S$. Write $\beta(f_i) = t_i h$ with elements $t_i \in S$. Thus, $h = \big(\sum_{i=1}^n s_i t_i\big) h$. Since $h$ is a non-zero divisor, the sum $\sum_{i=1}^n s_i t_i$ equals 1. In particular, there is an index $i$ for which $t_i$ does not belong to the maximal ideal of $S$. This implies that this $t_i$ is invertible in $S$, so that $h = t_i^{-1}\beta(f_i)$. In particular, for this $i$, the element $\beta(f_i)$ generates $\beta(I) \cdot S$.

(d) Let $f \in I$ be as in (c). By assumption, the image $\beta(f)$ is a non-zero divisor in $S$. For every $\ell \geq 0$ and every $h_\ell \in I^\ell$, there is an element $a_\ell \in S$ such that $\beta(h_\ell) = a_\ell \beta(f)^\ell$. Since $\beta(f)$ is a non-zero divisor, $a_\ell$ is unique with this property. For an arbitrary element $\sum_{\ell=0}^n h_\ell / g^\ell$ of $R[Ig^{-1}]$ with $h_\ell \in I^\ell$ set $\delta(\sum_{\ell=0}^n h_\ell / g^\ell) = \sum_{\ell=0}^n a_\ell$. This defines a ring homomorphism $\delta : R[Ig^{-1}] \to S$ that restricts to $\beta$ on $R$. By definition, $\delta$ is unique with this property. Let $p \subset R[Ig^{-1}]$ be the inverse image under $\delta$ of the maximal ideal of $S$. This is a prime ideal of $R[Ig^{-1}]$ which contains the maximal ideal of $R$. By the universal property of localization, $\delta$ induces a homomorphism of local rings $\gamma : R[Ig^{-1}]_p \to S$ that restricts to $\beta$ on $R$, i.e., satisfies $\gamma \circ \alpha = \beta$. By construction, $\gamma$ is unique. □

THEOREM 4.19. Let $X = \mathrm{Spec}(R)$ be an affine scheme and let $Z$ be a closed subscheme defined by an ideal $I$ of $R$. The blowup of $X$ along $Z$ defined by $\mathrm{Proj}(\widetilde{R})$ can be covered by affine charts as described in Def. 4.10.

PROOF. For $g \in I$ denote by $R[Ig^{-1}] \subset R_g$ the subring of the ring of quotients $R_g$ generated by homogeneous elements of degree 0 of the form $h/g^\ell$ with $h \in I^\ell$ and $\ell \in \mathbb{N}$. This gives an injective ring homomorphism $R[Ig^{-1}] \to \widetilde{R}_g$. Let now $g_1, \dots, g_k$ be generators of $I$. Then $X = \mathrm{Proj}(\widetilde{R})$ is covered by the principal open sets $\mathrm{Spec}(R[Ig_i^{-1}]) = \mathrm{Spec}(R[g_1/g_i, \dots, g_k/g_i])$. The chart expression $\mathrm{Spec}(R[g_1/g_i, \dots, g_k/g_i]) \to \mathrm{Spec}(R)$ of the blowup map $\pi : \widetilde{X} \to X$ follows now by computation. □

THEOREM 4.20. Let $X$ be an affine variety with coordinate ring $R = \mathbb{K}[X]$ and let $Z$ be a closed subvariety defined by the ideal $I$ of $R$ with generators $g_1, \dots, g_k$. The blowup of $X$ along $Z$ defined by $\mathrm{Proj}(\widetilde{R})$ of the Rees algebra $\widetilde{R}$ of $R$ equals the closure of the graph $\Gamma$ of $\gamma : X \setminus Z \to \mathbb{P}^{k-1}, a \mapsto (g_1(a) : \dots : g_k(a))$. If $g_1, \dots, g_k$ form a regular sequence, the closure is defined as a subvariety of $X \times \mathbb{P}^{k-1}$ by equations as indicated in Def. 4.10.

PROOF. (a) Let $U_j \subset \mathbb{P}^{k-1}$ be the affine chart given by $u_j \neq 0$, with isomorphism $U_j \simeq \mathbb{A}^{k-1}, (u_1 : \dots : u_k) \mapsto (u_1/u_j, \dots, u_k/u_j)$. The $j$-th chart expression of $\gamma$ equals $a \mapsto (g_1(a)/g_j(a), \dots, g_k(a)/g_j(a))$ and is defined on the principal open set $g_j \neq 0$ of $X$. The closure of the graph of $\gamma$ in $U_j$ is therefore given by $\mathrm{Spec}(R[g_1/g_i, \dots, g_k/g_i])$. The preceding theorem then establishes the required equality.

(b) If $g_1, \dots, g_k$ form a regular sequence, their only linear relations over $R$ are the trivial ones, so that

$$R[Ig_i^{-1}] = R[g_1/g_i, \dots g_k/g_i] \simeq R[t_1, \dots, t_k]/(g_i t_j - g_j, j = 1, \dots, k).$$

The $i$-th chart expression of $\pi : \widetilde{X} \to X$ is then given by the ring inclusion $R \to R[t_1, \dots, t_k]/(g_i t_j - g_j, j = 1, \dots, k)$. □

## *Examples*

EXAMPLE 4.21. For the cone $X$ in $\mathbb{A}^3$ of equation $x^2+y^2=z^2$, the line $Y$ in $\mathbb{A}^3$ defined by $x=y-z=0$ is a hypersurface of $X$ at each point. It is a Cartier divisor at any point $a\in Y\setminus\{0\}$ but it is not a Cartier divisor at 0. The double line $Y'$ in $\mathbb{A}^3$ defined by $x^2=y-z=0$ is a Cartier divisor of $X$ since it can be defined in $X$ by $y-z=0$. The subvariety $Y''$ in $\mathbb{A}^3$ consisting of the two lines defined by $x=y^2-z^2=0$ is a Cartier divisor of $X$ since it can be defined in $X$ by $x=0$.

EXAMPLE 4.22. For the surface $X: x^2y-z^2=0$ in $\mathbb{A}^3$, the subvariety $Y$ defined by $y^2-xz=x^3-yz=0$ is the singular curve parametrized by $t\mapsto(t^3,t^4,t^5)$. It is everywhere a Cartier divisor except at 0: there, the local ring of $Y$ in $X$ is $\mathbb{K}[x,y,z]_{(x,y,z)}/(x^2y-z^2,y^2-xz,x^3-yz)$, which defines a singular cubic. It is of codimension 1 in $X$ but not a complete intersection (i.e., cannot be defined in $\mathbb{A}^3$, as the codimension would suggest, by two but only by three equations). Thus it is not a Cartier divisor. At any other point $a=(t^3,t^4,t^5)$, $t\neq 0$ of $Y$, one has $\mathcal{O}_{Y,a}=S/(x^2y-z^2,y^2/x-z,x^3-yz)=S/(x^2y-z^2,y^2/x-z)$ with $S=\mathbb{K}[x,y,z]_{(x-t^3,y-t^4,z-t^5)}$ the localization of $\mathbb{K}[x,y,z]$ at $a$. Hence $Y$ is a Cartier divisor there.

EXAMPLE 4.23. Let $Z$ be one of the axes of the cross $X: xy=0$ in $\mathbb{A}^2$, e.g. the $x$-axis. At any point $a$ on the $y$-axis except the origin, $Z$ is Cartier: one has $\mathcal{O}_{X,a}=\mathbb{K}[x,y]_{(x,y-a)}/(xy)=\mathbb{K}[x,y]_{(x,y-a)}/(x)$ and the element $h=y$ defining $Z$ is a unit in this local ring. At the origin 0 of $\mathbb{A}^2$ the local ring of $X$ is $\mathbb{K}[x,y]_{(x,y)}/(xy)$ and $Z$ is locally defined by $h=y$ which is a zero-divisor in $\mathcal{O}_{X,0}$. Thus $Z$ is not Cartier in $X$ at 0.

EXAMPLE 4.24. Let $Z$ be the origin of $X=\mathbb{A}^2$. Then $Z$ is not a hypersurface and hence not Cartier in $X$.

EXAMPLE 4.25.$^\triangleright$ The (reduced) origin $Z=\{0\}$ in $X=V(xy,x^2)\subset\mathbb{A}^2$ is not a Cartier divisor in $X$.

EXAMPLE 4.26.$^\triangleright$ Are $Z=V(x^2)$ in $\mathbb{A}^1$ and $Z=V(x^2y)$ in $X=\mathbb{A}^2$ Cartier divisors?

EXAMPLE 4.27.$^\triangleright$ Let $Z=V(x^2,y)$ be the origin of $\mathbb{A}^2$ with non-reduced structure given by the ideal $(x^2,y)$. Then $Z$ is not a Cartier divisor in $X=V(xy,x^2)\subset\mathbb{A}^2$.

EXAMPLE 4.28. Let $X=\mathbb{A}^1$ be the affine line with coordinate ring $R=\mathbb{K}[x]$ and let $Z$ be the origin of $\mathbb{A}^1$. The Rees algebra $\widetilde{R}=\mathbb{K}[x,xt]\subset\mathbb{K}[x,t]$ with respect to the ideal defining $Z$ is isomorphic, as a graded ring, to a polynomial ring $\mathbb{K}[u,v]$ in two variables with $\deg u=0$ and $\deg v=1$. Therefore, the point blowup $\widetilde{X}$ of $X$ is isomorphic to $\mathbb{A}^1\times\mathbb{P}^0=\mathbb{A}^1$, and $\pi:\widetilde{X}\to X$ is the identity.

More generally, let $X=\mathbb{A}^n$ be $n$-dimensional affine space with coordinate ring $R=\mathbb{K}[x_1,\dots,x_n]$ and let $Z\subset X$ be a hypersurface defined by some non-zero $g\in\mathbb{K}[x_1,\dots,x_n]$. The Rees algebra $\widetilde{R}=\mathbb{K}[x_1,\dots,x_n,gt]\subset\mathbb{K}[x_1,\dots,x_n,t]$ with respect the ideal defining $Z$ is isomorphic, as a graded ring, to a polynomial ring $\mathbb{K}[u_1,\dots,u_n,v]$ in $n+1$ variables with $\deg u_i=0$ and $\deg v=1$. Therefore the blowup $\widetilde{X}$ of $X$ along $Z$ is isomorphic to $\mathbb{A}^n\times\mathbb{P}^0=\mathbb{A}^n$, and $\pi:\widetilde{X}\to X$ is the identity.

EXAMPLE 4.29. Let $X = \operatorname{Spec}(R)$ be an affine scheme and $Z \subset X$ be defined by some non-zero-divisor $g \subset R$. The Rees algebra $\widetilde{R} = R[gt] \subset R[t]$ with respect to the ideal defining $Z$ is isomorphic, as a graded ring, to $R[v]$ with $\deg r = 0$ for $r \in R$ and $\deg v = 1$. Therefore the blowup $\widetilde{X}$ of $X$ along $Z$ is isomorphic to $X \times \mathbb{P}^0 = X$, and $\pi : \widetilde{X} \to X$ is the identity.

EXAMPLE 4.30. Take $X = V(xy) \subset \mathbb{A}^2$ with coordinate ring $R = K[x,y]/(xy)$. Let $Z$ be defined in $X$ by $y = 0$. The Rees algebra of $R$ with respect to the ideal defining $Z$ equals $\widetilde{R} = R[yt] \cong K[x,y,u]/(xy,xu)$ with $\deg x = \deg y = 0$ and $\deg u = 1$.

EXAMPLE 4.31. Let $X = \operatorname{Spec}(R)$ be an affine scheme and let $Z \subset X$ be defined by some zero-divisor $g \neq 0$ in $R$, say $h \cdot g = 0$ for some non-zero $h \in R$. The Rees algebra $\widetilde{R} = R[gt] \subset R[t]$ with respect to the ideal defining $Z$ is isomorphic, as a graded ring, to $R[v]/(h \cdot v)$ with $\deg r = 0$ for $r \in R$ and $\deg v = 1$. Therefore, the blowup $\widetilde{X}$ of $X$ along $Z$ equals the closed subvariety of $X \times \mathbb{P}^0 = X$ defined by $h \cdot v = 0$, and $\pi : \widetilde{X} \to X$ is the inclusion map.

EXAMPLE 4.32. Take in the situation of the preceding example $X = V(xy) \subset \mathbb{A}^2$ and $g = y$, $h = x$. Then $R = \mathbb{K}[x,y]/(xy)$ and $\widetilde{R} = R[yt] \simeq \mathbb{K}[x,y,u]/(xy,xu)$ with deg $x =$ deg $y = 0$ and deg $u = 1$.

EXAMPLE 4.33. Let $X = \operatorname{Spec}(R)$ be an affine scheme and let $Z \subset X$ be defined by some nilpotent element $g \neq 0$ in $R$, say $g^k = 0$ for some $k \geq 1$. The Rees algebra $\widetilde{R} = R[gt] \subset R[t]$ with respect to the ideal defining $Z$ is isomorphic, as a graded ring, to $R[v]/(v^k)$ with $\deg r = 0$ for $r \in R$ and $\deg v = 1$. The blowup $\widetilde{X}$ of $X$ is the closed subvariety of $X \times \mathbb{P}^0 = X$ defined by $v^k = 0$.

EXAMPLE 4.34. Let $X = \mathbb{A}^2$ and let $Z$ be defined in $X$ by $I = (x,y)$. The Rees algebra $\widetilde{R} = \mathbb{K}[x,y,xt,yt] \subset \mathbb{K}[x,y,t]$ of $R = \mathbb{K}[x,y]$ with respect to the ideal defining $Z$ is isomorphic, as a graded ring, to the factor ring $\mathbb{K}[x,y,u,v]/(xv - yu)$, with $\deg x = \deg y = 0$ and $\deg v = \deg u = 1$. It follows that the blowup $\widetilde{X}$ of $X$ along $Z$ embeds naturally as the closed and regular subvariety of $\mathbb{A}^2 \times \mathbb{P}^1$ defined by $xv - yu = 0$, the morphism $\pi : \widetilde{X} \to X$ being given by the restriction to $\widetilde{X}$ of the first projection $\mathbb{A}^2 \times \mathbb{P}^1 \to \mathbb{A}^2$.

EXAMPLE 4.35. Let $X = \mathbb{A}^2$ and let $Z$ be defined in $X$ by $I = (x, y^2)$. The Rees algebra $\widetilde{R} = \mathbb{K}[x,y,xt,y^2t] \subset \mathbb{K}[x,y,t]$ of $R = \mathbb{K}[x,y]$ with respect to the ideal defining $Z$ is isomorphic, as a graded ring, to the factor ring $\mathbb{K}[x,y,u,v]/(xv - y^2u)$, with $\deg x = \deg y = 0$ and $\deg u = \deg v = 1$. It follows that the blowup $\widetilde{X}$ of $X$ along $Z$ embeds naturally as the closed and singular subvariety of $\mathbb{A}^2 \times \mathbb{P}^1$ defined by $xv - y^2u = 0$, the morphism $\pi : \widetilde{X} \to X$ being given by the restriction to $\widetilde{X}$ of the first projection $\mathbb{A}^2 \times \mathbb{P}^1 \to \mathbb{A}^2$.

EXAMPLE 4.36. Let $X = \mathbb{A}^3$ and let $Z$ be defined in $X$ by $I = (xy, z)$. The Rees algebra $\widetilde{R} = \mathbb{K}[x,y,z,xyt,zt] \subset \mathbb{K}[x,y,z,t]$ of $R = \mathbb{K}[x,y,z]$ with respect to the ideal defining $Z$ is isomorphic, as a graded ring, to $\mathbb{K}[u,v,w,r,s]/(uvs - wr)$, with $\deg u = \deg v = \deg w = 0$ and $\deg r = \deg s = 1$. It follows that the blowup $\widetilde{X}$ of $X$ along $Z$ embeds naturally as the closed and singular subvariety of $\mathbb{A}^3 \times \mathbb{P}^1$ defined by $uvs - wr = 0$, the morphism $\pi : \widetilde{X} \to X$ being given by the restriction to $\widetilde{X}$ of the first projection $\mathbb{A}^3 \times \mathbb{P}^1 \to \mathbb{A}^3$.

Example 4.37. Let $X = \mathbb{A}^1_{\mathbb{Z}}$ be the affine line over the integers and let $Z$ be defined in $X$ by $I = (x, p)$ for a prime $p \in \mathbb{Z}$. The Rees algebra $\widetilde{R} = \mathbb{Z}[x, xt, pt] \subset \mathbb{Z}[x, t]$ of $R = \mathbb{Z}[x]$ with respect to the ideal defining $Z$ is isomorphic, as a graded ring, to $\mathbb{Z}[u, v, w]/(xw - pv)$, with $\deg u = 0$ and $\deg v = \deg w = 1$. It follows that the blowup $\widetilde{X}$ of $X$ along $Z$ embeds naturally as the closed and regular subvariety of $\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}$ defined by $xw - pv$, the morphism $\pi : \widetilde{X} \to X$ being given by the restriction to $\widetilde{X}$ of the first projection $\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}} \to \mathbb{A}^1_{\mathbb{Z}}$.

Example 4.38.$^{\triangleright}$ Let $X = \mathbb{A}^2_{\mathbb{Z}_{20}}$ be the affine plane over the ring $\mathbb{Z}_{20} = \mathbb{Z}/20\mathbb{Z}$, and let $Z$ be defined in $X$ by $I = (x, 2y)$. The Rees algebra $\widetilde{R} = \mathbb{Z}_{20}[x, y, xt, 2yt] \subset \mathbb{Z}_{20}[x, y, t]$ of $R = \mathbb{Z}_{20}[x]$ with respect to $I$ is isomorphic, as a graded ring, to $\mathbb{Z}_{20}[u, v, w, z]/(xz - 2vw)$, with $\deg u = \deg v = 0$ and $\deg w = \deg z = 1$.

Example 4.39. Let $R = \mathbb{Z}_6$ with $\mathbb{Z}_6 = \mathbb{Z}/6\mathbb{Z}$, and let $Z$ be defined in $X$ by $I = (2)$. The Rees algebra $\widetilde{R} = \mathbb{Z}_6[2t] \subset \mathbb{Z}_6[t]$ of $R$ with respect to $I$ is isomorphic, as a graded ring, to $\mathbb{Z}_6 \oplus u \cdot \mathbb{Z}_3[u]$ with $\deg u = 1$.

Example 4.40. Let $R = \mathbb{K}[[x, y]]$ be a formal power series ring in two variables $x$ and $y$, and let $Z$ be defined in $X$ by $I$ be the ideal generated by $e^x - 1$ and $\ln(y + 1)$. The Rees algebra $\widetilde{R}$ of $R$ with respect to $I$ equals $R[(e^x - 1)t, \ln(y + 1)t] = \mathbb{K}[[x, y]][(e^x - 1)t, \ln(y + 1)t] \cong \mathbb{K}[[x, y]][u, v]/(\ln(y + 1)u - (e^x - 1)v)$ with $\deg u = \deg v = 1$.

Example 4.41. Let $X = \mathbb{A}^n$. The point blowup $\widetilde{X} \subset X \times \mathbb{P}^{n-1}$ of $X$ at the origin is defined by the ideal $(u_i x_j - u_j x_i, i, j = 1, \dots, n)$ in $\mathbb{K}[x_1, \dots, x_n, u_1, \dots, u_n]$. This is a graded ring, where $\deg x_i = 0$ and $\deg u_i = 1$ for all $i = 1, \dots, n$. The ring

$$\mathbb{K}[x_1, \dots, x_n, u_1, \dots, u_n]/(u_i x_j - u_j x_i, i, j = 1, \dots, n)$$

is isomorphic, as a graded ring, to $\widetilde{R} = \mathbb{K}[x_1, \dots, x_n, x_1 t, \dots, x_n t] \subset \mathbb{K}[x_1, \dots, x_n, t]$ with $\deg x_i = 0$, $\deg t = 1$. The ring $\widetilde{R}$ is the Rees-algebra of the ideal $(x_1, \dots, x_n)$ of $\mathbb{K}[x_1, \dots, x_n]$. Thus $\widetilde{X}$ is isomorphic to $\mathrm{Proj}\big(\bigoplus_{d \geq 0} (x_1, \dots, x_n)^d\big)$.

Example 4.42. Let $1 \leq k \leq n$. The $i$-th affine chart of the blowup $\widetilde{\mathbb{A}}^n$ of $\mathbb{A}^n$ in the ideal $I = (x_1, \dots, x_k)$ is isomorphic to $\mathbb{A}^n$, for $i = 1, \dots, k$, via the ring isomorphism

$$\mathbb{K}[x_1, \dots, x_n] \simeq \mathbb{K}[x_1, \dots, x_n, t_1, \dots, \widehat{t_i}, \dots t_k]/(x_i t_j - x_j, j = 1, \dots, k, j \neq i),$$

where $x_j \mapsto t_j$ for $j = 1, \dots, k$, $j \neq i$, respectively $x_j \mapsto x_j$ for $j = k + 1, \dots, n$ and $j = i$. The inverse map is given by $x_j \mapsto x_i x_j$ for $j = 1, \dots, k$, $j \neq i$, respectively $x_j \mapsto x_j$ for $j = k + 1, \dots, n$ and $j = i$, respectively $t_j \mapsto x_j$ for $j = 1, \dots, k$, $j \neq i$.

Example 4.43. Let $X = \mathbb{A}^2$ be the affine plane and let $g_1 = x^2, g_2 = xy, g_3 = y^3$ generate the ideal $I \subset \mathbb{K}[x, y]$. The $g_i$ do not form a regular sequence. The subvariety of $\mathbb{A}^2 \times \mathbb{P}^2$ defined by the equations $g_i u_j - g_j u_i = 0$ is singular, but the blowup of $\mathbb{A}^2$ in $I$ is regular.

Example 4.44. Compute the following blowups:
(a) $\mathbb{A}^2$ in the center $(x, y)(x, y^2)$,
(b) $\mathbb{A}^3$ in the centers $(x, yz)$ and $(x, yz)(x, y)(x, z)$,
(c) $\mathbb{A}^3$ in the center $(x^2 + y^2 - 1, z)$.
(d) The plane curve $x^2 = y$ in the origin.

Use affine charts and ring extensions to determine at which points the resulting varieties are regular or singular.

EXAMPLE 4.45. Blow up $\mathbb{A}^3$ in 0 and compute the inverse image of $x^2+y^2=z^2$.

EXAMPLE 4.46.$^\triangleright$ Blow up $\mathbb{A}^2$ in the point $(0,1)$. What is the inverse image of the lines $x+y=0$ and $x+y=1$?

EXAMPLE 4.47.$^\triangleright$ Compute the two chart transition maps for the blowup of $\mathbb{A}^3$ along the $z$-axis.

EXAMPLE 4.48.$^\triangleright$ Blow up the cone $X$ defined by $x^2+y^2=z^2$ in one of its lines.

EXAMPLE 4.49.$^\triangleright$ Blow up $\mathbb{A}^3$ in the circle $x^2+(y+2)^2-1=z=0$ and in the elliptic curve $y^2-x^3-x=z=0$.

EXAMPLE 4.50. Interpret the blowup of $\mathbb{A}^2$ in the ideal $(x,y^2)(x,y)$ as a composition of blowups in regular centers.

EXAMPLE 4.51. Show that the blowup of $\mathbb{A}^n$ along a coordinate subspace $Z$ equals the cartesian product of the point-blowup in a transversal subspace $V$ of $\mathbb{A}^n$ of complementary dimension (with respect to $Z$) with the identity map on $Z$.

EXAMPLE 4.52.$^\triangleright$ Show that the ideals $(x_1,\dots,x_n)$ and $(x_1,\dots,x_n)^m$ define the same blowup of $\mathbb{A}^n$ when taken as center.

EXAMPLE 4.53. Let $E$ be a normal crossings subvariety of $\mathbb{A}^n$ and let $Z$ be a subvariety of $\mathbb{A}^n$ such that $E\cup Z$ also has normal crossings (the union being defined by the product of ideals). Show that the inverse image of $E$ under the blowup of $\mathbb{A}^n$ along $Z$ has again normal crossings. Show by an example that the assumption on $Z$ cannot be dropped in general.

EXAMPLE 4.54. Draw a real picture of the blowup of $\mathbb{A}^2$ at the origin.

EXAMPLE 4.55. Show that the blowup $\widetilde{\mathbb{A}}^n$ of $\mathbb{A}^n$ with center a point is nonsingular.

EXAMPLE 4.56. Describe the geometric construction via secants of the blowup $\widetilde{\mathbb{A}}^3$ of $\mathbb{A}^3$ with center 0. What is $\pi^{-1}(0)$? For a chosen affine chart $W'$ of $\widetilde{\mathbb{A}}^3$, consider all cylinders $Y$ over a circle in $W'$ centered at the origin and parallel to a coordinate axis. What is the image of $Y$ under $\pi$ in $\mathbb{A}^3$?

EXAMPLE 4.57.$^\triangleright$ Compute the blowup of the Whitney umbrella $X=V(x^2-y^2z)\subset\mathbb{A}^3$ with center one of the three coordinate axes, respectively the origin 0.

EXAMPLE 4.58. Determine the locus of points of the Whitney umbrella $X=V(x^2-y^2z)$ where the singularities are normal crossings, respectively simple normal crossings. Blow up the complements of these loci and compare with the preceding example [Kol07] Ex. 3.6.1, p. 123, [BDMVP12] Thm. 3.4.

EXAMPLE 4.59. Let $Z$ be a regular center in $\mathbb{A}^n$, with induced blowup $\pi:\widetilde{\mathbb{A}}^n\to\mathbb{A}^n$, and let $a'$ be a point of $\widetilde{\mathbb{A}}^n$ mapping to a point $a\in Z$. Show that it is possible to choose local formal coordinates at $a$, i.e., a regular parameter system of $\widehat{\mathcal{O}}_{\mathbb{A}^n,a}$, so that the center is a coordinate subspace, and so that $a'$ is the origin of one of the affine charts of $\widetilde{\mathbb{A}}^n$. Is this also possible with a regular parameter system of the local ring $\mathcal{O}_{\mathbb{A}^n,a}$?

EXAMPLE 4.60. Let $X = V(f)$ be a hypersurface in $\mathbb{A}^n$, defined by $f \in \mathbb{K}[x_1, \dots, x_n]$, and let $Z$ be a regular closed subvariety which is contained in the locus $S$ of points of $X$ where $f$ attains its maximal order. Let $\pi : \widetilde{\mathbb{A}}^n \to \mathbb{A}^n$ be the blowup along $Z$ and let $X^s = V(f^s)$ be the strict transform of $X$, defined as the Zariski closure of $\pi^{-1}(X \setminus Z)$ in $\widetilde{\mathbb{A}}^n$. Show that for points $a \in Z$ and $a' \in E = \pi^{-1}(Z)$ with $\pi(a') = a$ the inequality $\mathrm{ord}_{a'} f' \leq \mathrm{ord}_a f$ holds. Do the same for subvarieties of $\mathbb{A}^n$ defined by arbitrary ideals.

EXAMPLE 4.61. Find an example of a variety $X$ for which the dimension of the singular locus increases under the blowup of a closed regular center $Z$ that is contained in the top locus of $X$ [Hau98] Ex. 9.

EXAMPLE 4.62. Let $\pi : (\widetilde{\mathbb{A}}^n, a') \to (\mathbb{A}^n, a)$ be the local blowup of $\mathbb{A}^n$ with center the point $a$, considered at a point $a' \in E$. Let $x_1, \dots, x_n$ be given local coordinates at $a$. Determine the coordinate changes in $(\mathbb{A}^n, a)$ which make the chart expressions of $\pi$ monomial (i.e., each component of a chart expression is a monomial in the coordinates).

EXAMPLE 4.63. Let $x_1, \dots, x_n$ be local coordinates at $a$ so that the local blowup $\pi : (\widetilde{\mathbb{A}}^n, a') \to (\mathbb{A}^n, a)$ is monomial with respect to them. Determine the formal automorphisms of $(\widetilde{\mathbb{A}}^n, a')$ which commute with the local blowup.

EXAMPLE 4.64.$^{\triangleright}$ Compute the blowup of $X = \mathrm{Spec}(\mathbb{Z}[x])$ in the ideals $(x, p)$ and $(px, pq)$ where $p$ and $q$ are primes.

EXAMPLE 4.65. Let $R = \mathbb{K}[x, y, z]$ be the polynomial ring in three variables and let $I$ be the ideal $(x, yz)$ of $R$. The blowup of $R$ along $I$ corresponds to the ring extensions:

$$R \hookrightarrow R\left[\frac{yz}{x}\right] \cong \mathbb{K}[s, t, u, v]/(sv - tu),$$
$$R \hookrightarrow R\left[\frac{x}{yz}\right] \cong \mathbb{K}[s, t, u, v]/(s - tuv).$$

The first ring extension defines a singular variety while the second one defines a non-singular one. Let $W = \mathbb{A}^3$ and $Z = V(I) \subset W$ be the affine space and the subvariety defined by $I$. The blowup of $R$ along $I$ coincides with the blowup $W'$ of $W$ along $Z$.

(a) Compute the chart expressions of the blowup maps.

(b) Determine the exceptional divisor.

(c) Apply one more blowup to $W'$ to get a non-singular variety $W''$.

(d) Express the composition of the two blowups as a single blowup in a properly chosen ideal.

(e) Blow up $\mathbb{A}^3$ along the three coordinate axes. Show that the resulting variety is non-singular.

(f) Show the same for the blowup of $\mathbb{A}^n$ along the $n$ coordinate axes.

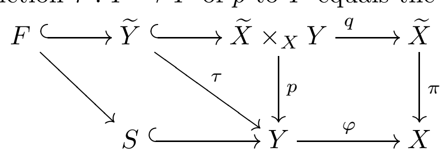
EXAMPLE 4.66. The zeroset in $\mathbb{A}^3$ of the non-reduced ideal $(x, yz)(x, y)(x, z) = (x^3, x^2y, x^2z, xyz, y^2z)$ is the union of the $y$- and the $z$-axis. Taken as center, the resulting blowup of $\mathbb{A}^3$ equals the composition of two blowups: The first blowup has center the ideal $(x, yz)$ in $\mathbb{A}^3$, giving a three-fold $W_1$ in a regular four-dimensional ambient variety with one singular point of local equation $xy = zw$. The second blowup is the point blowup of $W_1$ with center this singular point [Hau00] Prop. 3.5, [FW11, Lev01].

Example 4.67. + Consider the blowup $\widetilde{\mathbb{A}}^n$ of $\mathbb{A}^n$ in a monomial ideal $I$ of $\mathbb{K}[x, \dots, x_n]$. Show that $\widetilde{\mathbb{A}}^n$ may be singular. What types of singularities will occur? Find a natural saturation procedure $I \rightsquigarrow \overline{I}$ so that the blowup of $\mathbb{A}^n$ in $\overline{I}$ is regular and equal to a (natural) resolution of the singularities of $\widetilde{\mathbb{A}}^n$ [FW11].

Example 4.68. The *Nash modification* of a subvariety $X$ of $\mathbb{A}^n$ is the closure of the graph of the map which associates to each non-singular point its tangent space, taken as an element of the Grassmanian of $d$-dimensional linear subspaces of $\mathbb{K}^n$, where $d = \dim(X)$. For a hypersurface $X$ defined in $\mathbb{A}^n$ by $f = 0$, the Nash modification coincides with the blowup of $X$ in the Jacobian ideal of $f$ generated by the partial derivatives of $f$.

## 5. Lecture V: Properties of Blowup

Proposition 5.1. Let $\pi : \widetilde{X} \to X$ be the blowup of $X$ along a subvariety $Z$, and let $\varphi : Y \to X$ be a morphism, the base change. Denote by $p : \widetilde{X} \times_X Y \to Y$ the projection from the fibre product to the second factor. Let $S = \varphi^{-1}(Z) \subset Y$ be the inverse image of $Z$ under $\varphi$, and let $\widetilde{Y}$ be the Zariski closure of $p^{-1}(Y \setminus S)$ in $\widetilde{X} \times_X Y$. The restriction $\tau : \widetilde{Y} \to Y$ of $p$ to $\widetilde{Y}$ equals the blowup of $Y$ along $S$.

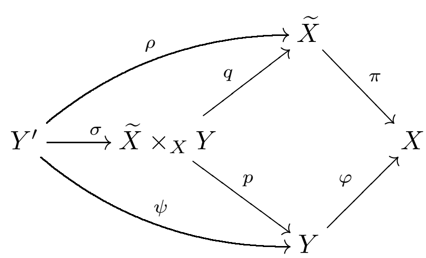



Proof. The assertion is best proven via the universal property of blowups. To show that $F = \tau^{-1}(S)$ is a Cartier divisor in $\widetilde{Y}$, consider the projection $q : \widetilde{X} \times_X Y \to \widetilde{X}$ onto the first factor. By the commutativity of the diagram,

$$F = p^{-1}(S) = p^{-1} \circ \varphi^{-1}(Z) = q^{-1} \circ \pi^{-1}(Z) = q^{-1}(E).$$

As $E$ is a Cartier divisor in $\widetilde{X}$ and $q$ is a projection, $F$ is locally defined by a principal ideal. The associated primes of $\widetilde{Y}$ are the associated primes of $Y$ not containing the ideal of $S$. Thus, the local defining equation of $E$ in $\widetilde{X}$ cannot pull back to a zero divisor on $\widetilde{Y}$. This proves that $F$ is a Cartier divisor.

To show that $\tau : \widetilde{Y} \to Y$ fulfills the universal property, let $\psi : Y' \to Y$ be a morphism such that $\psi^{-1}(S)$ is a Cartier divisor in $Y'$. This results in the following diagram.




Since $\psi^{-1}(S) = \psi^{-1}(\varphi^{-1}(Z)) = (\varphi \circ \psi)^{-1}(Z)$, there exists by the universal property of the blowup $\pi : \widetilde{X} \to X$ a unique map $\rho : Y' \to \widetilde{X}$ such that $\varphi \circ \psi = \pi \circ \rho$. By the universal property of fibre products, there exists a unique map $\sigma : Y' \to \widetilde{X} \times_X Y$ such that $q \circ \sigma = \rho$ and $p \circ \sigma = \psi$.

It remains to show that $\sigma(Y')$ lies in $\widetilde{Y} \subset \widetilde{X} \times_X Y$. Since $\psi^{-1}(S)$ is a Cartier divisor in $Y'$, its complement $Y' \setminus \psi^{-1}(S)$ is dense in $Y'$. From

$$Y' \setminus \psi^{-1}(S) = \psi^{-1}(Y \setminus S) = (p \circ \sigma)^{-1}(Y \setminus S) = \sigma^{-1}(p^{-1}(Y \setminus S))$$

follows that $\sigma(Y' \setminus \psi^{-1}(S)) \subset p^{-1}(Y \setminus S)$. But $\widetilde{Y}$ is the closure of $p^{-1}(Y \setminus S)$ in $\widetilde{X} \times_X Y$, so that $\sigma(Y') \subset \widetilde{Y}$ as required. □

Corollary 5.2. (a) Let $\pi : X' \to X$ be the blowup of $X$ along a subvariety $Z$, and let $Y$ be a closed subvariety of $X$. Denote by $Y'$ the Zariski closure of $\pi^{-1}(Y \setminus Z)$ in $X'$, i.e., the *strict transform* of $Y$ under $\pi$, cf. Def. 6.2. The restriction $\tau : Y' \to Y$ of $\pi$ to $Y'$ is the blowup of $Y$ along $Y \cap Z$. In particular, if $Z \subset Y$, then $\tau$ is the blowup $\widetilde{Y}$ of $Y$ along $Z$.

(b) Let $U \subset X$ be an open subvariety, and let $Z \subset X$ be a closed subvariety, so that $U \cap Z$ is closed in $U$. Let $\pi : X' \to X$ be the blowup of $X$ along $Z$. The blowup of $U$ along $U \cap Z$ equals the restriction of $\pi$ to $U' = \pi^{-1}(U)$.

(c) Let $a \in X$ be a point. Write $(X, a)$ for the germ of $X$ at $a$, and $(\widehat{X}, a)$ for the formal neighbourhood. There are natural maps

$$(X, a) \to X \quad \text{and} \quad (\widehat{X}, a) \to X$$

corresponding to the localization and completion homomorphisms $\mathcal{O}_X \to \mathcal{O}_{X,a} \to \widehat{\mathcal{O}}_{X,a}$. Take a point $a'$ above $a$ in the blowup $X'$ of $X$ along a subvariety $Z$ containing $a$. This gives local blowups of germs and formal neighborhoods

$$\pi_{a'} : (X', a') \to (X, a), \qquad \widehat{\pi}_{a'} : (\widehat{X}', a') \to (\widehat{X}, a).$$

The blowup of a local ring is not local in general; to get a local blowup one needs to localize also on $X'$.

(d) If $X_1 \to X$ is an isomorphism between varieties sending a subvariety $Z_1$ to $Z$, the blowup $X_1'$ of $X_1$ along $Z_1$ is canonically isomorphic to the blowup $X'$ of $X$ along $Z$. This also holds for local isomorphisms.

(e) If $X = Z \times Y$ is a cartesian product of two varieties, and $a$ is a given point of $Y$, the blowup $\pi : X' \to X$ of $X$ along $Z \times \{a\}$ is isomorphic to the cartesian product $\mathrm{Id}_Z \times \tau : Z \times Y' \to Z \times Y$ of the identity on $Z$ with the blowup $\tau : Y' \to Y$ of $Y$ in $a$.

Proposition 5.3. Let $\pi : X' \to X$ be the blowup of $X$ along a regular subvariety $Z$ with exceptional divisor $E$. Let $Y$ be a subvariety of $X$, and $Y^* = \pi^{-1}(Y)$ its preimage under $\pi$. Let $Y'$ be the Zariski closure of $\pi^{-1}(Y \setminus Z)$ in $X'$. If $Y$ is transversal to $Z$, i.e., $Y \cup Z$ has normal crossings at all points of the intersection $Y \cap Z$, also $Y^*$ has normal crossings at all points of $Y^* \cap E$. In particular, if $Y$ is regular and transversal to $Z$, also $Y'$ is regular and transversal to $E$.

Proof. Having normal crossings is defined locally at each point through the completions of local rings. The assertion is proven by a computation in local coordinates for which the blowup is monomial, cf. Prop. 5.4 below. □

Proposition 5.4. Let $W$ be a regular variety of dimension $n$ with a regular subvariety $Z$ of codimension $k$. Let $\pi : W' \to W$ denote the blowup of $W$ along $Z$, with exceptional divisor $E$. Let $V$ be a regular hypersurface in $W$ containing $Z$, let $D$ be a (not necessarily reduced) normal crossings divisor in $W$ having normal crossings with $V$. Let $a$ be a point of $V \cap Z$ and let $a' \in E$ be a point lying above $a$. There exist local coordinates $x_1, \ldots, x_n$ of $W$ at $a$ such that

(1) $a$ has components $a = (0, \ldots, 0)$.

(2) $V$ is defined in $W$ by $x_{n-k+1} = 0$.

(3) $Z$ is defined in $W$ by $x_{n-k+1} = \ldots = x_n = 0$.

(4) $D \cap V$ is defined in $V$ locally at $a$ by a monomial $x_1^{q_1} \cdots x_n^{q_n}$, for some $q = (q_1, \ldots, q_n) \in \mathbb{N}^n$ with $q_{n-k+1} = 0$.

(5) The point $a'$ lies in the $x_n$-chart of $W'$. The chart expression of $\pi$ in the $x_n$-chart is of the form

$$\begin{aligned} x_i &\mapsto x_i && \text{for } i \leq n-k \text{ and } i = n, \\ x_i &\mapsto x_i x_n && \text{for } n-k+1 \leq i \leq n-1. \end{aligned}$$

(6) In the induced coordinates of the $x_n$-chart, the point $a'$ has components $a' = (0, \ldots, 0, a'_{n-k+2}, \ldots, a'_{n-k+d}, 0, \ldots, 0)$ with non-zero entries $a'_j \in \mathbb{K}$ for $n-k+2 \leq j \leq n-k+d$, where $d$ is the number of components of $D$ whose strict transforms do not pass through $a'$.

(7) The strict transform (Def. 6.2) $V^s$ of $V$ in $W'$ is given in the induced coordinates locally at $a'$ by $x_{n-k+1} = 0$.

(8) The local coordinate change $\varphi$ in $W$ at $a$ given by $\varphi(x_i) = x_i + a'_i \cdot x_n$ makes the local blowup $\pi : (W', a') \to (W, a)$ monomial. It preserves the defining ideals of $Z$ and $V$ in $W$.

(9) If condition (4) is not imposed, the coordinates $x_1, \ldots, x_n$ at $a$ can be chosen with (1) to (3) and so that $a'$ is the origin of the $x_n$-chart.

Proof. [Hau10b]. □

Theorem 5.5. Any projective birational morphism $\pi : X' \to X$ is a blowup of $X$ in an ideal $I$.

Proof. [Har77], Chap. II, Thm. 7.17. □

## *Examples*

Example 5.6. Let $X$ be a regular subvariety of $\mathbb{A}^n$ and $Z$ a regular closed subvariety which is transversal to $X$. Show that the blowup $X'$ of $X$ along $Z$ is again a regular variety (this is a special case of Prop. 5.3).

Example 5.7.$^+$ Prove that plain varieties remain plain under blowup in regular centers [BHSV08] Thm. 4.3.

Example 5.8.$^+$ Is any rational and regular variety plain?

Example 5.9. Consider the blowup $\pi : W' \to W$ of a regular variety $W$ along a closed subvariety $Z$. Show that, for any chosen definition of blowup, the exceptional divisor $E = \pi^{-1}(Z)$ is a hypersurface in $W'$.

Example 5.10. The composition of two blowups $W'' \to W'$ and $W' \to W$ is a blowup of $W$ in a suitable center [Bod03].

Example 5.11. A *fractional ideal* $I$ over an integral domain $R$ is an $R$-submodule of $\mathrm{Quot}(R)$ such that $rI \subset R$ for some non-zero element $r \in R$. Blowups can be defined via Proj also for centers which are fractional ideals [Gro61]. Let $I$ and $J$ be two (ordinary) non-zero ideals of $R$. The blowup of $R$ along $I$ is isomorphic to the blowup of $R$ along $J$ if and only if there exist positive integers $k$, $\ell$ and fractional ideals $K$, $L$ over $R$ such that $JK = I^k$ and $IL = J^\ell$ [Moo01] Cor. 2.

EXAMPLE 5.12. ▷ Determine the equations in $X \times \mathbb{P}^2$ of the blowup of $X = \mathbb{A}^3$ along the image $Z$ of the monomial curve $(t^3, t^4, t^5)$ of equations $g_1 = y^2 - xz$, $g_2 = yz - x^3$, $g_3 = z^2 - x^2y$.

EXAMPLE 5.13. ▷ The blowup of the cone $X = V(x^2 - yz)$ in $\mathbb{A}^3$ along the $z$-axis $Z = V(x, y)$ is an isomorphism locally at all points outside 0, but not globally on $X$.

EXAMPLE 5.14. ▷ Blow up the non-reduced point $X = V(x^2)$ in $\mathbb{A}^2$ in the (reduced) origin $Z = 0$.

EXAMPLE 5.15. ▷ Blow up the subscheme $X = V(x^2, xy)$ of $\mathbb{A}^2$ in the reduced origin.

EXAMPLE 5.16. Blow up the subvariety $X = V(xz, yz)$ of $\mathbb{A}^3$ first in the origin, then in the $x$-axis, and determine the points where the resulting morphisms are local isomorphisms.

EXAMPLE 5.17. ▷ The blowups of $\mathbb{A}^3$ along the union of the $x$- with the $y$-axis, respectively along the cusp, with ideals $(xy, z)$ and $(x^3 - y^2, z)$, are singular.

## 6. Lecture VI: Transforms of Ideals and Varieties under Blowup

Throughout this section, $\pi : W' \to W$ denotes the blowup of a variety $W$ along a closed subvariety $Z$, with exceptional divisor $E = \pi^{-1}(Z)$ defined by the principal ideal $I_E$ of $\mathcal{O}_{W'}$. Let $\pi^* : \mathcal{O}_W \to \mathcal{O}_{W'}$ be the dual homomorphism of $\pi$. Let $X \subset W$ be a closed subvariety, and let $I$ be an ideal on $W$. Several of the subsequent definitions and results can be extended to the case of arbitrary birational morphisms, taking for $Z$ the complement of the open subset of $W$ where the inverse map of $\pi$ is defined.

DEFINITION 6.1. The inverse image $X^* = \pi^{-1}(X)$ of $X$ and the extension $I^* = \pi^*(I) = I \cdot \mathcal{O}_{W'}$ of $I$ are called the *total transform* of $X$ and $I$ under $\pi$. For $f \in \mathcal{O}_W$, denote by $f^*$ its image $\pi^*(f)$ in $\mathcal{O}_{W'}$. The ideal $I^*$ is generated by all transforms $f^*$ for $f$ varying in $I$. If $I$ is the ideal defining $X$ in $W$, the ideal $I^*$ defines $X^*$ in $W'$. In particular, the total transform of the center $Z$ equals the exceptional divisor $E$, and $W^* = W'$. In the category of schemes, the total transform will in general be non-reduced when considered as a subscheme of $W'$.

DEFINITION 6.2. The Zariski closure of $\pi^{-1}(X \setminus Z)$ in $W'$ is called the *strict transform* of $X$ under $\pi$ and denoted by $X^s$, also known as the *proper* or *birational transform*. The strict transform is a closed subvariety of the total transform $X^*$. The difference $X^* \setminus X^s$ is contained in the exceptional locus $E$. If $Z \subset X$ is contained in $X$, the strict transform $X^s$ of $X$ in $W'$ equals the blowup $\widetilde{X}$ of $X$ along $Z$, cf. Prop. 5.1 and its corollary. If the center $Z$ coincides with $X$, the strict transform $X^s$ is empty.

Let $I_U$ denote the restriction of an ideal $I$ to the open set $U = W \setminus Z$. Set $U' = \pi^{-1}(U) \subset W'$ and let $\tau : U' \to U$ be the restriction of $\pi$ to $U'$. The *strict transform* $I^s$ of the ideal $I$ is defined as $\tau^*(I_U) \cap \mathcal{O}_{W'}$. If the ideal $I$ defines $X$ in $W$, the ideal $I^s$ defines $X^s$ in $W'$. It equals the union of colon ideals

$$I^s = \bigcup_{i \geq 0} (I^* : I_E^i).$$

Let $h$ be an element of $\mathcal{O}_{W',a'}$ defining $E$ locally in $W'$ at a point $a'$. Then, locally at $a'$,

$$I^s = (f^s,\, f \in I),$$

where the strict transform $f^s$ of $f$ is defined at $a'$ and up to multiplication by invertible elements in $\mathcal{O}_{W',a'}$ through $f^* = h^k \cdot f^s$ with maximal exponent $k$. The value of $k$ is the order of $f^*$ along $E$, cf. Def. 8.4. By abuse of notation this is written as $f^s = h^{-k} \cdot f^* = h^{-\mathrm{ord}_Z f} \cdot f^*$.

Lemma 6.3. Let $f : R \to S$ be a ring homomorphism, $I$ an ideal of $R$ and $s$ an element of $R$, with induced ring homomorphism $f_s : R_s \to S_{f(s)}$. Let $I^e = f(I) \cdot S$ and $(I \cdot R_s)^e = f_s(I \cdot R_s) \cdot S_{f(s)}$ denote the respective extensions of ideals. Then $\bigcup_{i \geq 0}(I^e : f(s)^i) = (I \cdot R_s)^e \cap S$.

Proof. Let $u \in S$. Then $u \in \bigcup_{i \geq 0}(I^e : f(s)^i)$ if and only if $uf(s)^i \in I^e$ for some $i \geq 0$, say $uf(s)^i = \sum_j a_j f(x_j)$ for elements $x_j \in I$ and $a_j \in S$. Rewrite this as $u = \sum_j a_j f(\frac{x_j}{s^i})$. This just means that $u \in (I \cdot R_s)^e$. □

Remark 6.4. If $I$ is generated locally by elements $f_1, \dots, f_k$ of $\mathcal{O}_W$, then $I^s$ contains the ideal generated by the strict transforms $f_1^s, \dots, f_k^s$ of $f_1, \dots, f_k$, but the inclusion can be strict, see the examples below.

Definition 6.5. Let $\mathbb{K}[x] = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring over $\mathbb{K}$, considered with the natural grading given by the degree. Denote by $\mathrm{in}(g)$ the homogeneous form of lowest degree of a non-zero polynomial $g$ of $\mathbb{K}[x]$, called the *initial form* of $g$. Set $\mathrm{in}(0) = 0$. For a non-zero ideal $I$, denote by $\mathrm{in}(I)$ the ideal generated by all initial forms $\mathrm{in}(g)$ of elements $g$ of $I$, called the *initial ideal* of $I$. Elements $g_1, \dots, g_k$ of an ideal $I$ of $\mathbb{K}[x]$ are a *Macaulay basis* of $I$ if their initial forms $\mathrm{in}(g_1), \dots, \mathrm{in}(g_k)$ generate $\mathrm{in}(I)$. In [Hir64] III.1, Def. 3, p. 208, such a basis was called a *standard basis*, which is now used for a slightly more specific concept, see Rem. 6.7 below. By noetherianity of $\mathbb{K}[x]$, any ideal possesses a Macaulay basis.

Proposition 6.6. (Hironaka) The strict transform of an ideal under blowup in a regular center is generated by the strict transforms of the elements of a Macaulay basis of the ideal.

Proof. ([Hir64], III.2, Lemma 6, p. 216, and III.6, Thm. 5, p. 238) If $I \subset J$ are two ideals of $\mathbb{K}[[x]]$ such that $\mathrm{in}(I) = \mathrm{in}(J)$ then they are equal, $I = J$. This holds at least for degree compatible monomial orders, due to the Grauert-Hironaka-Galligo division theorem. Therefore it has to be shown that $\mathrm{in}(g_1^s), \dots, \mathrm{in}(g_k^s)$ generate $\mathrm{in}(I^s)$. But $\mathrm{in}(I^s) = (\mathrm{in}(I))^s$, and the assertion follows. □

Remark 6.7. The strict transform of a Macaulay basis at a point $a'$ of $W'$ need no longer be a Macaulay basis. This is however the case if the Macaulay basis is *reduced* and the sequence of its orders has remained constant at $a'$, cf. [Hir64] III.8, Lemma 20, p. 254. More generally, taking on $\mathbb{K}[x]$ instead of the grading by degree a grading so that all homogeneous pieces are one-dimensional and generated by monomials (i.e., a grading induced by a monomial order on $\mathbb{N}^n$), the initial form of a polynomial and the initial ideal are both monomial. In this case Macaulay bases are called *standard bases*. A standard basis $g_1, \dots, g_k$ is *reduced* if no monomial of the tails $g_i - \mathrm{in}(g_i)$ belongs to $\mathrm{in}(I)$. If the monomial order is degree compatible, i.e., the induced grading a refinement of the natural grading of $\mathbb{K}[x]$ by degree, the strict transforms of the elements of a standard basis of $I$ generate the strict transform of the ideal.

Definition 6.8. Let $I$ be an ideal on $W$ and let $c \geq 0$ be a natural number less than or equal to the order $d$ of $I$ along the center $Z$, cf. Def. ref844. Then $I^*$ has order $\geq c$ along $E$. There exists a unique ideal $I^!$ of $\mathcal{O}_{W'}$ such that $I^* = I_E^c \cdot I^!$, called the *controlled transform* of $I$ with respect to the *control* $c$. It is not defined for values of $c > d$. In case that $c = d$ attains the maximal value, $I^!$ is denoted by $I^\curlyvee$ and called the *weak transform* of $I$. It is written as $I^\curlyvee = I_E^{-d} \cdot I^* = I_E^{-\mathrm{ord}_Z I} \cdot I^*$.

Remark 6.9. The inclusions $I^* \subset I^\curlyvee \subset I^! \subset I^s$ are obvious. The components of $V(I^\curlyvee)$ which are not contained in $V(I^s)$ lie entirely in the exceptional divisor $E$, but can be strictly contained. For principal ideals, $I^\curlyvee$ and $I^s$ coincide. When the transforms are defined scheme-theoretically, the reduction $X^*_{red}$ of the total transform $X^*$ of $X$ consists of the union of $E$ with the strict transform $X^s$.

Definition 6.10. A *local flag* $\mathcal{F}$ on $W$ at $a$ is a chain $F_0 = \{a\} \subset F_1 \subset \ldots \subset F_n = W$ of regular closed subvarieties, respectively subschemes, $F_i$ of dimension $i$ of an open neighbourhood $U$ of $a$ in $W$. Local coordinates $x_1, \ldots, x_n$ on $W$ at $a$ are called *subordinate to the flag* $\mathcal{F}$ if $F_i = V(x_{i+1}, \ldots, x_n)$ locally at $a$. The flag $\mathcal{F}$ at $a$ is *transversal* to a regular subvariety $Z$ of $W$ if each $F_i$ is transversal to $Z$ at $a$ [Hau04, Pan06].

Proposition 6.11. Let $\pi : W' \to W$ be the blowup of $W$ along a center $Z$ transversal to a flag $\mathcal{F}$ at $a \in Z$. Let $x_1, \ldots, x_n$ be local coordinates on $W$ at $a$ subordinate to $\mathcal{F}$. At each point $a'$ of $E$ above $a$ there exists a unique local flag $\mathcal{F}'$ such that the coordinates $x'_1, \ldots, x'_n$ on $W'$ at $a'$ induced by $x_1, \ldots, x_n$ as in Def. 4.12 are subordinate to $\mathcal{F}'$ [Hau04] Thm. 1.

Definition 6.12. The flag $\mathcal{F}'$ is called the *transform of* $\mathcal{F}$ *under* $\pi$.

Proof. It suffices to define $F'_i$ at $a'$ by $x'_{i+1}, \ldots, x'_n$. For point blowups in $W$, the transform of $\mathcal{F}$ is defined as follows. The point $a' \in E$ is determined by a line $L$ in the tangent space $\mathrm{T}_a W$ of $W$ at $a$. Let $k \leq n$ be the minimal index for which $\mathrm{T}_a F_k$ contains $L$. For $i < k$, choose a regular $(i+1)$-dimensional subvariety $H_i$ of $W$ with tangent space $L + \mathrm{T}_a F_i$ at $a$. In particular, $\mathrm{T}_a H_{k-1} = \mathrm{T}_a F_k$. Let $H_i^s$ be the strict transform of $H_i$ in $W'$. Then set

$$\begin{aligned} F'_i &= E \cap H_i^s && \text{for } i < k, \\ F'_i &= F_i^s && \text{for } i \geq k, \end{aligned}$$

to get the required flag $\mathcal{F}'$ at $a'$ . □

## *Examples*

Example 6.13. Blow up $\mathbb{A}^2$ in 0 and compute the inverse image of the zerosets of $x^2 + y^2 = 0$, $xy = 0$ and $x(x - y^2) = 0$, as well as their strict transforms.

Example 6.14.$^\triangleright$ Compute the strict transform of $X = V(x^2 - y^3, xy - z^3) \subset \mathbb{A}^3$ under the blowup of the origin.

Example 6.15. Determine the total, weak and strict transform of $X = V(x^2 - y^3, z^3) \subset \mathbb{A}^3$ under the blowup of the origin. Clarify the algebraic and geometric differences between them.

Example 6.16. Blow up $\mathbb{A}^3$ along the curve $y^2 - x^3 + x = z = 0$ and compute the strict transform of the lines $x = z = 0$ and $y = z = 0$.

Example 6.17. Let $I_1$ and $I_2$ be ideals of $\mathbb{K}[x]$ of order $c_1$ and $c_2$ at 0. Take the blowup of $\mathbb{A}^n$ at zero. Show that the weak transform of $I_1^{c_2} + I_2^{c_1}$ is the sum of the weak transforms of $I_1^{c_2}$ and $I_2^{c_1}$.

Example 6.18.$^\triangleright$ For $W = \mathbb{A}^3$, a flag at a point $a$ consists of a regular curve $F_1$ through $a$ and contained in a regular surface $F_2$. Blow up the point $a$ in $W$, so that $E \simeq \mathbb{P}^2$ is the projective plane. The induced flag at a point $a'$ above $a$ depends on the location of $a$ on $E$: At the intersection point $p_1$ of the strict transform $C_1 = F_1^s$ of $F_1$ with $E$, the transformed flag $\mathcal{F}'$ is given by $F_1^s \subset F_2^s$. Along the intersection $C_2$ of $F_2^s$ with $E$, the flag $\mathcal{F}'$ is given at each point $p_2$ different from $p_1$ by $C_2 \subset F_2^s$. At any point $p_3$ not on $C_2$ the flag $\mathcal{F}'$ is given by $C_3 \subset E$, where $C_3$ is the projective line in $E$ through $p_1$ and $p_3$.

Example 6.19.$^\triangleright$ Blowing up a regular curve $Z$ in $W = \mathbb{A}^3$ transversal to $\mathcal{F}$ there occur six possible configurations of $Z$ with respect to $\mathcal{F}$. Denoting by $L$ the plane in the tangent space $\mathrm{T}_aW$ of $W$ at $a$ corresponding to the point $a'$ in $E$ above $a$, these are: (1) $Z = F_1$ and $L = \mathrm{T}_aF_2$, (2) $Z = F_1$ and $L \neq \mathrm{T}_aF_2$, (3) $Z \neq F_1$, $Z \subset F_2$ and $L = \mathrm{T}_aF_2$, (4) $Z \neq F_1$, $Z \subset F_2$ and $L \neq \mathrm{T}_aF_2$, (5) $Z \not\subset F_2$ and $\mathrm{T}_aF_1 \subset L$, (6) $Z \not\subset F_2$ and $\mathrm{T}_aF_1 \not\subset L$. Determine in each case the flag $\mathcal{F}'$.

Example 6.20. Let $\widetilde{\mathbb{A}}^n \to \mathbb{A}^n$ be the blowup of $\mathbb{A}^n$ in 0 and let $a'$ be the origin of the $x_n$-chart of $\widetilde{\mathbb{A}}^n$. Compute the total and strict transforms $g^*$ and $g^s$ for $g = x_1^d + \ldots + x_{n-1}^d + x_n^e$ for $e = d, 2d-1, 2d, 2d+1$ and $g = \prod_{i\neq j}(x_i - x_j)$.

Example 6.21. Determine the total, weak and strict transform of $X = V(x^2 - y^3, z^3) \subset \mathbb{A}^3$ under the blowup of $\mathbb{A}^3$ at the origin. Point out the geometric differences between the three types of transforms.

Example 6.22.$^\triangleright$ The inclusion $I^\curlyvee \subset I^s$ can be strict. Blow up $\mathbb{A}^2$ at 0, and consider in the $y$-chart of $\widetilde{\mathbb{A}}^2$ the transforms of $I = (x^2, y^3)$. Show that $I^\curlyvee = (x^2, y)$, $I^s = (x^2, 1) = \mathbb{K}[x, y]$.

Example 6.23.$^\triangleright$ Let $X = V(f)$ be a hypersurface in $\mathbb{A}^n$ and $Z$ a regular closed subvariety which is contained in the locus of points of $X$ where $f$ attains its maximal order. Let $\pi : \widetilde{\mathbb{A}}^n \to \mathbb{A}^n$ be the blowup along $Z$ and let $X' = V(f')$ be the strict transform of $X$. Show that for points $a \in Z$ and $a' \in E$ with $\pi(a') = a$ the inequality $\mathrm{ord}_{a'}f' \leq \mathrm{ord}_a f$ holds.

## 7. Lecture VII: Resolution Statements

Definition 7.1. A *non-embedded resolution* of the singularities of a variety $X$ is a non-singular variety $\widetilde{X}$ together with a proper birational morphism $\pi : \widetilde{X} \to X$ which induces a biregular isomorphism $\pi : \widetilde{X} \setminus E \to X \setminus \mathrm{Sing}(X)$ outside $E = \pi^{-1}(\mathrm{Sing}(X))$.

Remark 7.2. Requiring properness excludes trivial cases as e.g. taking for $\widetilde{X}$ the locus of regular points of $X$ and for $\pi$ the inclusion map. One may ask for additional properties: (a) Any automorphism $\varphi : X \to X$ of $X$ shall lift to an automorphism $\widetilde{\varphi} : \widetilde{X} \to \widetilde{X}$ of $\widetilde{X}$ which commutes with $\pi$, i.e., $\pi \circ \widetilde{\varphi} = \varphi \circ \pi$. (b) If $X$ is defined over $\mathbb{K}$ and $\mathbb{K} \subset \mathbb{L}$ is a field extension, any resolution of $X_{\mathbb{L}} = X \times_{\mathbb{K}} \mathrm{Spec}(\mathbb{L})$ shall induce a resolution of $X = X_{\mathbb{K}}$.

Definition 7.3. A *local non-embedded resolution* of a variety $X$ at a point $a$ is the germ $(\widetilde{X}, a')$ of a non-singular variety $\widetilde{X}$ together with a local morphism $\pi : (\widetilde{X}, a') \to (X, a)$ inducing an isomorphism of the function fields of $\widetilde{X}$ and $X$.

Definition 7.4. Let $X$ be an affine irreducible variety with coordinate ring $R = \mathbb{K}[X]$. A (local, ring-theoretic) *non-embedded resolution* of $X$ is a ring extension $R \hookrightarrow \widetilde{R}$ of $R$ into a regular ring $\widetilde{R}$ having the same quotient field as $R$.

Definition 7.5. Let $X$ be a subvariety of a regular ambient variety $W$. An *embedded resolution* of $X$ consists of a proper birational morphism $\pi : \widetilde{W} \to W$ from a regular variety $\widetilde{W}$ onto $W$ which is an isomorphism over $W \setminus \mathrm{Sing}(X)$ such that the strict transform $X^s$ of $X$ is regular and the total transform $X^* = \pi^{-1}(X)$ of $X$ has simple normal crossings.

Definition 7.6. A *strong resolution* of a variety $X$ is, for each closed embedding of $X$ into a regular variety $W$, a birational proper morphism $\pi : \widetilde{W} \to W$ satisfying the following five properties [EH02]:

*Embeddedness.* The variety $\widetilde{W}$ and the strict transform $X^s$ of $X$ are regular, and the total transform $X^* = \pi^{-1}(X)$ of $X$ in $\widetilde{W}$ has simple normal crossings.

*Equivariance.* Let $W' \to W$ be a smooth morphism and let $X'$ be the inverse image of $X$ in $W'$. The morphism $\pi' : \widetilde{W}' \to W'$ induced by $\pi : \widetilde{X} \to W$ by taking fiber product of $\widetilde{X}$ and $W'$ over $W$ is an embedded resolution of $X'$.

*Excision.* The restriction $\tau : \widetilde{X} \to X$ of $\pi$ to $X$ does not depend on the choice of the embedding of $X$ in $W$.

*Explicitness.* The morphism $\pi$ is a composition of blowups along regular centers which are transversal to the exceptional loci created by the earlier blowups.

*Effectiveness.* There exists, for all varieties $X$, a local upper semicontinuous invariant $\mathrm{inv}_a(X)$ in a well-ordered set $\Gamma$, depending only and up to isomorphism of the completed local rings of $X$ at $a$, such that: (a) $\mathrm{inv}_a(X)$ attains its minimal value if and only if $X$ is regular at $a$ (or has normal crossings at $a$); (b) the top locus $S$ of $\mathrm{inv}_a(X)$ is closed and regular in $X$; (c) blowing up $X$ along $S$ makes $\mathrm{inv}_a(X)$ drop at all points $a'$ above the points $a$ of $S$.

Remark 7.7. (a) Equivariance implies the *economy* of the resolution, i.e., that $\pi : X^s \to X$ is an isomorphism outside $\mathrm{Sing}(X)$. It also implies that $\pi$ commutes with open immersions, localization, completion, automorphisms of $W$ stabilizing $X$ and taking cartesian products with regular varieties.

(b) One may require in addition that the centers of a resolution are transversal to the inverse images of a given normal crossings divisor $D$ in $W$, the *boundary.*

Definition 7.8. Let $I$ be an ideal on a regular ambient variety $W$. A *log-resolution* of $I$ is a proper birational morphism $\pi : \widetilde{W} \to W$ from a regular variety $\widetilde{W}$ onto $W$ which is an isomorphism over $W \setminus \mathrm{Sing}(I)$ such that $I^* = \pi^{-1}(I)$ is a locally monomial ideal on $\widetilde{W}$.

*Examples*

Example 7.9. The blowup of the cusp $X : x^2 = y^3$ in $\mathbb{A}^2$ at 0 produces a non-embedded resolution. Further blouwps give an embedded resolution.

Example 7.10.$^\triangleright$ Determine the geometry at 0 of the hypersurfaces in $\mathbb{A}^4$ defined by the following equations:
(a) $x + x^7 - 3yw + y^7z^2 + 17yzw^5 = 0$,
(b) $x + x^5y^2 - 3yw + y^7z^2 + 17yzw^5 = 0$,
(c) $x^3y^2 + x^5y^2 - 3yw + y^7z^2 + 17yzw^5 = 0$.

Example 7.11. Let $X$ be the variety in $\mathbb{A}^3_{\mathbb{C}}$ given by the equation $(x^2 - y^3)^2 = (z^2 - y^2)^3$. Show that the map $\alpha : \mathbb{A}^3 \to \mathbb{A}^3 : (x, y, z) \to (u^2z^3, uyz^2, uz^2)$, with $u(x, y) = x(y^2 - 1) + y$, resolves the singularities of $X$. What is the inverse image $\alpha^{-1}(Z)$? Produce instructive pictures of $X$ over $\mathbb{R}$.

Example 7.12. Consider the inverse images of the cusp $X = V(y^2 - x^3)$ in $\mathbb{A}^2$ under the maps $\pi_x, \pi_y : \mathbb{A}^2 \to \mathbb{A}^2$, $\pi_x(x, y) = (x, xy)$, $\pi_y(x, y) = (xy, y)$. Factor the maximal power of $x$, respectively $y$, from the equation of the inverse image of $X$ and show that the resulting equation defines in both cases a regular variety. Apply the same process to the variety $E_8 = V(x^2 + y^3 + z^5) \subset \mathbb{A}^3$ repeatedly until all resulting equations define non-singular varieties.

Example 7.13. Let $R$ be the coordinate ring of an irreducible plane algebraic curve $X$. The integral closure $\widetilde{R}$ of $R$ in the field of fractions of $R$ is a regular ring and thus resolves $R$. The resulting curve $\widetilde{X}$ is the normalization of $X$ [Mum99] III.8, [dJP00] 4.4.

Example 7.14.$^\triangleright$ Consider a cartesian product $X = Y \times Z$ with $Z$ a regular variety. Show that a resolution of $X$ can be obtained from a resolution $Y'$ of $Y$ by taking the cartesian product $Y' \times Z$ of $Y'$ with $Z$.

Example 7.15. Let $X$ and $Y$ be two varieties (schemes, analytic spaces) with singular loci $\mathrm{Sing}(X)$ and $\mathrm{Sing}(Y)$ respectively. Suppose that $X'$ and $Y'$ are regular varieties (schemes, analytic spaces, within the same category as $X$ and $Y$) together with proper birational morphisms $\pi : X' \to X$ and $\tau : Y' \to Y$ which define resolutions of $X$ and $Y$ respectively, and which are isomorphisms outside $\mathrm{Sing}(X)$ and $\mathrm{Sing}(Y)$. There is a naturally defined proper birational morphism $f : X' \times Y' \to X \times Y$ giving rise to a resolution of $X \times Y$.

## 8. Lecture VIII: Invariants of Singularities

Definition 8.1. A *stratification* of an algebraic variety $X$ is a decomposition of $X$ into finitely many disjoint locally closed subvarieties $X_i$, called the *strata*,

$$X = \dot{\bigcup}_i X_i,$$

such that the *boundaries* $\overline{X}_i \setminus X_i$ of strata are unions of strata. This last property is called the *frontier condition*. Two strata are called *adjacent* if one lies in the closure of the other.

Definition 8.2. A *local invariant* on an algebraic variety $X$ is a function $\mathrm{inv}(X) : X \to \Gamma$ from $X$ to a well-ordered set $(\Gamma, \leq)$ which associates to each point $a \in X$ an element $\mathrm{inv}_a(X)$ depending only on the formal isomorphism class of $X$ at

$a$: If $(X,a)$ and $(X,b)$ are formally isomorphic, viz $\widehat{\mathcal{O}}_{X,a} \simeq \widehat{\mathcal{O}}_{X,b}$, then $\mathrm{inv}_a(X) = \mathrm{inv}_b(X)$. Usually, the ordering on $\Gamma$ will also be total: for any $c, d \in \Gamma$ either $c \leq d$ or $d \leq c$ holds. The invariant is *upper semicontinuous along a subvariety $S$ of $X$* if for all $c \in \Gamma$, the sets

$$\mathrm{top}_S(\mathrm{inv}, c) = \{a \in S,\ \mathrm{inv}_a(X) \geq c\}$$

are closed in $S$. If $S = X$, the map $\mathrm{inv}(X)$ is called *upper semicontinuous.*

Remark 8.3. The upper semicontinuity signifies that the value of $\mathrm{inv}_a(X)$ can only go up or remain the same when $a$ approaches a limit point. In the case of schemes, the value of $\mathrm{inv}(X)$ also has to be defined and taken into account at non-closed points of $X$.

Definition 8.4. Let $X$ be a subvariety of a not necessarily regular ambient variety $W$ defined by an ideal $I$, and let $Z$ be an irreducible subvariety of $W$ defined by the prime ideal $J$. The *order* of $X$ or $I$ in $W$ along $Z$ or with respect to $J$ is the maximal integer $k = \mathrm{ord}_Z(X) = \mathrm{ord}_Z(I)$ such that $I_Z \subset J_Z^k$, where $I_Z = I \cdot \mathcal{O}_{W,Z}$ and $J_Z = J \cdot \mathcal{O}_{W,Z}$ denote the ideals generated by $I$ and $J$ in the localization $\mathcal{O}_{W,Z}$ of $W$ along $Z$. If $Z = \{a\}$ is a point of $W$, the order of $X$ and $I$ at $a$ is denoted by $\mathrm{ord}_a(X) = \mathrm{ord}_a(I)$ or $\mathrm{ord}_{m_a}(X) = \mathrm{ord}_{m_a}(I)$.

Definition 8.5. Let $R$ be a local ring with maximal ideal $m$. Let $k \in \mathbb{N}$ be an integer. The $k$-th symbolic power $J^{(k)}$ of a prime ideal $J$ is defined as the ideal generated by all elements $x \in R$ for which there is an element $y \in R \setminus J$ such that $y \cdot x^k \in J^k$. Equivalently, $J^{(k)} = J^k \cdot R_J \cap R$.

Remark 8.6. The symbolic power is the smallest $J$-primary ideal containing $J^k$. If $J$ is a complete intersection, the ordinary power $J^k$ and the symbolic power $J^{(k)}$ coincide [ZS75] IV, [Hoc73], §12, [Pel88].

Proposition 8.7. Let $X$ be a subvariety of a not necessarily regular ambient variety $W$ defined by an ideal $I$, and let $Z$ be an irreducible subvariety of $W$ defined by the prime ideal $J$. The order of $X$ along $Z$ is the maximal integer $k$ such that $I \subset J^{(k)}$.

Proof. This follows from the equality $J^k \cdot R_J = (J \cdot R_J)^k$. □

Proposition 8.8. Let $R$ be a noetherian local ring with maximal ideal $m$, and let $\widehat{R}$ denote its completion with maximal ideal $\widehat{m} = m \cdot \widehat{R}$. Let $I$ be an ideal of $R$, with completion $\widehat{I} = I \cdot \widehat{R}$. Then $\mathrm{ord}_m(I) = \mathrm{ord}_{\widehat{m}}(\widehat{I})$.

Proof. If $I \subset m^k$, then also $\widehat{I} \subset \widehat{m}^k$, and hence $\mathrm{ord}_m(I) \leq \mathrm{ord}_{\widehat{m}}(\widehat{I})$. Conversely, $m = \widehat{m} \cap R$ and $\bigcap_{i \geq 0}(I + m^i) = \widehat{I} \cap R$ by Lemma 2.26. Hence, if $\widehat{I} \subset \widehat{m}^k$, then $I \subset \widehat{I} \cap R \subset \widehat{m}^k \cap R = m^k$, so that $\mathrm{ord}_m(I) \geq \mathrm{ord}_{\widehat{m}}(\widehat{I})$. □

Definition 8.9. Let $X$ be a subvariety of a regular variety $W$, and let $a$ be a point of $W$. The *local top locus* $\mathrm{top}_a(X)$ of $X$ at $a$ with respect to the order is the stratum $S$ of points of an open neighborhood $U$ of $a$ in $W$ where the order of $X$ equals the order of $X$ at $a$. The *top locus* $\mathrm{top}(X)$ of $X$ with respect to the order is the (global) stratum $S$ of points of $W$ where the order of $X$ attains its maximal value. For $c \in \mathbb{N}$, define $\mathrm{top}_a(X,c)$ and $\mathrm{top}(X,c)$ as the local and global stratum of points of $W$ where the order of $X$ is at least $c$.

REMARK 8.10. The analogous definition holds for ideals on $W$ and can be made for other local invariants. By the upper semicontinuity of the order, the local top locus of $X$ at $a$ is locally closed in $W$, and the top locus of $X$ is closed in $W$.

PROPOSITION 8.11. The order of a variety $X$ or an ideal $I$ in a regular variety $W$ at points of $W$ defines an upper semicontinuous local invariant on $W$.

PROOF. In characteristic zero, the assertion follows from the next proposition. For the case of positive characteristic, see [Hir64] III.3, Cor. 1, p. 220. □

PROPOSITION 8.12. Over fields of zero characteristic, the local top locus $\mathrm{top}_a(I)$ of an ideal $I$ at $a$ is defined by the vanishing of all partial derivatives of elements of $I$ up to order $o-1$, where $o$ is the order of $I$ at $a$.

PROOF. In zero characteristic, a polynomial has order $o$ at a point $a$ if and only if all its partial derivatives up to order $o-1$ vanish at $a$. □

PROPOSITION 8.13. Let $X$ be a subvariety of a regular ambient variety $W$. Let $Z$ be a non-singular subvariety of $W$ and $a$ a point on $Z$ such that locally at $a$ the order of $X$ is constant along $Z$, say equal to $d = \mathrm{ord}_a X = \mathrm{ord}_Z X$. Consider the blowup $\pi : W' \to W$ of $W$ along $Z$ with exceptional divisor $E = \pi^{-1}(Z)$. Let $a'$ be a point on $E$ mapping under $\pi$ to $a$. Denote by $X^*$, $X^\curlyvee$ and $X^s$ the total, weak and strict transform of $X$ respectively (Def. 6.1 and 6.2). Then, locally at $a'$, the order of $X^*$ along $E$ is $d$, and

$$\mathrm{ord}_{a'} X^s \leq \mathrm{ord}_{a'} X^\curlyvee \leq d.$$

PROOF. By Prop. 5.4 there exist local coordinates $x_1, \ldots, x_n$ of $W$ at $a$ such that $Z$ is defined locally at $a$ by $x_1 = \ldots = x_k = 0$ for some $k \leq n$, and such that $a'$ is the origin of the $x_1$-chart of the blowup. Let $I \subset \mathcal{O}_{W,a}$ be the local ideal of $X$ in $W$ at $a$, and let $f$ be an element of $I$. It has an expansion $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$ with respect to the coordinates $x_1, \ldots, x_n$, with coefficients $c_\alpha \in \mathbb{K}$.

Set $\alpha_+ = (\alpha_1, \ldots, \alpha_k)$. Since the order of $X$ is constant along $Z$, the inequality $|\alpha_+| = \alpha_1 + \ldots + \alpha_k \geq d$ holds whenever $c_\alpha \neq 0$. There is an element $f$ with an exponent $\alpha$ such that $c_\alpha \neq 0$ and such that $|\alpha| = |\alpha_+| = d$. The total transform $X^*$ and the weak transform $X^\curlyvee$ are given locally at $a'$ by the ideal generated by all

$$f^* = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x_1^{|\alpha_+|} x_2^{\alpha_2} \ldots x_n^{\alpha_n},$$

respectively

$$f^\curlyvee = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x_1^{|\alpha_+|-d} x_2^{\alpha_2} \ldots x_n^{\alpha_n},$$

for $f$ varying in $I$. The exceptional divisor $E$ is given locally at $a'$ by the equation $x_1 = 0$. This implies that, locally at $a'$, $\mathrm{ord}_E f^* \geq d$ for all $f \in I$ and $\mathrm{ord}_E f^* = d$ for the special $f$ chosen above with $|\alpha| = |\alpha_+| = d$. Therefore $\mathrm{ord}_E X^* = d$ locally at $a'$.

Since $\mathrm{ord}_{a'} f^\curlyvee \leq |\alpha_-| \leq d$ for the chosen $f$, it follows that $\mathrm{ord}_{a'} X^\curlyvee \leq d$. The ideal $I^s$ of the strict transform $X^s$ contains the ideal $I^\curlyvee$ of the weak transform $X^\curlyvee$, thus also $\mathrm{ord}_{a'} X^s \leq \mathrm{ord}_{a'} X^\curlyvee$. □

COROLLARY 8.14. If the order of $X$ is globally constant along $Z$, the order of $X^*$ along $E$ is globally equal to $d$.

Definition 8.15. Let $X$ be a subvariety of a regular variety $W$, and let $W' \to W$ be a blowup with center $Z$. Denote by $X'$ the strict or weak transform in $W'$. A point $a' \in W'$ above $a \in Z$ is called *infinitesimally near to a* or *equiconstant* if $\mathrm{ord}_{a'}X' = \mathrm{ord}_a X$.

Definition 8.16. Let $X$ be a variety defined over a field $\mathbb{K}$, and let $a$ be a point of $W$. The *Hilbert-Samuel function* $\mathrm{HS}_a(X) : \mathbb{N} \to \mathbb{N}$ of $X$ at $a$ is defined by

$$\mathrm{HS}_a(X)(k) = \dim_{\mathbb{K}}(m_{X,a}^k / m_{X,a}^{k+1}),$$

where $m_{X,a}$ denotes the maximal ideal of the local ring $\mathcal{O}_{X,a}$ of $X$ at $a$. If $X$ is a subvariety of a regular variety $W$ defined by an ideal $I$, with local ring $\mathcal{O}_{X,a} = \mathcal{O}_{W,a}/I$, one also writes $\mathrm{HS}_a(I)$ for $\mathrm{HS}_a(X)$.

Remark 8.17. The Hilbert-Samuel function does not depend on the embedding of $X$ in $W$. There exists a univariate polynomial $P(t) \in \mathbb{Q}[t]$, called the *Hilbert-Samuel polynomial* of $X$ at $a$, such that $HS_a(X)(k) = P(k)$ for all sufficiently large $k$ [Ser00]. The Hilbert-Samuel polynomial provides local information on the singularity of $X$ at a point as e.g. the multiplicity and the local dimension.

Theorem 8.18. The Hilbert-Samuel function of a subvariety $X$ of a regular variety $W$ defines an upper semicontinuous local invariant on $X$ with respect to the lexicographic ordering of integer sequences.

Proof. [Ben70] Thm. 4, p. 82, cf. also [Hau04]. □

Theorem 8.19. Let $\pi : W' \longrightarrow W$ be the blowup of $W$ along a non-singular center $Z$. Let $I$ be an ideal of $\mathcal{O}_W$. Assume that the Hilbert-Samuel function of $I$ is constant along $Z$, and denote by $I^s$ the strict transform of $I$ in $W'$. Let $a' \in E$ be a point in the exceptional divisor mapping under $\pi$ to $a$. Then

$$\mathrm{HS}_{a'}(I^s) \leq \mathrm{HS}_a(I)$$

holds with respect to the lexicographic ordering of integer sequences.

Proof. [Ben70] Thm. 0, [Hau04]. □

Theorem 8.20. (Zariski-Nagata, [Hir64] III.3, Thm. 1, p. 218) Let $S \subset T$ be closed irreducible subvarieties of a closed subvariety $X$ of a regular ambient variety $W$. The order of $X$ in $W$ along $T$ is less than or equal to the order of $X$ in $W$ along $S$.

Remark 8.21. In the case of schemes, the assertion says that if $X$ is embedded in a regular ambient scheme $W$, and $a, b$ are points of $X$ such that $a$ lies in in the closure of $b$, then $\mathrm{ord}_b X \leq \mathrm{ord}_a X$.

Proof. The proof goes in several steps and relies on the resolution of curves. Let $R$ be a regular local ring, $m$ its maximal ideal, $p$ a prime ideal in $R$ and $I \neq 0$ a non-zero ideal in $R$. Denote by $\nu_p(I)$ the maximal integer $\nu \geq 0$ such that $I \subset p^\nu$. Recall that $\mathrm{ord}_p(I)$ is the maximal integer $n$ such that $IR_p \subset p^n R_p$ or, equivalently, $I \subset p^{(\nu)}$, where $p^{(\nu)} = p^\nu R_p \cap R$ denotes the $\nu$-th symbolic power of $p$. Thus $\nu_p(I) \leq \mathrm{ord}_p(I)$ and the inequality can be strict if $p$ is not a complete intersection, in which case the usual and the symbolic powers of $p$ may differ. Write $\nu(I) = \nu_m(I)$ for the maximal ideal $m$ so that $\nu_p(I) \leq \nu(I)$ and $\nu_p(I) \leq \nu(IR_p) = \mathrm{ord}_p(I)$.

The assertion of the theorem is equivalent to the inequality $\nu(IR_p) \leq \nu(I)$, taking for $R$ the local ring $\mathcal{O}_{W,S}$ of $W$ along $S$, for $I$ the ideal of $R$ defining $X$ in $W$ along $S$ and for $p$ the ideal defining $T$ in $W$ along $S$.

If $R/p$ is regular, then $\nu_p(I) = \nu(IR_p)$ because $p^nR_p \cap R = p^n$: The inclusion $p^n \subseteq p^nR_p \cap R$ is clear, so suppose that $x \in p^nR_p \cap R$. Choose $y \in p^n$ and $s \notin p$ such that $xs = y$. Then $n \leq \nu_p(y) = \nu_p(xs) = \nu_p(x) + \nu_p(s) = \nu_p(x)$, hence $x \in p^n$. It follows that $\nu(IR_p) \leq \nu(I)$.

It remains to prove the inequality in the case that $R/p$ is not regular. Since $R$ is regular, there is a chain of prime ideals $p = p_0 \subset \ldots \subset p_k = m$ in $R$ with $\dim(R_{p_i}/p_{i-1}R_{p_i}) = 1$ for $1 \leq i \leq n$. By induction on the dimension of $R/p$, it therefore suffices to prove the inequality in the case $\dim(R/p) = 1$. Since the order remains constant under completion of a local ring by Prop. 8.8, it can be assumed that $R$ is complete.

By the resolution of curve singularities there exists a sequence of complete regular local rings $R = R_0 \to R_1 \to \ldots \to R_k$ with prime ideals $p_0 = p$ and $p_i \subset R_i$ with the following properties:

(1) $R_{i+1}$ is the blowup of $R_i$ with center the maximal ideal $m_i$ of $R_i$.
(2) $(R_{i+1})_{p_{i+1}} = (R_i)_{p_i}$.
(3) $R_k/p_k$ is regular.

Set $I_0 = I$ and let $I_{i+1}$ be the weak transform of $I_i$ under $R_i \to R_{i+1}$. Set $R' = R_k$, $I' = I_k$ and $p' = p_k$. The second condition on the blowups $R_i$ of $R_{i-1}$ implies $\nu(IR_p) = \nu(I'R'_{p'})$. By Prop. 8.13 one knows that $\nu(I') \leq \nu(I)$. Further, since $R'/p'$ is regular, $\nu(I'R'_{p'}) \leq \nu(I')$. Combining these inequalities yields $\nu(IR_p) \leq \nu(I)$. □

PROPOSITION 8.22. An upper semicontinuous local invariant $\mathrm{inv}(X) : X \to \Gamma$ with values in a totally well ordered set $\Gamma$ induces, up to refinement, a stratification of $X$ with strata $X_c = \{a \in X, \mathrm{inv}_a(X) = c\}$, for $c \in \Gamma$.

PROOF. For a given value $c \in \Gamma$, let $S = \{a \in X, \mathrm{inv}_a(X) \geq c\}$ and $T = \{a \in X, \mathrm{inv}_a(X) = c\}$. The set $S$ is a closed subset of $X$. If $c$ is a maximal value of the invariant on $X$, the set $T$ equals $S$ and is thus a closed stratum. If $c$ is not maximal, let $c' > c$ be an element of $\Gamma$ which is minimal with $c' > c$. Such elements exist since $\Gamma$ is well-ordered. As $\Gamma$ is totally ordered, $c'$ is unique. Therefore

$$S' = \{a \in X, \mathrm{inv}_a(X) \geq c'\} = \{a \in X, \mathrm{inv}_a(X) > c\} = S \setminus T$$

is closed in $X$. Therefore $T$ is open in $S$ and hence locally closed in $X$. As $S$ is closed in $X$ and $T \subset S$, the closure $\overline{T}$ is contained in $S$. It follows that the boundary $\overline{T} \setminus T$ is contained in $S'$ and closed in $X$. Refine the strata by replacing $S'$ by $\overline{T} \setminus T$ and $S' \setminus \overline{T}$ to get a stratification. □

## *Examples*

EXAMPLE 8.23. The singular locus $S = \mathrm{Sing}(X)$ of a variety is closed and properly contained in $X$. Let $X_1 = \mathrm{Reg}(X)$ be the set of regular points of $X$. It is open and dense in $X$. Repeat the procedure with $X \setminus X_1 = \mathrm{Sing}(X)$. By noetherianity, the process eventually stops, yielding a stratification of $X$ in regular strata. The strata of locally minimal dimension are closed and non-singular. The frontier condition holds, since the regular points of a variety are dense in the variety, and hence $\overline{X}_1 = X = X_1 \dot\cup \mathrm{Sing}(X)$.

EXAMPLE 8.24. There exists a threefold $X$ whose singular locus $\mathrm{Sing}(X)$ consists of two components, a non-singular surface and a singular curve meeting the surface at a singular point of the curve. The stratification by iterated singular loci as in the preceding example satisfies the frontier condition.

EXAMPLE 8.25. $^{\triangleright}$ Give an example of a variety whose stratification by the iterated singular loci has four different types of strata.

EXAMPLE 8.26. Find interesting stratifications for three three-folds.

EXAMPLE 8.27. The stratification of an upper semicontinuous invariant need not be finite. Take for $\Gamma$ the set $X$ underlying a variety $X$ with the trivial (partial) ordering $a \leq b$ if and only if $a = b$.

EXAMPLE 8.28. Let $X$ be the non-reduced scheme defined by $xy^2 = 0$ in $\mathbb{A}^2$. The order of $X$ at points of the $y$-axis outside 0 is 1, at points of the $x$-axis outside 0 it is 2, and at the origin it is 3. The local embedding dimension equals 1 at points of the $y$-axis outside 0, and 2 at all points of the $x$-axis.

EXAMPLE 8.29. Determine the stratification given by the order for the following varieties. If the smallest stratum is regular, blow it up and determine the stratification of the strict transform. Produce pictures and describe the geometric changes.
(a) Cross: $xyz = 0$,
(b) Whitney umbrella: $x^2 - yz^2 = 0$,
(c) Kolibri: $x^3 + x^2z^2 - y^2 = 0$,
(d) Xano: $x^4 + z^3 - yz^2 = 0$,
(e) Cusp & Plane: $(y^2 - x^3)z = 0$.

EXAMPLE 8.30. $^{\triangleright}$ Consider the order $\mathrm{ord}_Z I$ of an ideal $I$ in $\mathbb{K}[x_1, \ldots, x_n]$ along a closed subvariety $Z$ of $\mathbb{A}^n$, defined as the order of $I$ in the localization of $\mathbb{K}[x_1, \ldots, x_n]_J$ of $K[x_1, \ldots, x_n]$ at the ideal $J$ defining $Z$ in $\mathbb{A}^n$. Express this in terms of the symbolic powers $J^{(k)} = J^k \cdot \mathbb{K}[x_1, \ldots, x_n]_J \cap \mathbb{K}[x_1, \ldots, x_n]$ of $J$. Give an example to show that $\mathrm{ord}_Z I$ need not coincide with the maximal power $k$ such that $I \subset J^k$. If $J$ defines a complete intersection, the ordinary powers $J^k$ and the symbolic powers $J^{(k)}$ coincide. This holds in particular when $Z$ is a coordinate subspace of $\mathbb{A}^n$ [ZS75] IV, §12, [Hoc73],[Pel88].

EXAMPLE 8.31. $^{\triangleright}$[2] The ideal $I = (y^2 - xz, yz - x^3, z^2 - x^2y)$ of $\mathbb{K}[x, y, z]$ has symbolic square $I^{(2)}$ which strictly contains $I^2$.

EXAMPLE 8.32. $^{\triangleright}$ The variety defined by $I = (y^2 - xz, yz - x^3, z^2 - x^2y)$ in $\mathbb{A}^3$ has parametrization $t \mapsto (t^3, t^4, t^5)$. Let $f = x^5 + xy^3 + z^3 - 3x^2yz$. For all maximal ideals $m$ which contain $I$, $f \in m^2$ and thus, $\mathrm{ord}_m f \geq 2$. But $f \notin I^2$. Since $xf \in I^2$ and $x$ does not belong to $I$, it follows that $f \in I^2 R_I$ and thus $\mathrm{ord}_I f \geq 2$, in fact, $\mathrm{ord}_I f = 2$.

EXAMPLE 8.33. The order of an ideal depends on the embedding of $X$ in $W$. If $X$ is not minimally embedded locally at $a$, the order of $X$ at $a$ is 1 and not significant for measuring the complexity of the singularity of $X$ at $a$.

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[2] This example is due to Macaulay and was kindly communicated by M. Hochster.

Example 8.34. Associate to a stratification of a variety the so called *Hasse diagram*, i.e., the directed graph whose nodes and edges correspond to strata, respectively to the adjacency of strata. Determine the Hasse diagram for the surface $X$ given in $\mathbb{A}^4$ as the cartesian product of the cusp $C : x^2 = y^3$ with the node $D : x^2 = y^2+y^3$. Then project $X$ to $\mathbb{A}^3$ by means of $\mathbb{A}^4 \to \mathbb{A}^3, (x, y, z, w) \mapsto (x, y+z, w)$ and compute the Hasse diagram of the image $Y$ of $X$ under this projection.

Example 8.35. Show that the order of a hypersurface, the dimension and the Hilbert-Samuel function of a variety, the embedding-dimension of a variety and the local number of irreducible components are invariant under local formal isomorphisms, and determine whether they are upper or lower semicontinuous. How does each of these invariants behave under localization and completion?

Example 8.36. Take $\mathrm{inv}_a(X) = \dim_a(X)$, the dimension of $X$ at $a$. It is constant on irreducible varieties, and upper semicontinuous on arbitrary ones, because at an intersection point of several components, $\dim_a(X)$ is defined as the maximum of the dimensions of the components.

Example 8.37. Take $\mathrm{inv}_a(X) =$ the number of irreducible components of $X$ passing through $a$. If the components carry multiplicities as e.g. a divisor, one may alternatively take the sum of the multiplicities of the components passing through $a$. Both options produce an upper semicontinuous local invariant.

Example 8.38. Take $\mathrm{inv}_a(X) = \mathrm{embdim}_a(X) = \dim \mathrm{T}_a X$, the local embedding dimension of $X$ at $a$. It is upper semicontinuous. At regular points, it equals the dimension of $X$ at $a$, at singular points it exceeds this dimension.

Example 8.39.$^{\triangleright}$ Consider for a given coordinate system $x, y_1, \ldots, y_n$ on $\mathbb{A}^{n+1}$ a polynomial of order $c$ at 0 of the form

$$g(x, y_1, \ldots, y_n) = x^c + \sum_{i=0}^{c-1} g_i(y) \cdot x^i.$$

Express the order and the top locus of $g$ nearby 0 in terms of the orders of the coefficients $g_i$.

Example 8.40. Take $\mathrm{inv}_a(X) = \mathrm{HS}_a(X)$, the Hilbert-Samuel function of $X$ at $a$. The lexicographic order on integer sequences defines a well-ordering on $\Gamma = \{\gamma : \mathbb{N} \to \mathbb{N}\}$. Find two varieties $X$ and $Y$ with points $a$ and $b$ where $\mathrm{HS}_a(X)$ and $\mathrm{HS}_b(Y)$ only differ from the fourth entry on.

Example 8.41. Take $\mathrm{inv}_a(X) = \nu_a^*(X)$ the increasingly ordered sequence of the orders of a minimal Macaulay basis of the ideal $I$ defining $X$ in $W$ at $a$ [Hir64] III.1, Def. 1 and Lemma 1, p. 205. It is upper semicontinuous but does not behave well under specialization [Hir64] III.3, Thm. 2, p. 220 and the remark after Cor. 2, p. 220, see also [Hau98], Ex. 12.

Example 8.42. Let a monomial order $<_\varepsilon$ on $\mathbb{N}^n$ be given, i.e., a total ordering with minimal element 0 which is compatible with addition in $\mathbb{N}^n$. The *initial ideal* $\mathrm{in}(I)$ of an ideal $I$ of $\mathbb{K}[[x_1, \ldots, x_n]]$ with respect to $<_\varepsilon$ is the ideal generated by all *initial monomials* of elements $f$ of $I$, i.e., the monomials with minimal exponent with respect to $<_\varepsilon$ in the series expansion of $f$. The initial monomial of 0 is 0.

The initial ideal is a monomial ideal in $\mathbb{K}[[x_1,\ldots,x_n]]$ and depends on the choice of coordinates. If the monomial order $<_\varepsilon$ is compatible with degree, $\mathrm{in}(I)$ determines the Hilbert-Samuel function $\mathrm{HS}_a(I)$ of $I$ [Hau04].

Order monomial ideals totally by comparing their increasingly ordered unique minimal monomial generator system lexicographically, where any two monomial generators are compared with respect to $<_\varepsilon$. If two monomial ideals have generator systems of different length, complete the sequences of their exponents by a symbol $\infty$ so as to be able to compare them. This defines a well-order on the set of monomial ideals.

Take for $\mathrm{inv}_a(X)$ the minimum $\min(I)$ or the maximum $\max(I)$ of the initial ideal of the ideal $I$ of $X$ at $a$, the minimum and maximum being taken over all choices of local coordinates, say regular parameter systems of $\mathbb{K}[[x_1,\ldots,x_n]]$. Both exist and define local invariants which are upper semicontinuous with respect to localization [Hau04] Thms. 3 and 8.

(a) The minimal initial ideal $\min(I) = \min_x\{\mathrm{in}(I)\}$ over all choices of regular parameter systems of $\mathbb{K}[[x_1,\ldots,x_n]]$ is achieved for almost all regular parameter systems.

(b)$^+$ Let $I$ be an ideal in $\mathbb{K}[x_1,\ldots,x_n]$. For a point $a=(a_1,\ldots,a_n)\in\mathbb{A}^n$ denote by $I_a$ the induced ideal in $\mathbb{K}[[x_1-a_n,\ldots,x_n-a_n]]$. The minimal initial ideal $\min(I_a)$ is upper semicontinuous when the point $a$ varies.

(c) Compare the induced stratification of $\mathbb{A}^n$ with the stratification by the Hilbert-Samuel function of $I$.

(d) Let $Z$ be a regular center inside a stratum of the stratification induced by the initial ideal $\mathrm{in}(I)$, and consider the induced blowup $\widetilde{\mathbb{A}}^n \to \mathbb{A}^n$ along $Z$. Let $a\in Z$ and $a'\in W'$ be a point above $a$. Assume that the monomial order $<_\varepsilon$ is compatible with degree. Show that $\min_a(I)$ and $\max_a(I)$ do not increase when passing to the strict transform of $I_a$ at $a'$ [Hau04] Thm. 6.

Example 8.43.$^\triangleright$ Let $(W',a')\to(W,a)$ be the composition of two monomial point blowups of $W=\mathbb{A}^2$ with respect to coordinates $y,z$, defined as follows. The first is the blowup of $\mathbb{A}^2$ with center 0 considered at the origin of the $y$-chart, the second has as center the origin of the $y$-chart and is considered at the origin of the $z$-chart. Show that the order of the strict transform $g'(y,z)$ at $a'$ of any non zero polynomial $g(y,z)$ in $W$ is at most the half of the order of $g(y,z)$ at $a$.

Example 8.44. (B. Schober) Let $\mathbb{K}$ be a non perfect field of characteristic 3, let $t\in\mathbb{K}\setminus\mathbb{K}^2$ be an element which is not a square. Stratify the hypersurface $X: x^2+y(z^2+tw^2)=0$ in $\mathbb{A}^4$ according to its singularities. Show that this stratification does not provide suitable centers for a resolution.

Example 8.45. Let $I=(x^2+y^{17})$ be the ideal defining an affine plane curve singularity $X$ with singular locus the origin of $W=\mathbb{A}^2$. The order of $X$ at 0 is 2. The blowup $\pi: W'\to W$ of $W$ at the origin with exceptional divisor $E$ is covered by two affine charts, the $x$- and the $y$-chart. The total and strict transform of $I$ in the $x$-chart are as follows:

$$\begin{aligned} I^* &= (x^2+x^{17}y^{17}),\\ I^s &= (1+x^{15}y^{17}). \end{aligned}$$

At the origin of this chart, the order of $I^s$ has dropped to zero, so the strict transform $X^s$ of $X$ does not contain this point. Therefore it suffices to consider the complement of this point in $E$, which lies entirely in the $y$-chart. There, one obtains the following transforms:

$$\begin{aligned} I^* &= (x^2y^2 + y^{17}), \\ I^s &= (x^2 + y^{15}). \end{aligned}$$

The origin $a'$ of the $y$-chart is the only singular point of $X^s$. The order of $X^s$ at $a'$ has remained constant equal to 2. Find a local invariant of $X$ which has improved at $a'$. Make sure that it does not depend on any choices.

Example 8.46. The ideal $I = (x^2+y^{16})$ has in the $y$-chart of the point blowup of $\mathbb{A}^2$ at 0 the strict transform $I^s = (x^2+y^{14})$. If the ground field has characteristic 2, the $y$-exponents 16 and 14 are irrelevant to measure an improvement of the singularity because the coordinate change $x \mapsto x + y^8$ transforms $I$ into $(x^2)$.

Example 8.47. The ideal $I = (x^2 + 2xy^7 + y^{14} + y^{17})$ has in the $y$-chart the strict transform $I^s = (x^2 + 2xy^6 + y^{12} + y^{15})$. Here the drop of the $y$-exponent of the last monomial from 17 to 15 is significant, whereas the terms $2xy^7 + y^{14}$ can be eliminated in any characteristic by the coordinate change $x \mapsto x + y^7$.

Example 8.48. The ideal $I = (x^2+xy^9) = (x)(x+y^9)$ defines the union of two non-singular curves in $\mathbb{A}^2$ which have a common tangent line at their intersection point 0. The strict transform is $I^s = (x^2 + xy^8)$. The degree of tangency, viz the intersection multiplicity, has decreased.

Example 8.49. Let $X$ and $Y$ be two non-singular curves in $\mathbb{A}^2$, meeting at one point $a$. Show that there exists a sequence of point blowups which separates the two curves, i.e., so that the strict transforms of $X$ and $Y$ do not intersect.

Example 8.50. Take $I = (x^2 + g(y))$ where $g$ is a polynomial in $y$ with order at least 3 at 0. The strict transform under point blowup in the $y$-chart is $I^s = x^2 + y^{-2}g(y)$, with order at 0 equal to 2 again. This suggests to take the order of $g$ as a secondary invariant. In characteristic 2 it may depend on the choice of coordinates.

Example 8.51. Take $I = (x^2 + xg(y) + h(y))$ where $g$ and $h$ are polynomials in $y$ of order at least 1, respectively 2, at 0. The order of $I$ at 0 is 2. The strict transform equals $I^s = (x^2 + xy^{-1}g(y) + y^{-2}h(y))$ of order 2 at 0. Here it is less clear how to detect a secondary invariant which represents an improvement.

Example 8.52. Take $I = (x^2 + y^3z^3)$ in $\mathbb{A}^3$, and apply the blowup of $\mathbb{A}^3$ in the origin. The strict transform of $I$ in the $y$-chart equals $I^s = (x^2 + y^4z^3)$ and the singularity seems to have gotten worse.

Example 8.53.$^\triangleright$ Let $X$ be a surface in three-space, and $S$ its top locus. Assume that $S$ is singular at $a$, and let $X'$ be the blowup of $X$ in $a$. Determine the top locus of $X'$.

## 9. Lecture IX: Hypersurfaces of Maximal Contact

Proposition 9.1. (Zariski) Let $X$ be a subvariety of a regular variety $W$, defined over a field of arbitrary characteristic. Let $W' \to W$ be the blowup of $W$ along a regular center $Z$ contained in the top locus of $X$, and let $a$ be a point of $Z$. There exists, in a neighbourhood $U$ of $a$, a regular closed hypersurface $V$ of $U$ whose strict transform $V^s$ in $W'$ contains all points $a'$ of $W'$ lying above $a$ where the order of the strict transform $X^s$ of $X$ in $W'$ has remained constant equal to the order of $X$ along $Z$.

Proof. Choose local coordinates $x_1, \ldots, x_n$ of $W$ at $a$. The associated graded ring of $\mathcal{O}_{W,a}$ can be identified with $\mathbb{K}[x_1, \ldots, x_n]$. Let $\mathrm{in}(I) \subset \mathbb{K}[x_1, \ldots, x_n]$ denote the ideal of initial forms of elements of $I$ at $a$. Apply a linear coordinate change so that generators of $\mathrm{in}(I)$ are expressed with the minimal number of variables, say $x_1, \ldots, x_k$, for some $k \leq n$. Choose any $1 \leq i \leq k$ and define $V$ in $W$ at $a$ by $x_i = 0$. It follows that the local top locus of $X$ at $a$ is contained in $V$. Hence $Z \subset V$, locally at $a$. Let $a'$ be a point of $W'$ above $a$ where the order of $X$ has remained constant. By Prop. 5.4 the local blowup $(W', a') \to (W, a)$ can be made monomial by a suitable coordinate change. The assertion then follows by computation, cf. Ex. 9.8 and [Zar44]. □

Definition 9.2. Let $X$ be a subvariety of a regular variety $W$, and let $a$ be a point of $W$. A *hypersurface of maximal contact for $X$ at $a$* is a regular closed hypersurface $V$ of an open neighborhood $U$ of $a$ in $W$ such that

(1) $V$ contains the local top locus $S$ of $X$ at $a$, i.e., the points of $U$ where the order of $X$ equals the order of $X$ at $a$.

(2) The strict transform $V^s$ of $V$ under any blowup of $U$ along a regular center $Z$ contained in $S$ contains all points $a'$ above $a$ where the order of $X^s$ has remained constant equal to the order of $X$ at $a$.

(3) Property (2) is preserved in any sequence of blowups with regular centers contained in the successive top loci of the strict transforms of $X$ along which the order of $X$ has remained constant.

$$\begin{array}{ccccccccc} a' \in & E & \cap & V^s & \subset & U' & \subset & W' \\ \downarrow & \downarrow & & & & & & \downarrow \pi \\ a \in & Z & \subset & V & \subset & U & \subset & W \end{array}$$

Definition 9.3. Assume that the characteristic of the ground field is zero. Let $X$ be a subvariety of a regular variety $W$ defined locally at a point $a$ of $W$ by the ideal $I$. Let $o$ be the order of $X$ at $a$. An *osculating hypersurface for $X$ at $a$* is a regular closed hypersurface $V$ of a neighbourhood $U$ of $a$ in $W$ defined by a derivative of order $o-1$ of an element $f$ of order $o$ of $I$ [EH02].

Remark 9.4. The element $f$ has necessarily order $o$ at $a$, and its $(o-1)$-st derivative has order 1 at $a$, so that it defines a regular hypersurface at $a$. The concept is due to Abhyankar and Zariski [AZ55]. Abhyankar called the local isomorphism constructing an osculating hypersurface $V$ from a given regular hypersurface $H$ of $W$ *Tschirnhaus transformation*. If $H$ is given by $x_n = 0$ for some coordinates $x_1, \ldots, x_n$, this transformation eliminates from $f$ all monomials whose $x_n$-exponent is $o-1$. The existence of osculating hypersurfaces was exploited systematically by Hironaka in his proof of characteristic zero resolution [Hir64].

For each point $a$ in $X$, osculating hypersurfaces contain locally at $a$ the local top locus $S = \mathrm{top}_a(X)$, and their strict transform contain the equiconstant points above $a$.

Proposition 9.5. (Abhyankar, Hironaka) Let $X$ be a subvariety of a regular variety $W$, and let $a$ be a point of $W$. For ground fields of characteristic zero there exist, locally at $a$ in $W$, hypersurfaces of maximal contact for $X$. Any osculating hypersurface $V$ at $a$ has maximal contact with $X$ at $a$.

PROOF. [EH02]. □

REMARK 9.6. The assertion of the proposition does not hold over fields of positive characteristic: R. Narasimhan, a student of Abhyankar, gave an example of a hypersurface $X$ in $\mathbb{A}^4$ over a field of characteristic 2 whose top locus is not contained at 0 in any regular local hypersurface, and Kawanoue describes a whole family of such varieties [Nar83, Kaw14], [Hau98], Ex. 8. See also Ex. 12.1 below. In Narasimhan's example, there is a sequence of point blowups for which there is no regular local hypersurface $V$ of $\mathbb{A}^4$ at 0 whose strict transforms contain all points where the transforms of $X$ have order 2 as at the beginning [Hau03] II.14, Ex. 2, p. 388.

REMARK 9.7. The existence of hypersurfaces of maximal contact in zero characteristic suggests to associate to $X$ locally at a point $a$ a variety $Y$ defined by an ideal $J$ in the hypersurface $V$ and to observe the behaviour of $X$ under blowup by means of the behaviour of $Y$ under the induced blowup: the transform of $Y$ under the blowup of $V$ along a center $Z$ of $W$ locally contained in $V$ should equal the variety $Y'$ which is associated in a similar manner as $Y$ to $X$ to the strict transform $X^s$ of $X$ in $V^s$ at points $a'$ above $a$ where the order of $X^s$ has remained constant. The variety $Y$ or the ideal $J$ and their transforms may then help to measure the improvement of $X^s$ at $a'$ by looking at their respective orders. This is precisely the way how the proof of resolution in zero characteristic proceeds. The reasoning is also known as *descent in dimension*. The main problem in this approach is to define properly the variety $Y$, respectively the ideal $J$, and to show that the local construction is independent of the choice of $V$ and patches to give a global resolution algorithm.

$$\begin{array}{ccccccc} a' \in & E & \subset W' & \rightsquigarrow & V' & \supset Y' \\ \downarrow & \downarrow & \downarrow \pi & & \downarrow & \downarrow \pi_{|Y'} \\ a \in & Z & \subset W & \rightsquigarrow & V & \supset Y \end{array}$$

## *Examples*

EXAMPLE 9.8.$^{\triangleright}$ Let $\pi : (W', a') \to (W, a)$ be a local blowup and let $x_1, \ldots, x_n$ be local coordinates on $W$ at $a$ such that $\pi$ is monomial. Assume that $x_1$ appears in the initial form of an element $f \in \mathcal{O}_{W,a}$, and let $V \subset W$ be the local hypersurface at $a$ defined by $x_1 = 0$. If the order of the strict transform $f^s$ of $f$ at $a'$ has remained constant equal to the order of $f$ at $a$, the point $a'$ belongs to the strict transform $V^s$ of $V$.

EXAMPLE 9.9. Let the characteristic of the ground field be different from 3. Apply the second order differential operator $\partial = \frac{\partial^2}{\partial x^2}$ to $f = x^3 + x^2yz + z^5$ so that $\partial f = 6x + 2yz$. This defines a hypersurface of maximal contact for $f$ at 0. Replacing in $f$ the variable $x$ by $x - \frac{1}{3}yz$ gives

$$g = x^3 - \frac{1}{3}xy^2z^2 + \frac{2}{27}y^3z^3 + z^5.$$

The term of degree 2 in $x$ has been eliminated, and $x = 0$ defines an osculating hypersurface for $g$ at 0.

Example 9.10. $\triangleright$ Assume that the characteristic is 0. Let $X \subset \mathbb{A}^n$ be a hypersurface defined locally at the origin by a polynomial $f = x_n^d + \sum_{i=0}^{d-1} a_i(y)x_n^i$ where $y = (x_1, \ldots, x_{n-1})$ and $\mathrm{ord}_0 a_i(y) \geq d - i$. Make the change of coordinates $x_n \mapsto x_n - \frac{1}{d}a_{d-1}(y)$. Show that after this change, the hypersurface defined by $x_n = 0$ has maximal contact with $X$ at the origin. What prevents this technique from working in positive characteristic?

Example 9.11. $\triangleright$ Consider the hypersurface $X \subset \mathbb{A}^3$ given by the equation $x^2y + xy^2 - x^2z + y^2z - xz^2 - yz^2 = 0$. Show that the hypersurface $V$ given by $x = 0$ does not have maximal contact with $X$ at 0. In particular, consider the blowup of $\mathbb{A}^3$ in the origin. Find a point $a'$ on the exceptional divisor that lies in the $x$-chart of the blowup such that the strict transform of $X$ has order 3 at $a'$. Then show that the strict transform of $V$ does not contain this point.

Example 9.12. Hypersurfaces of maximal contact are only defined locally. They need not patch to give a globally defined hypersurface of maximal contact on $W$. Find an example for this.

Example 9.13. Consider $f = x^4 + y^4 + z^6$, $g = x^4 + y^4 + z^{10}$ and $h = xy + z^{10}$ under point blowup. Determine, according to the characteristic of the ground field, the points where the orders of $f$, $g$ and $h$ have remained constant.

Example 9.14. $\triangleright$ Let $f = x^c + g(y_1, \ldots, y_m) \in K[[x, y_1, \ldots, y_m]]$ be a formal power series with $g$ a series of order $\geq c$ at 0. Show that there exists in any characteristic a formal coordinate change maximizing the order of $g$.

Example 9.15. Let $f = x^c + g(y_1, \ldots, y_m) \in K[x, y_1, \ldots, y_m]$ be a polynomial with $g$ a polynomial of order $\geq c$ at 0. Does there exist a local coordinate change in $\mathbb{A}^{1+m}$ at 0 maximizing the order of $g$?

Example 9.16. Let $f$ be a polynomial or power series in $n$ variables $x_1, \ldots, x_n$ of order $c$ at 0. Assume that the ground field is infinite. There exists a linear coordinate change after which $f(0, \ldots, 0, x_n)$ has order $c$ at 0. Such polynomials and series, called *$x_n$-regular of order $c$* at 0, appear in the Weierstrass preparation theorem, which was frequently used by Abhyankar in resolution arguments.

Example 9.17. $\triangleright$ Let $(W', a') \to (W, a)$ be a composition of local blowups in regular centers such that $a'$ lies in the intersection of $n$ exceptional components where $n$ is the dimension of $W$ at $a$. Let $f \in \mathcal{O}_{W,a}$ and assume that the characteristic is zero. Show that the order of $f$ has dropped between $a$ and $a'$.

Example 9.18. $^+$ Show the same in positive characteristic.

## 10. Lecture X: Coefficient Ideals

Definition 10.1. Let $I$ be an ideal in a regular variety $W$, let $a$ be a point of $W$ with open neighbourhood $U$, and let $V$ be a regular closed hypersurface of $U$ containing $a$. Let $x_1, \ldots, x_n$ be coordinates on $U$ such that $V$ is defined in $U$ by $x_n = 0$. The restrictions of $x_1, \ldots, x_{n-1}$ to $V$ form coordinates on $V$ and will be abbreviated by $x'$. For $f \in \mathcal{O}_U$, denote by $\sum a_{f,i}(x') \cdot x_n^i$ the expansion of $f$ with respect to $x_n$, with coefficients $a_{f,i} = a_{f,i}(x') \in \mathcal{O}_V$. The *coefficient ideal of $I$ in $V$ at $a$* is the ideal $J_V(I)$ on $V$ defined by

$$J_V(I) = \sum_{i=0}^{o-1} (a_{f,i}, f \in I)^{\frac{o!}{o-i}},$$

where $o$ denotes the order of $I$ at $a$.

Remark 10.2. The coefficient ideal is defined on whole $V$. It depends on the choice of the coordinates $x_1, \dots, x_n$ on $U$, even so the notation only refers to $V$. Actually, $J_V(I)$ depends on the choice of a section $\mathcal{O}_{V,a} \to \mathcal{O}_{U,a}$ of the map $\mathcal{O}_{U,a} \to \mathcal{O}_{V,a}$ defined by restriction to $V$. The same definition applies to stalks of ideals in $W$ at points $a$, giving rise to an ideal, also denoted by $J_V(I)$, in the local ring $\mathcal{O}_{V,a}$. The weights $\frac{o!}{o-i}$ in the exponents are chosen so as to obtain a systematic behavior of the coefficient ideal under blowup, cf. Prop. 10.6 below. The chosen algebraic definition of the coefficient ideal is modelled so as to commute with blowups [EH02], but is less conceptual than definitions through differential operators proposed and used by Encinas-Villamayor, Bierstone-Milman, Włodarczyk, Kawanoue-Matsuki and Hironaka [EV00, EV98, BM97, Wło05, KM10, Hir03].

Proposition 10.3. The passage to the coefficient ideal $J_V(I)$ of $I$ in $V$ commutes with taking germs along the local top locus $S = \mathrm{top}_a(I) \cap V$ of points of $V$ where the order of $I$ in $W$ is equal to the order of $I$ in $W$ at $a$: Let $x_1, \dots x_n$ be coordinates of $W$ at $a$, defined on an open neighborhood $U$ of $a$, and let $V$ be closed and regular in $U$, defined by $x_n = 0$. The stalks of $J_V(I)$ at points $b$ of $S$ inside $U$ coincide with the coefficient ideals of the stalks of $I$ at $b$.

Proof. Clear from the definition of coefficient ideals. □

Corollary 10.4. In the above situation, for any fixed closed hypersurface $V$ in $U \subset W$ open, the order of $J_V(I)$ at points of $S \cap V$ is upper semicontinuous along $S$, locally at $a$.

Remark 10.5. In general, $V$ need not contain, locally at $a$, the top locus of $I$ in $W$. This can, however, be achieved in zero characteristic by choosing for $V$ an osculating hypersurface, cf. Prop. 10.9 below. In this case, the order of $J_V(I)$ at points of $S = \mathrm{top}_a(I)$ does not depend on the choice of the hypersurface, cf. Prop. 10.13. In arbitrary characteristic, a local hypersurface $V$ will be chosen separately at each point $b \in S$ in order to maximize the order of $J_V(I)$ at $b$, cf. Prop. 10.11. In this case, the order of $J_V(I)$ at $b$ does not depend on the choice of $V$, and its upper semicontinuity as $b$ moves along $S$ holds again, but is more difficult to prove [Hau04].

Proposition 10.6. The passage to the coefficient ideal $J_V(I)$ of $I$ at $a$ commutes with blowup: Let $\pi : W' \to W$ be the blowup of $W$ along a regular center $Z$ contained locally at $a$ in $S = \mathrm{top}_a(I)$. Let $V$ be a local regular hypersurface of $W$ at $a$ containing $Z$ and such that $V^s$ contains all points $a'$ above $a$ where the order of the weak transform $I^\curlyvee$ has remained constant equal to the order of $I$ at $a$. For any such point $a'$, the coefficient ideal $J_{V^s}(I^\curlyvee)$ of $I^\curlyvee$ equals the controlled transform $J_V(I)^! = I_E^{-c} \cdot J_V(I)^*$ of $J_V(I)$ with respect to the control $c = o!$ with $o = \mathrm{ord}_a(I)$.

PROOF. Write $V'$ for $V^s$, and let $h = 0$ be a local equation of $E \cap V'$ in $V'$. The weak transform $I^\curlyvee$ of $I$ is generated by the elements $f^\curlyvee = h^{-o} \cdot f^*$ for $f$ varying in $I$, where $^*$ denotes the total transform. The coefficients $a_{f,i}$ of the monomials $x_n^i$ of the expansion of an element $f$ of $I$ of order $o$ at $a$ in the coordinates $x_1, \ldots, x_n$ satisfy $a_{f^\curlyvee,i} = h^{i-o} \cdot (a_{f,i})^*$. Then

$$J_{V'}(I^\curlyvee) = J_{V'}(\sum_i a_{f^\curlyvee,i} \cdot x_n^i,\ f^\curlyvee \in I^\curlyvee)$$

$$= J_{V'}(\sum_i a_{h^{-o}\cdot f^*,i} \cdot x_n^i,\ f \in I)$$

$$= J_{V'}(\sum_i h^{-o} \cdot (a_{f,i} \cdot x_n^i)^*,\ f \in I)$$

$$= \sum_{i<o} h^{-o!} \cdot (a_{f,i}^*,\ f \in I)^{o!/(o-i)}$$

$$= h^{-o!} \cdot (\sum_{i<o} (a_{f,i},\ f \in I)^{o!/(o-i)})^*$$

$$= h^{-o!} \cdot (J_V I)^* = (J_V I)^!.$$

This proves the claim. □

REMARK 10.7. The definitions of the coefficient ideal used in [EV00, EV98, BM97, Wło05, KM10, Hir03] produce a weaker commutativity property with respect to blowups, typically only for the radicals of the coefficient ideals.

REMARK 10.8. The order of the coefficient ideal $J_V(I)$ of $I$ is not directly suited as a secondary invariant when the order of $I$ remains constant since, by the proposition, the coefficient ideal passes under blowup to its controlled transform, and thus its order may increase. In order to get a practicable secondary invariant it is appropriate to decompose $J_V(I)$ and $(J_V(I))^!$ into products of two ideals: the first factor is a principal monomial ideal supported by the exceptional locus, the second, possibly singular factor, is an ideal called the *residual factor*, and supposed to pass under blowup in the factorization to its weak transform. Choosing suitably the exceptional monomial factor it can be shown that such factorizations always exist [EH02]. In this situation the order of the residual factor does not increase under blowup by Prop. 8.13 and can thus serve as a secondary invariant whenever the order of the ideal $I$ remains constant under blowup.

PROPOSITION 10.9. Assume that the characteristic of the ground field is zero. Let $I$ have order $o$ at a point $a \in W$, and let $V$ be a regular hypersurface for $I$ at $a$, with coefficient ideal $J = J_V(I)$. The locus $\mathrm{top}_a(J, o!)$ of points of $V$ where $J$ has order $\geq o!$ coincides with $\mathrm{top}_a(I)$,

$$\mathrm{top}_a(J, o!) = \mathrm{top}_a(I).$$

PROOF. Choose local coordinates $x_1, \ldots, x_n$ in $W$ at $a$ so that $V$ is defined by $x_n = 0$. Expand the elements $f$ of $I$ with respect to $x_n$ with coefficients $a_{f,i} \in \mathcal{O}_{V,a}$, and choose representatives of them on a suitable neighbourhood of $a$. Let $b$ be a point in a sufficiently small neighborhood of $a$. Then, by the upper semicontinuity of the order, $b$ belongs to $\mathrm{top}_a(I)$ if and only if $\mathrm{ord}_b I \geq o$, which is equivalent to $\sum_{i<o} a_{f,i} \cdot x_n^i$ having order $\geq o$ at $b$ for all $f \in I$. This, in turn, holds if and only if

$a_{f,i}$ has order $\geq o-i$ at $b$, say $a_{f,i}^{\frac{o!}{o-i}}$ has order $\geq o!$ at $b$. Hence $b \in \mathrm{top}_a(I)$ if and only if $b \in \mathrm{top}_a(J_V(I), o!)$. □

Corollary 10.10. Assume that the characteristic of the ground field is zero. Let $a$ be a point in $W$ and set $S = \mathrm{top}_a(I)$. Let $U$ be a neighbourhood of $a$ on which there exists a closed regular hypersurface $V$ which is osculating for $I$ at all points of $S \cap U$. Let $J_V(I)$ be the coefficient ideal of $I$ in $V$.

(a) The top locus $\mathrm{top}_a(J_V(I))$ of $J_V(I)$ on $V$ is contained in $\mathrm{top}_a(I)$.

(b) The blowup of $U$ along a regular locally closed subvariety $Z$ of $\mathrm{top}_a(J_V(I))$ $Z$ commutes with the passage to the coefficient ideals of $I$ and $I^\curlyvee$ in $V$ and $V^s$.

Proof. Assertion (a) is immediate from the proposition, and (b) follows from Prop. 10.6. □

Proposition 10.11. (Encinas-Hauser) Assume that the characteristic of the ground field is zero. The order of the coefficient ideal $J_V(I)$ of $I$ at $a$ with respect to an osculating hypersurface $V$ at $a$ attains the maximal value of the orders of the coefficient ideals over all local regular hypersurfaces. In particular, it is independent of the choice of $V$.

Proof. Choose local coordinates $x_1, \ldots, x_n$ in $W$ at $a$ such that the appropriate derivative of the chosen element $f \in I$ is given by $x_n$. Let $o$ be the order of $f$ at $a$. The choice of coordinates implies that the expansion of $f$ with respect to $x_n$ has a monomial $x_n^o$ with coefficient 1 and no monomial in $x_n$ of degree $o-1$. Any other local regular hypersurface $U$ is obtained from $V$ by a local isomorphism $\varphi$ of $W$ at $a$. Assume that the order of $J_V(\varphi^*(I))$ is larger than the order of $J_V(I)$. Let $g = \varphi^*(f)$. This signifies that the order of all coefficients $a_{g,i}^{\frac{o!}{o-i}}$ is larger than the order of $J_V(I)$. Therefore $\varphi^*$ must eliminate from $f$ the terms of $a_{f,i}$ for which $a_{f,i}^{\frac{o!}{o-i}}$ has order equal to the order of $J_V(I)$. But then $\varphi^*$ produces from $x_n^o$ a non-zero coefficient $a_{g,o-1}$ such that $a_{g,o-1}^{o!}$ has order equal to the order of $J_V(I)$, contradiction. □

Remark 10.12. This result suggests to consider in positive characteristic as a substitute for hypersurfaces of maximal contact local regular hypersurfaces which maximize the order of the associated coefficient ideal. Such hypersurfaces are used in recent approaches to resolution of singularities in positive characteristic [Hir12, Hau10a, HW14], relying on the work of Moh on the behaviour of the coefficient ideal in this situation [Moh87].

Proposition 10.13. Let the characteristic of the ground field be arbitrary. The supremum in $\mathbb{N} \cup \{\infty\}$ of the orders of the coefficient ideals $J_V(I)$ of $I$ in local regular hypersurfaces $V$ in $W$ at $a$ is realized by a formal local regular hypersurface $V$ in $W$ at $a$ (i.e., $V$ is defined by an element of the complete local ring $\widetilde{\mathcal{O}}_{W,a}$). If the supremum is finite, it can be realized by a local regular hypersurface $V$ in $W$ at $a$.

Proof. If the supremum is finite, the existence of some $V$ realizing this value is obvious. If the supremum is infinite, one uses the completeness of $\widehat{\mathcal{O}}_{W,a}$ to construct $V$, see [EH02, Hau04]. □

Definition 10.14. A formal local regular hypersurface $V$ realizing the supremum of the order of the coefficient ideal $J_V(I)$ of $I$ at $a$ is called a *hypersurface of weak maximal contact of $I$ at $a$*. If the supremum is finite, it will always be assumed to be a local hypersurface.

Proposition 10.15. (Zariski) Let $V$ be a formal local regular hypersurface in $W$ at $a$ of weak maximal contact with $I$ at $a$. Let $\pi : W' \to W$ be the blowup of $W$ along a closed regular center $Z$ contained locally at $a$ in $S = \mathrm{top}_a(I)$. The points $a' \in W'$ above $a$ for which the order of the weak transform $I^\curlyvee$ of $I$ at $a'$ has not decreased are contained in the strict transform $V^s$ of $V$.

Proof. By definition of weak maximal contact, the variable $x_n$ defining $V$ in $W$ at $a$ appears in the initial form of some element $f$ of $I$ of order $o = \mathrm{top}_a(I)$ at $a$, cf. Ex. 10.23 below. The argument then goes analogously to the proof of Prop. 9.1. □

Remark 10.16. In characteristic zero and if $V$ has been chosen osculating at $a$, the hypersurface $V'$ is again osculating at points $a'$ above $a$ where $\mathrm{ord}_{a'}(I') = \mathrm{ord}_a(I)$, hence it has again weak maximal contact with $I'$ at such points $a'$. In positive characteristic this is no longer true, cf. Ex. 10.24.

## *Examples*

Example 10.17. Determine in all characteristics the points of the blowup $\widetilde{\mathbb{A}}^2$ of $\mathbb{A}^2$ at 0 where the strict transform of $g = x^4 + kx^2y^2 + y^4 + 3y^7 + 5y^8 + 7y^9$ under the blowup of $\mathbb{A}^2$ at 0 has order 4, for any $k \in \mathbb{N}$.

Example 10.18. Determine for all characteristics the maximal order of the coefficient ideal of $I = (x^3 + 5y^3 + 3(x^2y^2 + xy^4) + y^6 + 7y^7 + y^9 + y^{10})$ in regular local hypersurfaces $V$ at 0.

Example 10.19. Same as before for $I = (xy + y^4 + 3y^7 + 5y^8 + 7y^9)$.

Example 10.20.$^\triangleright$ Compute the coefficient ideal of $I = (x^5 + x^2y^4 + y^k)$ in the hypersurfaces $x = 0$, respectively $y = 0$. According to the value of $k$ and the characteristic, which hypersurface is osculating or has weak maximal contact?

Example 10.21.$^\triangleright$ Consider $f = x^2 + y^3z^3 + y^7 + z^7$. Show that $V \subset \mathbb{A}^3$ defined by $x = 0$ is a local hypersurface of weak maximal contact for $f$. Blow up $\mathbb{A}^3$ at the origin. How does the order of the coefficient ideal of $f$ in $V$ behave under these blowups at the points where the order of $f$ has remained constant? Factorize suitably the controlled transform of the coefficient ideal with respect to the exceptional factor and observe the behaviour of the order of the residual factor.

Example 10.22.$^\triangleright$ Assume that, for a given coordinate system $x_1, \ldots, x_n$ in $W$ at $a$, the hypersurface $V$ defined by $x_n = 0$ is osculating for a polynomial $f$ and that the coefficient ideal of $f$ in $V$ is a principal monomial ideal. Show that there is a sequence of blowups in coordinate subspaces of the induced affine charts which eventually makes the order of $f$ drop.

Example 10.23. The variable $x_n$ defining a hypersurface $V$ in $W$ at $a$ of weak maximal contact with an ideal $I$ appears in the initial form of some element $f$ of $I$ of order $o = \mathrm{ord}_a(I)$ at $a$.

EXAMPLE 10.24. In positive characteristic, a hypersurface of weak maximal contact for an ideal $I$ need not have again weak maximal contact after blowup with the weak transform $I^\curlyvee$ at points $a'$ where the order of $I^\curlyvee$ has remained constant.

EXAMPLE 10.25. Compute in the following situations the coefficient ideal of $I$ in $W$ at $a$ with respect to the given local coordinates $x$, $y$, $z$ in $\mathbb{A}^3$ and the hypersurface $V$. Determine in each case whether the order of the coefficient ideal is maximal. If not, find a coordinate change which maximizes it.

(a) $a = 0 \in \mathbb{A}^1$, $x$, $V : x = 0$, $I = (x)$ and $I = (x + x^2)$.
(b) $a = 0 \in \mathbb{A}^2$, $x, y$, $V : x = 0$, $I = (x)$, $I = (x + y^2)$, $I = (y + x^2)$, $I = (xy)$.
(c) $a = (1, 0, 0) \in \mathbb{A}^3$, $x, y, z$, $V : y + z = 0$, $I = (x^2)$, $I = (xy)$, $I = (x^3 + z^3)$.
(d) $a = 0 \in \mathbb{A}^3$, $x, y, z$, $V : x = 0$, $I = (xyz)$, $I = (x^2 + y^3 + z^5)$.
(e) $a = 0 \in \mathbb{A}^2$, $x, y$, $V : x = 0$, $I = (x^2 + y^4, y^4 + x^2)$.

EXAMPLE 10.26. Blow up in each of the preceding examples the origin and determine the points of the exceptional divisor $E$ where the order of the weak transform $I^\curlyvee$ of $I$ has remained constant. Check at these points whether commutativity holds for the descent to the coefficient ideal and its controlled transform.

EXAMPLE 10.27. $^\triangleright$ Show that the maximum of the order of the coefficient ideal $J_V(I)$ of an ideal $I$ over all regular parameter systems of $\widehat{\mathcal{O}}_{\mathbb{A}^n,0}$ is attained (it might be $\infty$).

EXAMPLE 10.28. Show that the supremum of the order of the coefficient ideal of an ideal $I$ in $W$ at a point $a$ can be realized, if the supremum is finite, by a regular system of parameters of the local ring $\mathcal{O}_{W,a}$ without passing to the completion.

EXAMPLE 10.29. $^+$ Show that this maximum, when taken at any point of the top locus $S$ of $\mathbb{A}^n$ where $I$ has maximal order $o$, defines an upper semicontinuous function on $S$.

EXAMPLE 10.30. $^\triangleright$ Let $V$ be the hypersurface $x_n = 0$ of $\mathbb{A}^n$ and let $V' \to V$ be the blowup with regular center $Z$ in $V$. Let $I$ be an ideal of order $o$ at a point $a$ of $Z$, with weak transform $I^\curlyvee$. Assume that $\mathrm{ord}_{a'} I^\curlyvee = \mathrm{ord}_a I$ at the origin $a'$ of an $x_j$-chart for a $j < n$. Compare the controlled transform of the coefficient ideal $J_V(I)$ of $I$ with respect to the control $c = o!$ with the coefficient ideal $J_{V'}(I^\curlyvee)$ of $I^\curlyvee$.

EXAMPLE 10.31. Show that, in characteristic zero, the top locus of an ideal $I$ of $W$, when taken locally at a point $a$, is contained in a local regular hypersurface $V$ through $a$. Does this hypersurface maximize the order of the associated coefficient ideal?

EXAMPLE 10.32. Consider $f = x^3 + y^2 z$ in $\mathbb{A}^3$ and the point blowup of $\mathbb{A}^3$ at the origin. Find, according to the characteristic, at all points of the exceptional divisor a hypersurface of weak maximal contact for the strict transform of $f$.

EXAMPLE 10.33. Consider surfaces defined by polynomials $f = x^o + y^a z^b \cdot g(y, z)$ where $y^a z^b$ is considered as an exceptional monomial factor of the coefficient ideal in the hypersurface $V$ defined by $x = 0$ (up to raising the coefficient ideal to the power $c = o!$). Assume that $a + b + \mathrm{ord}_0 g \geq o$. Give three examples where the order of $g$ at 0 is not maximal over all choices of local hypersurfaces at 0, and indicate the coordinate changes which make it maximal.

EXAMPLE 10.34. Consider surfaces defined by polynomials $f = x^o + y^a z^b \cdot g(y,z)$ where $y^a z^b$ is considered again as an exceptional monomial factor of the coefficient ideal in the hypersurface $V$ defined by $x = 0$. Assume that $a + b + \mathrm{ord}_0 g \geq o$. Compute the strict transform $f' = x^o + y^{a'} z^{b'} \cdot g'(y,z)$ of $f$ under point blowup at points where the order of $f$ has remained equal to $o$. Find three examples where the order of $g'$ is not maximal over all local coordinate choices.

EXAMPLE 10.35. For a flag of local regular subvarieties $V_{n-1} \supset \ldots \supset V_1$ at $a$ in $W = \mathbb{A}^n$ one gets from an ideal $I = I_n$ in $W$ a chain of coefficient ideals $J_{n-1}, \ldots, J_1$ in $V_{n-1}, \ldots, V_1$ respectively, defined recursively as follows. Assume that $J_{n-1}, \ldots, J_{i+1}$ have been constructed and that $J_{i+1}$ can be decomposed into $J_{i+1} = M_{i+1} \cdot I_{i+1}$ with prescribed monomial factor $M_{i+1}$ and some residual factor $I_{i+1}$. Then set $J_i = J_{V_i}(I_{i+1})$, the coefficient ideal of $I_{i+1}$ in $V_i$ at $a$. Show that the lexicographic maximum of the vector of orders of the ideals $J_i$ at $a$ over all choices of flags at $a$ admitting the above factorizations of the ideals $J_i$ can be realized stepwise, choosing first a local hypersurface $V_{n-1}$ in $W$ at $a$ maximizing the order of $J_{n-1}$ at $a$, then a local hypersurface $V_{n-2}$ in $V_{n-1}$ at $a$ maximizing the order of $J_{n-2}$, and iterating this process.

## 11. Lecture XI: Resolution in Zero Characteristic

The inductive proof of resolution of singularities in characteristic zero requires a more detailed statement about the nature of the resolution:

THEOREM 11.1. Let $W$ be a regular ambient variety and let $E \subset W$ be a (possibly empty) divisor with normal crossings. Assume that the characteristic of the ground field is 0. Let $J$ be an ideal on $W$, together with a decomposition $J = M \cdot I$ into a principal monomial ideal $M$, the monomial factor of $J$, supported on a normal crossings divisor $D$ transversal to $E$, and an ideal $I$, the residual factor of $J$. Let $c_+ \geq 1$ be a given number, the control of $J$.

There exists a sequence of blowups of $W$ along regular centers $Z$ transversal to $E$ and $D$ and their total transforms, contained in the locus $\mathrm{top}(J, c_+)$ of points where $J$ and its controlled transforms with respect to $c_+$ have order $\geq c_+$, and satisfying the requirements *equivariance* and *excision* of a strong resolution so that the order of the controlled transform of $J$ with respect to $c_+$ drops eventually at all points below $c_+$.

DEFINITION 11.2. In the situation of the theorem, with prescribed divisor $D$ and control $c_+$, the ideal $J$ is called *resolved* with respect to $D$ and $c_+$ if the order of $J$ at all points of $W$ is $< c_+$.

REMARK 11.3. Once the order of the controlled transform of $J$ has dropped below $c_+$, induction on the order can be applied to find an additional sequence of blowups which makes the order of the controlled transform of $J$ drop everywhere to 0. At that stage, the controlled transform of $J$ has become the whole coordinate ring of $W$, and the total transform of $J$, which differs from the controlled transform by a monomial exceptional factor, has become a monomial ideal supported on a normal crossings divisor $D$ transversal to $E$. This establishes the existence of a strong embedded resolution of $J$, respectively of the singular variety $X$ defined by $J$ in $W$.

PROOF. The technical details can be found in [EH02], and motivations are given in [Hau03]. The main argument is the following.

The resolution process has two different stages: In the first, a sequence of blowups is chosen through a local analysis of the singularities of $J$ and by induction on the ambient dimension. The order of the weak transforms of $I$ will be forced to drop eventually below the maximum of the order of $I$ at the points $a$ of $W$. By induction on the order one can then apply additional blowups until the order of the weak transform of $I$ has become equal to 0. At that moment, the weak transform of $I$ equals the coordinate ring/structure sheaf of the ambient variety, and the controlled transform of $J$ has become a principal monomial ideal supported on a suitable transform of $D$, which will again be a normal crossings divisor meeting the respective transform of $E$ transversally.

For simplicity, denote this controlled transform again by $J$. It is a principal monomial ideal supported by exceptional components. The second stage of the resolution process makes the order of $J$ drop below $c_+$. The sequence of blowups is now chosen globally according to the multiplicities of the exceptional factors appearing in $J$. This is a completely combinatorial and quite simple procedure which can be found in many places in the literature [Hir64, EV00, EH02].

The first part of the resolution process is much more involved and will be described now. The centers of blowup are defined locally at the points where the order of $I$ is maximal. Then it is shown that the definition does not depend on the local choices and thus defines a global, regular and closed center in $W$. Blowing up $W$ along this center will improve the singularities of $I$, and finitely many further blowups will make the order of the weak transform of $I$ drop.

The local definition of the center goes as follows. Let $S = \mathrm{top}(I)$ be the stratum of points in $W$ where $I$ has maximal order. Let $a$ be a point of $S$ and denote by $o$ the order of $I$ at $a$. Choose an osculating hypersurface $V$ for $I$ at $a$ in an open neighbourhood $U$ of $a$ in $W$. This is a closed regular hypersurface of $U$ given by a suitable partial derivative of an element of $I$ of order $o-1$. The hypersurface $V$ maximizes the order of the coefficient ideal $J_{n-1} = J_V(I)$ over all choices of local regular hypersurfaces, cf. Prop. 10.11. Thus $\mathrm{ord}_a(J_{n-1})$ does not depend on the choice of $V$. There exists a closed stratum $T$ in $S$ at $a$ of points $b \in S$ where the order of $J_{n-1}$ equals the order of $J_{n-1}$ at $a$.

The ideal $J_{n-1}$ on $V$ together with the new control $c = o!$ can now be resolved by induction on the dimension of the ambient space, applying the respective statement of the theorem. The new normal crossings divisor $E_{n-1}$ in $V$ is defined as $V \cap E$. For this it is necessary that the hypersurface $V$ can be chosen transversal to $E$. This is indeed possible, but will not be shown here. Also, it is used that $J_{n-1}$ admits again a decomposition $J_{n-1} = M_{n-1} \cdot I_{n-1}$ with $M_{n-1}$ a principal monomial ideal supported on some normal crossings divisor $D_{n-1}$ on $V$, and $I_{n-1}$ an ideal, the residual factor of $J_{n-1}$.

There thus exists a sequence of blowups of $V$ along regular centers transversal to $E_{n-1}$ and $D_{n-1}$ and their total transforms, contained in the locus $\mathrm{top}(J_{n-1}, c)$ of points where $J_{n-1}$ and its controlled transforms with respect to $c$ have order $\geq c$, and satisfying the requirements of a strong resolution so that the order of the weak transform of $I_{n-1}$ drops eventually at all points below the maximum of the order of $I_{n-1}$ at points $a$ of $V$.

All this is well defined on the open neighbourhood $U$ of $a$ in $W$, but depends a priori on the choice of the hypersurface $V$, since the centers are chosen locally in each $V$. It is not clear that the local choices patch to give a globally defined center. One can show that this is indeed the case, even though the local ideals $J_{n-1}$ do depend on $V$. The argument relies on the fact that the centers of blowup are constructed as the maximum locus of an invariant associated to $J_{n-1}$. By induction on the ambient dimension, such an invariant exists for each $J_{n-1}$: In dimension 1, it is just the order. In higher dimension, it is a vector of orders of suitably defined coefficient ideals. One then shows that the invariant of $J_{n-1}$ is independent of the choice of $V$ and defines an upper semicontinuous function on the stratum $S$. Its maximum locus is therefore well defined and closed. Again by induction on the ambient dimension, it can be assumed that it is also regular and transversal to the divisors $E_{n-1}$ and $D_{n-1}$.

By the assertion of the theorem in dimension $n-1$, the order of the weak transform of $J_{n-1}$ can be made everywhere smaller than $c = o!$ by a suitable sequence of blowups. The sequence of blowups also transforms the original ideal $J = M \cdot I$, producing controlled transforms of $J$ and weak transforms of $I$. The order of $I$ cannot increase in this sequence, if the centers are always chosen inside $S$. To achieve this inclusion a technical adjustment of the definition of coefficient ideals is required which will be omitted here. If the order of $I$ drops, induction applies. So one is left to consider points where possible the order of the weak transform has remained constant. There, one will use the commutativity of blowups with the passage to coefficient ideals, Prop. 10.6: The coefficient ideal of the final transform of $I$ will equal the controlled transform of the coefficient ideal $J_{n-1}$ of $I$. But the order of this controlled transform has dropped below $c$, hence, by Prop. 10.9, also the order of the weak transform of $I$ must have dropped. This proves the existence of a resolution. $\square$

## *Examples*

Example 11.4. In the situation of the theorem, take $W = \mathbb{A}^2$, $E = \emptyset$, and $J = (x^2y^3) = 1 \cdot I$ with control $c_+ = 1$. The ideal $J$ is monomial, but not resolved yet, since it is not supported on the exceptional divisor. It has order 5 at 0, order 3 along the $x$-axis, and order 2 along the $y$-axis. Blow up $\mathbb{A}^2$ in 0. The controlled transform in the $x$-chart is $J^! = (x^4y^3)$ with exceptional factor $I_E = (x^4)$ and residual factor $I_1 = (y^3)$, the strict transform of $J$. The controlled transform in the $y$-chart is $J^! = (x^2y^4)$ with exceptional factor $I_E = (y^4)$ and residual factor $I_1 = (x^2)$, the strict transform of $J$. One additional blowup resolves $J$.

Example 11.5. In the situation of the preceding example, replace $E = \emptyset$ by $E = V(x+y)$, respectively $E = V(x+y^2)$, and resolve the ideal $J$.

Example 11.6. Take $W = \mathbb{A}^3$, $E = V(x+z^2)$, and $J = (x^2y^3)$ with control $c_+ = 1$. The variety $X$ defined by $J$ is the union of the $xz$- and the $yz$-plane. The top locus of $J$ is the $z$-axis, which is tangent to $E$. Therefore it is not allowed to take it as the center of the first blowup. The only possible center is the origin. Applying this blowup, the top locus of the controlled transform of $J$ and the total transform of $E$ have normal crossings, so that the top locus can now be chosen as the center of the next blowup. Resolve the singularities of $X$.

REMARK 11.7. The non-transversality of the candidate center of blowup with already existing exceptional components is known as the *transversality problem*. The preceding example gives a first instance of the problem, see [EH02, Hau03] for more details.

EXAMPLE 11.8. Take $W = \mathbb{A}^3$, $E = D = \emptyset$ and $J = (x^2+yz)$. In characteristic different from 2, the origin of $\mathbb{A}^3$ is the unique isolated singular point of the cone $X$ defined by $J$. The ideal $J$ has order 2 at 0. After blowing up the origin, the singularity is resolved and the strict transform of $J$ defines a regular surface. It is transversal to the exceptional divisor.

EXAMPLE 11.9. Take $W = \mathbb{A}^3$, $E = D = \emptyset$, $J = (x^2 + y^a z^b)$ with $a, b \in \mathbb{N}$. According to the values of $a$ and $b$, the top locus of $J$ is either the origin, the $y$- or $z$-axis, or the union of the $y$- and the $z$-axis. If $a + b \geq 2$, the $yz$-plane is a hypersurface of maximal contact for $J$ at 0. The coefficient ideal is $J_1 = (y^a z^b)$ (up to raising it to the square). This is a monomial ideal, but not supported yet on exceptional components. Resolve $J$.

EXAMPLE 11.10. Take $W = \mathbb{A}^4$, $E = D = V(yz)$, $J = (x^2 + y^2z^2(y + w))$. The order of $J$ at 0 is 2. The $yzw$-hyperplane has maximal contact with $J$ at 0, with coefficient ideal $J_1 = y^2z^2(y + w)$, in which the factor $M_1 = (y^2z^2)$ is exceptional and $I_1 = (y + w)$ is residual. Resolve $J$. The top locus of $I_1$ is the plane $V(y + w) \subset \mathbb{A}^3$, hence not contained in the top locus of $J$. This technical complication is handled by introducing an intermediate ideal, the *companion ideal* [Vil07, EH02].

EXAMPLE 11.11. Compute the first few steps of the resolution process for the three surfaces defined in $\mathbb{A}^3$ by the polynomials $x^2 + y^2z$, $x^2 + y^3 + z^5$ and $x^3 + y^4z^5 + z^{11}$.

EXAMPLE 11.12. Prove with all details the embedded resolution of plane curves in characteristic zero according to the above description.

EXAMPLE 11.13. Resolve the following items according to Thm. 11.1, taking into account different characteristics of the ground field.
(a) $W = \mathbb{A}^2$, $E = \emptyset$, $J = I = (x^2y^3)$, with control $c_+ = 1$.
(b) $W = \mathbb{A}^2$, $E = V(x)$, $J = (x^2y^3)$, $c_+ = 1$.
(c) $W = \mathbb{A}^2$, $E = V(xy)$, $J = (x^2y^3)$, $c_+ = 1$.
(d) $W = \mathbb{A}^3$, $E = V(x + z^2)$, $J = (x^2y^3)$, $c_+ = 1$.
(e) $W = \mathbb{A}^3$, $E = V(y + z)$, $J = (x^3 + (y + z)z^2)$, $c_+ = 3$.
(f) $W = \mathbb{A}^3$, $E = \emptyset$, $J = I = (x^2 + yz)$, $c_+ = 2$.
(g) $W = \mathbb{A}^3$, $E = \emptyset$, $J = I = (x^3 + y^2z^2)$, $c_+ = 3$.
(h) $W = \mathbb{A}^3$, $E = \emptyset$, $J = I = (x^2 + xy^2 + y^5)$, $c_+ = 2$.
(i) $W = \mathbb{A}^3$, $E = \emptyset$, $J = I = (x^2 + xy^3 + y^5)$, $c_+ = 2$.
(j) $W = \mathbb{A}^3$, $E = V(x + y^2)$, $J = (x^3 + (x + y^2)y^4z^5)$, $c_+ = 3$.
(k) $W = \mathbb{A}^3$, $E = \emptyset$, $J = I = (x^2 + y^2z^2(y + z))$, $c_+ = 12$.
(l) $W = \mathbb{A}^3$, $E = V(yz)$, $J = (x^2 + y^2z^2(y + z))$, $c_+ = 2$.
(m) $W = \mathbb{A}^4$, $E = V(yz)$, $J = (x^2 + y^2z^2(y + w))$, $c_+ = 2$.

EXAMPLE 11.14. Consider $J = I = (x^2 + y^7 + yz^4)$ in $\mathbb{A}^3$, with $E = \emptyset$ and $c_+ = 2$. Consider the sequence of three point blowups with the following centers. First the origin of $\mathbb{A}^3$, then the origin of the $y$-chart, then the origin of the $z$-chart.

On the last blowup, consider the midpoint between the origin of the $y$- and the $z$-chart. Show that $J$ is resolved at this point if the characteristic is zero or $> 2$.

EXAMPLE 11.15.$^\triangleright$ What happens in the preceding example if the characteristic is equal to 2? Describe in detail the behaviour of the first two components of the resolution invariant under the three local blowups.

EXAMPLE 11.16.$^\triangleright$ A combinatorial version of resolution is known as Hironaka's polyhedral game: Let $N$ be an integral convex polyhedron in $\mathbb{R}^n_+$, i.e., the positive convex hull $N = \text{conv}(S + \mathbb{R}^n_+)$ of a finite set $S$ of points in $\mathbb{N}^n$. Player $A$ chooses a subset $J$ of $\{1, \ldots, n\}$, then player $B$ chooses an element $j \in J$. After these moves, $S$ is replaced by the set $S'$ of points $\alpha$ defined by $\alpha'_i = \alpha_i$ if $i \neq j$ and $\alpha'_j = \sum_{k \in J} \alpha_k$, giving rise to a new polyhedron $N'$. Player $A$ has won if after finitely many rounds the polyhedron has become an orthant $\alpha + \mathbb{R}^n_+$. Player $B$ can never win, but only prevent player $A$ from winning. Show that player $A$ has a winning strategy, first with and then without using induction on $n$ [Spi83, Zei06].

## 12. Lecture XII: Positive Characteristic Phenomena

The existence of the resolution of varieties of arbitrary dimension over a field of positive characteristic is still an open problem. For curves, there exist various proofs [Kol07] Chap. 1. For surfaces, the first proof of non-embedded resolution was given by Abhyankar in his thesis [Abh56]. Later, he proved embedded resolution for surfaces, but the proof is scattered over several papers which sum up to over 500 pages [Abh59, Abh64, Abh66b, Abh66a, Abh67, Abh98]. Cutkosky was able to shorten and simplify the argumens substantially [Cut11]. An invariant similar to the one used by Abhyankar was developed independently by Zeillinger and Wagner [Zei05, Wag09, HW14]. Lipman gave an elegent proof of non-embedded resolution for arbitrary two-dimensional schemes [Lip78, Art86]. Hironaka proposed an invariant based on the Newton polyhedron which allows to prove embedded resolution of surfaces which are hypersurfaces [Hir84, Cos81, Hau00]. Hironaka's invariant seems to be restricted to work only for surfaces. The case of higher codimensional surfaces was settled by Cossart-Jannsen-Saito, extending Hironaka's invariant [CJS09]. Different proofs were recently proposed by Benito-Villamayor and Kawanoue-Matsuki [BV12, KM12].

For three-folds, Abhyankar and Cossart gave partial results. Quite recently, Cossart-Piltant proved non-embedded resolution of three-folds by a long case-by-case study [CP08, CP09, CP12, CP13]. See also [Cut09].

Programs and techniques for resolution in arbitrary dimension and characteristic have been developed quite recently, among others, by Hironaka, Teissier, Kuhlmann, Kawanoue-Matsuki, Benito-Bravo-Villamayor, Hauser-Schicho, Cossart [Hir12, Tei03, Kuh00, Kaw14, KM10, BV12, BV14, BV10, BV11, Hau10b, HS12, Cos11].

REMARK 12.1. Let $\mathbb{K}$ be an algebraically closed field of prime characteristic $p > 0$, and let $X$ be an affine variety defined in $\mathbb{A}^n$. The main problems already appear in the hypersurface case. Let $f = 0$ be an equation for $X$ in $\mathbb{A}^n$. Various properties of singularities used in the characteristic zero proof fail in positive characteristic:

(a) The top locus of $f$ of points of maximal order need not be contained locally in a regular hypersurface. Take the variety in $\mathbb{A}^4$ defined by $f = x^2 + yz^3 + zw^3 + y^7 w$ over a field of characteristic 2 [Nar83, Mul83, Kaw14, Hau98].

(b) There exist sequences of blowups for which the sequence of points above a given point $a$ where the order of the strict transforms of $f$ has remained constant are eventually not contained in the strict transforms of any regular local hypersurface passing through $a$.

(c) Derivatives cannot be used to construct hypersurfaces of maximal contact.

(d) The characteristic zero invariant is no longer upper semicontinuous when translated to positive characteristic.

EXAMPLE 12.2. A typical situation where the characteristic zero resolution invariant does not work in positive characteristic is as follows. Take the polynomial $f = x^2 + y^7 + yz^4$ over a field of characteristic 2. There exists a sequence of blowups along which the order of the strict transforms of $f$ remains constant equal to 2 but where eventually the order of the residual factor of the coefficient ideal of $f$ with respect to a hypersurface of weak maximal contact increases.

The hypersurface $V$ of $W = \mathbb{A}^3$ defined by $x = 0$ produces – up to raising the ideal to the square – as coefficient ideal of $f$ the ideal on $\mathbb{A}^2$ generated by $y^7 + yz^4$. Its order at the origin is 5.

Blow up $\mathbb{A}^3$ at the origin. In the $y$-chart $W'$ of the blowup, the total transform of $f$ is given by

$$f^* = y^2(x^2 + y^5 + y^3z^4),$$

with $f' = x^2 + y^3(y^2 + z^4)$ the strict transform of $f$. It has order 2 at the origin of $W'$. The generator of the coefficient ideal of $f'$ in $V' : x = 0$ decomposes into a monomial factor $y^3$ and a residual factor $y^2 + z^4$. The order of the residual factor at the origin of $W'$ is 2. Blow up $W'$ along the $z$-axis, and consider the $y$-chart $W''$, with strict transform

$$f'' = x^2 + y(y^2 + z^4),$$

of $f$. The residual factor of the coefficient ideal in $V'' : x = 0$ equals again $y^2 + z^4$. Blow up $W''$ at the origin and consider the $z$-chart $W'''$, with strict transform

$$f''' = x^2 + yz(y^2 + z^2).$$

The origin of $W'''$ is the intersection point of the two exceptional components $y = 0$ and $z = 0$. The residual factor of the coefficient ideal in $V''' : x = 0$ equals $y^2 + z^2$, of order 2 at the origin of $W'''$.

Blow up the origin of $W'''$ and consider the affine chart $W^{(iv)}$ given by the coordinate transformation

$$x \mapsto xz, y \mapsto yz + z, \text{ and } z \mapsto z.$$

The origin of this chart is the midpoint of the new exceptional component. The strict transform of $f'''$ equals

$$f^{(iv)} = x^2 + (y + 1)z^2y^2,$$

which, after the coordinate change $x \mapsto x + yz$, becomes

$$f^{(iv)} = x^2 + y^3z^2 + y^2z^2.$$

The order of the strict transforms of $f$ has remained constant equal to 2 along the sequence of local blowups. The order of the residual factor of the associated coefficient ideal has decreased from 5 to 2 in the first blowup, then remained constant until the last blowup, where it increased from 2 to 3.

Remark 12.3. The preceding example shows that the order of the residual factor of the coefficient ideal of the defining ideal of a singularity with respect to a local hypersurface of weak maximal contact is not directly suited for an induction argument as in the case of zero characteristic. For surfaces, practicable modifications of this invariant are studied in [HW14].

Definition 12.4. A hypersurface singularity $X$ at a point $a$ of affine space $W = \mathbb{A}^n_{\mathbb{K}}$ over a field $\mathbb{K}$ of characteristic $p$ is called *purely inseparable of order $p^e$ at $a$* if there exist local coordinates $x_1, \ldots, x_n$ on $W$ at $a$ such that $a = 0$ and such that the local equation $f$ of $X$ at $a$ is of the form

$$f = x_n^{p^e} + F(x_1, \ldots, x_{n-1})$$

for some $e \geq 1$ and a polynomial $F \in \mathbb{K}[x_1, \ldots, x_{n-1}]$ of order $\geq p^e$ at $a$.

Proposition 12.5. For a purely inseparable hypersurface singularity $X$ at $a$, the polynomial $F$ is unique up to multiplication by units in the local ring $\mathcal{O}_{W,a}$ and the addition of $p^e$-th powers in $\mathbb{K}[x_1, \ldots, x_{n-1}]$.

Proof. Multiplication by units does not change the local geometry of $X$ at $a$. A coordinate change in $x_n$ of the form $x_n \mapsto x_n + a(x_1, \ldots, x_{n-1})$ with $a \in \mathbb{K}[x_1, \ldots, x_{n-1}]$ transforms $f$ into $f = x_n^{p^e} + a(x_1, \ldots, x_{n-1})^{p^e} + F(x_1, \ldots, x_{n-1})$. This implies the assertion. □

Definition 12.6. Let affine space $W = \mathbb{A}^n$ be equipped with an exceptional normal crossings divisor $E$ produced by earlier blowups with multiplicities $r_1, \ldots, r_n$. Let $x_1, \ldots, x_n$ be local coordinates at a point $a$ of $W$ such that $E$ is defined at $a$ by $x_1^{r_1} \cdots x_n^{r_n} = 0$. Let $f = x_n^{p^e} + F(x_1, \ldots, x_{n-1})$ define a purely inseparable singularity $X$ of order $p^e$ at the origin $a = 0$ of $\mathbb{A}^n$ such that $F$ factorizes into

$$F(x_1, \ldots, x_{n-1}) = x_1^{r_1} \cdots x_{n-1}^{r_{n-1}} \cdot G(x_1, \ldots, x_{n-1}).$$

The *residual order of $X$ at $a$ with respect to $E$* is the maximum of the orders of the polynomials $G$ at $a$ over all choices of local coordinates such that $f$ has the above form [Hir12, Hau10b].

Remark 12.7. The residual order can be defined for arbitrary singularities [Hau10b]. In characteristic zero, the definition coincides with the second component of the local resolution invariant, defined by the choice of an osculating hypersurface or, more generally, of a hypersurface of weak maximal contact and the factorization of the associated coefficient ideal.

Remark 12.8. In view of the preceding example, one is led to investigate the behaviour of the residual order under blowup at points where the order of the singularity remains constant. Moh showed that it can increase at most by $p^{e-1}$ [Moh87]. Abhyankar seems to have observed already this bound in the case of surfaces. He defines a correction term $\varepsilon$ taking values equal to 0 or $p^{e-1}$ which is added to the residual order according to the situation in order to make up for the occasional increases of the residual order [Abh67, Cut11]. A similar construction has been proposed by Zeillinger and Hauser-Wagner [Zei05, Wag09, HW14]. This allows, at least for surfaces, to define a secondary invariant after the order of the singularity, the *modified residual order of the coefficient ideal*, which does not increase under blowup. The problem then is to handle the case where the order of the singularity and the modified residual order remain constant. It is not clear how to define a third invariant which manifests the improvement of the singularity.

Remark 12.9. Following ideas of Giraud, Cossart has studied the behaviour of the order of the Jacobian ideal of $f$, defined by certain partial derivatives of $f$. Again, it seems that hypersurfaces of maximal contact do not exist for this invariant [Gir75, Cos11]. There appeared promising recent approaches by Hironaka, using the machinery of differential operators in positive characteristic, by Villamayor and collaborators using instead of the restriction to hypersurfaces of maximal contact projections to regular hypersurfaces via elimination algebras, and by Kawanoue-Matsuki using their theory of idealistic filtrations and differential closures. None of these proposals has been able to produce an invariant or a resolution strategy which works in positive characteristic for all dimensions.

Remark 12.10. Another approach consists in analyzing the singularities and blowups for which the residual order increases under blowup. This leads to the notion of kangaroo singularities:

Definition 12.11. A hypersurface singularity $X$ defined at a point $a$ of affine space $W = \mathbb{A}^n_{\mathbb{K}}$ over a field $\mathbb{K}$ of characteristic $p$ by a polynomial equation $f = 0$ is called *a kangaroo singularity* if there exists a local blowup $\pi : (\widetilde{W}, a') \to (W, a)$ of $W$ along a regular center $Z$ contained in the top locus of $X$ and transversal to an already existing exceptional normal crossings divisor $E$ such that the order of the strict transform of $X$ remains constant at $a'$ but the residual order of the strict transform of $f$ increases at $a'$. The point $a'$ is then called *kangaroo point of $X$ above $a$*.

Remark 12.12. Kangaroo singularities can be defined for arbitrary singularities. They have been characterized in all dimensions by Hauser [Hau10a, Hau10b]. However, the knowledge of the algebraic structure of these singularities did not yet give any hint how to overcome the obstruction caused by the increase of the residual order.

Proposition 12.13. If a polynomial $f = x_n^{p^e} + F(x_1, \dots, x_{n-1}) = x_n^{p^e} + x_1^{r_1} \cdots x_{n-1}^{r_{n-1}} \cdot G(x_1, \dots, x_{n-1})$ defines a kangaroo singularity of order $p^e$ at 0 the following conditions must hold: (a) the sum of the $r_i$ and of the order of $G$ at 0 is divisible by $p^e$; (b) the sum of the residues of the exceptional multiplicities $r_i$ modulo $p^e$ is bounded by $(m-1) \cdot p^e$ with $m$ the number of exceptional multiplicities not divisible by $p^e$; (c) the initial form of $F$ equals a specific homogeneous polynomial which can be explicitly described. Any kangaroo point $a'$ of $X$ above $a$ lies outside the strict transform of the components of the exceptional divisor at $a$ whose multiplicities are not a multiple of $p^e$.

Remark 12.14. A more detailed description of kangaroo singularities and a further discussion of typical characteristic $p$ phenomena can be found in [Hau10a, Hau10b].

## *Examples*

Example 12.15. Prove the resolution of plane curves in arbitary characteristic by using the order and the residual order as the resolution invariants.

Example 12.16. Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$. Develop a significant notion of resolution for elements of the quotient of rings $\mathbb{K}[x, y]/\mathbb{K}[x^p, y^p]$. Then prove that such a resolution always exists.

Example 12.17. ▷ Consider the polynomial $f = x^2 + yz^3 + zw^3 + y^7w$ on $\mathbb{A}^4$ over a ground field of characteristic 2. Its maximal order is 2, and the respective top locus is the image of the monomial curve $(t^{32}, t^7, t^{19}, t^{15}), t \in \mathbb{K}$. The image curve has embedding dimension 4 at 0 and cannot be embedded locally at 0 into a regular hypersurface of $\mathbb{A}^4$. Hence there is no hypersurface of maximal contact with $f$ locally at the origin.

Example 12.18. ▷ Find a surface $X$ in positive characteristic and a sequence of point blowups starting at $a \in X$ so that some of the points above $a$ where the order of the weak transforms of $X$ remains constant eventually leave the transforms of any local regular hypersurface passing through $a$.

Example 12.19. ▷ Show that $f = x^2 + yz^3 + zw^3 + y^7w$ has in characteristic 2 top locus $\mathrm{top}(f)$ equal to the parametrized curve $(t^{32}, t^7, t^{19}, t^{15})$ in $\mathbb{A}^4$ [Nar83, Mul83, Kaw14].

Example 12.20. ▷ Show that $f$ is not contained in the square of the ideal defining the parametrized curve $(t^{32}, t^7, t^{19}, t^{15})$.

Example 12.21. ▷ Find the defining ideal for the image of the monomial curve $(t^{32}, t^7, t^{19}, t^{15})$ in $\mathbb{A}^4$. What is the local embedding dimension at 0?

Example 12.22. ▷ Show that $f = x^2 + yz^3 + zw^3 + y^7w$ admits in characteristic 2 at the point 0 no local regular hypersurface of permanent maximal contact (i.e., whose successive strict transforms contain all points where the order of $f$ has remained constant in any sequence of blowups with regular centers inside the top locus).

Example 12.23. Consider $f = x^2 + y^7 + yz^4$ in characteristic 2. Show that there exists a sequence of point blowups for which $f$ admits at the point 0 no local regular hypersurface whose transforms have weak maximal contact with the transforms of $f$ as long as the order of $f$ remains equal to 2.

Example 12.24. ▷ Define the $p$-th order derivative of polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ for $\mathbb{K}$ a field of characteristic $p$.

Example 12.25. ▷ Construct a surface of order $p^5$ in $\mathbb{A}^3$ over a field of characteristic $p$ for which the residual order increases under blowup.

Example 12.26. ▷ Show that for the polynomial $f = x^p + y^pz$ over a field of characteristic $p$ and taking $E = \emptyset$ the residual order of $f$ along the (closed) points of the $z$-axis is not equal to its value at the generic point.

Example 12.27. ▷ Let $y_1, \ldots, y_m$ be fixed coordinates, and consider a homogeneous polynomial $G(y) = y^r \cdot g(y)$ with $r \in \mathbb{N}^m$ and $g(y)$ homogeneous of degree $k$. Let $G^+(y)$ be the polynomial obtained from $G$ by the linear coordinate change $y_i \to y_i + y_m$ for $i = 1, \ldots, m-1$. Show that the order of $G^+$ along the $y_m$-axis is at most $k$.

Example 12.28. Express the assertion of the preceding example through the invertibility of a matrix of multinomial coefficients.

Example 12.29. ▷ Consider $G(y, z) = y^r z^s \sum_{i=0}^{k} \binom{k+r}{i+r} y^i (tz - y)^{k-i}$. Compute for $t \in \mathbb{K}^*$ the polynomial $G^+(y, z) = G(y + tz, z)$ and its order with respect to $y$ modulo $p$-th power polynomials.

EXAMPLE 12.30.▷ Determine all homogeneous polynomials $G(y,z) = y^r z^s g(y,z)$ so that $G^+(y,z)$ has order $k+1$ with respect to $y$ modulo $p$-th power polynomials.

EXAMPLE 12.31. + Find a new systematic proof for the embedded resolution of surfaces in three-space in arbitrary characteristic.

EXAMPLE 12.32. Let $G(x)$ be a polynomial in one variable over a field $\mathbb{K}$ of characteristic $p$, of degree $d$ and order $k$ at 0. Let $t \in K$, and consider the equivalence class $\overline{G}$ of $G(x+t)$ in $K[x]/K[x^p]$ (i.e., consider $K(x+t)$ modulo $p$-th power polynomials). What is the maximal order of $\overline{G}$ at 0? Describe all examples where this maximum is achieved.

## 13. Discussion of selected examples

The comments and hints below were compiled by Stefan Perlega and Valerie Roitner.

EX. 1.1. Let $X$ be defined by $27x^2y^3z^2+(x^2+y^3-z^2)^3 = 0$ and let $Y = C\times C$ be the cartesian product of the cusp $C$ defined by $x^3 - y^2 = 0$ with itself. The surface $Y$ can be parametrized by $(s,t) \to (s^3, s^2, t^3, t^2)$. Composing this map with the projection $(x,y,z,w) \mapsto (x, -y+w, z)$ from $\mathbb{A}^4$ to $\mathbb{A}^3$ gives $(s,t) \mapsto (s^3, t^2 - s^2, t^3)$. Substitution into the equation of $X$ gives 0,

$$27s^6t^6(t^2-s^2)^3 + (s^6 + (t^2-s^2)^3 - t^6)^3 =$$

$$= 27(s^6t^{12} - 3s^8t^{10} + 3s^{10}t^8 - s^{12}t^6) + 27(s^{12}t^6 - 3s^{10}t^8 + 3s^8t^{10} - s^6t^{12}) = 0.$$

Therefore the image of $Y$ under the projection lies inside $X$. It remains to show that every point in $X$ is obtained in this way. The restriction $Y \to X$ of the projection $\mathbb{A}^4 \to \mathbb{A}^3$ is a finite map (as can e.g. be checked by using a computer algebra program), hence the image of $Y$ is closed in $X$. As it has dimension two and $X$ is an irreducible surface, the image of $Y$ is whole $X$.

EX. 1.2. Asides from the symmetries in the text, replacing $x$ with $\pm\sqrt{-1}\cdot x$ and $z$ with $\pm\sqrt{-1}\cdot z$ gives a new symmetry.

In characteristic 2 the defining polynomial of $X$ equals $f = x^2y^3z^3 + (x^2 + y^3 + z^3)^3$ and there appears an additional symmetry which is given by interchanging $x$ with $z$.

EX. 1.3. Replacing $z$ by $\sqrt{-1}\cdot z$ gives the equation $g = -27x^2y^3z^2 + (x^2+y^3+z^2)^3 = 0$, cf. Figure 3. Symmetries of this surface are e.g. given by replacing $x$ by $-x$ or $z$ by $-z$ or by interchanging $x$ with $z$. Also sending $(x,y,z)$ to $(t^3x, t^2y, t^3z)$ with $t \in \mathbb{K}$ gives a symmetry.

The partial derivatives of $g$ give the equations for $\mathrm{Sing}(X)$,

$$x\cdot(-9y^3z^2 + (x^2+y^3+z^2)^3) = 0,$$

$$y^2\cdot(-9x^2z^2 + (x^2+y^3+z^2)^3) = 0,$$

$$z\cdot(-9y^3x^2 + (x^2+y^3+z^2)^3) = 0.$$

Therefore the singular locus of $X$ has six components, given by the equations

$$x = y = z = 0,$$

$$x = y^3 + z^2 = 0,$$

$$z = x^2 + y^3 = 0,$$
$$y = x^2 + z^2 = 0,$$
$$z - x = y^3 - z^2 = 0,$$
$$z + x = y^3 - z^2 = 0.$$

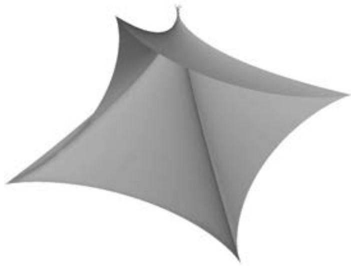

Figure 3: The surface of equation $27x^2y^3z^2 = (x^2 + y^3 + z^2)^3$.

Ex. 1.4. The point $a = (0, 1, 1)$ lies in the component $S$ of $\mathrm{Sing}(X)$ defined by $g = x - y^3 + z^2 = 0$. Since $(\partial_x g(a), \partial_y g(a), \partial_z g(a)) = (1, 3, 2)$, the component $S$ is regular at $a$. The curve given by the parametrization $(0, t^2, t^3)$ lies in $S$ and passes through $a$ for $t = 1$. Its tangent vector at $t = 1$ is $(0, 2, 3)$, which is a normal vector to the plane $P$. Therefore $P$ intersects $S$ transversally at $a$.

The intersection $X \cap P$ is given by the equations $y = \frac{1}{2}(5 - 3z)$ and $h(x, z) = f(x, \frac{1}{2}(5-3z), z) = 27x^2z^2\frac{1}{8}(5-3z)^3 + (x^2 + \frac{1}{8}(5-3z)^3 - z^2)^3 = 0$. Computing the points where $\partial_x h = 0$ and $\partial_z h = 0$ gives (among others) the solution $x = 0$, $z = 1$, whence $y = 1$. Therefore $X \cap P$ has a singularity at $(0, 1, 1)$. The Taylor expansion of $h$ at $(x, z) = (0, 1)$ is given by

$$\left(-\frac{2197}{8}w^3 + O(w^4)\right) + \left(27 - \frac{135w}{2} + 93w^2 + O(w^3)\right)x^2 + O(x^4),$$

where $w = z - 1$.

Ex. 1.6. The blowup $\widetilde{X}$ of $X$ in the origin of $\mathbb{A}^3$ is defined in $X \times \mathbb{P}^2$ by the equations

$$xu_2 - yu_1 = xu_3 - zu_1 = yu_3 - zu_2 = 0,$$

where $(u_1 : u_2 : u_3)$ are projective coordinates on $\mathbb{P}^2$. The blowup map $\pi : \widetilde{X} \to X$ is the restriction to $\widetilde{X}$ of the projection $X \times \mathbb{P}^2 \to X$.

The equation of the $x$-chart of the blowup of $X$ is obtained by replacing $(x, y, z)$ with $(x, xy, xz)$ in the equation defining $X$ and by factoring the appropriate power of the polynomial $x$ defining the exceptional divisor. This gives the equation

$$C_x : 27x^7y^3z^2 + (x^2 + x^3y^3 - x^2z^2) = x^6 \cdot (27xy^3z^2 + (1 + xy^3 - z^2)^3) = 0,$$

so that the patch of $\widetilde{X}$ in the $x$-chart is defined by $27xy^3z^2 + (1 + xy^3 - z^2)^3 = 0$ in $\mathbb{A}^3$. See Figure 4. Similarly, the $y$- and the $z$-chart are obtained by replacing $(x, y, z)$ with $(xy, y, yz)$ and $(xz, yz, z)$ respectively and give the equations

$$C_y : y^6 \cdot (27x^2yz^2 + (x^2 + y - z^2)^3) = 0$$

and

$$C_z : z^6 \cdot (27x^2y^3z + (x^2 + y^3z - 1)^3) = 0.$$

The blowup $\widetilde{Y}$ of $Y = C \times C$ in the origin of $\mathbb{A}^4$ is defined in $Y \times \mathbb{P}^3$ by

$$xu_2 - yu_1 = xu_3 - zu_1 = xu_4 - wu_1 = yu_3 - zu_2 = yu_4 - wu_2 = zu_4 - wu_3 = 0.$$

The $x$-chart of $\widetilde{Y}$ is obtained by replacing $(x, y, z, w)$ with $(x, xy, xz, xw)$ in the equations defining $Y$ and by then factoring the appropriate powers of $x$. It is hence given in $\mathbb{A}^4$ by the equations $1 - xy^3 = z^2 - xw^3 = 0$. The other charts are obtained similarly: in the $y$-chart, $x^2 - y = z^2 - yw^3 = 0$; in the $z$-chart, $x^2 - y^3z = 1 - zw^3 = 0$; in the $w$-chart, $x^2 - y^3w = z^2 - w = 0$.

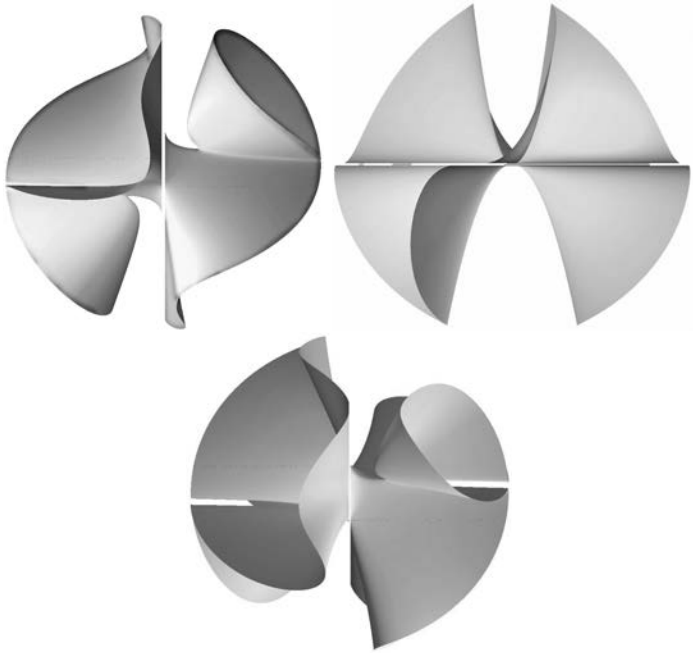

Figure 4: Chart expressions of the blowup of the surface Camelia.

Ex. 2.34. Since $\mathcal{O}_{X,a} \subseteq \widehat{\mathcal{O}}_{X,a}$ and $\widehat{m}_{X,a} = m_{X,a} \cdot \widehat{\mathcal{O}}_{X,a}$ every regular system of parameters $x_1, \ldots, x_n$ of $\mathcal{O}_{X,a}$ is also a generator system of $\widehat{m}_{X,a}$ in $\widehat{\mathcal{O}}_{X,a}$. But there exists a subset $S$ of $\{x_1, \ldots, x_n\}$ such that $S$ is a regular system of parameters of $\widehat{\mathcal{O}}_{X,a}$. The converse does not hold, since in general a regular system of parameters of $\widehat{\mathcal{O}}_{X,a}$ need not belong to $\mathcal{O}_{X,a}$.

Ex. 2.38. The image of $f : (x, y) \mapsto (xy, y)$ is $f(\mathbb{A}^2) = (\mathbb{A}^2 \setminus (\mathbb{A}^1 \times \{0\})) \cup \{(0, 0)\}$. This is a constructible and dense subset of $\mathbb{A}^2$. The inverse $f^{-1} : (x, y) \mapsto (\frac{x}{y}, y)$ of $f$ is defined on the complement of the $x$-axis. It cannot be extended to the origin 0 of $\mathbb{A}^2$, since $f$ contracts the $x$-axis of $\mathbb{A}^2$ onto 0.

Ex. 2.40. It is easily checked that the inverse of $\varphi_{ij}$ is $\varphi_{ji}$:

$$[\varphi_{ji} \circ \varphi_{ij}(x_1, \ldots, x_n)]_k = \begin{cases} \frac{x_k}{x_j} \cdot x_j = x_k, & k \neq i, j, \\ \frac{1}{x_j} \cdot x_i x_j = x_i, & k = i, \\ \frac{1}{1/x_j} = x_j, & k = j, \end{cases}$$

where $[-]_k$ denotes the $k$-th component. Therefore $\varphi_{ji} \circ \varphi_{ij}$ is the identity. The domain $U_{ij}$ of $\varphi_{ij}$ is $\mathbb{A}^n \setminus V(x_j)$, a dense open subset of $\mathbb{A}^n$. The image $\varphi_{ij}(U_{ij})$ equals $\mathbb{A}^n \setminus V(x_i)$. It is open, too. Since all components of $\varphi_{ij}$ are rational and the denominators do not vanish on $U_{ij}$, the maps $\varphi_{ij}$ induce biregular maps $U_{ij} \to U_{ji}$.

Ex. 2.41. Elliptic curves admit an additive group structure. It therefore suffices to restrict to the origin $a = 0 \in \mathbb{A}^2$ and to show that the curve is formally isomorphic at 0 to $(\widehat{\mathbb{A}}^1, 0)$. But $\mathbb{K}[[x,y]]/(y^2 - x^3 + x) \simeq \mathbb{K}[[y]]$ since, by the implicit function theorem for formal power series, the residue class of $x$ in $\mathbb{K}[[x,y]]/(y^2 - x^3 + x)$ can be expressed as a series in $y$. To show that $X$ is nowhere locally biregular to the affine line is much harder.

Ex. 2.42. A morphism $f : X \to Y$ of algebraic varieties is proper if it is separated and universally closed, i.e., if for any variety $Z$ and any morphism $h : Z \to Y$ the induced morphism $g : X \times_Y Z \to Y, g(x,z) = f(x) = h(z)$ is closed. In the example, already the image $\pi(X) = \mathbb{A}^1 \setminus \{0\}$ of $X$ is not closed.

Ex. 2.43. A formal isomorphism between $(\widehat{X}, 0)$ and $(\widehat{Y}, 0)$ would send the the jacobian ideals generated by the partial derivatives of $x^3 - y^2$ and $x^5 - y^2$ to each other. This is impossible. On the other hand, the formal isomorphism defined by $(x,y) \mapsto (x\sqrt[3]{1+x^2}, y)$ sends $(\widehat{X}, 0)$ onto $(\widehat{Z}, 0)$. It is an isomorphism by the inverse function theorem for formal power series.

Ex. 3.27. The maximal ideal of $\mathbb{K}[[x]]$ is generated by $x$. Since $\sqrt{1+x}$ is a unit in $\mathbb{K}[[x]]$ (its inverse is $(1+x)^{-1/2} = \sum_{k\geq 0} \binom{-1/2}{k}(-1)^k x^k$), also $x \cdot \sqrt{1+x}$ generates this ideal. Therefore $x \cdot \sqrt{1+x}$ is a regular system of parameters in $\mathbb{K}[[x]]$. It does not stem from a regular parameter system in $\mathbb{K}[x]_{(x)}$ since $\sqrt{1+x}$ cannot be written as a quotient of polynomials.

Ex. 3.28. The maximal ideal of the local ring $\mathcal{O}_{\mathbb{A}^2,0} = \mathbb{K}[x,y]_{(x,y)}$ is generated by $x$ and $y$. As $x^2 + 1$ is invertible in $\mathbb{K}[x,y]_{(x,y)}$, also $y^2 - x^3 - x = y^2 - x(x^2+1)$ and $y$ generate this ideal. They hence form a regular system of parameters of $\mathcal{O}_{\mathbb{A}^2,0}$. For the remaining assertions, see ex. 2.41.

Ex. 3.31. Computing the derivatives of $x^2 - y^2 z$ with respect to $x$, $y$ and $z$ and setting them zero shows that the singular locus of $X$ is the $z$-axis. Let $a = (0,0,t)$ with $t \neq 0$ be a point of the $z$-axis outside the origin. Then $\widehat{\mathcal{O}}_{X,a} = \mathbb{C}[[x,y,z]]/(x^2 - y^2(z-t))$. Since $\sqrt{z-t}$ is for $t \neq 0$ a formal power series in $z$ the quotient can be rewritten as $\mathbb{C}[[x,y,z]]/(x - y\sqrt{z-t})(x + y\sqrt{z-t})$. The product $(x - y\sqrt{z-t})(x + y\sqrt{z-t}) = 0$ defines two smooth formal surfaces intersecting each other transversally. They are formally isomorphic to the union of the two planes defined by $(x - y\sqrt{-t})(x + y\sqrt{-t}) = 0$. Hence $a$ is a normal crossings point of $X$. But it is not a simple normal crossings point since, globally, $X$ consists of only one component which is moreover singular at $a$.

The formal neighborhood of 0 is $\widehat{\mathcal{O}}_{X,0} = K[[x,y,z]]/(x^2 - y^2 z)$. Since $x^2 - y^2 z$ is irreducible in $K[[x,y,z]]$ and $X$ is singular at 0 the origin is not a normal crossings point of $X$.

Ex. 3.34. Over $\mathbb{C}$ or any finite field $\mathbb{F}_q$ with $q$ congruent to 1 modulo 4 there exists a square root of $-1$, hence the variety defined by $x^2+y^2=(x+\sqrt{-1}y)(x-\sqrt{-1}y)=0$ has normal crossings at the origin. Since both components are regular at 0 it also has simple normal crossings.

The surface defined by $x^2+y^2+z^2=0$ does not have normal crossings over $\mathbb{R}$ or $\mathbb{C}$, but it has normal crossings over a field of characteristic 2, since $x^2+y^2+z^2=(x+y+z)^2$ in characteristic 2 so that the variety is a double plane. The same holds for the variety defined by $x^2+y^2+z^2+w^2=0$. The variety $xy(x-y)=0$ has three components passing through 0, but as $\mathbb{A}^2$ has only two coordinate subspaces the variety does not have normal crossings at 0. The same argument works for $xy(x^2-y)=0$. The variety defined by $(x-y)z(z-x)=0$ has both normal and simple normal crossings at the origin over any field.

Ex. 3.35. The visualization of the surface looks as follows, cf. figure 5.

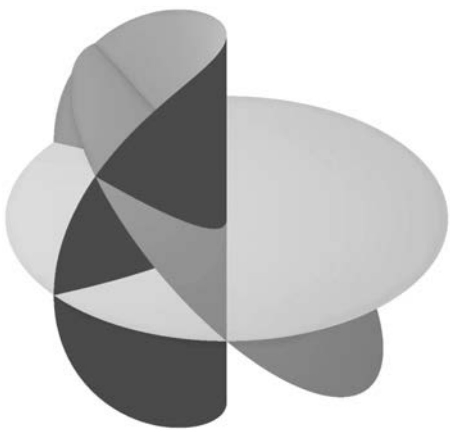

Figure 5: The zeroset of $(x-y^2)(x-z)z=0$ in $\mathbb{A}^3$.

Ex. 3.38. A point $a$ is a regular point of a variety if the local ring is regular, i.e., if its maximal ideal can be generated by as many elements as the Krull dimension indicates. For a cartesian product $X\times Y$, the local ring at a point $(a,b)$ is the tensor product of the local rings of the two factors,

$$\mathcal{O}_{X\times Y,(a,b)}=\mathbb{K}[X\times Y]_{m_{(a,b)}}\simeq\mathbb{K}[X]_{m_a}\otimes\mathbb{K}[Y]_{m_b}=\mathcal{O}_{X,a}\otimes\mathcal{O}_{Y,b}$$

and the same holds for the respective maximal ideals,

$$m_{(a,b)}/m_{(a,b)}^2\simeq m_a/m_a^2\otimes m_b/m_b^2.$$

Therefore $(a,b)\in X\times Y$ is regular if and only if $a$ is regular in $X$ and $b$ is regular in $Y$. Consequently, $\mathrm{Sing}(X\times Y)=(\mathrm{Sing}(X)\times Y)\cup(X\times\mathrm{Sing}(Y))$.

Ex. 3.39. A rather simple example for such a variety is the surface *Tülle* defined by $xz(x+z-y^2)=0$ in $\mathbb{A}^3$. The pairwise intersections of its three components are the $y$-axis, respectively the two parabolas defined by $x=z-y^2=0$ and $z=x-y^2=0$, and are therefore regular. The parabolas are tangent to the $y$-axis at 0. Said differently, the intersection of all three components, which is set-theoretically just the origin $0\in\mathbb{A}^3$, is scheme-theoretically singular (i.e., the sum of the three ideals $(x)$, $(z)$ and $(x+z-y^2)$ is not the ideal $(x,y,z)$ defining the origin in $\mathbb{A}^3$). Because of this, the surface is not mikado at 0.

Ex. 4.25. For $X = V(xy, x^2)$ and $a$ the origin, the local ring is $\mathcal{O}_{X,a} = \mathbb{K}[x,y]/(xy,x^2)_{(x,y)} \simeq \mathbb{K}[x,y]_{(x,y)}/(xy,x^2)$. As a scheme, $X$ equals the $y$-axis together with an embedded point at 0, since $(xy,x^2)$ has the primary decomposition $(xy,x^2) = (x) \cap (x^2,y)$. As $Z = \{0\}$ cannot be defined in $X$ by a single equation, it is not a Cartier divisor.

Ex. 4.26. The local ring $\mathcal{O}_{X,a}$ of $X = \mathbb{A}^1$ at the origin is $\mathcal{O}_{X,a} = K[x]_{(x)}$. The monomial $x^2$ is not a zero-divisor in $\mathcal{O}_{X,a}$, hence $Z = V(x^2)$ is a Cartier divisor in $\mathbb{A}^1$. Similarly, $x^2y$ is not a zero divisor in $\mathcal{O}_{\mathbb{A}^2,a} = K[x,y]_{(x-a_1,y-a_2)}$ for any $a = (a_1,a_2) \in \mathbb{A}^2$ and $Z = V(x^2y)$ is a Cartier divisor in $\mathbb{A}^2$.

Ex. 4.27. As $Z = V(x^2,y)$ is defined in $X = V(x^2,xy)$ by $\overline{y} = 0$ where the residue class $\overline{y}$ of $y$ in $\mathcal{O}_{X,0} = K[x,y]_{(x,y)}/(x^2,xy)$ is a zero divisor, it follows that $Z$ is not a Cartier divisor in $X$.

Ex. 4.38. The Rees algebra of the ideal $I = (x, 2y)$ in $\mathbb{Z}_{20}[x,y]$ is given by

$$\widetilde{R} = \bigoplus_{k\geq 0} I^k t^k = \bigoplus_{k\geq 0} (xt, 2yt)^k = \mathbb{Z}_{20}[x,y,xt,2yt]$$

with $\deg t = 1$. It is isomorphic to $\mathbb{Z}_{20}[u,v,w,z]/(uz-2vw)$ with $\deg u = \deg v = 0$ and $\deg w = \deg z = 1$.

Ex. 4.46. The blowup of $\mathbb{A}^2$ in $Z = \{(0,1)\}$ is given by the closure $\overline{\Gamma}$ of the graph of $\gamma : \mathbb{A}^2 \setminus \{(0,1)\} \to \mathbb{P}^1, (a,b) \mapsto (a : b-1)$ in $\mathbb{A}^2 \times \mathbb{P}^1$ together with the projection $\pi : \overline{\Gamma} \to X, (a,b,(c:d)) \mapsto (a,b)$. More explicitly,

$$\overline{\Gamma} = \{(a,b,(a:b-1)),\ (a,b) \in \mathbb{A}^2\} \cup \{(0,1)\} \times \mathbb{P}^1.$$

The line $L$ in $\mathbb{A}^2$ defined by $x+y=0$ does not contain the point $(0,1)$, therefore its preimage in $\overline{\Gamma}$ is $\pi^{-1}(L) = \{((a,-a),(a:-a-1)),\ a \in \mathbb{A}^1\}$.

The line $L'$ in $\mathbb{A}^2$ defined by $x+y=1$ contains the point $(0,1)$, therefore its preimage in $\overline{\Gamma}$ is $\pi^{-1}(L') = \{(a, 1-a, (a:-a)),\ a \in \mathbb{A}^1 \setminus \{0\}\} \cup \{(0,1)\} \times \mathbb{P}^1$.

Ex. 4.47. The chart expressions of the blowup map of $\mathbb{A}^3$ along the $z$-axis are given by $\pi_1(x,y,z) = (x,xy,z)$ and $\pi_2(x,y,z) = (xy,y,z)$. Their inverses are $\pi_1^{-1}(x,y,z) = (x, \frac{y}{x}, z)$ and $\pi_2^{-1}(x,y,z) = (\frac{x}{y}, y, z)$. The chart transition maps are given by the compositions

$$\pi_1^{-1} \circ \pi_2(x,y,z) = \left(xy, \frac{1}{x}, z\right)$$

and

$$\pi_2^{-1} \circ \pi_1(x,y,z) = \left(\frac{1}{y}, xy, z\right).$$

Ex. 4.48. Let the chosen line $Z$ in the cone $X$ be defined by $x = y - z = 0$. It requires two equations, therefore it is not a Cartier divisor. The polynomials $x$ and $y-z$ form a regular sequence in $\mathbb{K}[X] = \mathbb{K}[x,y,z]/(x^2+y^2-z^2)$. Therefore the blowup $\widetilde{X}$ of $X$ along $Z$ is given in $X \times \mathbb{P}^1$ by the equation $xv - (y-z)u =$ where $(u : v)$ are projective coordinates in $\mathbb{P}^1$.

An alternate way to compute the blowup is by applying first the linear coordinate change $u = x, w = y + z$ and $t = y - z$. The equation $x^2 + y^2 = z^2$ of $X$ transforms into $Y : u^2 + 2wt = 0$ and $x = y - z = 0$ becomes $u = t = 0$. The latter equations define the $w$-axis, which is now the center of blowup. The resulting chart expressions of the blowup map are given by $\pi_1(u, w, t) = (ut, w, t)$ and $\pi_2(u, w, t) = (u, w, wt)$. This gives the chart descriptions of the total transform $Y^*$ of $Y$ via

$$Y_1^* = V(u^2t^2 + 2wt) = V(t) \cup V(u^2t + 2w),$$
$$Y_2^* = V(u^2 + 2uwt) = V(u) \cup V(u + 2wt),$$

respectively. Substituting backwards gives

$$X_1^* = V(y - z) \cup V(x^2(y - z) + 2(y + z)),$$
$$X_2^* = V(x) \cup V(x + 2(y^2 - z^2)),$$

where the second components denote the chart expressions of the strict transform $X^s$ of $X$ in the blowup $\widetilde{\mathbb{A}}^3$ of $\mathbb{A}^3$ along $Z$, i.e., of the blowup $\widetilde{X}$ of $X$ along $Z$, cf. Def. 6.2 as well as Prop. 5.1 together with its corollary.

Ex. 4.49. The polynomials $z$ and $x^2+(y+2)^2-1$ defining the circle in $\mathbb{A}^3$ form a regular sequence. Hence, if $(u : v)$ denote projective coordinates in $\mathbb{P}^1$, the blowup $\widetilde{X}$ of $X$ along the circle is given in $X \times \mathbb{P}^1$ by the equation $uz-v(x^2+(y+2)^2-1) = 0$ together with the projection $\pi : \widetilde{X} \to X$ on the first factor.
The polynomials $z$ and $y^2 - x^3 - x$ defining the elliptic curve in $\mathbb{A}^3$ form a regular sequence, too. The blowup $\widetilde{X}$ of $X$ along this curve is given in $X \times \mathbb{P}^1$ by the equation $uz - v(y^2 - x^3 - x) = 0$ together with the projection $\pi : \widetilde{X} \to X$ on the first factor.

Ex. 4.52. The ideals $I = (x_1, \dots, x_n)$ and $J = (x_1, \dots, x_n)^m = I^m$ induce isomorphic Rees algebras $\widetilde{R} = \oplus_{k \geq 0} I^k$ and $\widetilde{S} = \oplus_{k \geq 0} J^k = \oplus_{k \geq 0} I^{mk}$, hence define the same blowups of $\mathbb{A}^n$. See [Moo01] for more details on the characterization of ideals producing the same blowup.

Ex. 4.57. Since the centers of the blowups are given by coordinate subspaces, the definition of blowup via affine charts can be used. For the blowup in the origin the three chart expressions of the blowup map are given by $\pi_x(x, y, z) = (x, xy, xz)$, $\pi_y(x, y, z) = (xy, y, yz)$ and $\pi_z(x, y, z) = (xz, yz, z)$. This gives for the total transforms of $X$ the expressions

$$X_x^* = V(x^2 - x^3y^2z) = V(x^2) \cup V(1 - xy^2z),$$
$$X_y^* = V(x^2y^2 - y^3z) = V(y^2) \cup V(x^2 - yz),$$
$$X_z^* = V(x^2y^2 - y^2z^3) = V(z^2) \cup V(1 - xy^2z).$$

The blowup $\widetilde{X}$ of $X$ is given by gluing the three charts $V(1 - xy^2z)$, $V(x^2 - yz)$, and $V(1 - xy^2z)$ of the strict transform $X^s$ of $X$ according to Def. 4.12. Observe that the origin of the $y$-chart $V(x^2 - yz)$ has the same singularity as $X$ at 0.

The blowup of $X$ along the $x$-axis yields for the blowup map the chart expressions $\pi_y(x, y, z) = (x, y, yz)$ and $\pi_z(x, y, z) = (x, yz, z)$. As the $x$-axis is not contained in $X$, the total transform $X^*$ and the strict transform $X^s = \widetilde{X}$ of $X$ coincide, cf. Def. 6.2. This gives for $\widetilde{X}$ the chart expressions

$$\widetilde{X}_y = V(x^2 - y^3 z),$$
$$\widetilde{X}_z = V(x^2 - y^2 z^3).$$

For the blowup of $X$ along the $y$-axis, the total transform has chart expressions

$$X^*_x = V(x) \cup V(x - y^2 z),$$
$$X^*_z = V(z) \cup V(x^2 z - y^2),$$

where the secondly listed components are the charts of the strict transform. Again, the chart $V(x^2 z - y^2)$ has, up to permutation of the variables, the same singularity as $X$ at 0. The blowup of $X$ along the $z$-axis gives accordingly

$$X^*_x = V(x^2) \cup V(1 - y^2 z),$$
$$X^*_y = V(y^2) \cup V(x^2 - z).$$

Ex. 4.64. The blowup of $X = \mathrm{Spec}(\mathbb{Z}[x])$ along $I = (x, p)$ is covered by two charts with coordinate rings $\mathbb{Z}[x, x/p] \simeq \mathbb{Z}[x, u]/(x - pu) \simeq \mathbb{Z}[u]$ and $\mathbb{Z}[x, p/x] \simeq \mathbb{Z}[x, u]/(p - xu)$, respectively. Observe that the second chart is not equal to the affine line $\mathbb{A}^1_{\mathbb{Z}}$ over $\mathbb{Z}$, see also [EH00].

For the blowup of $X$ along $I = (px, pq)$ one cannot use the equations of Def. 4.10 since $px$ and $pq$ do not form a regular sequence in $\mathbb{Z}[x]$. Similarly as before, the affine charts have coordinate rings $\mathbb{Z}[x, x/q] \simeq \mathbb{Z}[x, u]/(x - qu) \simeq \mathbb{Z}[u]$ and $\mathbb{Z}[x, q/x] \simeq \mathbb{Z}[x, u]/(q - xu)$, respectively.

Ex. 5.12. The equations $g_1 = y^2 - xz, g_2 = yz - x^3$ and $g_3 = z^2 - x^2 y$ do not form a regular sequence, since they admit the non-trivial linear relations $z \cdot g_1 - y \cdot g_2 + x \cdot g_3 = 0$ and $x^2 \cdot g_1 - z \cdot g_2 + y \cdot g_3 = 0$. Therefore one cannot use the equations of Def. 4.10 to describe the blowup of $\mathbb{A}^3$ along the curve $Z$ defined by $g_1 = g_2 = g_3 = 0$.

Ex. 5.13. Consider the map $\gamma : X \setminus Z \to \mathbb{P}^1, (x, y, z) \mapsto (x : y)$. For $x = y = 0$ and $z \neq 0$ there lies only one point in the closure $\overline{\Gamma}$ of the graph of $\gamma$ in $X \times \mathbb{P}^1$, while for $z = 0$ the set of limit points forms a projective line $\mathbb{P}^1$. The blowup is a local isomorphism outside 0 since $Z$ is locally a Cartier divisor in $X$ at these points (being locally a regular curve in a regular surface). Above $0 \in X$ the blowup map $\pi : \widetilde{X} \to X$ is not a local isomorphism, since $\pi$ contracts all limit points to 0, or, alternatively, because the blowup $\widetilde{X}$ is regular, while $X$ is singular at 0.

Ex. 5.14. The blowup map $\pi : \widetilde{\mathbb{A}}^2 \to \mathbb{A}^2$ with center the origin has the chart expressions $(x, y) \mapsto (x, xy)$ and $(x, y) \mapsto (xy, y)$. The total transform $X^* = \pi^{-1}(X)$ of $X = V(x^2)$ in $\widetilde{\mathbb{A}}^2$ has therefore charts defined in $\mathbb{A}^2$ by $x^2 = 0$, respectively $x^2 y^2 = 0$, with exceptional divisors given by $x = 0$ and $y = 0$. Hence the strict transform $X^s = \widetilde{X}$ of $X$ lies only in the $y$-chart and is defined there by $x^2 = 0$.

Ex. 5.15. The same computation as in the preceding example applies and shows that $\widetilde{X}$ lies entirely in the $y$ chart. It is defined there by the ideal $(x^2, x) = (x)$, equals hence the $y$-axis of this chart.

Ex. 5.17. The blowup of $\mathbb{A}^3$ along the union of two coordinate axes is discussed in Ex. 4.36 and 4.66.

The blowup $\widetilde{\mathbb{A}}^3$ of $\mathbb{A}^3$ along the cusp $(x^3 - y^2, z)$ is defined in $\mathbb{A}^3 \times \mathbb{P}^1$ by $uz - v(x^3 - y^2) = 0$, since $x^3 - y^2$ and $z$ form a regular sequence. It follows that $\widetilde{\mathbb{A}}^3$ is singular at 0.

Ex. 6.14. By Prop. 6.6, the defining ideal of the strict transform of $X$ is generated by the strict transforms of the elements of a Macaulay basis of the ideal $(x^2 - y^3, xy - z^3)$. Notice that the given generators are not a Macaulay basis since their initial forms are $x^2$ and $xy$, which do not generate the initial form of $y(x^2 - y^3) - x(xy - z^3) = xz^3 - y^4$. By adding this element, the Macaulay basis $x^2 - y^3, xy - z^3, xz^3 - y^4$ is obtained.

The chart expressions of the strict transform of $X$ can now be computed from this Macaulay basis,

$$X_x^s = V(1 - xy^3, y - xz^3, z^3 - y^4),$$

$$X_y^s = V(x^2 - y, x - yz^3, xz^3 - 1),$$

$$X_z^s = V(x^2 - y^3 z, xy - z, x - y^4).$$

Ex. 6.18 and 6.19. The transformation of flags under blowups is explicitly described in [Hau04] p. 5–8.

Ex. 6.22. In the $y$-chart, the total transform of $I = (x^2, y^3)$ is given by the ideal $I^* = (x^2y^2, y^3)$. Factoring out the maximal power of the monomial defining the exceptional divisor, $I^* = (y^2)(x^2, y)$ is obtained. Thus, the weak transform of $I$ is given by the ideal $I^\curlyvee = (x^2, y)$. On the other hand, it is easy to see that $x^2, y^3$ is a Macaulay basis for $I$. Thus, by Prop. 6.6, the strict transform of $I$ is generated by the strict transforms of these generators. Therefore, $I^s = (x^2, 1) = \mathbb{K}[x, y]$.

Ex. 6.23. This result is proved in [Hau03], p. 345.

Ex. 7.10. For the first two equations, the implicit function theorem shows that the zerosets are regular at 0. This gives a parametrization by formal power series. The zeroset of the third equation is singular at 0 and one cannot use the implicit function theorem to describe it at 0. To give a parametrization requires to construct first a resolution, which, in the present case, is very tedious.

Ex. 7.14. Consider the case that the resolution of $Y$ is achieved by a sequence of blowups:

$$Y' = Y_n \xrightarrow{\pi_{n-1}} Y_{n-1} \xrightarrow{\pi_{n-2}} \cdots \xrightarrow{\pi_0} Y_0 = Y.$$

Denote by $Z_i \subseteq Y_i$ the center of the blowup $\pi_i : Y_{i+1} \to Y_i$. Since $Z$ is regular, the singular locus of $Y \times Z$ is $\mathrm{Sing}(Y) \times Z$. By the base change property for blowups, Prop. 5.1, the blowup of $Y_i \times Z$ along the center $Z_i \times Z$ equals $Y_{i+1} \times Z$. This gives a new sequence

$$Y' \times Z = Y_n \times Z \xrightarrow{\widetilde{\pi}_{n-1}} Y_{n-1} \times Z \xrightarrow{\widetilde{\pi}_{n-2}} \cdots \xrightarrow{\widetilde{\pi}_0} Y_0 \times Z = Y \times Z$$

where $\widetilde{\pi}_i : Y_{i+1} \times Z \to Y_i \times Z$ is the blowup along the center $Z_i \times Z \subseteq Y_i \times Z$. It is checked that the morphism $Y' \times Z \to Y \times Z$ is a resolution of the singularities of $Y \times Z$.

Ex. 8.25. Consider the variety $X = V((x^2 - y^3)(z^2 - w^3)) \subseteq \mathbb{A}^4$ over a field of characteristic zero. The stratification of $X$ by the iterated singular loci is as follows.

$$\mathrm{Sing}(X) = V(x,y) \cup V(z,w) \cup V(x^2 - y^3, z^2 - w^3),$$
$$\mathrm{Sing}^2(X) = V(x,y,z^2 - w^3) \cup V(z,w,x^2 - y^3),$$
$$\mathrm{Sing}^3(X) = V(x,y,z,w).$$

Ex. 8.30–8.32. Let $J \subseteq R$ be an ideal and $I \subseteq R$ a prime ideal. The order of $J$ along $I$ is defined as

$$\mathrm{ord}_I J = \max\{k \in \mathbb{N},\ JR_I \subseteq I^k R_I\}$$

where $R_I$ is the localization of $R$ in $I$. If $I^{(k)} = I^k R_I \cap R$ denotes the $k$-th symbolic power of $I$, the order can also be expressed as

$$\mathrm{ord}_I J = \max\{k \in \mathbb{N},\ J \subseteq I^{(k)}\}$$

without making explicit use of the localization [ZS75].

Now consider the example $R = \mathbb{K}[x,y,z]$, $I = (y^2 - xz, yz - x^3, z^2 - x^2y)$. It can be checked that $I$ is a prime ideal of $R$ but not a complete intersection (i.e., not generated by a regular sequence, cf. Ex. 5.12). Also consider the principal ideal $J$ in $R$ that is generated by $f = x^5 + xy^3 + z^3 - 3x^2yz$. The order of $J$ along $I$ can be determined as follows. First notice that $f \notin I^2$ since $f$ has order 3 at the origin, but all elements in $I^2$ have at least order 4 at the origin. On the other hand,

$$xf = x^6 + x^2y^3 + xz^3 - 3x^3yz = (x^3 - yz)^2 - (y^2 - xz)(z^2 - x^2y) \in I^2.$$

Since $x \notin I$, this implies that $J \cdot R_I \subseteq I^2 R_I$, and in particular, $\mathrm{ord}_I J \geq 2$. Thus, $f$ is an example for an element in $R$ that is contained in the symbolic power $I^{(2)}$, but not in $I^2$. It remains to show that $\mathrm{ord}_I J = 2$. By Thm. 8.20 it suffices to find a point $a$ that lies on the curve $V(I)$ for which $\mathrm{ord}_a f = 2$. Such a point is for instance $a = (1,1,1) \in V(I)$.

Ex. 8.39. Consider the polynomial $g = x^c + \sum_{i=0}^{c-1} g_i(y) \cdot x^i$ at the origin $a = 0$ of $\mathbb{A}^{1+n}$. The order of $g$ at $a$ equals the minimum of $c$ and all values $\mathrm{ord}_a g_i + i$, for $0 \leq i < c$. Assume that $\mathrm{ord}_0 g = c$ and also that $g_{c-1} = 0$. If the characteristic of the ground field is zero, this can be achieved by a change of coordinates $x \mapsto x + \frac{1}{c} \cdot g_{c-1}(y)$, compare with rem. 9.4 and Ex. 9.10. The defining ideal $I$ of the top locus of $V(g)$ is generated by the derivatives $\frac{\partial^{i+|\alpha|}}{\partial x^i \partial y^\alpha} g$ where $i \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$ and $i + |\alpha| < c$. In particular, if the characteristic of the ground field is zero, $\frac{\partial^{c-1}}{\partial x^{c-1}} g = c! \cdot x \in I$ and thus $x \in I$. This allows to express $I$ in the form:

$$I = \left( x, \frac{\partial^{|\alpha|}}{\partial y^\alpha} g_i(y),\ \text{ for } \alpha \in \mathbb{N}^n, i < c, |\alpha| < c - i \right).$$

Ex. 8.43. Assume that $\mathrm{ord}\, g = d$ and $g = \sum_{i,j} c_{ij} y^i z^j$. The order will always be taken at the origin of the respective charts. Let $g'$ be the strict transform of $g$

under the first blowup. Then

$$g' = \sum_{i,j} c_{ij} y^{i+j-d} z^j.$$

Set $d' = \operatorname{ord} g'$ and notice that $d' = \min\{i + 2j - d,\, c_{ij} \neq 0\}$. Let $g''$ be the strict transform of $g'$ under the second blowup. Then

$$g'' = \sum_{i,j} c_{ij} y^{i+j-d} z^{i+2j-d-d'}.$$

Set $d'' = \operatorname{ord} g''$ and notice that $d'' = \min\{2i + 3j - 2d - d',\, c_{ij} \neq 0\}$. It is clear that $d'' \leq d' \leq d$. If $d' \leq \frac{d}{2}$, then $d'' \leq \frac{d}{2}$ follows. So assume that $d' > \frac{d}{2}$. By assumption, there exists a pair $(i, j) \in \mathbb{N}^2$ such that $i + j = d$ and $c_{ij} \neq 0$. Thus,

$$d'' \leq \underbrace{2i + 3j}_{=2d+j} - 2d \underbrace{-d'}_{<-\frac{d}{2}} < \underbrace{j}_{\leq d} - \frac{d}{2} \leq \frac{d}{2}.$$

Ex. 8.53. The situation is described in detail in [Hau00], Prop. 4.5, p. 354, and [Zar44], Thm. 1 and Lemma 3.2, p. 479. If the center is a smooth curve, see [Hau00], Prop. 4.6, p. 354, and [Zar44], Thm. 2, p. 484 and its corollary, p. 485.

Ex. 9.8. Let $a = 0$ and pass to the completion $\widehat{\mathcal{O}}_{W,a} \simeq \mathbb{K}[[x_1, \ldots, x_n]]$. Assume that $\operatorname{ord}_a f = c$ and that $f$ has the expansion $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$. The initial form of $f$ is $\sum_{|\alpha|=c} c_\alpha x^\alpha$. It is assumed that the blowup is monomial; thus, $a'$ is the origin of the $x_i$-chart for some $i \leq n$. If $i > 1$, then $a'$ is contained in the strict transform of $V^s$. It remains to show that, if $a'$ is the origin of the $x_1$-chart, the order of the strict transform of $f$ is smaller than $c$ .

For this, notice that the strict transform of $f$ at the origin of the $x_1$-chart is given by $f' = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x_1^{|\alpha| - c - \alpha_1} x^\alpha$ where $\alpha_1$ is the first component of $\alpha$. The monomials of this expansion are distinct, so there can be no cancellation between them. By assumption, $x_1$ appears in the initial form of $f$; thus, there is an exponent $\alpha' \in \mathbb{N}^n$ such that $|\alpha'| = c$, $\alpha'_1 > 0$ and $c_{\alpha'} \neq 0$. This implies that the expansion of $f'$ contains the non-zero monomial $c_{\alpha'} x_1^{-\alpha'_1} x^\alpha$. Therefore $\operatorname{ord}_{a'} f' < c = \operatorname{ord}_a f$.

Ex. 9.10. Assume that the characteristic of the ground field is zero. The coordinate change $x_n \mapsto x_n - \frac{1}{d} \cdot a_{d-1}(y)$ transforms $f$ into

$$f \mapsto \left(x_n - \frac{1}{d} \cdot a_{d-1}(y)\right)^d + \sum_{i=0}^{d-1} a_i(y) \left(x_n - \frac{1}{d} \cdot a_{d-1}(y)\right)^i.$$

Notice that the coefficient of $x_n^d$ in this new expansion is 1 and the coefficient of $x_n^{d-1}$ is $-\binom{d}{1} \frac{1}{d} a_{d-1}(y) + a_{d-1}(y) = 0$. Thus, there are polynomials $\widetilde{a}_i(y)$ such that, in the new coordinates,

$$f = x_n^d + \sum_{i=0}^{d-2} \widetilde{a}_i(y) x_n^i.$$

Consequently, $\frac{\partial^{d-1}}{\partial x_n^{d-1}}(f) = d! \cdot x_n$. Thus, the variety defined by $x_n = 0$ defines an osculating hypersurface for $X$ at the origin. By Prop. 9.5, every osculating hypersurface has maximal contact.

Ex. 9.11. The defining equation for the strict transform of $X$ in the $x$-chart is given by

$$f' = y + y^2 - z + y^2 z - z^2 - yz^2 = ((y+1) - (z+1))(y+1)(z+1).$$

The order of $f'$ at the point $a' = (0, -1, -1)$ is 3. Since the exceptional divisor is given by the equation $x = 0$ in the $x$-chart, $a'$ lies on it. The strict transform $V^s$ of $V$ coincides with the complement of the $x$-chart, so $a'$ cannot lie on $V^s$.

Ex. 9.14. The relevant case appears when $c = p > 0$ where $p$ is the characteristic of the ground field. Let $g(y) = \sum_{\alpha \in \mathbb{N}^m} g_\alpha y^\alpha$ be the expansion of $g$ with respect to the coordinates $y_1, \ldots, y_m$. It decomposes into

$$g(y) = \underbrace{\sum_{\alpha \in p \cdot \mathbb{N}^m} g_\alpha y^\alpha}_{g_1(y)} + \underbrace{\sum_{\alpha \in \mathbb{N}^m \setminus p \cdot \mathbb{N}^m} g_\alpha y^\alpha}_{g_2(y)} .$$

If the ground field is assumed to be perfect, $g_1(y)$ is a $p$-th power. Thus, there is a formal power series $\widetilde{g}(y)$ with $\operatorname{ord} \widetilde{g} \geq 1$ such that $\widetilde{g}^p = g_1$. Apply the change of coordinates $x \mapsto x - \widetilde{g}$. This transforms $f$ into

$$f = (x - \widetilde{g})^p + g_1 + g_2 = x^p - \widetilde{g}^p + g_1 + g_2 = x^p + g_2.$$

Notice that $\operatorname{ord} g_2 \geq \operatorname{ord} g$. To show that this order is maximal, it suffices to consider an arbitrary change of coordinates $z \mapsto z + h(y)$ where $h \in \mathbb{K}[[y_1, \ldots, y_m]]$ is any power series with $\operatorname{ord} h \geq 1$. This transforms the equation into

$$f = x^p + h^p + g_2.$$

Notice that there can be no cancellation between the terms in the expansions of $h^p$ and $g_2$. Thus, $\operatorname{ord}(h^p + g_2) \leq \operatorname{ord} g_2$.

Ex. 9.17. Denote the sequence of local blowups by

$$(W', a') = (W_m, a_m) \overset{\pi_{m-1}}{\longrightarrow} \cdots \overset{\pi_0}{\longrightarrow} (W_0, a_0) = (W, a).$$

Set $f^{(0)} = f$ and let $f^{(i)}$ be the strict transform of $f$ after $i$ blowups. Assume that the center of each local blowup $\pi_i : (W_{i+1}, a_{i+1}) \to (W_i, a_i)$ is contained in the top locus of $f^{(i)}$ and has normal crossings with the exceptional divisors produced by previous blowups. Let $Z$ be a local hypersurface of maximal contact for $f$ at $a$. Assume that $\operatorname{ord}_{a_i} f^{(i)} = \operatorname{ord}_a f$ for $i = 1, \ldots, m$. Then, by definition of maximal contact, $a_i \in Z_i$ for $i = 0, \ldots, m$, where $Z_i$ denotes the strict transform of $Z$ after $i$ blowups. Since all centers are contained in the top locus of $f^{(i)}$, they are also contained in $Z_i$. By repeatedly applying Prop. 5.3, the union of all exceptional divisors produced by the blowups $\pi_{m-1}, \ldots, \pi_0$ with $Z' = Z_m$ is a normal crossings divisor. Thus, $a'$ is contained in $n+1$ hypersurfaces that form a normal crossings divisor. But this is impossible in an ambient space of dimension $n$.

Ex. 10.20. Set $f = x^5 + x^2 y^4 + y^k$. Denote by $V_x$ and $V_y$ the regular local hypersurfaces defined by $x = 0$, respectively $y = 0$. If $k \geq 5$, then the order of $f$ at the origin is 5 and the derivative $\partial_x^4 f = 5! \cdot x$ defines the hypersurface $V_x$. Thus, $V_x$ is osculating. If $k = 4$, then the order of $f$ at the origin is 4. The derivative $\partial_y^3 f = 4! \cdot y(1 + x^2)$ defines a regular parameter. Since $1 + x^2$ is a unit in the local ring $\mathcal{O}_{\mathbb{A}^2,0}$, the hypersurface $V_y$ is osculating. If $k < 4$, taking the differential with respect to $y$ shows again that $V_y$ is an osculating hypersurface.

If $k \geq 5$, then $J_{V_x}(I) = (y^{160}, y^{24k})$ and $J_{V_y}(I) = (x^{240}, x^{120}) = (x^{120})$. Thus, both $V_x$ and $V_y$ have weak maximal contact if $k = 5$, while only $V_x$ has weak maximal contact if $k > 5$. If $2 \leq k \leq 4$, then $J_{V_x}(I) = (y^{4\frac{k!}{k-2}}, y^{k!})$ and $J_{V_y}(I) = (x^{5(k-1)!})$. Since $k! < 5(k-1)!$, only $V_y$ has weak maximal contact in this case.

Ex. 10.21. Assume that the characteristic of the ground field is not equal to 2. Then $\partial_x f = 2x$ and the hypersurface $V$ has weak maximal contact with $f$ by Prop. 10.11. The coefficient ideal $J_V(f)$ of $f$ with respect to $V$ is generated by $y^3z^3 + y^7 + z^7$ and has order 6 (up to raising the generator to the required power).

Restrict to the case of the blowup of $\mathbb{A}^2$ at the origin and the study of $f$ at the origin of the $y$-chart. The strict transform of $f$ is given there by $f' = x^2 + y^4(z^3 + y + yz^7)$. The points where the order of $f'$ has remained constant are exactly the points of the $z$-axis. Let $V'$ denote the strict transform of $V$. Then $J_{V'}(f')$ is generated by $y^4(z^3 + y + yz^7)$ where $y^4$ is the exceptional factor (again, up to taking powers). The order of the residual factor has dropped to 1 at the origin and to 0 at all other points of the $z$-axis.

Ex. 10.22. Assume that $a = 0$, $\mathrm{ord}_a f = c$ and that $J_V(f) = (y^\alpha)$ where $y = (x_1, \ldots, x_{n-1})$ and $\alpha \in \mathbb{N}^{n-1}$. Now let $L \subseteq \{1, \ldots, n-1\}$ be a subset such that $\sum_{i \in L} \alpha_i \geq c!$ but $\sum_{i \in L \setminus \{j\}} \alpha_i < c!$ for all $j \in L$. This is possible since $\mathrm{ord}_a J_V(f) = |\alpha| \geq c!$ by the definition of coefficient ideals.

Blow up $\mathbb{A}^n$ in the center given by $x_n = 0$ and $x_i = 0$ for all $i \in L$. Let $a'$ be a point over $a$ at which the order of the strict transform of $f$ has remained constant. Since $V$ is osculating, $a'$ is not contained in the $x_n$-chart. So let $j \in L$ be such that $a'$ is contained in the $x_j$-chart. The transform of the coefficient ideal in this chart is

$$J_{V'}(f') = J_V(f)^! = (x_j^{\sum_{i \in L \setminus \{j\}} \alpha_i - c!} y^\alpha).$$

But since $\sum_{i \in L \setminus \{j\}} \alpha_i - c! < 0$ by assumption, it follows that $\mathrm{ord}_{a'} J_{V'}(f') < \mathrm{ord}_a J_V(f)$. But the order of the coefficient ideal is at least $c!$ as long as the order of the strict transform of $f$ remains equal to $c$. Thus, by iterating this procedure, after finitely many steps the order of the strict transform of $f$ has to drop.

Ex. 10.27. Assume that $\mathrm{ord}\, I = c$. Let $x_1, \ldots, x_n$ be a regular system of parameters for $\widehat{\mathcal{O}}_{\mathbb{A}^n,0} = \mathbb{K}[[x_1, \ldots, x_n]]$. Define a map $\pi : \mathbb{N}^n \to \mathbb{N} \cup \{\infty\}$ in the following way:

$$\pi(\alpha_1, \ldots, \alpha_n) = \begin{cases} \frac{c!}{c - \alpha_n} \sum_{i=1}^{n-1} \alpha_i & \text{if } \alpha_n < c, \\ \infty & \text{if } \alpha_n \geq c. \end{cases}$$

Let $V$ be the regular local hypersurface defined by $x_n = 0$. Elements $f \in I$ have expansions $f = \sum_{\alpha \in \mathbb{N}^n} c_{f,\alpha} x^\alpha$. Then, by the definition of coefficient ideals,

$$\mathrm{ord}(J_V(I)) = \min\{\pi(\alpha), \alpha \in \mathbb{N}^n : \text{there is an } f \in I \text{ such that } c_{f,\alpha} \neq 0\}.$$

Now define a monomial order $<_\varepsilon$ on $\mathbb{K}[x_1,\dots,x_n]$ in the following way: Set $x^\alpha <_\varepsilon x^\beta$ if and only if $\pi(\alpha)<\pi(\beta)$ or $\pi(\alpha)=\pi(\beta)$ and $\alpha <_{lex} \beta$ where $<_{lex}$ denotes the lexicographic order on $\mathbb{N}^n$.

By [Hau04], Thm. 3, p. 10, there exists a regular system of parameters $x_1,\dots,x_n$ for $\widehat{\mathcal{O}}_{\mathbb{A}^n,0}$ such that the initial ideal of $I$ with respect to $<_\varepsilon$ is maximal (again, with respect to $<_\varepsilon$) over all choices of regular systems of parameters for $\widehat{\mathcal{O}}_{\mathbb{A}^n,0}$. In particular, let $y_1,\dots,y_n$ be another regular system of parameters and let each $f\in I$ have the expansion $f=\sum_{\alpha\in\mathbb{N}^n} c'_{f,\alpha}y^\alpha$ with respect to these parameters. Then there exist elements $g\in I$ and $\widetilde{\alpha}\in\mathbb{N}^n$ such that $c'_{g,\widetilde{\alpha}}\neq 0$ and

$$\pi(\widetilde{\alpha})\leq \min\{\pi(\alpha),\alpha\in\mathbb{N}^n, \text{ there is an } f\in I \text{ such that } c_{f,\alpha}\neq 0\}.$$

Let $V'$ be the regular local hypersurface defined by $y_n=0$. Then the last statement implies that $\operatorname{ord}(J_{V'}(I))\leq \operatorname{ord}(J_V(I))$. Thus, the regular system of parameters $x_1,\dots,x_n$ maximizes the order of the coefficient ideal.

Ex. 10.30. Let $I$ be an ideal with $\operatorname{ord}_a I=\operatorname{ord}_{a'} I^\curlyvee$ where $a'$ is the origin of the $x_j$-chart for some $j<n$ and $I^\curlyvee$ denotes the weak transform of $I$. Elements $f\in I$ have expansions $f=\sum_{i\geq 0} f_i(y)x_n^i$ where $y=(x_1,\dots,x_{n-1})$. Then

$$J_V(I)=(f_i^{\frac{o!}{o-i}},\, i<o, f\in I).$$

The weak transform of $I$ is given by $I^\curlyvee=(f^\curlyvee, f\in I)$ where

$$f^\curlyvee = x_j^{-o}f^* = x_j^{-o}\sum_{i\geq 0} f_i^* x_n^i x_j^i = \sum_{i\geq 0}\underbrace{(f_i^* x_j^{i-o})}_{=:f_i'} x_n^i.$$

Here, $f^*$ and $f_i^*$ denote the total transforms. Thus, the coefficient ideal of $I^\curlyvee$ with respect to the strict transform $V'$ of $V$ is

$$J_{V'}(I^\curlyvee)=(f_i'^{\frac{o!}{o-i}} : i<o, f\in I)=(f_i^{*\frac{o!}{o-i}}x_j^{(i-o)\frac{o!}{o-i}} : i<o, f\in I)$$
$$x_j^{-o!}(f_i^{*\frac{o!}{o-i}} : i<o, f\in I)=x_j^{-o!}(J_V(I))^*=J_V(I)^!$$

where $J_V(I)^!$ denotes the controlled transform with control $o!$.

Ex. 11.14 and 11.15. These examples are discussed in detail in Lecture XII.

Ex. 12.17–12.22. Let $\mathbb{K}$ be a field of characteristic 2. Consider the ring homomorphism $\phi:\mathbb{K}[x,y,z,w]\to\mathbb{K}[t]$ given by $\phi(x)=t^{32}$, $\phi(y)=t^7$, $\phi(z)=t^{19}$, $\phi(w)=t^{15}$. Let $I\subseteq\mathbb{K}[x,y,z,w]$ be its kernel. Its zeroset is an irreducible curve $C$ in $\mathbb{A}^4$.

Now let $f$ be the polynomial $f=x^2+yz^3+zw^3+y^7w$. The partial derivatives of $f$ have the form:

$$\frac{\partial}{\partial x}f=0,$$
$$\frac{\partial}{\partial y}f=z^3+y^6w,$$
$$\frac{\partial}{\partial z}f=yz^2+w^3,$$
$$\frac{\partial}{\partial w}f=zw^2+y^7.$$

It is easy to check that $f$ and all of its first derivatives are contained in $I$. Thus, by the Jacobian criterion, $\operatorname{ord}_I f \geq 2$. Since the order of $f$ in the origin is 2, it is possible to conclude that $\operatorname{ord}_I f = 2$. In other words, the top locus of $f$ contains the curve $C$ that is parametrized by $t \mapsto (t^{32}, t^7, t^{19}, t^{15})$. It can be checked, e.g. via any computer algebra system, that the top locus of $f$ is itself an irreducible curve. Thus, it coincides with the curve $C$.

Now suppose that there is a regular local hypersurface at the origin that contains the top locus of $f$ and hence the curve $C$. This hypersurface has an equation $h = 0$ in which at least one variable must appear linearly. But since $h \in I$, this is only possible if one of the numbers $32, 7, 19, 15$ can be written as an $\mathbb{N}$-linear combination of the others. This is not the case so that there is no regular local hypersurface containing the top locus of $f$.

Let $V$ be any regular local hypersurface at the origin. Since it does not contain the curve $C$, there is a sequence of point blowups that separates the strict transforms of $V$ and $C$. Since the variety $X$ defined by $f = 0$ has order 2 along the curve $C$ and the point blowups are isomorphisms over all but one point of $C$, the order of the strict transform of $X$ under these blowups will again be 2 along the strict transform of $C$. So there eventually is a point $a'$ at which the strict transform of $X$ has order 2, but which is not contained in the strict transform of $V$. Thus, $V$ does not have maximal contact with $X$. Since $V$ was chosen arbitrarily, there can be no hypersurface that has maximal contact with $X$.

Ex. 12.24. Let $\alpha \in \mathbb{N}^n$. Define the differential operator $\partial_{x^\alpha}$ on $\mathbb{K}[x_1, \ldots, x_n]$ as the linear extension of $\partial_{x^\alpha} x^\beta = \binom{\beta}{\alpha} x^{\beta-\alpha}$.

In particular, consider the differential operator $\partial_{x_i^p}$ for $i \in \{1, \ldots, n\}$. If $n \in \mathbb{N}$ has the $p$-adic expansion $n = \sum_{i \geq 0} n_i p^i$ with $0 \leq n_i < p$, then $\partial_{x_i^p} x_i^n = \binom{n}{p} x_i^{n-p} = n_1 x_i^{n-p}$. Similarly, $\partial_{x_i^{p^k}} x_i^n = n_k x_i^{n-p^k}$ for any $k \in \mathbb{N}$.

Notice that $\partial_{x_i^p}$ is not a derivation since it does not fulfill the Leibniz rule: $\partial_{x_i^p}(x_i^p) = 1$, but

$$x_i \underbrace{\partial_{x_i^p} x_i^{p-1}}_{0} + x_i^{p-1} \underbrace{\partial_{x_i^p} x_i}_{0} = 0.$$

For more details on differential operators in positive characteristic, see [Kaw07] Chap. 1, pp. 838–851.

Ex. 12.25. Consider $f = z^{3^5} + x^{4 \cdot 3^4} y^{3^4} (x^{3^4} + y^{3^4} + x^{300})$ as a polynomial over a field of characteristic 3. Assume that the exceptional locus is given locally by $V(xy)$. Then its residual order with respect to the local hypersurface defined by $z = 0$ is $3^4$. Now consider the blowup of $\mathbb{A}^2$ at the origin and let $a'$ be the point $(0, 1, 0)$ in the $x$-chart. The strict transform of $f$ has the equation

$$f' = z^{3^5} + x^{3^5} (y^{3^4} + 1)(y^{3^4} - 1 + x^{219}) = z^{3^5} + x^{3^5} (y^{2 \cdot 3^4} - 1 + x^{219}(1 + y^{3^4})).$$

By making the change of coordinates $z \mapsto z + x$, this transforms into

$$f' = z^{3^5} + x^{3^5} (y^{2 \cdot 3^4} + x^{219}(1 + y^{3^4})).$$

The exceptional divisor is given locally by $V(x)$. Thus, the residual order of $f'$ with respect to the hypersurface defined by $z = 0$ is $2 \cdot 3^4 > 3^4$.

Ex. 12.26. The equation of $f$ in local coordinates $x, y, z$ at a point $(0,0,t)$ on the $z$-axis is obtained by making a translation $z \mapsto z+t$. Thus, the equation $x^p + y^p(z+t) = x^p + y^p z + y^p t$ is obtained. Assume that the ground field is perfect and let $\lambda$ be the $p$-th root of $t$. Then one can write $x^p + y^p z + y^p t = (x+\lambda y)^p + y^p z$. Thus, the residual order of $f$ at every point of the $z$-axis is $p+1$.

Now consider the generic point $P=(x,y)$ of the $z$-axis. The order of the coefficient ideal along $P$ is $p$.

Ex. 12.27. Observe that

$$G^+(y) = \prod_{i=1}^{m-1} (y_i + y_m)^{r_i} y_m^{r_m} g^+(y).$$

Let $I$ denote the ideal $(y_1, \dots, y_{m-1})$. Notice that $I$ defines a complete intersection. By [Hoc73] 2.1, p. 57, [Pel88], Prop. 1.8, p. 359, the order of a polynomial $f$ along $I$ can be expressed as the largest power of $I$ that contains $f$. In particular, it fulfills that $\mathrm{ord}_I(f \cdot g) = \mathrm{ord}_I f + \mathrm{ord}_I g$ for polynomials $f, g$. Now compution yields

$$\begin{aligned} \mathrm{ord}_I G^+(y) &= \sum_{i=1}^{m-1} r_i \cdot \underbrace{\mathrm{ord}_I(y_i + y_m)}_{0} + r_m \cdot \underbrace{\mathrm{ord}_I y_m}_{0} + \mathrm{ord}_I g^+(y) \\ &\leq \mathrm{ord}\, g^+(y) = \mathrm{ord}\, g(y) = k. \end{aligned}$$

Ex. 12.29 and 12.30. This is explained in detail in [Hau10a].

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Clay Mathematics Proceedings
Volume **20**, 2014

# On the behavior of the multiplicity on schemes: Stratification and blow ups

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Contents



2010 *Mathematics Subject Classification.* Primary 14E15.
*Key words and phrases.* Multiplicity, integral closure, Rees algebras.
The authors are partially supported by MTM2009-07291.

## Introduction

The objective of these notes is to discuss some techniques of commutative algebra, and to show how they apply to the problem of resolution of singularities. This will lead us to study, as starting point, finite extension of rings and integral closure of ideals.

Among the algebraic techniques that will show up in our discussion is the theory of *elimination*. The ring of functions of an algebraic variety can be expressed, at least locally, as a finite extension of a regular ring. In this setting the discriminant of a monic polynomial with coefficients in a regular ring is one example of elimination. In fact, if the variety is a hypersurface given by such a monic polynomial, then the discriminant, which provides information on the singularities, can be expressed as a function on the coefficients of the polynomial, and hence it does not involve the variable. This form of elimination will be the stating point to address resolution of singularities by induction on the number of variables involved.

A first step in the study of the resolution of singularities of algebraic varieties could be to consider the case of curves. The resolution of singularities of a curve is given by a unique curve. In fact, it is obtained from the normalization of the ring of functions. This uniqueness fails in higher dimensions, and there are many ways in which one can resolve the singularities of an algebraic variety of dimension greater than one.

There is, however, a common strategy in the so called *algorithmic*, or *constructive, resolutions*. In this approach the aim is to construct a resolution of singularities by blowing up successively along closed and regular centers.

Roughly speaking, an algorithm for resolution of singularities is a procedure which stratifies any variety into regular and locally closed sets, and provides, in addition, a closed stratum which is the natural center to be blown up. The blow up of the variety at such center is a variety, and the algorithm defines again a stratification into regular locally closed sets, among which, one is the center to be blown up in the next step. Moreover, it is required that after blowing up finitely many times at the centers given by the algorithm one comes to a regular variety. In this sense, an algorithm of resolution will enable us to *construct* a resolution.

Let us remark here that algorithms which lead to constructive resolution of singularities are known to exist, but only for the class of varieties over fields of characteristic zero. In addition, even within the class of varieties of characteristic zero, one can establish different algorithms of resolution of singularities. In other words, and this is important to keep in mind, there are different ways to obtain *stratifications of algebraic varieties* which will lead to resolution of singularities. Moreover, there is no finite list of properties, to be required on a resolution of singularities, which characterizes or privileges a unique algorithm.

This lack of uniqueness reflects the complexity of birational theory in dimension bigger than 2, yet it is not to be taken as a weakness. By fixing an algorithm of resolution we are giving a precise procedure to stratify varieties, and to construct, for each variety, a particular resolution of singularities.

Once we have fixed an algorithm, then we can ask if it fulfills some *natural properties*. As it resolves the singularities of any variety $X$, it resolves, in particular, the singularities of an open set in $X$. It is natural to require that the latter resolution be the restriction of the former.

Another natural property of constructive resolution is the compatibility with isomorphisms. For example, suppose given two varieties, each one of them defined over some field of characteristic zero. Both are, in particular, abstract schemes, and suppose we are given an isomorphism between both underlying schemes. The property of *compatibility with isomorphisms* states that *any* such isomorphism maps each stratum of one scheme isomorphically into a stratum of the other. Namely, if $X$ and $Y$ are varieties, and if $\Theta : X \to Y$ is any isomorphism between the underlying schemes, we require on the stratification that any such isomorphism should map each stratum on $X$ to a stratum on $Y$. Since we are to blow up at a smooth closed center given by a closed stratum, this will ensure that the isomorphism can be lifted to an isomorphism of the two varieties obtained by blowing up. Moreover, a property of any constructive resolution is that any such isomorphism can be lifted all the way to the resolution of singularities of both varieties.

This latter property is also significant when considering isomorphisms of a variety into itself. In fact, a consequence of this is that if a group is acting algebraically on a variety, then the group action can be lifted to the resolution of singularities. This property is expressed by saying that any constructive resolution (i.e., any resolution constructed by an algorithm) is *equivariant*. This concept will be considered along these notes (see also [**Vi2**]).

So, despite the fact that there is no uniqueness in algorithmic resolution, every time one fixes an algorithm there are many natural properties which are fulfilled. This flexibility can be used to produce an algorithm of resolution which is well suited to a particular setting. This occurs, for example, when we want to study the behavior of the resolution of singularities for the members of a given family of singular schemes. In fact, every time we fix an algorithm one can naturally stratify any family of schemes, as for example a Hilbert Scheme, into locally closed subsets corresponding to equi-resolvable members of the family.

**How do algorithms of resolution arise?** So far we have mentioned some natural properties expected from an algorithm, but we have given no indication about how algorithms of resolution of singularities arise. Recall that we conceive an algorithm as *a procedure to stratify varieties, with some prescribed properties*. One way in which a stratification can be achieved is by fixing a totally ordered set, say $(\Delta, \geq)$, and assigning to any variety $X$ an upper-semi-continuous function, say $f_X : X \to \Delta$. The level sets of this function will stratify $X$ into locally closed sets.

Take for example $\Delta = \mathbb{N}^{\mathbb{N}}$ with the lexicographic order. Consider, at each closed point $\xi \in X$, the graph of the Hilbert-Samuel function at the local ring $\mathcal{O}_{X,\xi}$. This is a function from $\mathbb{N}$ to $\mathbb{N}$, so the graph is an element of $\Delta = \mathbb{N}^{\mathbb{N}}$. One can extend this function to non-closed points, say $\mathrm{HS}_X : X \to \Delta$, in such a way that it is upper-semi-continuos.

The stratification on $X$ defined by this function is called the *Hilbert-Samuel stratification* (each stratum is called a Hilbert-Samuel stratum). Let $\Theta : X \to X'$ be an isomorphism of abstract schemes between two varieties. This defines an isomorphism of local rings between $\mathcal{O}_{X,\xi}$ and $\mathcal{O}_{X',\xi}$ for $\xi' = \Theta(\xi)$. In particular $\mathrm{HS}_X(\xi) = \mathrm{HS}_{X'}(\xi')$, or say, the two functions are compatible with the isomorphism, so $\Theta$ maps the level sets on $X$ isomorphically into the level sets in $X'$.

In these notes we shall also draw attention to a different stratification, in which $\Delta = \mathbb{N}$ is ordered in the usual way, and for any $\xi \in X$ we set $\mathrm{M}_X(\xi)$ to be the multiplicity at the local ring $\mathcal{O}_{X,\xi}$. We shall study the function $\mathrm{M}_X : X \to \mathbb{N}$ which is also upper-semi-continuous. Of course each stratum is given by the points of $X$ with the same multiplicity.

The same argument used before shows that this function is also compatible with isomorphisms: for any isomorphism $\Theta : X \to X'$, $\mathrm{M}_X(\xi) = \mathrm{M}_{X'}(\xi')$ for any $\xi \in X$, so isomorphisms map the level sets on $X$ isomorphically into those of $X'$.

Recall that an algorithm of resolution is a procedure to stratify varieties into regular locally closed varieties. The Hilbert-Samuel strata are not regular in general. However, one way in which algorithms of resolution appear is by taking a suitable refinement of the previous functions $\mathrm{HS}_X : X \to \Delta$. Namely, a refinement of the Hilbert-Samuel stratification.

We produce a refinement by giving a new totally ordered set, say $(\Delta', \geq)$, and by assigning to a variety $X$ an upper-semi-continuous function, say $f_X : X \to \Delta \times \Delta'$, where $\Delta \times \Delta'$ is ordered lexicographically, and the first coordinate of the value $f_X(\xi)$ is $\mathrm{HS}(\xi)$ for any $\xi \in X$. One readily checks that the level sets of $f_X$ are included in the level sets of $\mathrm{HS}_X$ for any $X$.

This is the achievement of constructive resolution. It shows that such refinement can be done in a way that it still preserves very natural properties, such as the compatibility with isomorphisms, and moreover, in such a way that it leads to a resolution of singularities for any variety of characteristic zero by blowing up successively at the closed stratum.

The precise definition of a totally ordered set $\Delta'$, and the definition of the functions $f_X : X \to \Delta \times \Delta'$ may vary, giving rise to different algorithms. We refer here to [**Cu1**], [**EV1**], or to [**Vi1**], for a detailed discussion of this refinement, and for the natural properties such as the compatibility of the functions $f_X$ with isomorphisms.

In this presentation we show how to obtain resolutions of singularities by taking a refinement of the multiplicity function $\mathrm{M}_X : X \to \Delta = \mathbb{N}$ ([**Vi7**]), where each stratum is given by the points with the same multiplicity.

We shall also stress here on an interesting feature of constructive resolution. Namely that the refinement of the stratification defined by the multiplicity, which leads to constructive resolutions, and the refinement of the Hilbert-Samuel functions mentioned above, are in fact very similar. For example both refinements make use of the same totally ordered set $\Delta'$.

**Hilbert-Samuel vs. Multiplicity.** Recall that in the Hilbert-Samuel stratification we fix the totally ordered set $\Delta = \mathbb{N}^{\mathbb{N}}$, and then a function $\mathrm{HS}_X : X \to \mathbb{N}^{\mathbb{N}}$ is assigned to each variety $X$. Let max $\mathrm{HS}_X$ denote the biggest value achieved, and let $\underline{\mathrm{Max}}\ \mathrm{HS}_X$ be the maximum stratum.

There is a fundamental result used in those algorithms of resolution which arise as a refinement of the Hilbert-Samuel stratification. It is called *Bennett's Theorem* and asserts that whenever $X \leftarrow X_1$ is the blow up at a smooth center $Y$ included in a Hilbert-Samuel stratum, then $\max \mathrm{HS}_X \geq \max \mathrm{HS}_{X_1}$ (see [**Be**]).

A *second theorem, due to Hironaka*, states that if one can construct, for any variety $X$, a sequence of blow ups as above, say:

$$X \leftarrow X_1 \leftarrow \cdots \leftarrow X_r, \tag{0.1}$$

in such a way that $\max \mathrm{HS}_X > \max \mathrm{HS}_{X_r}$, then one can resolve the singularities of any variety by iterating this procedure finitely many times.

In these notes both results will be adapted to the case in which the totally ordered set is $\mathbb{N}$, and the function assigned to $X$ is $\mathrm{M}_X : X \to \mathbb{N}$. The analog of the first result, namely of Bennett's Theorem, says that whenever $X \leftarrow X_1$ is the blow up at a center $Y$ included in a stratum defined by the multiplicity, say $\underline{\mathrm{Max}}\ \mathrm{M}_X$, then $\max \mathrm{M}_X \geq \max \mathrm{M}_{X_1}$. This is a *Theorem of Dade* ([**D**]), but we include some indications on this line. As for the second result, it is unnecessary in our setting: If one can construct a sequence $X \leftarrow X_1 \leftarrow \cdots \leftarrow X_r$, in such a way that $\max \mathrm{M}_X > \max \mathrm{M}_{X_r}$, then at some point one comes to the case in which $\max \mathrm{M}_{X_r} = 1$, which already ensures regularity.

Given a variety $X$, neither of the functions $\mathrm{HS}_X$ or $\mathrm{M}_X$ have the desired property of stratifying into locally closed sets which are *regular*, a necessary condition required on an algorithm of resolution of singularities. So in both cases we will have to present, what we have called, a refinement of these functions. As was already mentioned, the refinement of the functions $\mathrm{HS}_X$, and that of the function $\mathrm{M}_X$, undergo a very similar construction. We will indicate the basic reasons that justify this analogy in the following paragraphs.

**Reformulation Theorem.** The refinement of the Hilbert-Samuel stratification can be achieved via the following theorem of Hironaka:

THEOREM 0.1. [**Hi5**] **Reformulation Theorem.** *Suppose given an embedding of varieties $X \subset W$, where $W$ is smooth over a perfect field. Then the problem of constructing a sequence of blow ups over $X$ as stated in (0.1), so that max $HS_X >$ max $HS_{X_r}$, is equivalent to the problem of reduction of the order of an ideal in the smooth variety $W$ by a sequence of blow ups.*

The precise meaning of this reformulation requires some clarifications. Let us simply indicate that Hironaka attaches to $X \subset W$ an ideal $J$ in the smooth scheme $W$, together with a non-negative integer $b$. He shows that if one can produce a sequence of blow ups at regular centers, together with *weak transforms*[1] *of* $J$:

$$\begin{array}{ccccccc} W_0 = W & \leftarrow & W_1 & \leftarrow & \dots & \leftarrow & W_r \\ J_0 = J & & J_1 & & & \dots & J_r \end{array} \tag{0.2}$$

such that if $J_r$ has order less than $b$ at any point of $W_r$ then:

A) The sequence (0.2) induces a sequence of blow ups over $X$ at regular centers

$$\begin{array}{ccccccc} W_0 = W & \leftarrow & W_1 & \leftarrow & \dots & \leftarrow & W_r \\ \cup & & \cup & & \dots & & \\ X & \leftarrow & X_1 & \leftarrow & \dots & \leftarrow & X_r; \end{array} \tag{0.3}$$

B) This latter sequence fulfills the requirement of (0.1), namely that

$$\max \mathrm{HS}_X > \max \mathrm{HS}_{X_r}.$$

In other words,

> *The problem of lowering the maximum value of $HS_X$ is equivalent to that of lowering the maximum order of a weak transform of $J$ below $b$.*

This result has a similar formulation for the case in which one considers the functions $\mathrm{M}_X$, instead of the functions $\mathrm{HS}_X$.

**Reformulation Theorem and the obstruction in positive characteristic.** Bennett's Theorem holds in a very ample context, with no assumption on the characteristic. And so does Hironaka's reformulation theorem. However the construction of a sequence as that in (0.2) is only known in characteristic zero. A similar obstruction occurs when considering the multiplicity.

**Reformulation Theorem, local presentations and refinements of stratifications.** With the same notation as above, we attach a *local presentation*, i.e., a pair $(J, b)$ in a smooth $W$, to a neighborhood of each point in either $\underline{\mathrm{Max}}\ \mathrm{HS}_X$ or $\underline{\mathrm{Max}}\ \mathrm{M}_X$. Then the object of interest is the (closed) set of points in $W$ where the order of $J$ is greater or equal to $b$. This closed set is referred to as the *singular locus of the pair* and denoted by Sing $(J, b)$.

Now the next step is to construct a sequence like (0.2). To this end, new upper semi-continuous functions are defined now on Sing $(J, b)$. These upper-semi continuous functions can be shown to stratify Sing $(J, b)$ in smooth strata. And these stratifications, in turn, are shown to induce smooth stratifications of both, $\underline{\mathrm{Max}}\ \mathrm{HS}_X$ via the local embedding of $X$, or in $\underline{\mathrm{Max}}\ \mathrm{M}_X$.

[1]If $W_i \leftarrow W_{i+1}$ is the blow up at a smooth $Y$ with $\mathrm{ord}_Y J_i \geq b$ with exceptional divisor $H_{i+1}$, then the *weak transform of* $J_i$, $J_{i+1}$, is defined as $J_i \mathcal{O}_{W_{i+1}} \mathcal{I}(H_{i+1})^{-b}$.

The upper-semi continuous functions are defined at each step of sequence (0.2), and their maximum strata determines the center to blow up. The most important ingredient to define such functions is the so called *Hironaka's order function*:

$$\text{(0.4)} \qquad \begin{array}{rrcl} \operatorname{ord}_{(J,b)}: & \operatorname{Sing}(J,b) & \longrightarrow & \mathbb{Q} \\ & \xi & \longmapsto & \frac{\nu_\xi(J)}{b}, \end{array}$$

where $\nu_\xi(J)$ denotes the order of $J$ at the regular local ring $\mathcal{O}_{W,\xi}$. All the other functions involved in the resolution process derive from this particular function.

In these notes, we will be using *Rees algebras* instead of of pairs as local presentations, since we find them more convenient to handle in some contexts. For instance, the *Canonicity Theorem*, which we will refer to in the next lines, has a more natural formulation using the language of Rees algebras.

**The Reformulation Theorem and the local-global problem in resolution.** The Reformulation Theorem 0.1 is of local nature. Namely, given $X \subset W$, the assignment of the ideal $J$ together with the number $b$ is made locally, in a neighborhood $U \subset W$ of each point in the maximum stratum of the Hilbert-Samuel function. Similarly when considering local presentations for the Multiplicity. This means that:

(1) A smooth stratification of Sing $(J, b)$ induces a smooth stratification only in a neighborhood $U \cap X \subset U$ of a point in $\underline{\text{Max}}\ \text{HS}_X$ or $\underline{\text{Max}}\ \text{M}_X$;
(2) A sequence (0.2), induces a sequence (0.3) only locally in a neighborhood $U \cap X \subset U$ of each point of $X$ in the maximum stratum.

But to resolve the singularities of $X$ we need to obtain a (globally defined) smooth centers to blow up, and reach ultimately a resolution of singularities of $X$. As we will see, the Canonicity Theorem for Rees algebras settles this question in a simple manner.

Also, one would like to know that both, the stratification and the resolution, do not depend on the particular choice of the embedding $X \subset W$. This issue will be addressed in Part III of these notes.

**Reformulation Theorem and natural properties.** Once we have indicated that the lowering of both, max $\text{HS}_X$ and max $\text{M}_X$ is addressed via local presentations, it remains to check that the process of reduction (0.2) and (0.3) is compatible with isomorphisms $\varphi : X \to Y$. As it has been indicated above, for all $\xi \in X$, $\text{HS}_X(\xi) = \text{HS}_X(\varphi(\xi))$ and $\text{M}_X(\xi) = \text{M}_X(\varphi(\xi))$. On the other hand, notice that local presentations for any of the functions on $X$ induce, via $\varphi$, local presentations for the corresponding function on $Y$. Thus, the refinement of the stratifications, and the lowering of either max $\text{HS}_X$ or max $\text{M}_X$, induced by local presentations on $X$ lead, via $\varphi$, to a refinement and lowering of the corresponding functions on $Y$.

**On the organization of these notes.** The notes are organized in four parts. Part I is divided in two blocks. In the first one we discuss the main properties of the multiplicity, in connexion with the notion of integral closure of ideals. This is done in Sections 1-5. In the second block, entitled "Multiplicity and local presentations via elimination", we address the Reformulation Theorem in the context of the multiplicity. Here the reformulation on a smooth scheme $W$ is called a *local presentation.* This part will require some elementary results of algebraic elimination (see Sections 6-8).

Rees algebras are well known in Commutative Algebra (see [**V**]). Here they appear as well suited tools to describe the local presentations. This will be discussed in Part II where we state the Theorem of Resolution of Rees algebras. Moreover, the Reformulation Theorem, and more precisely, the pair $(J, b)$ and the construction of the sequence (0.2), will be expressed in terms of a Rees algebra.

Part II is divided in four blocks. The first one corresponds to Sections 9-11, where Rees algebras are presented from scratch, and we establish the so called *Canonicity Theorem* mentioned before. The second block is devoted to presenting the main invariants behind constructive resolution of Rees algebras (see Sections 12-14).

In the third block, we discuss a form of elimination in the class of Rees algebras over smooth schemes. This will be essential for inductive arguments. In fact, this form of elimination enables us to extend the previous invariants by induction on the dimension (Sections 15-17).

The fourth block, consists of three sections. Section 18 contains the Theorem of Resolution of Rees algebras in characteristic zero (Theorem 18.9), and an example is discussed in Section 19. Finally, in Section 20, some applications of Theorem 18.9 are given.

In Part III we address a question that rises from the Reformulation Theorem. Recall that Hironaka attaches to a scheme $X$, and to an immersion in a smooth scheme, say $X \subset W$, an ideal $J$ over $W$ and a pair $(J, b)$. In Part II we have replaced $(J, b)$ by an algebra on $W$. If $X \subset W'$ is another inclusion in a smooth scheme $W'$, we would get a new algebra now over $W'$. This already motivates two results:

a) A notion of identification between two Rees algebras, defined maybe over different smooth schemes.

b) If two Rees algebras are identified, as above, we want them to undergo equivalent resolutions.

This result (b) should say, for example, that the sequence of blow-ups over $X$ that appear in the lower row of (0.3) should be independent of the embedding of $X$ in the smooth scheme $W$.

The notion of identification in (a) will be addressed in Section 21. The problem in (b) is treated in the rest of Part III, where the main result is Theorem 26.5.

Finally in Part IV we present the proof of constructive resolution over fields of characteristic zero (see Theorem 30.7). The approach to this proof, which differs from that in previous presentations, follows from the previous discussions, particularly from those in Part III.

Appendix A (Sections 31-36) includes some results concerning the notion of transversal morphisms $X \to W$. These are needed in Part I.

We have mentioned, but only in passing, some of the natural properties that hold in constructive resolution of singularities. We refer here to [**BrEV**] for an expository presentation of these and other properties of algorithms of resolution. Among them there is the so called *compatibility with étale topology*, so Appendix A includes an introduction to étale morphisms and a discussion of their role both in the study of the multiplicity and of the Hilbert-Samuel function at a local ring.

Appendix B (Section 37) is devoted to treat some technical aspects concerning the construction of the resolutions functions $f_{\mathscr{B}}$ from Theorem 18.9, and to show how they evolve from the statement in Proposition 18.2.

**Final comments.** In these notes, the notation that we use for the invariants is consistent with that used in Hironaka's work, in [**Cu1**], and in previous publications of our research team. Rees algebras, as they appear in our discussions, have also been treated and thoroughly studied by other authors (see [**K**], [**KM**], [**W**]). This volume includes contributions of H. Kawanoue and J. Schicho which are also related to this subject.

**Acknowledgments.** We would like to acknowledge the constructive criticism received from the referees. In addition, our thanks to A. Nobile, whose suggestions have been useful for the development of Part III, and to C. Abad who helped us simplify some arguments. We also thank A. Benito, S. Encinas, M.L. García-Escamilla, and T. Preu, among others.

## Part I: Multiplicity

Along these notes an affine algebraic variety $V$, over a field $k$, will be an object attached to a reduced and equidimensional $k$-algebra $D$. To be precise, the points of $V$ are the maximal ideals in $D$, and if a point $\xi$ is in correspondence with the maximal ideal $m_\xi$, then the ring of local functions, say $\mathcal{O}_{V,\xi}$, is the localization $D_{m_\xi}$.

In this first part we present and discuss the notion of multiplicity of a variety $V$ at a point $\xi$. When $k$ is the field of complex numbers, $V$ can be endowed with the usual topology, and we begin by motivating the concept of multiplicity, within this context, in terms of finite coverings in the topological sense.

The discussion will lead us to consider a class of rings which is more ample than that of $k$-algebras of finite type. For example, given $D$ and $m_\xi$ as above, the completion of $D_{m_\xi}$ is no longer the localization of a $k$-algebra of finite type. And complete rings will arise naturally in the coming discussions.

The more general notion of affine scheme is better suited for this purpose than affine varieties. However we will have to impose conditions on the rings involved. We begin by expressing the multiplicity at a point by using ramified covers, and then we reformulate this notion in terms of a finite extension of a regular ring. Given a finite extension of rings, say $S \subset B$, where $S$ is a regular ring, then a theorem of Zariski, formulated in 3.1, enables us to the study of the multiplicity at the prime ideals of $B$. The extension defines $\mathrm{Spec}(B) \to \mathrm{Spec}(S)$, is a finite morphism on a regular scheme, and the goal of this first part is to present techniques of elimination to reformulate the study of the multiplicity on $\mathrm{Spec}(B)$ by equations on the regular space $\mathrm{Spec}(S)$. When $B$ is the ring of functions of an affine chart of a variety over a field, this technique will lead to resolution of singularities, at least for fields of characteristic zero.

### 1. Preliminaries

An algebraic variety, say $X$, is usually considered, at least locally, together **with an immersion in a regular space** $V$. If $X$ is an affine variety over a field $k$, then there is a local immersion, say $X \subset \mathbb{A}^n_k$, in an affine $n$-dimensional space. It is through this local immersion that $X$ is expressed as the common zeros of a finite set of polynomials.

There is yet another way to present a variety $X$ **in which it is mapped onto a regular space** $V$. In algebraic geometry a *branched covering* is a morphism between algebraic varieties $X \to V$ both being of the same dimension, and the typical fiber being of dimension zero. This latter approach is useful in the study of local properties of $X$, in which we fix a point $\xi \in X$, and construct a branched cover, $X \to V$ from a neighborhood of $\xi$ onto a regular space $V$.

In the case of complex analytic varieties, given $\xi \in X$, one can construct an analytic branched cover $(X, \xi) \stackrel{\delta}{\to} (\mathbb{C}^d, 0)$. There is a hypersurface $D \subset \mathbb{C}^d$, defined locally at the origin, called the *ramification locus*, so that over the open set $\mathbb{C}^d \setminus D$ the morphism is a covering space. Namely, it resembles the topological notion of a covering space, or say, it satisfies the local triviality condition: for $y \in \mathbb{C}^d \setminus D$, there is a neighborhood $U$ so that $\delta^{-1}(U)$ is a union of sections, and each section is homeomorphic over $U$, in the topological setting, and isomorphic over $U$ as analytic varieties (see [**He**]). Locally at a point $\xi \in X$, one can define different coverings as above. The number of sections (the number of points in the general fiber) is not intrinsic to $(X, \xi)$, and depends on the covering. Consider for instance $\mathbb{C}^2 \supset X = \{X^2 + Y^3 = 0\} \to \mathbb{C}^1$ by projecting on the first coordinate. In this case $D = 0 \in \mathbb{C}^1$, and the general fiber has 3 points, whereas by projecting on the second coordinate the general fiber has 2 points.

Given $X$ of dimension $d$, the *multiplicity of* $X$ *at point* $\xi \in X$ is defined as the smallest number of points in the general fiber, for the different local morphisms $(X, \xi) \to (\mathbb{C}^d, 0)$.

QUESTION 1.1. Let $X$ be an algebraic variety over a field $k$. Can one construct finite coverings as above in a neighborhood of a point in an algebraic variety over $k$?

QUESTION 1.2. If so, what do we mean by a general fiber?

**1.3. The formal setting and the generic fiber.** Let $X$ be a $d$-dimensional variety defined over a field $k$, and let $\xi \in X$ be a point. Consider the $d$-dimensional local ring $(\mathcal{O}_{X,\xi}, m_\xi, k(\xi))$.

Observe that if $\{x_1, \dots, x_d\} \subset \mathcal{O}_{X,\xi}$ is a system of parameters, the quotient $\mathcal{O}_{X,\xi}/\langle x_1, \dots, x_d\rangle$ is a finite dimensional vector space over over the residue field $k(\xi)$. A particular feature of complete local rings is that there is an inclusion

$$S = k(\xi)[[x_1, \dots, x_d]] \subset \hat{\mathcal{O}}_{X,\xi} \tag{1.1}$$

which is a finite extension of rings ([**ZS**, Corollary 2, pg. 259]).

Conversely, if $\{x_1, \dots, x_d\} \subset \mathcal{O}_{X,\xi}$ is a set of non-invertible elements and

$$S = k(\xi)[[x_1, \dots, x_d]] \subset \hat{\mathcal{O}}_{X,\xi}$$

is finite, then it can be checked that $\{x_1, \dots, x_d\} \subset \mathcal{O}_{X,\xi}$ is a system of parameters (i.e., the quotient $\mathcal{O}_{X,\xi}/\langle x_1, \dots, x_d\rangle$ is a finite dimensional $k(\xi)$-vector space). See [**Ma2**, §14].

So there are many ways to express the complete ring $\hat{\mathcal{O}}_{X,\xi}$ as a finite extension of a regular ring.

A remarkable fact for the case of complex analytic varieties is that there no need to consider the completion to achieve this result: If $\{x_1, \dots, x_d\}$ is a system of parameters in the local ring of the analytic variety associated to $X$ at $\xi$, $\mathcal{O}^{(h)}_{X,\xi}$, then

$$S = \mathbb{C}\langle\langle x_1, \dots, x_d\rangle\rangle \subset \mathcal{O}^{(h)}_{X,\xi} \tag{1.2}$$

is already a finite extension of rings. Here $S$ denotes the local ring of analytic functions at $(\mathbb{C}^d, 0)$. This extension defines, locally, a finite morphism of analytic varieties $(X, \xi) \to (\mathbb{C}^d, 0)$, which depends on the choice of the system of parameters.

Observe that in both cases, (1.1) and (1.2), $S$ is a domain. Let $K$ be the quotient field of $S$. Then, setting $B = \hat{\mathcal{O}}_{X,\xi}$ as in (1.1), or $B = \mathcal{O}_{X,\xi}$ as in (1.2), and localizing the inclusion $S \subset B$ one obtains that

$$K \subset B \otimes_S K \tag{1.3}$$

which is a finite extension of the field $K$.

It turns out that in the complex analytic case, the number of points in the general fiber of $(X, \xi) \to (\mathbb{C}^d, 0)$ is given by the dimension of this vector space over $K$, say $\dim_K(K \otimes_S B)$. We shall take this fact as guide for our forthcoming discussion.

In fact, in the algebraic case, the finite extension in (1.1) induces

$$\operatorname{Spec}(B) \to \operatorname{Spec}(k(\xi)[[x_1, \dots, x_d]]). \tag{1.4}$$

and we claim that the integer $\dim_K(K \otimes_S B)$ is again the number of points in a *general* fiber.

We sketch a proof of this claim, at least for the case in which $k$ is a field of characteristic zero: recall that $B = \hat{\mathcal{O}}_{X,\xi}$, is the completion of the local reduced and equidimensional ring $\mathcal{O}_{X,\xi}$. This latter local ring is excellent, and hence the completion, namely $B$, is also reduced and equidimensional ([**Gr**, 7.8.9, (vii) and (x)]). Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ denote the minimal primes of $B$. Since $B$ is reduced, the map $B \to B/\mathfrak{p}_1 \oplus \dots B/\mathfrak{p}_s$ is injective. By assumption

$$d = \dim_{\mathrm{Krull}} B = \dim_{\mathrm{Krull}}(B/\mathfrak{p}_i), \text{ for } i = 1, \dots, s.$$

In particular, the induced finite homomorphism $S = k(\xi)[[x_1, \dots, x_d]] \to B/\mathfrak{p}_i$ must be an inclusion.

A localization of a reduced ring is reduced, and $B/\mathfrak{p}_i \otimes_S K = L_i$ is a field. Thus,

$$B \otimes_S K = L_1 \oplus \cdots \oplus L_s$$

is a direct sum of fields. Let $\overline{K}$ be an algebraic closure of $K$. Then

$$\dim_K(K \otimes_S B) = \dim_{\overline{K}}(\overline{K} \otimes_S B).$$

Finally, if $k$ is a field of characteristic zero, each $L_i$ is a finite separable extension of $K$, and hence $\overline{K} \otimes_S L_i$ is a direct sum of $r$ copies of $\overline{K}$, where $r = \dim_{\overline{K}}(\overline{K} \otimes_S L_i)$. This leads to the fact that $\dim_K(K \otimes_S B)$ is the number of points in the scheme $\operatorname{Spec}(B \otimes_S \overline{K})$. $\operatorname{Spec}(B \otimes_S K)$ is called the *fiber over the generic point* of (1.4), whereas $\operatorname{Spec}(B \otimes_S \overline{K})$ is known as the *geometric fiber* over the generic point. So the integer $\dim_K(K \otimes_S B)$ is the number of points in the geometric fiber over the generic point.

**1.4. A preliminary geometric approach to the notion of multiplicity.** Let $X$ be an algebraic variety of dimension $d$, defined over a field $k$, and let $\xi \in X$ be a point with residue field $k(\xi)$. Given a system of parameters $\mathcal{P} = \{x_1, \dots, x_d\} \subset \mathcal{O}_{X,\xi}$, consider the finite extension $S = k(\xi)[[x_1, \dots, x_d]] \subset \hat{\mathcal{O}}_{X,\xi}$, and define

$$n_{\mathcal{P}} := \dim_K(K \otimes_S \hat{\mathcal{O}}_{X,\xi}).$$

Definition 1.5. The *multiplicity of $X$ at $\xi$*, or the *multiplicity of the local ring* $\mathcal{O}_{X,\xi}$, is the smallest integer of the form $n_{\mathcal{P}}$ when $\mathcal{P}$ runs through all systems of parameters in $\mathcal{O}_{X,\xi}$.

Recall that a system of parameters in $\mathcal{O}_{X,\xi}$ is also a system of parameters in its completion $\hat{\mathcal{O}}_{X,\xi}$. So we are looking at all finite morphisms

$$\operatorname{Spec}(\hat{\mathcal{O}}_{X,\xi}) \to \operatorname{Spec}(k(\xi)[[x_1, \dots, x_d]])$$

when $\{x_1, \dots, x_d\} \subset \mathcal{O}_{X,\xi}$ runs through all systems of parameters in $\mathcal{O}_{X,\xi}$. Each choice enables us to represents $\operatorname{Spec}(\hat{\mathcal{O}}_{X,\xi})$ as a ramified cover of a regular ring of the form $\operatorname{Spec}(k(\xi)[[x_1, \dots, x_d]]$.

The multiplicity of the local ring $\mathcal{O}_{X,\xi}$, with completion $\hat{\mathcal{O}}_{X,\xi}$, is an integer. Moreover, and following the discussion in the last paragraph, over fields of characteristic zero the multiplicity is defined as the smallest number of points in the geometric fiber over the generic point, for all finite covers $\operatorname{Spec}(\hat{\mathcal{O}}_{X,\xi}) \to \operatorname{Spec}(k(\xi)[[x_1, \dots, x_d]])$.

Question 1.6. How do we decide whether, given a system of parameters $\mathcal{P} = \{x_1, \dots, x_d\} \subset \mathcal{O}_{X,\xi}$, the number $n_{\mathcal{P}} = \dim_K(K \otimes_S \hat{\mathcal{O}}_{X,\xi})$ is the multiplicity of the local ring?

Question 1.7. Is it possible to define the multiplicity of $\mathcal{O}_{X,\xi}$ without passing to the completion?

Question 1.8. If the answer to Question 1.7 is yes, can we find finite covers, as in the previous discussion, in the category of $k$-algebras of finite type?

A positive answer to this last question would allow us to construct finite covers in a **neighborhood** of a point of an algebraic variety. See Appendix A for an approach on how to construct such finite covers in the context of $k$-algebras of finite type.

## 2. The algebraic definition of multiplicity

We have defined the multiplicity at a point in a variety, say $\xi \in X$, as a positive integer assigned to the local ring $\mathcal{O}_{X,\xi}$. We now present an alternative definition of multiplicity, which applies to a wider class of local rings. In fact the multiplicity is an invariant assigned to any local noetherian ring. This alternative approach will allow us to answer the questions 1.6 and 1.7.

**2.1. On local rings of dimension zero.** Fix a local neotherian $(A, m)$ and a module $M$. An increasing chain of submodules of $M$, say

$$M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_r$$

is said to be a chain of length $r$ if $M_i/M_{i-1} \neq 0$, $1 \leq i \leq r$.

Recall that a local ring $(A, m)$ is zero dimensional if and only if there is a positive integer $n$ so that $m^n = 0$. If in addition we assume that $m^{n-1} \neq 0$ it follows that there is a principal ideal in $A$, say $I_1$, included in $m^{n-1}$, which is isomorphic, as $A$ module, to the field $A/m$. As a quotient of a zero dimensional ring is zero dimensional, it follows that the neotherian ring $A$ admits an increasing sequence of ideals, which must be finite, say

$$l_0 = 0 \subset I_1 \subset I_2 \subset \cdots \subset I_s = A,$$

where $l_i/I_{i-1} = A/m$, $1 \leq i \leq s$. The Jordan-Holder Theorem says that any two chains of ideals with this property must have the same length $s$, called the length of the local ring. More generally, given a finitely generated $A$-module $M$, there is a chain of submodules

$$M_0 = 0 \subset M_1 \subset M_2 \subset \cdots \subset M_L = M,$$

where $M_i/M_{i-1} = A/m$, for $1 \leq i \leq L$, and two chains with this property have the same length, say

$$L = \lambda(M),$$

which we refer to as *the length of* $M$. When $A$ is a field $L = \lambda(M)$ is the dimension of the vector space.

If we now fix a local ring $(A, m)$ of dimension zero and consider the class of finitely generated $A$ modules, then the previous theorem ensures that given a short exact sequence of finitely generated $A$-modules,

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

one has that:

$$\lambda(M_2) = \lambda(M_1) + \lambda(M_3). \tag{2.1}$$

So the length is a function defined on the class of finitely generated modules over $A$, and takes values in the integers. In addition in satisfies the additive property (2.1) for modules in a short exact sequence.

We take this as a starting point to introduce the notion of multiplicity for local rings of any dimension, and also for any finitely generated modules over a local ring.

Once we fix a local ring, the multiplicity will be a function from the class of finitely generated modules over the ring, with values in the integers, and with an additive behavior to be discussed.

The multiplicity of a local ring $(A, m)$ of dimension zero will be the length $\lambda(A)$, and the same will hold for finitely generated modules.

**2.2. The Hilbert-Samuel Function and multiplicity.** Let $(R, m, k)$ be a noetherian local ring of dimension $d$, and let $J$ be an $m$-primary ideal. For each positive integer $r$ the ideal $J^r$ is $m$-primary. Therefore $R/J$, and $R/J^r$ are zero dimensional for each positive integer $r$. The *Hilbert-Samuel function* of $J$ in $R$ is

$$\begin{aligned} HS_{R,J} : \mathbb{N} &\longrightarrow \mathbb{N} \\ n &\longmapsto \lambda(R/J^n). \end{aligned}$$

It can be shown that there is a polynomial of degree $d$, and rational coefficients, say $P_{R,J}(x)$, so that $HS_{R,J}(n) = P_{R,J}(n)$ for $n$ large enough. This is referred to as the *Hilbert polynomial* (see [**AtM**]). Moreover, the leading coefficient of $P_{R,J}(x)$ multiplied by $d!$ is a positive integer, called the *multiplicity of $R$ at $J$*. So

$$P_{R,J}(x) = \frac{e_R(J)}{d!}x^d + a_{d-1}x^{d-1} + \ldots + a_0 \in \mathbb{Q}[x].$$

Of particular interest is the case $J = m$. The multiplicity of the local ring $(R, m, k)$ is denoted by $e_R(m)$. However a good comprehension of the properties of the multiplicity requires a understanding of this notion for arbitrary $m$-primary ideal. In particular, for the primary ideals spanned by a system of parameters.

Definition 2.3. Let $B$ be a noetherian ring. A prime ideal $q \subset B$ is said to be an *n-fold point of* Spec$(B)$, or an *n-fold prime*, when $e_{B_q}(qB_q) = n$.

**2.4. Multiplicity: some facts and some examples.** Given a local ring $(R, m, k)$ and an element $f \in R$, define

$$\nu_R(f) = r \in \mathbb{Z}$$

if $f \in m^r \setminus m^{r+1}$. If $(R, m, k)$ is a regular local ring we refer to $\nu_R(f)$ as the order of $f$ at the local ring.

Some important properties of the multiplicity are the following:

- The multiplicity of a local regular ring is one.
- If $(R, m, k)$ is a local regular ring, and if $f \in m$ has order $\nu_R(f) = n \geq 1$, then the local ring $(R/fR, m/fR, k)$ has multiplicity $n$. For instance, consider the ideal $m = \langle x_1, x_2 \rangle$ in $k[x_1, x_2]$. The local ring $k[x_1, x_2]_m$ is regular, so it has multiplicity 1, and the local ring $k[x_1, x_2]_m/\langle x_1^3 + x_2^7 \rangle$ has multiplicity 3.
- The multiplicity of a local ring $(R, m, k)$ is the same as the multiplicity of its completion $(\hat{R}, \hat{m}, k)$. In fact, $R/m^n = \hat{R}/\hat{m}^n$ for any $n$, and hence both have the same Hilbert-Samuel function.
- Similarly, note that if $J$ is primary for the maximal ideal of $(R, m, k)$, then $J\hat{R}$ is primary for the maximal ideal of $(\hat{R}, \hat{m})$ and

$$e_R(J) = e_{\hat{R}}(J\hat{R}).$$

**2.5. Relating the two definitions of multiplicity.** Given a point in a variety, say $\xi \in X$, we have introduced the notion of multiplicity in 1.4. There, the multiplicity was expressed in terms of a finite extension of rings, involving the completion of $\mathcal{O}_{X,\xi}$ and a ring of formal power series. On the other hand in 2.2 the multiplicity for an arbitrary local noetherian ring, say $(A, m)$, was defined and it coincides with that of the completion $(\hat{A}, \hat{m})$. In this part we will simply formulate two claims, which ensure that that the two definitions agree for a local ring $\mathcal{O}_{X,\xi}$. Although the proofs of the claims will be postponed, they will enable us extract a first significant consequence for the study of singularities on a variety: $X$ is regular at a point $\xi$, or say $\mathcal{O}_{X,\xi}$ is regular, if and only if it has multiplicity one.

This fundamental result applies also for a wide class of reduced and equidimensional schemes of finite type over a field, the multiplicity at a point is one if and only if the point is regular (2.10). Firstly we shall discuss the additivity of the multiplicity in 2.8, which will indicate way we restrict our study of the multiplicity at points on an equidimensional schemes.

Let $(R, m, k)$ be a local noetherian ring of dimension $d$. If $J \subset R$ is an $m$-primary ideal, then $\lambda(R/m^r) \leq \lambda(R/J^r)$, for any positive integer $r$, or say $HS_{R,m}(n) \leq HS_{R,J}(n)$ for all $n$. On the other hand both polynomials $P_{R,m}(n)$ and $P_{R,J}(n)$ have the same degree, and hence

$$e_R(m) \leq e_R(J). \tag{2.2}$$

Claim 2.6. *Let* $(\hat{R}, \hat{m}, k)$ *be the completion of* $(R, m, k)$*, and assume that the residue field* $k$ *is included in* $\hat{R}$*. Let* $\{x_1, \ldots, x_d\} \subset R$ *be a system of parameters and let* $J = \langle x_1, \ldots, x_d\rangle$*. Then*

$$e_R(J) = [\hat{R} \otimes_S K : K], \tag{2.3}$$

*where* $S = k[[x_1, \ldots, x_d]]$*,* $K$ *denotes the quotient field of* $S$ *and* $[\hat{R} \otimes_S K : K]$ *is the dimension of the* $K$ *vector space* $\hat{R} \otimes_S K$*.*

The proof of Claim 2.6 will be addressed in 3.2. Together with (2.2) it says that

$$e_R(m) \leq e_R(J) = [\hat{R} \otimes_S K : K],$$

for all ideals $J$ generated by a system of parameters. We now formulate a Second Claim (see 4.6):

Claim 2.7. One can find a system of parameters, say $\{x_1, \ldots, x_d\}$, so that

$$e_R(m) = e_R(\langle x_1, \ldots, x_d\rangle). \tag{2.4}$$

Note that both claims together would show that the integer introduced in the Definition 1.5 is $e_R(m)$. In fact, they say that:

$$e_R(m) = \min\{e_R(J) : J \text{ is spanned by a system of parameters in } R\}. \tag{2.5}$$

**2.8. Multiplicity for finitely generated modules.** Let $A$ be a ring. Recall that an $A$-module $M$ is an abelian group, together with a homomorphism $\phi : A \to End_{\mathbb{Z}}(M)$, mapping $a \in A$ to $h_a : M \to M$, $h_a(m) = a.m$ for $m \in M$. The kernel of $\phi$ is usually denoted by $\mathrm{Ann}(M)$. Since $\phi$ factors though $A/Ann(M)$, one can naturally view $M$ as a module over $A/\mathrm{Ann}(M)$. The closed set $V(\mathrm{Ann}(M))$, of primes in $A$ containing this ideal, is also called the *support of* $M$. It is characterized as the collection of primes $p$ in $A$ for which $M_p$ is non-zero.

Now let $(R, m, k)$ be a noetherian local ring of dimension $d$, and let $J$ be an $m$-primary ideal. Given a finitely generated $R$-module $N$, the length $l(N/J^nN)$ is also given by a polynomial say $P_{N,J}(x)$, for $x = n$ sufficiently big. The degree of $P_{N,J}(x)$, say $d'$, is the dimension of the support of $N$. So $d' \leq d$.

The leading coefficient of $P_{N,J}(x)$ can be expressed as $\frac{e_N(J)}{d'!}$, for a positive integer $e_N(J)$ which is called *the multiplicity of* $N$ *relative to the open ideal* $J$. We shall define the *$d$-dimensional multiplicity of a module* $e^{(d)}_N(J)$ to be zero if $d' < d = \dim R$, and to be $e_N(J)$ when $d' = d$.

**2.9. The multiplicity is additive.** An important property of the $d$-dimensional multiplicity is its additive behavior. With the same notation as before, and given a short exact sequence of finitely generated $R$-modules, say

$$0 \to N_1 \to N_2 \to N_3 \to 0,$$

the following property holds:

$$e^{(d)}_{N_2}(J) = e^{(d)}_{N_1}(J) + e^{(d)}_{N_3}(J). \tag{2.6}$$

In other words, the coefficient in degree $d$ of the polynomial corresponding to $N_2$ is the sum of those of $N_1$ and $N_3$ (see [**Ma1**, Proposition 12 D, pg. 74]).

**2.10. Equidimensional rings: from multiplicity one to regularity.** Let $D$ denote the ring of functions of an affine variety $X$, (i.e., a finitely generated $k$-algebra which is reduced and equidimensional). Fix a maximal ideal $m_\xi \subset D$ in correspondence with a point $\xi \in X$ (so that $\mathcal{O}_{X,\xi} = R = D_{m_\xi}$), and let $k'$ be the residue field. Let $\{x_1, \dots, x_d\} \subset \mathcal{O}_{X,\xi}$ be a system of parameters so that the conditions in (2.4) hold for $J = \langle x_1, \dots, x_d \rangle$ in $B = \hat{\mathcal{O}}_{X,\xi}$, and let $S = k'[[x_1, \dots, x_d]] \subset B$. Then

$$e_R(m) = e_R(J) = [B \otimes_S K : K] \tag{2.7}$$

where $K$ denotes the quotient field of $S$.

Recall that if a local excellent ring is reduced and equidimensional, the same holds for the completion ([**Gr**, 7.8.3, (vii) and (x)] or [**Ma1**, §34]). This applies, in particular, for the local ring $B = \hat{\mathcal{O}}_{X,\xi}$. Therefore, if $\{p_1, \dots, p_s\}$ denote the minimal ideals, then:

- $B \to B/p_1 \oplus \cdots \oplus B/p_s$ is injective, and
- each homomorphism $B/p_i \subset B/p_i \otimes_S K$ is injective, for $1 \leq i \leq s$.

The second statement follows from the assumption that each domain $B/p_i$ is $d$-dimensional, and finite over $S$. It says that all minimal primes in $B$ dominate $S$ at the prime zero. So one concludes, in addition, that

$$B \otimes_S K = (B/p_1 \oplus \cdots \oplus B/p_s) \otimes_S K$$

and hence, that $B \to B \otimes_S K$ is injective.

Assume finally that the multiplicity of the local ring $\mathcal{O}_{V,x}$ is one. In this case $B \otimes_S K = K$ in (2.7). Now $K$ is the quotient field of $S$, and $B$ is finite over $S$. As regular rings are normal, we conclude that $B = S$, and hence $\hat{\mathcal{O}}_{X,\xi}$, and also $\mathcal{O}_{X,\xi}$, are regular.

**Equidimensionality is a necessary condition.** The hypothesis of equidimensionality was used in the previous characterization of regular local rings in a variety. The following example of sub-schemes in $\mathbb{A}^3$ illustrates the necessity of the hypothesis for more general schemes over $k$. Consider the inclusion $X\langle Y, Z\rangle \subset \langle X \rangle$ in $k[X, Y, Z]$, which defines a surjection of the quotient rings, say $B_1 \to B_2$, where both are reduced $k$-algebras of finite type, but $B_1$ is not equidimensional. Consider the exact sequence

$$0 \to J \to B_1 \to B_2 \to 0$$

localized at the origin. Observe that $B_1$ and $B_2$ are two dimensional, and that $J$ is supported in a closed set of smaller dimension. Using the additive property (2.6) one checks that $B_1$ has multiplicity one at the origin, but it is not regular at this point.

**2.11. Multiplicity and stratification.** Given an algebraic variety $X$, a function $\text{mult}_X : X \to \mathbb{N}$ is defined by setting $\text{mult}_X(\xi)$ the multiplicity of $\mathcal{O}_{X,\xi}$. The discussion in these notes will show that it is upper semi-continuous, and the image is a finite set $\{n_1 < n_2 < \cdots < n_r\}$ (see [**A**] and [**Vi7**, Theorem 6.12]). Let $(X)_{n_i}$ denotes each stratum (each level set). These level sets may not be regular, and the problem is to define new invariants, in some natural way, which enable us to refine this stratification into regular strata. Moreover, we want to define such stratification so that the multiplicity drops after blowing up successively along the closed regular stratum.

**Some examples of stratifications defined by the multiplicity.**

EXAMPLE 2.12. Let $X = \{Z^3 + (T^2 + Y^3)^4 = 0\} \subset \mathbb{C}^3$.

- $(X)_3 = \{Z = 0, T^2 + Y^3 = 0\}$: stratum corresponding to the set of points where the multiplicity is three.
- $(X)_1 = X \setminus (X)_3$: stratum corresponding to the set of points where the multiplicity is one.

EXAMPLE 2.13. Let $X = \{Z^3 + (T^2 + Y^3)^2\} \subset \mathbb{C}^3$.

- $(X)_3 = (0, 0, 0)$: stratum corresponding to the set of points where the multiplicity is three.
- $(X)_2 = \{Z = 0, T^2 + Y^3 = 0\} \setminus \{(0, 0, 0)\}$: stratum corresponding to the set of points where the multiplicity is two.
- $(X)_1 = X \setminus ((X)_3 \cup (X)_2)$: stratum corresponding to the set of points where the multiplicity is one.

Observe that in both examples one obtains a stratification into locally closed sets. This is a general property of the multiplicity for general algebraic varieties, not only in case of a hypersurface.

**Goal:** Find local presentations for the stratum of maximum multiplicity of an arbitrary algebraic variety $X$. Recall here the role of local presentations in the Reformulation Theorem discussed in the Introduction.

## 3. Zariski's multiplicity formula for finite extensions

The purpose of this section is to present Zariski's Multiplicity formula for finite morphisms, and to discuss some of its consequences.

THEOREM 3.1. [**ZS**, Theorem 24, pg. 297] *Let $(R, m, k)$ be a local noetherian domain, and let $B$ be a finite extension of $R$. Let $K$ denote the quotient field of $R$, and $L = K \otimes_R B$. Let $\mathcal{Q}_1, \dots, Q_r$ denote the maximal ideals of the semi-local ring $B$, and assume that* $\dim B_{Q_i} = \dim R$, *for $i = 1, \dots, r$. Then*

$$e_R(m)[L : K] = \sum_{1 \leq i \leq r} e_{B_{Q_i}}(mB_{Q_i})[k_i : k], \tag{3.1}$$

*where for $i = 1, \dots, r$, $k_i$ denotes the residue field of $B_{Q_i}$, $[k_i : k] = \dim_k k_i$, and $[L : K] = \dim_K L$.*

SKETCH OF THE PROOF. Let $d$ denote the dimension of the local ring $(R, m)$. Recall the definition of the integer $e^{(d)}_M(m)$ attached to a finitely generated $R$-module $M$. We prove the theorem by showing that both sides in the equality (3.1) coincide with $e^{(d)}_B(m)$. In fact, here $B$ is a finite module over $R$, so there is a Hilbert-Samuel function attached to $B$, defined by the length $\lambda_{R/m^r}(B/m^rB)$ for all $r \in \mathbb{Z} \geq 0$. Note that there is an inclusion of a free $R$-module $R^{[L:K]}$ in $B$, so that

$$0 \to R^{[L:K]} \to B \to N \to 0, \tag{3.2}$$

is an exact sequence of $R$-modules with $N \otimes_R K = 0$. Since $N$ is supported in smaller dimension, the additive formula in (2.6) shows that the left hand side of the equality in (3.1) is $e^{(d)}_B(m)$.

As for the term in the right, consider the length of $B/m^nB$, for all positive integers $n$, as a module over $R/m^n$. Note here that $B/m^nB$ is an artinian semi-local ring, and that

$$B/m^nB = B_{Q_1}/m^nB_{Q_1} \oplus \cdots \oplus B_{Q_r}/m^nB_{Q_1}.$$

Each local ring $B_{Q_1}/m^nB_{Q_1}$ is artinian, and it is also finite over $R/m^n$. One readily checks that the length of the artinian ring $B_{Q_1}/m^nB_{Q_1}$ relates to the length of the $R/m^n$-module $B_{Q_1}/m^nB_{Q_1}$ by the formula:

$$\lambda_{R/m^n}(B_{Q_1}/m^nB_{Q_1}) = \lambda_{B_{Q_1}/m^nB_{Q_1}}(B_{Q_1}/m^nB_{Q_1}) \cdot [k_1 : k].$$

The proof finally follows by looking at the leading coefficients of the corresponding Hilbert-Samuel polynomials. □

**3.2. Zariski's formula and a proof of Claim 2.6.** Let $(R, m, k)$ be a local ring containing its residue field $k$. Let $\{y_1, \ldots, y_d\}$ be a regular system of parameters, and let $J = \langle y_1, \ldots, y_d\rangle$. Consider the finite extension

$$S = k[[y_1, \ldots, y_d]] \subset \hat{R}.$$

As $J$ is $\hat{m}$-primary in $\hat{R}$, we observe that $\hat{m}$ is the only prime ideal dominating $\langle y_1, \ldots, y_d\rangle \subset S$. The multiplicity of the local regular ring $S$ is one, in addition, since the local rings $\hat{R}$ and $S$ have the same residue fields, Theorem 3.1 asserts that

$$[\hat{R} \otimes_S K, K] = e_J(\hat{R}).$$

Finally, the claim in 2.6 follows from the fact that $e_J(R) = e_J(\hat{R})$. □

## 4. Multiplicity and integral closure of ideals

Hilbert-Samuel polynomials where defined for $m$-primary ideals in a local noetherian ring. Let $(R, m, k)$ be a noetherian local ring of dimension $d$. If $I \subset J$ are two primary ideals

$$e_R(I) \geq e_R(J). \tag{4.1}$$

Question 4.1. When does the equality hold?

The question is particularly relevant when we want to find a system of parameters $\{x_1, \ldots, x_d\}$ as in (2.4). Namely, so that

$$e_R(\langle x_1, \ldots, x_d\rangle) = e_R(m). \tag{4.2}$$

The following theorem of Rees (see Theorem 4.2 below) will be a useful tool to address these questions. First we recall the notion of integral closure of ideals.

**On integral closure of ideals, and the Claim 2.7.** Let $A$ be a ring, and let $I \subset A$ be an ideal. An element $f \in A$ is said to be *integral over $I$* if satisfies an equation of the form

$$(f)^n + a_1(f)^{n-1} + \cdots + a_n = 0$$

for some $n \in \mathbb{Z}_{\geq 1}$, and some $a_j \in I^j$, for $j = 1, \ldots, n$. The set of all elements in $A$ that are integral over $I$ form an ideal called the *integral closure of $I$ in $A$*, which is usually denoted by $\overline{I}$. Obviously $I \subset \overline{I}$. If $J \subset A$ is an ideal and $I \subset J \subset \overline{I}$, then $J$ is said to be an *integral extension of $I$*. We also say that $I$ *is a reduction of* $J$.

Theorem 4.2. [**Re**, D. Rees] *Let $(R, m, k)$ be an excellent equidimensional ring, and let $I \subset J$ be primary ideals for the maximal ideal $m$. Then $e_R(I) = e_R(J)$ if and only if $I$ and $J$ have the same integral closure in $R$.*

Thus Theorem 4.2 says that a system of parameters $\{x_1, \ldots, x_d\}$ is so that

$$e_R(\langle x_1, \ldots, x_d\rangle) = e_R(m),$$

if and only if $\langle x_1, \ldots, x_d\rangle$ is a reduction of the maximal ideal. This relates to the Claim 2.7 and a simple criterion to establish this condition will be discussed below.

**Some examples.**

EXAMPLE 4.3. Let $(R, m, k) = (k[[x, y]]/\langle y^3+x^4y+x^7\rangle, \langle x, y\rangle, k)$. Observe that $R$ is equidimensional and that the ideal $J = \langle x\rangle$ is a reduction of $m$. Therefore, by Theorem 4.2, $e_R(m) = e_R(J)$. Now, to compute $e_R(J)$, we use the statement in Claim 2.6. Observe that the extension $k[[x]] \subset k[[x, y]]/\langle y^3 + x^4y + x^7\rangle$ is finite, and that the rank at the generic point of $\mathrm{Spec}(k[[x]])$ is three. So, $e_R(J) = 3$, and therefore the multiplicity of $R$ is 3 as well.

EXAMPLE 4.4. Let $(R, m, k) = (k[[x, y]]/\langle x^2 - y^3\rangle, \langle x, y\rangle, k)$ and compare the finite extensions:

(1) $k[[x]] \subset k[[x, y]]/\langle x^2 - y^3\rangle$;
(2) $k[[y]] \subset k[[x, y]]/\langle x^2 - y^3\rangle$.

It can be checked that the rank at the generic point of $\mathrm{Spec}(k[[x]])$ is 3, while the rank at the generic point of $\mathrm{Spec}(k[[y]])$ is 2. Observe that $\langle y\rangle$ is a reduction of $m$, while $\langle x\rangle$ is not.

**4.5. System of parameters and reductions of the maximal ideal.** Given a local noetherian ring $(R, m, k)$ of dimension $d$, there is a lot of information encoded in the graded ring $\mathrm{Gr}_m(R)$. To begin with, $\mathrm{Gr}_m(R)$ is also a $d$-dimensional ring. The homogeneous term in degree one, namely $m/m^2$, is a finite dimensional vector space over the residue field $k$, so the ring is a finitely generated $k$-algebra with a graded structure. Recall that $\dim_K(m/m^2) \geq d$ and that equality holds when $R$ is regular. Note that an element $\overline{x} \in m/m^2$ can be lifted (non-uniquely) to an element $x$ in $m$.

CLAIM 4.6. With the same assumptions as above, a generic choice of $d$ elements in $m/m^2$, $\{\overline{x}_1, \ldots, \overline{x}_d\} \subset m/m^2$, lifts to a set $\{x_1, \ldots, x_d\}$ in $m$ with two properties:

(1) $\{x_1, \ldots, x_d\}$ is a system of parameters in $(R, m, k)$.
(2) $\langle x_1, \ldots, x_d\rangle \subset m$ is an integral extension.

The reader can find in the Theorem 34.2 a precise formulation of this statement, particularly to clarify what we understand by a *generic choice of elements in* $m/m^2$. The previous examples in 4.4 already illustrate the geometric meaning of the claim.

## 5. Multiplicity and finite morphisms on a regular scheme

Given a noetherian ring $B$ one would like to extract conclusions on the behavior of the multiplicity along all prime ideals in $B$. Namely, at the localization of $B$ at the prime ideals. So for instance, if $l$ denotes the highest multiplicity arising in this way, one would like to study the subset of points in $\mathrm{Spec}(B)$ which have multiplicity $l$.

We will give here a first step in this direction, but under some particular additional assumptions. In fact we will fix a subring $S \subset B$ , where $S$ will be a regular domain with quotient field $K$, and such that $S \subset B$ is a finite extension of rings. In this case $B \otimes_S K$ will be a finite extension of $K$. Let $n = \dim_K(B \otimes_S K)$. Our discussion will show, for example, that $n$ will be an upper bound for the multiplicity, namely that $e_{B_P}(PB_P) \leq n$ for any prime $P$ in $B$.

The strategy is to extract information from the finite morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(S)$. As in our discussion we will make use of Zariski's Multiplicity formula and also of the Theorem of Rees, some further condition will be imposed on $B$ as these theorems require that the localizations of $B$ at prime ideals be equidimensional.

Our ultimate goal is to study the behavior of the multiplicity along points of a variety, and a variety can be treated as a reduced and equidimensional scheme of finite type over a field. The rest of this Part I will be devoted to the applications of the results of this section in this latter context. Namely for a variety together with a finite dominant morphism on a regular variety. In this context it will be shown there is a closed set of points of highest multiplicity.

**5.1. Further applications of Zariski's multiplicity formula.** Fix a finite extension $S \subset B$ as above, where $S$ is a regular domain, and $B$ is an excellent equidimensional ring.

**Claim.** *The conditions in Theorem* 3.1 *will be fulfilled for $S_m \subset B \otimes_S S_m$, for any prime ideal $m$ in $S$.*

In fact, let $\{q_1, \dots, q_s\}$ denote the minimal prime ideals in the equidimensional ring $B$, and note that $S \subset B/q_i$ for $1 \leq i \leq s$. Fix a prime $\overline{Q}$ in $B/q_i$, and set $m = \overline{Q} \cap S$. Since $S$ is normal, the local rings $(B/q_i)_{\overline{Q}}$ and $S_m$ have the same dimension. This ensures that the localization of the semi-local ring $B \otimes_S S_m$ at any maximal ideal has the same dimension as $S_m$, which enables us to use Zariski's Formula.

Let $Q_1, \dots, Q_r$ denote the maximal ideals in $B \otimes_S S_m$. As the multiplicity of $S_m$ is one,

$$[B \otimes_S K : K] = \sum_{1 \leq i \leq r} e_{B_{Q_i}}(mB_{Q_i})[k_i : k],$$

Fix an index $i_0$, $1 \leq i_0 \leq r$. Note that $e_{B_{Q_{i_0}}}(mB_{Q_{i_0}}) \geq e_{B_{Q_{i_0}}}(Q_{i_0}B_{Q_{i_0}})$, so

$$[B \otimes_S K : K] \geq e_{B_{Q_{i_0}}}(Q_{i_0}B_{Q_{i_0}}).$$

Moreover, if $[B \otimes_S K : K] = e_{B_{Q_{i_0}}}(Q_{i_0}B_{Q_{i_0}})$, then:

(1) $e_{B_{Q_{i_0}}}(mB_{Q_{i_0}}) = e_{B_{Q_{i_0}}}(Q_{i_0}B_{Q_{i_0}})$.
(2) $[k_{i_0} : k] = 1$.
(3) $r = 1$, so there is a unique prime dominating $S$ at $m$.

As we assume that $B$ is equidimensional and excellent, the theorem of Rees applies, and the condition in (1) can be reformulated by saying that $mB_{Q_{i_0}}$ is a reduction of $Q_{i_0}B_{Q_{i_0}}$. Further information can be extracted when $B$ is reduced, namely when

$$B \to B/q_1 \oplus \cdots \oplus B/q_s$$

is injective. This last condition ensures that

$$B \to L = B \otimes_S K,$$

is an inclusion, a condition which was used in our characterization of regular points in terms of the multiplicity (2.10).

We will be concerned here with the problem of resolution of singularities for varieties. In particular, with a $k$-algebra $B$ which is reduced and equidimensional. These are also excellent schemes so all the previous conditions hold.

DEFINITION 5.2. Fix a finite extension $S \subset B$, where $S$ is a regular domain and $B$ is an excellent equidimensional ring. We shall say that the finite morphism

$$\mathrm{Spec}(B) \to \mathrm{Spec}(S)$$

is finite transversal at a point $Q$ in $\mathrm{Spec}(B)$, if

$$e_{B_Q}(QB_Q) = [B \otimes_S K : K],$$

where $K$ denotes the quotient field of $S$.

**A characterization of finite transversal morphisms.** Some direct consequences can be extracted from the previous discussion.

COROLLARY 5.3. *Fix $S \subset B$ as above. Let $L = B \otimes_S K$, and let $P$ be a prime ideal in $B$. Then $e_{B_P}(PB_P) \leq \dim_K L = n$, and the following conditions (1) and (2) are equivalent:*

(1) $e_{B_P}(PB_P) = \dim_K L = n$.
(2) *Let $\mathfrak{p} = S \cap P$.*
  (a) *$P$ is the only prime in $B$ dominating $\mathfrak{p}$ (i.e., $B_P = B \otimes_S S_{\mathfrak{p}}$).*
  (b) $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} = B_P/PB_P$.
  (c) *$PB_P$ is the integral closure of $\mathfrak{p}B_P$ in $B_P$.*

REMARK 5.4. With the same hypotheses and notation as above, observe that the three conditions 2a, 2b, and 2c, can be replaced by:

(a') $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} = B_P/PB_P$.
(b') The prime $PB_{\mathfrak{p}}$ is the integral closure of $\mathfrak{p}B_{\mathfrak{p}}$ in $B_{\mathfrak{p}}$.

In fact, this last condition implies that $B_{\mathfrak{p}} = B_P$.

COROLLARY 5.5. *With the same assumptions and notation as in Corollary 5.3, let $n = \dim_K B \otimes_S K$, and let*

$$\delta : Spec(B) \to Spec(S)$$

*be the finite morphism given by $S \subset B$. Denote by $F$ be the set of $n$-fold points of* Spec$(B)$. *Then:*

(1) *If $P \in F$ and $\delta(P) = \mathfrak{p}$, the dimensions and residue fields of the local rings $B_P$ and $S_{\mathfrak{p}}$ are the same. In addition $\mathfrak{p}B_P$ is a reduction of $PB_P$.*
(2) *$\delta$ defines a set theoretical bijection between points of $F$ and points of the image, say $\delta(F)$ in* Spec$(S)$. *So given $P \in F$, $P$ is the only prime ideal in $B$ dominating $S$ at $\delta(P) = \mathfrak{p}$.*

PROOF. The statements in (1) and (2) follow easily from Theorem 3.1. □

This last Corollary says, in particular, that the finite morphism $\delta : \mathrm{Spec}(B) \to \mathrm{Spec}(S)$ establishes a set-theoretical bijection between $F$ (the set of $n$-fold points) and $\delta(F)$. As finite morphisms are proper morphism this shows that $F$ will be closed in $\mathrm{Spec}(B)$ if and only if $\delta(F)$ is closed in $\mathrm{Spec}(B)$. Moreover, in such case $F$ and $\delta(F)$ will be homeomorphic.

In the rest of Part 1 we study the set $\delta(F)$ in $\mathrm{Spec}(S)$, when $\delta : Spec(B) \to Spec(S)$ arises from a morphism of algebraic varieties over a field. In this context we will show that $\delta(F)$ is closed in Spec$(S)$, and we shall provide a suitable description of this closed set.

## Multiplicity and local presentations via elimination

In this part the objective is to study the stratification defined on a variety $X$ when considering the subsets of equimultiple points. This will be done under an extra assumption, namely that there be a finite morphism $X \to V$, from $X$ to some regular variety $V$. Later we will show how such finite morphisms can be constructed, at least locally, in étale topology. This last assertion will be discussed starting in 31.

### 6. A first approach to the case of hypersurfaces

Assume that $X$ is a hypersurface embedded in some smooth $d$-dimensional scheme $W$ of finite type over a perfect field $k$. Let $n$ denote the highest multiplicity at points of $X$. Fix a point $\xi \in W$, where the ideal $I(X)$ has order $n$ (thus the multiplicity of $X$ at $\xi$ is $n$).

Claim 6.1. (Simplified form) Assume that at a suitable (étale) neighborhood of $\xi$ there is a smooth morphism to some $(d-1)$-dimensional smooth scheme $W \to V$, so that:

(1) $W$ is locally of the form $V \times \mathbb{A}^1_k$, and $W \to V$ is the natural projection;
(2) Let $\mathbb{A}^1 = \mathrm{Spec}(k[Z])$. Then $I(X)$ is spanned by a monic polynomial of the form

$$f(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_1 Z + a_0 \in \mathcal{O}_V[Z].$$

The Claim 6.1 is a simplified form of what we need. The precise formulation and the proof will be addressed in 36.1. At any rate, this justifies the interest in studying the points of multiplicity $n$ of a hypersurface defined by a monic polynomial of degree $n$ (same integer $n$). Thus, in this section we take as starting point a finitely generated smooth $k$-algebra, $S$, and a monic polynomial $f(Z) \in S[Z]$. We will focus on the closed set, say $F_n$, of $n$-fold points of the hypersurface $\{f(Z) = 0\}$ in Spec$(S[Z])$.

Let $B = S[Z]/\langle f(Z)\rangle$, consider the natural morphism

$$\beta : \mathrm{Spec}(S[Z]) \to \mathrm{Spec}(S),$$

and let

$$\delta : \mathrm{Spec}(B) \to \mathrm{Spec}(S)$$

be the restriction of the former. So here $X = \mathrm{Spec}(B)$ is a hypersurface in the smooth scheme Spec$(S[Z])$ and one readily checks that the maximal multiplicity is at most $n$. So in the conditions of 6.1 we observe that $F_n(X)$ is a closed subset in $X = \mathrm{Spec}(B)$, and we consider

$$\begin{array}{c} F_n \subset X = \mathrm{Spec}(B) \\ \Big\downarrow{\scriptstyle \delta} \\ \delta(F_n) \subset V = \mathrm{Spec}(S) \end{array} \tag{6.1}$$

As $\delta : \mathrm{Spec}(B) \to \mathrm{Spec}(S)$ is finite, $\delta(F_n)$, is closed in Spec$(S)$.

Since $S$ is a regular ring, for each prime $\mathfrak{p} \subset S$, $S_{\mathfrak{p}}$ is a regular local ring, and

$$\nu_{\mathfrak{p}} : S \to \mathbb{Z}$$

will denote the order at $S_{\mathfrak{p}}$. Here $S$ is a smooth algebra over a field, in particular it has the following property: Given $h \in S$ and a positive integer $m$, the set

$$\{\mathfrak{p} \text{ such that } \nu_{\mathfrak{p}}(h) \geq m\}$$

is a closed subset in Spec(S). We aim to find a natural description of $\delta(F_n)$ in terms of closed sets defined as above (see Theorem 7.2 and Proposition 8.4). In fact we will come to a direct proof of the fact that $\delta(F_n)$ is closed from the natural description that we introduce below.

**6.2. Finite transversality and stability under blow ups.** Fix the notation as above, where $B$ is defined by a monic polynomial of degree $n$. Let $K$ the quotient field of the domain $S$. Then $B$ is a free $S$-module of rank $n$, and therefore $\dim_K(B \otimes_S K) = n$. This shows that $\delta : \mathrm{Spec}(B) \to \mathrm{Spec}(S)$ is finite-transversal at the $n$-fold points, $F_n$, in $\mathrm{Spec}(B)$ (see Definition 5.2).

Under these conditions the Multiplicity Formula of Theorem 3.1 shows that $F_n$ is mapped bijectively into its image (see Corollary 5.5). In other words, $\delta(F_n)$ gives a faithful image of $F_n$. In the conditions of 6.1 $F_n$ is closed and homeomorphic to $\delta(F_n)$. We claim that this homeomorphism establishes a one to one correspondence between smooth closed sub-schemes.

We will prove our claim only in one direction, which is as much as will be needed along these notes: Let $Y \subset \delta(F_n)$ be a smooth irreducible closed subscheme, and let $\mathfrak{p}$ denote its generic point. So there is a unique prime, say $P$ in $B$, dominating $S$ at $\mathfrak{p}$. Now $S/\mathfrak{p} \to B/P$ is a finite extension, and both domains have the same quotient fields (Corollary 5.5, 1)). Since $S/\mathfrak{p}$ is regular, it is normal in its quotient field. Therefore $S/\mathfrak{p} = B/P$, and hence $P$ defines a regular center, say $Y'$ in $\mathrm{Spec}(B)$, in natural correspondence with $Y$.

A particular feature of the blow up at regular centers included in the set of $n$-fold points of $X$, as is the case for $Y'$, is that there is a commutative diagram of blow ups and finite-transversal projections

$$\begin{array}{ccc} X & \longleftarrow & X_1 \supset F_n(X_1) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta'} \\ V & \longleftarrow & V_1 \supset \delta'(F_n(X_1)) \end{array} \tag{6.2}$$

where the upper horizontal map is the blow up at $Y'$, and the lower one is the blow up at $Y$. This is called the *stability of finite-transversality*, and it provides a natural notion of transformation of the data $F_n \subset X$, and $\delta : X \to \mathrm{Spec}(S) = V$ under blow ups at smooth centers contained in $F_n$ (see Proposition 7.5 and the discussion that follows Proposition 8.4).

**6.3. Searching for local presentations.** In the case in which $k$ is a field of characteristic zero, we will show that a local presentation of $\delta(F_n)$ can constructed at $V$. Local presentations will appear first in an apparently different manner then that in the introduction:

- We will find $\{g_1, \ldots . g_s\}$ in $S$, and integers, $\{m_1, \ldots . m_s\}$, so that:

  $\delta(F_n) = \{x \in V/\nu_x(g_1) \geq m_1\} \cap \cdots \cap \{x \in V/\nu_x(g_s) \geq m_s\} \subset V.$

- Moreover, this local presentation is *naturally attached* to the $n$-fold points of $X$. Namely, if $Y' \subset F_n(X)$ is a smooth closed subscheme (if $Y \subset \delta(F_n(X))$ is smooth), then there is natural transform of this local presentation given by the blow up of $V$ at $\delta(Y)$,

$$
\begin{array}{ccc}
X & \longleftarrow & X_1 \\
\delta \downarrow & & \downarrow \delta' \\
V & \longleftarrow & V_1 \\
\cap\{x \in V/\nu_x(g_i) \geq m_i\} & & \cap\{x \in V_1/\nu_x(g_i^{(1)}) \geq m_i\}
\end{array}
\tag{6.3}
$$

and the property is that for *any choice of* $Y \subset \delta(F_n(X))$,

$$\delta'(F_n(X_1)) = \cap\{x \in V_1/\nu_x(g_i^{(1)}) \geq m_i\}(\subset V_1) \text{ for the diagram in (6.2).}$$

Let us recall the definition of the terms involved in this last formula. Let $\mathfrak{p}$ denote the generic point of $Y$, and note that $\nu_{\mathfrak{p}}(g_i) \geq m_i$, for $1 \leq i \leq s$. So if $H \subset V_1$ denotes the exceptional hypersurface introduced by the blow up, then $g_i \mathcal{O}_{V_1} = g_i^{(1)} I(H)^{m_1}$, for $1 \leq i \leq s$.

The construction of such a presentation will be done in steps, and it will make use of a *form of elimination*. We shall begin by discussing the case in which $X$ is a hypersurface (see Section 7), and later we address the general case (see Section 8).

**6.4. Example: when $n = 2$.** Suppose $n = 2$, and let $f(Z) = Z^2 + a_1 Z + a_2$ with $a_1, a_2 \in S$, and $S$ is regular. In this case the discriminant, $a_1^2 - 4a_2 \in S$ describes the image under $\delta$ of the ramified locus in $\mathrm{Spec}(S)$. Notice that $a_1^2 - 4a_2 \in S$ is a weighted homogeneous polynomial of degree 2 if we assign weight one to $a_1$ and weight two to $a_2$. It is not hard to check that if the characteristic of $S$ is not 2, then the closed subset of $\mathrm{Spec}(S)$ where the discriminant has order at least two, is the image of the two fold points via $\delta : \mathrm{Spec}(S[Z]/\langle f(Z)\rangle) \to \mathrm{Spec}(S)$. Namely,

$$\delta(F_2(X)) = \{\mathfrak{p} \in V, \nu_{\mathfrak{p}}(a_1^2 - 4a_2) \geq 2\}.$$

This will be our local presentation for the case $n = 2$.

**6.5. How do we start searching?** Let $K$ be a field, and consider a monic polynomial $f(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_{n-1} Z + a_n \in K[Z]$. If $K_1$ is a decomposition field of $f(X)$, then

$$f(Z) = (Z - \theta_1) \cdots (Z - \theta_n) \in K_1[Z],$$

and the coefficients $a_i \in K$ can be expressed in terms of the elements $\{\theta_1, \ldots, \theta_n\}$ in $K_1$. In fact each coefficient $a_i$ is obtained from a symmetric function in $n$ variables, evaluated in $(\theta_1, \ldots, \theta_n)$. So one can set

$$\mathbb{Z}[a_1, \ldots, a_n] = \mathbb{Z}[s_{n,1}(\theta_1, \ldots, \theta_n), \ldots, s_{n,n}(\theta_1, \ldots, \theta_n)], \tag{6.4}$$

where the $s_{n,i}$ are the symmetric functions in $n$ variables. Recall that if $S_n$ denotes the permutation group of $n$, it acts on the polynomial ring with $n$ variables and

$$\mathbb{Z}[X_1, \ldots, X_n]^{S_n} = \mathbb{Z}[s_{n,1}, \ldots, s_{n,n}].$$

Consider a change of variable of the form $Z_1 = Z - \lambda$, for $\lambda \in K$. Then $f(Z) = g(Z_1)$ in $K[Z] = K[Z_1]$, i.e.,

$$f(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_{n-1} Z + a_n = f_1(Z_1) = Z_1^n + b_1 Z_1^{n-1} + \ldots + b_{n-1} Z_1 + b_n.$$

In addition, $K_1[Z] = K_1[Z_1]$, and

$$f(Z) = (Z - \theta_1) \cdots (Z - \theta_n) = f_1(Z_1) = (Z_1 - \beta_1) \cdots (Z_1 - \beta_n)$$

where $\beta_i = \theta_i - \lambda$ for $i = 1, \dots, n$. Each coefficient $b_i$ of $f_1(Z_1)$ is obtained by the evaluation of the symmetric function $s_{n,i}$ in $(\beta_1, \dots, \beta_n)$.

Our goal is to obtain polynomial expressions on the coefficients, say $G(s_1, \dots, s_n) \in \mathbb{Z}[s_1, \dots, s_n]$, so that

$$G(a_1, \dots, a_n) = G(b_1, \dots, b_n) \tag{6.5}$$

every time when $f_1(Z_1)$ is $f(Z)$ expressed in a variable of the form $Z_1 = Z - \lambda$, for some choice of $\lambda \in K$. The reason for restricting attention to this particular type of change of variables will be justified in the following section.

A first observation is that for any such change of variable we get

$$\theta_i - \theta_j = \beta_i - \beta_j, \ 1 \leq i, j \leq n.$$

So the differences of roots are invariants by this kind of change. Note that

$$\mathbb{Z}[\theta_i - \theta_j]_{1 \leq i,j \leq n} \subset \mathbb{Z}[\theta_1, \dots, \theta_n].$$

In particular, if we could define in some natural way a subring, say $C \subset \mathbb{Z}[\theta_i - \theta_j]_{1 \leq i,j \leq n}$, so that

$$C \subset \mathbb{Z}[a_1, \dots, a_n], \tag{6.6}$$

then any element in $C$ would provide a polynomial expression in the coefficients, say $G(a_1, \dots, a_n)$, so that $G(a_1, \dots, a_n) = G(b_1, \dots, b_n)$ if $f_1(Z_1) = f(Z)$ is obtained by a change of variables as above.

## 7. The case of hypersurfaces: the universal setting

Assume, as in the previous section, that $S$ is a smooth domain over a perfect field $k$, with quotient field $K$, and let $f(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_1 Z + a_0 \in S[Z]$ be a monic polynomial. Consider $B = S[Z]/\langle f(Z) \rangle$ together with the finite morphism $\delta : \mathrm{Spec}(B) \to \mathrm{Spec}(S)$ defined by $S \subset B$.

We shall follow the arguments exposed in Section 6, to which we will add some additional conditions. In fact, we will be looking for weighted equations on $S$ that describe the image by $\delta$ of the closed set of $n$-fold points of $B$ (see 6.3).

It is natural to expect that there be significant information concerning this matter encoded in the coefficients of $f(Z)$. The discriminant is a first example of this fact (see Example 6.4).

Our search we will start with the *universal case.* Let $k$ be a field as before, and consider the polynomial ring in $n$ variables $k[Y_1, \dots, Y_n]$. The *universal polynomial* of degree $n$, is

$$F_n(Z) = (Z - Y_1) \cdots (Z - Y_n) = Z^n - s_{n,1} Z^{n-1} + \ldots + (-1)^n s_{n,n} \in k[Y_1, \dots, Y_n, Z],$$

where for $i = 1, \dots, n$, $s_{n,i} \in k[Y_1, \dots, Y_n]$ denotes the $i$-th symmetric polynomial in $n$ variables.

The diagram

$$\begin{array}{ccc} \mathrm{Spec}\,(k[s_{n,1}, \dots, s_{n,n}][Z]/\langle F_n(Z) \rangle) & \longrightarrow & \mathrm{Spec}(k[s_{n,1}, \dots, s_{n,n}][Z]) \\ & \searrow & \downarrow \\ & & \mathrm{Spec}(k[s_{n,1}, \dots, s_{n,n}]) \end{array} \tag{7.1}$$

illustrates the universal situation. To clarify this latter property note that $\delta$ : Spec$(B) \to$ Spec$(S)$ is obtained from the specialization given by

$$\begin{aligned} \Theta : k[s_{n,1}, \ldots, s_{n,n}] &\longrightarrow S \\ (-1)^i s_{n,i} &\longmapsto a_i, \end{aligned} \tag{7.2}$$

an homomorphism of $k$-algebras. In other words, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(k[s_{n,1}, \ldots, s_{n,n}][Z]/\langle F_n(Z)\rangle) & \longleftarrow & \mathrm{Spec}(B) \\ \downarrow{\scriptstyle \overline{\alpha}} & & \downarrow{\scriptstyle \delta} \\ \mathrm{Spec}(k[s_{n,1}, \ldots, s_{n,n}]) & \longleftarrow & \mathrm{Spec}(S) \end{array} \tag{7.3}$$

which, in addition, is a fiber product.

**First observation.** Our search for weighted equations on the coefficients of $F_n$ has led us to work on the ring $k[s_{n,1}, \ldots, s_{n,n}]$. Note that if $S_n$ denotes the group of permutation of $n$ elements, then

$$k[Y_1, \ldots, Y_n]^{S_n} = k[s_{n,1}, \ldots, s_{n,n}] \tag{7.4}$$

which is again a polynomial ring.

**Second observation.** Changes on the variable $Z$, as the ones considered in 6.5, should not change the image of the set of $n$-fold points of the hypersurface $\{F_n = 0\}$ by $\overline{\alpha}$. Therefore, we will search for functions on the subring $k[Y_i - Y_j]_{1\leq i,j\leq n} \subset k[Y_1, \ldots, Y_n]$. Notice that the permutation group $S_n$ also acts on this subring. So there is an inclusion

$$k[Y_i - Y_j]^{S_n}_{1\leq i,j\leq n} \subset k[Y_1, \ldots, Y_n]^{S_n} = k[s_{n,1}, \ldots, s_{n,n}]. \tag{7.5}$$

As $S_n$ is a finite group, the algebra in the left is finitely generated. So set

$$k[G_{m_1}, \ldots, G_{m_r}] := k[Y_i - Y_j]^{S_n}_{1\leq i,j\leq n}. \tag{7.6}$$

In addition, and since $S_n$ acts linearly in $k[Y_1, \ldots, Y_n]$ (hence preserving the degree of this graded ring), we can take each generator $G_{m_i}$ as a homogeneous polynomial in $k[Y_1, \ldots, Y_n]$. Let

$$m_i = \mathrm{degree}\, G_{m_i} \tag{7.7}$$

where $k[Y_1, \ldots, Y_n]$ is graded in the usual way. The inclusion (7.5) yields an expression, say

$$G_{m_i} = G_{m_i}(s_{n,1}, \ldots, s_{n,n}) \tag{7.8}$$

where $G_{m_i}(s_{n,1}, \ldots, s_{n,n})$ is a weighted homogeneous of degree $m_i$ in $k[s_{n,1}, \ldots, s_{n,n}] \subset k[Y_1, \ldots, Y_n]$.

**Conclusion.** The morphism $\Theta : k[s_{n,1}, \ldots, s_{n,n}] \to S$ maps $G_{m_i}(s_{n,1}, \ldots, s_{n,n})$ to the element $G_{m_i}(a_1, \ldots, a_n) \in S$. Let $\lambda \in S$ and set $S[Z] = S[Z_1]$ where $Z_1 = Z - \lambda$, and let

$$f(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_{n-1} Z + a_n = g(Z_1) = Z_1^n + b_1 Z_1^{n-1} + \ldots + b_{n-1} Z_1 + b_n.$$

If $K$ denotes the quotient field of the domain $S$, and if $K_1$ is decomposition field of $f(Z) \in S[Z] \subset K[Z]$, then

$$G_{m_i}(a_1, \ldots, a_n) = G_{m_i}(b_1, \ldots, b_n) \in S, \text{ (see (6.5))}. \tag{7.9}$$

Remark 7.1. Let $k[G_1, \dots, G_s]$ be a graded ring generated by homogeneous elements $G_i$, and let $m_i = \deg(G_i)$ for $i = 1, \dots, s$. Let $(R, m, k)$ be a local regular $k$-algebra, and let $\Theta : k[G_1, \dots, G_s] \to R$ be a homomorphism of $k$-algebras. If $\Theta(G_i)$ has order $\geq m_i$, $i = 1, \dots, s$, then one checks that for any homogeneous element $G \in k[G_1, \dots, G_s]$ of degree $d$, the image $\Theta(G)$ has order $\geq d$ at $R$.

Theorem 7.2. [**Vi4**, Theorem 1.16] **Local presentation via elimination.** *Let $k$ be either a field of characteristic zero, or a field of positive characteristic $p$ coprime with $n$. Let $S$ be a smooth $k$-algebra, let $f(Z) = Z^n + a_1 Z^{n-1} + \dots + a_{n-1}Z + a_n \in S[Z]$, and let $B = S[Z]/\langle f(Z)\rangle$. This defines a diagram of morphisms*

$$\begin{array}{ccc} \operatorname{Spec}(B) & \hookrightarrow & \operatorname{Spec}(S[Z]) \\ & \searrow^{\delta} & \downarrow \beta \\ & & \operatorname{Spec}(S). \end{array} \tag{7.10}$$

*Denote by $F_n$ the set of $n$-fold points of $\{f(Z) = 0\} \subset \operatorname{Spec}(S[Z])$. Consider the morphism defined by specialization*

$$\begin{aligned} \Theta : k[s_{n,1}, \dots, s_{n,n}] &\longrightarrow S \\ s_{n,i} &\longmapsto (-1)^i a_i. \end{aligned}$$

*Then:*

$$\delta(F_n) = \bigcap_{1 \leq j \leq r} \{x \in \operatorname{Spec}(S) : \nu_x(G_{m_j}(a_1, \dots, a_n)) \geq m_j\}, \tag{7.11}$$

*for $G_{m_j}$ as in (7.8) and $m_j$ as in (7.7).*

Proof. First note that if the characteristic of $k$ is zero (or if the characteristic does not divide $n$), then

$$k[Y_1, \dots, Y_n] = (k[Y_i - Y_j]_{1 \leq i,j \leq n})[s_{n,1}],$$

where $s_{n,1} = Y_1 + Y_2 + \dots + Y_n$. Since $s_{n,1}$ is an invariant by the action of the group $S_n$, we conclude that

$$k[Y_1, \dots, Y_n]^{S_n} = (k[Y_i - Y_j]^{S_n}_{1 \leq i,j \leq n})[s_{n,1}],$$

or, in other words,

$$k[s_{n,1}, \dots, s_{n,n}] = k[G_{m_1}, \dots, G_{m_r}][s_{n,1}].$$

This gives an expression of $k[Y_1, \dots, Y_n]^{S_n} = k[s_{n,1}, \dots, s_{n,n}]$ by two different collection of homogeneous generators. Therefore each $G_{m_1}$ is a weighted homogeneous polynomial in $s_{n,1}, \dots, s_{n,n}$, and conversely, each $s_{n,i}$ is a weighted homogeneous in $G_{m_1}, \dots, G_{m_r}, s_{n,1}$.

Now let $f(Z) = Z^n + a_1 Z^{n-1} + \dots + a_{n-1}Z + a_n \in S[Z]$ be as in the Theorem and assume that for $j = 1, \dots, r$, $\nu_\xi(G_{m_j}(a_1, \dots, a_n)) \geq m_j$ at a point $\xi \in \operatorname{Spec}(S)$. We claim that $\xi \in \delta(F_n)$. Over fields of characteristic zero one can always choose $\lambda \in S$ so that setting $Z_1 = Z - \lambda$

$$f(Z) = Z^n + a_1 Z^{n-1} + \dots + a_{n-1}Z + a_n = g(Z_1) = Z_1^n + b_1 Z_1^{n-1} + \dots + b_{n-1}Z_1 + b_n,$$

with $\nu_\xi(b_1) \geq 1$. If we could prove that

$$\nu_\xi(b_i) \geq i, \ i = 1, \dots, n, \tag{7.12}$$

then it would follow that $\xi \in \delta(F_n)$. Notice that (7.9) ensures that, for $j = 1, \dots, r$, $\nu_\xi(G_{m_j}(b_1, \dots, b_n)) \geq m_j$, and that (7.12) follow from Remark 7.1.

Suppose now that $\xi \in \delta(F_n)$. We want to show that then $\nu_\xi(G_{m_j}(a_1, \dots, a_n)) \geq m_j$ for $j = 1, \dots, r$.

Zariski's formula for multiplicities ensures that there is a unique point, say $\beta$, in the fiber over $\xi$, and that the local rings, $B_\beta$ and $S_\xi$, have the same residue fields (see Corollary 5.5). In particular the class of $Z$ in the residue field of $B_\beta$ is also the class of some element $\lambda \in S$, at the residue field of $S_\xi$. Set $Z_1 = Z - \lambda$, and

$$f(Z) = g(Z_1) = Z_1^n + b_1 Z_1^{n-1} + \dots + b_{n-1} Z_1 + b_n.$$

Let $m_\xi$ denote the maximal ideal in $S_\xi$, and let $k'$ denotes the residue field $S_\xi/m_\xi$. The uniqueness and rationality of the point $\beta$ in the fiber shows that class of $g(Z_1)$ in $k'[Z_1]$ is $Z_1^n$. Therefore $g(Z_1) \in (m_\xi + \langle Z_1 \rangle)^n$ (use here that $\beta$ is an $n$-fold point of $B$), and this occurs if and only if $b_i \in m_\xi^i$. Remark 7.1 together with the equalities in (7.9) show that $\nu_\xi(G_{m_j}(a_1, \dots, a_n)) \geq m_j$ , $j = 1, \dots, r$. □

Remark 7.3. Theorem 7.2 does not hold in positive characterisitc as the following example shows. Suppose $k$ is a field of characteristic two, and let $f(Z) = Z^2 + X^2 Z + Y^3 \in k[X,Y][Z]$. Then it can be checked that $k[G_{m_1}, \dots, G_{m_r}] = k[s_{2,1}^2]$, where $s_{2,1} = X^2$, and thus the conclusion does not hold in this case. The form of elimination to be considered in positive characteristic, although weaker than in characteristic zero, will be discussed in 16.5.

**7.4. A few more changes of variables.** We keep the same notation as in the previous sections, so $S$ is a regular domain, $f(Z) = Z^n + a_1 Z^{n-1} + \dots + a_{n-1} Z + a_n \in S[Z]$ is a monic polynomial, and $B = S[Z]/\langle f(Z) \rangle$. We were interested on the image of $F_n$, the set of $n$-fold points of $B$, by the finite morphism $\delta : \mathrm{Spec}(B) \to \mathrm{Spec}(S)$. Theorem 7.2 asserts that $\delta(F_n)$ can be described using suitable weighted equations on the coefficients of $f(Z)$. The functions involved in these equations,

$$\{G_{m_i} = G_{m_i}(s_{n,1}, \dots, s_{n,n}); i = 1, \dots, r\}$$

(see (7.8)), where obtained as functions on the coefficients of $f(z)$ that are invariant under changes of variables of the form $Z_1 = Z - \lambda$ for $\lambda \in S$. However, there are other changes of variables that we have not taken into account:

(1) Let $u \in S$ be a unit. Then $S[Z] = S[Z_1]$ with $Z_1 = uZ$.
(2) Let $v \in S \setminus 0$. Then set $a_i = v^i a'_i$ for $i = 1, \dots, n$.

The replacement in (1) is also a change of variable. The replacement in (2) is very natural when taking blow ups at smooth centers included in the set $F_n$ of $n$-fold points, as indicated in the diagram (6.3).

**Case 1:** In this case $Z = \frac{Z_1}{u}$, and

$$\begin{aligned} f(Z) &= Z^n + a_1 Z^{n-1} + \dots + a_{n-1} Z + a_n \\ &= \left(\frac{Z_1}{u}\right)^n + a_1 \left(\frac{Z_1}{u}\right)^{n-1} + \dots + a_{n-1}\left(\frac{Z_1}{u}\right) + a_n \in S[Z] \end{aligned}$$

which is not monic in $Z_1$, but the associated polynomial

$$u^n f(Z_1) = Z_1^n + u a_1 Z_1^{n-1} + \dots + u^{n-1} a_{n-1} Z + u^n a_n$$

is monic. The weighted homogeneous expression of $G_{m_i} = G_{m_i}(s_{n,1}, \ldots, s_{n,n})$ ensures that

$$G_{m_i}(ua_1, \ldots, u^n a_n) = u^{m_i} G_{m_i}(a_1, \ldots, a_n).$$

Note that, in particular, the ideal spanned by $G_{m_i}(a_1, \ldots, a_n)$ in $S$ is intrinsic to $\{f(Z) = 0\}$ and independent of any change of variable in $S[Z]$.

**Case 2:** In this case set formally $Z_1 = \frac{Z}{v}$. Note that this defines a change of variables in $K[Z]$ where $K$ denotes the the fraction field of $S$. Thus,

$$\left(\frac{1}{v}\right)^n f(Z) = Z_1^n + a_1' Z_1^{n-1} + \ldots + a_{n-1}' Z_1 + a_n'$$

The same argument used for case 1, shows that

$$G_{m_i}(a_1', \ldots, a_n') = \left(\frac{1}{v}\right)^{m_i} G_{m_i}(a_1, \ldots, a_n).$$

**Local presentations for a hypersurface.**

PROPOSITION 7.5. *Let the hypotheses and notation be as in Theorem* 7.2. *Then the description in* (7.11) *is a local presentation for* $F_n$. *In other words, let* $Y' \subset \delta(F_n)$ *be a regular center. Then:*

(1) *There is a unique regular center* $Y \subset F_n$ *so that* $\delta(Y) = Y' \subset \delta(F_n)$*;*
(2) *Consider the blow ups at* $Y$*, respectively at* $\delta(Y)$*,*

$$\mathrm{Spec}(B) \longleftarrow X_1; \qquad \mathrm{Spec}(S) \longleftarrow V_1,$$

*and let*

$$f(Z^{(1)}) = (Z^{(1)})^n + a_1^{(1)}(Z^{(1)})^{n-1} + \ldots + a_n^{(1)}$$

*be a strict transform of* $f(Z)$ *in some open set* $U \subset V_1[Z^{(1)}]$. *Then there is a commutative diagram of blow ups and finite projections:*

(7.13)
$$\begin{array}{ccc} \mathrm{Spec}(B) & \longleftarrow & X_1 \\ \downarrow \delta & & \downarrow \delta' \\ \mathrm{Spec}(S) & \longleftarrow & V_1 \end{array}$$

(3) *Let* $G^{(1)}_{m_j}(a_1^{(1)}, \ldots, a_n^{(1)})$ *be an* $m_j$*-weighted transform of* $G_{m_j}(a_1, \ldots, a_n)$ *for* $j = 1, \ldots, r$ *(see* 6.3*). Then*

$$\delta'(F_n(X_1) = \cap\{\eta \in V_1/\nu_\eta((G^{(1)}_{m_j}(a_1^{(1)}, \ldots, a_n^{(1)}))) \geq m_j\} \subset V_1.$$

*In other words, for any choice of* $Y$ *as above, the weighted transform of the local description of* $\delta(F_n)$ *given in* (7.11) *is a local description of* $\delta'(F_n(X_1))$*,*

(7.14)
$$\begin{array}{ccc} V & \longleftarrow & V_1 \\ \cap\{\xi \in V/\nu_\xi(G_{m_j}(a_1, \ldots, a_n)) \geq m_j\} & & \cap\{\eta \in V_1/\nu_\eta((G^{(1)}_{m_j}(a_1^{(1)}, \ldots, a_n^{(1)}))) \geq m_j\} \end{array}$$

Proof. The assertion in (1) was proved in 6.2. It says that $Y$ and $\delta(Y)$ are isomorphic. For the proof of part (2), let $\operatorname{Spec}(B) \leftarrow T$ denote the blow up at $Y$, and let $\operatorname{Spec}(S) \leftarrow R$ denote the blow up at the regular center $\delta(Y)$.

There is a natural commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(B) & \longleftarrow & T \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta'} \\ \operatorname{Spec}(S) & \longleftarrow & R \end{array} \tag{7.15}$$

where $\delta' : T \to R$ is finite (see [**Vi4**, Theorem 1.16], [**Vi7**, Theorem 6.8]). In addition, there is a suitable affine cover of $R$ and $Z$ so that the restriction of $\delta'$ is a finite map of the form $\operatorname{Spec}(S'[Z]/\langle g(Z)\rangle) \to \operatorname{Spec}(S')$, where $g(Z_1) \in S'[Z_1]$ is a monic polynomial of degree $n$, and a *strict transform* of $f(Z) \in S[Z]$.

With regard to (3), we argue as in 6.2. Let $Y_1 \subset \delta(F_n(X)) \subset \operatorname{Spec}(S)$ be a regular closed subscheme, with generic point $\mathfrak{p}$. Then there is a unique prime say $P \subset B$ dominating $S$ at $\mathfrak{p}$. Let $Y \subset F_n(X)$ be the irreducible subscheme with generic point $P$. The discussion in 6.2 shows that $B/P = S/\mathfrak{p}$.

As a consequence, the class of $Z$ in $B/P$ is the class of an element $s \in S$ in the quotient ring $S/\mathfrak{p}$. This ensures that, after a change of variables of the form $Z - s$, we may assume that $P$ contains $Z$ and dominates $S$ at $\mathfrak{p}$, or, in other words, that $P$ can be identified with $\langle Z, \mathfrak{p}\rangle$. This implies that $\nu_{\mathfrak{p}}(a_i) \geq i$ for $i = 1, \dots, n$. Moreover, since ordinary powers and symbolic powers coincide in a regular ring (a fact that can be checked at the completion of a regular local ring), one has that $a_i \in \mathfrak{p}^i$ for $i = 1, \dots, n$. Now the assertion in (3) follows by taking the strict transform of this polynomial, and the discussion in Case 2 of 7.4. □

## 8. The general case: Reduced schemes

**8.1. Finite extensions of a regular ring.** In the previous discussion we have fixed a finite extension, say $S \subset B$, where $B$ was defined by a monic polynomial of degree $n$ in $S[Z]$, and we have studied the points of multiplicity $n$ of this ring. We now mention some results which hold in a more setting, where $B$ is a finite extension of a regular ring $S$.

Let $S$ be a regular domain with quotient field $K$. We will draw attention to finite extensions of $S$, say $S \subset B$, so that the map $B \to B \otimes_S K$ is injective.

(1) One example arises when considering a monic polynomial $f(Z) = Z^n + a_1 Z^{n-1} + \cdots + a_n \in S[Z]$. Then the ring $B = S[Z]/\langle f(Z)\rangle$ is a finite and free $S$-module. Hence the map $B \to B \otimes_S K$ is injective.
(2) Suppose that $S \subset B$ is finite, and that $B$ is a reduced and pure dimensional $S$-algebra. Let $\{\mathfrak{q}_1, \dots, \mathfrak{q}_s\}$ be the minimal prime ideals in $B$. Since $B$ is reduced, $B \to \oplus B/\mathfrak{q}_i$ is injective. As $B$ is equidimensional we also conclude that $\mathfrak{q}_i \cap S = 0$, thus $S \subset B/\mathfrak{q}_i$, for $i = 1, \dots, s$.

Each $B/\mathfrak{q}_i$ is a domain and a finite extension of $S$, hence $B/\mathfrak{q}_i \subset L_i = (B/\mathfrak{q}_i) \otimes_S K$ is the inclusion of $B/\mathfrak{q}_i$ in the quotient field $L_i$ , and $L_i$ is a finite field extension of $K$. Thus also in this case $B \to B \otimes_S K$ is injective, and the artinian ring $B \otimes_S K$ is a direct sum of fields. Moreover

$$B \otimes_S K = B_{\mathfrak{q}_1} \oplus \cdots \oplus B_{\mathfrak{q}_s} = L_1 \oplus \cdots \oplus L_s.$$

We now assign a monic polynomial in $S[X]$ to each element $\theta \in B$. Consider the subring $S[\theta] \subset B$. Note that $S[\theta] \subset B \otimes_S K$, and

$$S[\theta] \otimes_S K = K[\theta] = K[X]/\langle f_\theta(X)\rangle (\subset B \otimes_S K)$$

for some monic polynomial $f_\theta(X) \in K[X]$, which we call the minimal polynomial of $\theta$ (over $K$).

CLAIM 8.2. The minimal polynomial of an element $\theta \in B$ has coefficients in $S$ (i.e., $f_\theta(X) \in S[X]$).

PROOF. Let $f_\theta(X) = (g_1)^{r_1} \cdots (g_m)^{r_m}$, where each $g_i = g_i(X)$ is an irreducible monic polynomial in $K[X]$. So $K[\theta]$ has $m$ maximal ideals, say $\mathcal{M}_1, \dots, \mathcal{M}_m$, one for each irreducible factor $g_i$. As $K[\theta] \subset B \otimes_S K$ is a finite extension, for each index $i$, $1 \leq i \leq m$, there is a prime, say $q_i$ in $B \otimes_S K$, dominating $K[\theta]$ at $\mathcal{M}_i$. Note finally that $g_i(X)$ is the minimal polynomial of say $\overline{\theta}$, the class of $\theta$ in $B \otimes_S K/q_i = L_i$.

In our setting the prime ideals in $B \otimes_S K$ can be identified with the minimal primes of $B$. Note that the class $\overline{\theta} \in B/\mathfrak{q}_i \subset L_i$, is integral over the subring $S$. Since $S$ is normal, the irreducible polynomial $g_i$ is in $S[X]$, and this proves claim. □

**8.3. Local presentations via elimination.** Let $B$ be an excellent equidimensional reduced scheme, finite over a regular domain $S$. Say $B = S[\theta_1, \dots, \theta_r]$. Let $n = \dim_K(B \otimes_S K)$ so that points in $\mathrm{Spec}(B)$ have at most multiplicity $n$ (see Corollary 5.3). Let

$$\delta : X = \mathrm{Spec}(B) \to V = \mathrm{Spec}(S) \tag{8.1}$$

be the associated finite morphism, and let $F_n(X)$ be the set of $n$-fold points in $X$. Let

$$f_i(Z) = Z^{d_i} + a_1^{(i)} Z^{n_i - 1} + \ldots + a_{n_i-1}^{(i)} Z + a_{d_i}^{(i)} \in S[Z] \tag{8.2}$$

be the minimal polynomial of $\theta_i$ for $i = 1, \dots, r$. For each index $i$, consider the elements $\{g_{(i,1)}, g_{(i,2)}, \dots, g_{(i,N_i)}\}$ in $S$, together with the positive integers, say $\{m_{(i,1)}, m_{(i,2)}, \dots, m_{(i,N_i)}\}$, so that

$$\{x \in Spec(S), \nu_x(g_{(i,j)}) \geq m_{(i,j)},\ j = 1, \dots, N_i\}$$

is the local presentation in $\mathrm{Spec}(S)$ obtained from $f_i(Z) \in S[Z]$.

PROPOSITION 8.4. **Local presentation via elimination.** *Fix $B = S[\theta_1, \dots, \theta_M]$ as before. Assume that $B$ contains a field $k$ of characteristic zero, and let $f_i(Z)$ in (8.2) be the minimal polynomial of $\theta_i$, $1 \leq i \leq M$.*

*Let $\delta : X = Spec(B) \to V = Spec(S)$ be the finite morphism defined by the inclusion $S \subset B$, and let $F_n(X)$ be the set of $n$-fold points of* $\mathrm{Spec}(B)$. *Then*

$$\delta(F_n(X)) = \bigcap_{1 \leq i \leq r} \{x \in Spec(S), \nu_x(g_{(i,j)}) \geq m_{(i,j)},\ j = 1, \dots, N_i\}. \tag{8.3}$$

The previous proposition makes use of Proposition 8.7 stated below. It expresses $\delta(F_n(X))$ as an intersection of closed sets in $V = \mathrm{Spec}(S))$, and hence $F_n(X)$ is closed in $X$, and homeomorphic to $\delta(F_n(X))$ (see the remarks after Corollary 5.5). Moreover, this expression is naturally compatible with blow ups at smooth centers included in the closed set of $n$-fold points. Namely, given any sequence of blow ups with smooth centers $Y_i \subset F_n(X_i)$, one has:

$$
\begin{array}{ccccc}
X = \mathrm{Spec}(B) & \longleftarrow & \cdots & \longleftarrow & X_s \\
\downarrow \delta & & & & \downarrow \delta_n \\
V = \mathrm{Spec}(S) & \longleftarrow & \cdots & \longleftarrow & V_s
\end{array}
\tag{8.4}
$$

and for $m = 0, \dots s$, and in addition:

$$\delta_m(F_n(X_m)) = \bigcap \{x \in V_m, \nu_x(g^{(m)}_{(i,j)}) \geq m_{(i,j)},\ j = 1, \dots, N_i\} \subset V_m \tag{8.5}$$

where $g^{(m)}_{(i,j)}$ is the transform of $g_{(i,j)}$ (see Theorem 6.8 and Corollary 6.9 in [**Vi7**]).

This result does not hold in positive characteristic (see Remark 7.3).

**8.5. Local presentations via closed immersions.** We add here some indications to results related to the previous discussion, but now in positive characteristic too: when $B$ and $S$ are, in addition, algebras of finite type over a perfect field. In this case we will show how local presentations arise for schemes embedded in a smooth scheme. Let us first draw attention to algebras of the form $S[Z]/\langle f(Z)\rangle$, where $S$ is smooth over a field $k$, and $f(Z) \in S[Z]$ is a monic polynomial defining a hypersurface in the smooth scheme Spec$(S[Z])$. Given a prime ideal $\mathfrak{p}$ in $S$, $\nu_{\mathfrak{p}}$ will denote the order function defined on $S$ by the local regular ring $S_{\mathfrak{p}}$. The next result follows from the proof of Theorem 7.2, it is formulated explicitly here as it holds in positive characteristic, and plays a key role in the proof of Proposition 8.7.

PROPOSITION 8.6. *Fix $S$ as above, and an algebra $B = S[Z]/\langle f(Z)\rangle$, where*

$$f(Z) = Z^s + c_1 Z^{s-1} + \cdots + c_s \in S[Z].$$

*A prime ideal $\mathfrak{p}$ in $S$ is the image of an $s$-fold point if and only if there is an element $\lambda \in S_{\mathfrak{p}}$, so that setting $Z_1 = Z - \lambda$ and*

$$f(Z) = g(Z_1) = Z_1^s + c'_1 Z_1^{s-1} + \cdots + c'_s \in S_{\mathfrak{p}}[Z_1] (= S_{\mathfrak{p}}[Z]) \text{ then} \tag{8.6}$$

$$\nu_{\mathfrak{p}}(c'_j) \geq j,\ j = 1, \dots, s. \tag{8.7}$$

The proof follows from Corollary 5.5 (or see Proposition 5.4 in [**Vi7**]). This result has applications in the study of finite extensions of rings arising in algebraic geometry, with no condition on the characteristic of the underlying field. Fix $S \subset B$ and a presentation of $B$, say $\{\theta_1, \dots, \theta_N\}$ (i.e., $B = S[\theta_1, \dots, \theta_N]$). Let $f_1(Z), \dots, f_N(Z) \in K[Z]$ denote the minimal polynomials of $\{\theta_1, \dots, \theta_N\}$ corresponding to $S \subset B$. Let $d_i$ denote the degree of $f_i(Z)$. One can reorder $\{\theta_1, \dots, \theta_N\}$, and assume that there is an integer $M$, $1 \leq M \leq N$, so that $d_i \geq 2$ for $1 \leq i \leq M$, and $d_i = 1$ for $M+1 \leq i \leq N$. So $\theta_i \in S$ for $M+1 \leq i \leq N$, and

$$B = S[\theta_1, \dots, \theta_M] \text{ and } d_i = \deg(f_i(Z)) \geq 2\ , i = 1, \dots, M. \tag{8.8}$$

PROPOSITION 8.7. [**Vi7**, Proposition 5.7] *Fix $B = S[\theta_1, \dots, \theta_M]$ as above. Then, if $\dim_K(B \otimes_s K) = n$, a point $\mathfrak{p} \in Spec(S)$ is the image of an $n$-fold point of $Spec(B)$, if and only if $\mathfrak{p}$ is the image of a point of multiplicity $d_i(\geq 2)$ in $Spec(S[Z]/\langle f_i(Z)\rangle)$, for every index $i = 1, \dots, M$.*

As $S$ is smooth over $k$, the same holds for $S[Z_1, \dots, Z_M]$, obtained from $S$ by adding $M$ variables. Let $J$ denote the kernel of the surjective homomorphism of $S$-algebras $S[Z_1, \dots, Z_M] \to B$, mapping $Z_i$ to $\Theta_i$, for $i = 1, \dots, M$. Set $V = \mathrm{Spec}(S)$, $V' = \mathrm{Spec}(S[Z_1, \dots, Z_M])$, so that there is a closed immersion

(8.9) $X = \mathrm{Spec}(B) \subset V' = \mathrm{Spec}(S[Z_1, \dots, Z_M])$ inducing $\delta : X = \mathrm{Spec}(B) \to V$,

where the latter is a finite morphism. Recall that each $f_i(Z) \in S[Z]$ is a monic polynomial defined in terms of $\Theta_i \in B$, for $i = 1, \dots, M$. For each $1 \le i \le M$, set

$$f_i(Z_i) = Z_i^{d_i} + a_1^{(i)} Z_i^{d_i - 1} + \dots + a_{d_i}^{(i)} \in S[Z_i] \subset S[Z_1, \dots, Z_M],$$

and note that $\langle f_1(Z_1), \dots, f_M(Z_M) \rangle \subset J$. Set $H_i = V(f_i(Z_i)) \subset V'$, so that

$$X \subset H_1 \cap H_2 \cap \dots, \cap H_M \subset V'.$$

Proposition 8.8. **Presentations II.** *With the setting as in (8.8) and (8.9).*

(1) *A point $q \in X$ is a point of multiplicity $n$ if and only if each hypersurface $H_i$ has multiplicity $d_i$ at $q \in V'$, or say:*

$F_n(X) = F_{d_1}(H_1) \cap \dots \cap F_{d_M}(H_M)$, *and furthermore (8.3) holds for $\delta(F_n(X))$.*

(2) *If $X \leftarrow X^{(1)}$ and $V' \leftarrow V^{(1)}$ are the blow ups at a smooth center $Y \subset F_n(X)$ (included in the set of points of multiplicity $n$ in $X$), then*

$$F_n(X^{(1)}) = F_{d_1}(H_1^{(1)}) \cap \dots \cap F_{d_M}(H_M^{(1)})$$

*where $H_i^{(1)}$ denotes the strict transform of $H_i$, $i = 1, \dots, M$. Moreover, the same holds after applying any sequence of blow ups over $X$ at regular centers in the set of points of multiplicity $n$.*

Proof. The claim in (1) follows essentially from Propositions 8.6 and 8.7, whereas (2) follows from the natural compatibility of local presentations with blow ups at regular equimultiple centers ([**Vi7**], Theorem 6.8). □

Remark 8.9. The local presentation in $V' = \mathrm{Spec}(S[Z_1, \dots, Z_M])$ is given by $f_i(Z_i) = Z_i^{d_i} + a_1^{(i)} Z_i^{d_i - 1} + \dots + a_{d_i}^{(i)} \in S[Z_i] \subset S[Z_1, \dots, Z_M]$, and the $d_i$, $i = 1, \dots, M$. In fact

$$F_n(X) = \bigcap_{1 \le i \le r} \{x \in V', \nu_x(f_i) \ge d_i,\ i = 1, \dots, N_i\}. \tag{8.10}$$

If $(X \subset V') \longleftarrow (X^{(1)} \subset V^{(1)})$ is defined by blowing up at a regular center $Y \subset F_n(X)$, then $F_n(X^{(1)}) = \bigcap_{1 \le i \le r} \{x \in V', \nu_x(f_i^{(1)}) \ge d_i,\ i = 1, \dots, N_i\}$, where $f_i^{(1)}$ denotes a strict transform of $f_i$. Similarly for any sequence of blow-ups, say

$$(X \subset V') \longleftarrow (X^{(1)} \subset V^{(1)}) \dots \longleftarrow (X^{(s)} \subset V^{(s)}) \tag{8.11}$$

at centers included in the $n$-fold points. Namely

$$F_n(X^{(s)}) = \bigcap_{1 \le i \le r} \{x \in V', \nu_x(f_i^{(s)}) \ge d_i,\ i = 1, \dots, N_i\}, \tag{8.12}$$

In characteristic zero there is a link between Propositions 8.8 and 8.4: the sequence (8.11) induces a diagram as in (8.4), and

$$\delta_s(F_n(X^{(s)})) = \bigcap \{x \in V_s, \nu_x(g_{(i,j)}^{(s)}) \ge m_{(i,j)},\ j = 1, \dots, N_i\} \subset V_s. \tag{8.13}$$

The link between the expressions (8.12) and (8.13) already suggests the study of elimination in terms of local presentations. In Part II local presentations are expressed by Rees algebras, and a refined form of elimination will be treated in that context.

## Part II: Local presentations, Rees algebras and their resolution

Let $X$ be an algebraic variety over a perfect field $k$. There is a form of simplification of singularities, which leads to the resolution of singularities of $X$ when $k$ is a field of characteristic zero. In general, at least over perfect fields, this form of simplification has led to resolution in small dimension, and we hope that this strategy will play a relevant role in the open problem of resolution of singularities. We proceed as follows:

(1) Write:

$$X = (X \setminus \mathrm{Sing}(X)) \bigsqcup \mathrm{Sing}(X).$$

(2) Next, stratify $\mathrm{Sing}(X)$ into locally closed sets. To this end we use invariants as the Hilbert-Samuel function, or the Multiplicity, and we find a stratification by considering the (locally closed) subsets where either of these functions are constant:

$$\mathrm{Sing}(X) = F_1 \sqcup F_2 \sqcup \cdots \sqcup F_r.$$

(3) The key point is that, in general, we need to refine this stratification to obtain regular locally closed sets,

$$F_i = G_1^{(i)} \sqcup G_2^{(i)} \sqcup \cdots \sqcup G_{r_i}^{(i)}.$$

**Main idea:** In the following sections we will introduce Rees algebras over smooth schemes. Then we will see that the local presentations from Proposition 8.8 and the statement of the Reformulation Theorem 0.1 can be expressed in terms of Rees algebras. This together with techniques from elimination theory will enable us to go from (2) to (3) above.

**In addition, we can:**

- Define functions that measure the improvement after blowing up.
- In characteristic zero, achieve resolution; in positive characteristic "simplify" (in the sense of Theorem 18.7) the singularities.

**Conclusion:** Let $X$ be an algebraic variety defined over a perfect field. The local presentations in Propositions 8.4 and 8.8 are defined under the assumptions in (8.1) and (8.9), where a finite morphism is defined from $X$ to a regular variety $V$. We will prove in Appendix A that such assumptions hold, locally at any point of $X$, but in a neighborhood in the sense of étale topology. We will assume here that these conditions hold for $X$, and we will express the local presentations in Propositions 8.4 and 8.8 in terms of suitably defined $\mathcal{O}_V$-Rees algebras. The notion of singular locus and that of resolution of a Rees algebra will be introduced, and we will see that, to obtain a refinement like (3) it suffices to find refinements of the singular loci of Rees algebras defined on smooth schemes. In addition, when the characteristic is zero, there is an algorithm that resolves Rees algebras. This, as we will see, in turn induces a sequence of blowing ups on $X$ that leads to a lowering of the maximum multiplicity, and thus to a desingularization by iteration of the process (see the Reformulation Theorem 0.1 in the Introduction). In fact, in Section 14 we show that the reduction of the order of an ideal stated in Theorem 0.1 can be expressed in terms of Rees algebras.

The following sections will be devoted to the study of Rees algebras, their properties and their resolution. The main result in this part is Theorem 18.9.

# The basics of Rees algebras and the Canonicity Theorem

## 9. The basics

Definition 9.1. Let $D$ be a Noetherian ring, and let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of ideals in $D$ satisfying the following conditions:

(1) $I_0 = D$;
(2) $I_k \cdot I_l \subset I_{k+l}$.

The graded subring $\mathcal{G} = \bigoplus_{n\geq 0} I_n W^n$ of the polynomial ring $D[W]$ is said to be a *$D$-Rees algebra*, or a Rees algebra over $D$, if it is a finitely generated $D$-algebra.

A Rees algebra can be described by giving a finite set of generators, say $\{f_1W^{n_1}, \dots, f_sW^{n_s}\}$,

$$\mathcal{G} = D[f_1W^{n_1}, \dots, f_sW^{n_s}] \subset D[W]$$

with $f_i \in D$ for $i = 1\dots, s$. An element $g \in I_n$ will be of the form $g = F_n(f_1, \dots, f_s)$ for some weighted homogeneous polynomial of degree $n$ in $s$-variables, $F_n(Y_1, \dots, Y_s)$, where $Y_i$ has weight $n_i$ for $i = 1, \dots, s$.

Example 9.2. The typical example of a Rees algebra is the *Rees ring of an ideal* $J \subset D$, say $\mathcal{G} = D[JW] = \oplus_n J^n W^n$.

Given algebras $\mathcal{G} = D[f_1W^{n_1}, \dots, f_sW^{n_s}]$ and $\mathcal{G}' = D[g_1W^{m_1}, \dots, g_tW^{m_t}]$, we will use $\mathcal{G} \odot \mathcal{G}'$ to denote the smallest algebra containing both $\mathcal{G}$ and $\mathcal{G}'$, i.e.,

$$\mathcal{G} \odot \mathcal{G}' := D[f_1W^{n_1}, \dots, f_sW^{n_s}, g_1W^{m_1}, \dots, g_tW^{m_t}]. \tag{9.1}$$

Example 9.3. Following a slightly different pattern, also the graded algebra $\mathcal{G} = D[JW^b] = D \oplus 0W \oplus \dots \oplus 0W^{b-1} \oplus JW^b \oplus 0W^{b+1} \oplus \dots \oplus 0W^{2b-1} \oplus J^2W^{2b} \oplus 0W^{2b+1} \oplus \dots$ is a Rees algebra, which we will refer to as an *almost-Rees ring*, and will be denoted by $\mathcal{G}_{(J,b)}$.

The notion of Rees algebra extends to noetherian schemes in the obvious manner: a sequence of sheaves of ideals $\{I_n\}_{n\geq 0}$ on a scheme $V$, defines a sheaf of Rees algebras $\mathcal{G}$ over $V$ if $I_0 = \mathcal{O}_V$, and $I_k \cdot I_l \subset I_{k+l}$ for all non-negative integers $k, l$, and if there is an affine open cover $\{U_i\}$ of $V$, such that $\mathcal{G}(U_i) \subset \mathcal{O}_V(U_i)[W]$ is an $\mathcal{O}_V(U_i)$-Rees algebra in the sense of Definition 9.1.

**9.4. The motivation.** In what follows we will consider $D$ to be a smooth algebra (of finite type) over a field $k$. Rees algebras appear in smooth schemes as a way of formulating (or reformulating) local presentations. Essentially a local presentation of a closed set $\mathcal{C}$ in Spec$(D)$ is given by a collection $f_, \dots, f_r \in D$, and a positive integer $n_i$ for each $f_i$, so that

$$\mathcal{C} = \bigcap_{1\leq i\leq r} \{x \in \mathrm{Spec}(D) : \nu_x(f_i) \geq n_i\}. \tag{9.2}$$

Our two main examples of local presentations are given in (8.5) and in (8.10). From the previous data, of elements $f_, \dots, f_r \in D$ and integers $n_1, \dots, n_r$, we will define

$$\mathcal{G} = D[f_1W^{n_1}, \dots, f_sW^{n_s}] \subset D[W].$$

Then we will show how the previous closed set $\mathcal{C}$ can be naturally assigned to this Rees algebra $\mathcal{G}$. Furthermore, we will show how $\mathcal{G}$ enables us to:

- Stratify this closed set in smooth strata;

- Have an easy-to-handle law of transformation under blow ups, that is well suited to define transforms of local presentations.

And that is why Rees algebras are useful to address problems of resolution.

**9.5. Rees algebras and integral closure.** Since we are interested in applying the theory of Rees algebras to problems of resolution, it will be natural not to distinguish between two Rees algebras when they have the same integral closure. Since integral closure is compatible with open restrictions, we may assume to be working on a smooth ring $D$ over a perfect field $k$, with quotient field $K(D)$. We shall consider the integral closure of a $D$-Rees algebra $\mathcal{G} \subset D[W]$, or equivalently in $K(D)[W]$, and will denote it by $\overline{\mathcal{G}}$.

The integral closure of a $D$-Rees algebra is a $D$-Rees algebra again: on the one hand, [**HuS**, Theorem 2.3.2] ensures that the integral closure of a Rees algebra is a graded ring; on the other, the fact that Rees algebras are, by definition, finitely generated over an excellent ring, guarantees that their integral closure is finitely generated too (see [**Gr**, 7.8.3.ii), vi)]).

**9.6. The Veronese action on a Rees algebra.** Given a natural number $M$, the *$M$-th Veronese* action on a Rees algebra $\mathcal{G} = D \oplus I_1 W \oplus I_2 W^2 \oplus \ldots \oplus I_n W^n \oplus \ldots$, is defined as

$$\mathbb{V}_M(\mathcal{G}) := \bigoplus_{k \geq 0} I_{Mk} W^{Mk}.$$

Since $\mathcal{G}$ is finitely generated, $\mathbb{V}_M(\mathcal{G}) \subset \mathcal{G}$ is a finite extension for any choice of $M$. In fact, it can be shown that for suitable choices of $N$, the Rees algebras $\mathbb{V}_N(\mathcal{G})$ are almost-Rees rings (see Example 9.3 above). In particular, any Rees algebra is finite over an almost-Rees ring (see [**EV3**, Remark 1.3], [**Vi6**, 2.3], or [**BrG-EV**, Lema 1.7]).

**9.7. The singular locus of a Rees algebra.** Let $V$ be a smooth scheme over a perfect field $k$, and let $\mathcal{G} = \oplus_n I_n W^n$ be a sheaf of $\mathcal{O}_V$-Rees algebras. Then the *singular locus of* $\mathcal{G}$, Sing $\mathcal{G}$, is

$$\operatorname{Sing} \mathcal{G} := \bigcap_{n \in \mathbb{N}_{>0}} \{\xi \in V : \nu_\xi(I_n) \geq n\},$$

where $\nu_\xi(I_n)$ denotes the usual order of $I_n$ in the regular local ring $\mathcal{O}_{V,\xi}$. Observe that Sing $\mathcal{G}$ is a closed subset in $V$. If $\mathcal{G}$ is generated by $f_1 W^{n_1}, \ldots, f_s W^{n_s}$ on an affine open set $U \subset V$, then it can be shown that

$$\operatorname{Sing} \mathcal{G} \cap U = \bigcap_{i=1}^{s} \{\xi \in V : \nu_\xi(f_i) \geq n_i\} \tag{9.3}$$

(see [**EV3**, Proposition 1.4]).

**9.8. The order of a Rees algebra at a point** [**EV3**, 6.3]**.** Consider a Rees algebra $\mathcal{G} = \bigoplus_{n \geq 0} I_n W^n$ on a smooth scheme $V$, let $\xi \in$ Sing $\mathcal{G}$, and assume that $fW^n \in I_n W^n$ in an open neighborhood of $\xi$. Then set

$$\operatorname{ord}_{fW^n}(\xi) = \frac{\nu_\xi(f)}{n} \in \mathbb{Q},$$

where, as before, $\nu_x(f)$ denotes the order of $f$ in the regular local ring $\mathcal{O}_{V,\xi}$. Notice that $\mathrm{ord}_\xi(f) \geq 1$ since $\xi \in \mathrm{Sing}\ \mathcal{G}$. Now define

$$\mathrm{ord}_{\mathcal{G}}(\xi) = \inf_{n\geq 1}\left\{\frac{\nu_\xi(I_n)}{n}\right\}.$$

If $\mathcal{G}$ is generated by $\{f_1W^{n_1}, \dots, f_sW^{n_s}\}$ on an affine neighborhood of $\xi$ then it can be shown that

$$\mathrm{ord}_{\mathcal{G}}(\xi) = \min\{\mathrm{ord}_{f_iW^{n_i}}(\xi) : i = 1, \dots, s\}, \tag{9.4}$$

and therefore, since $x \in \mathrm{Sing}\ \mathcal{G}$, $\mathrm{ord}_{\mathcal{G}}(\xi) \in \mathbb{Q} \geq 1$.

EXAMPLE 9.9. Let $H \subset V$ be a hypersurface, and let $b$ be a non-negative integer. Then the singular locus of the Rees algebra generated by $\mathcal{I}(H)$ in degree $b$, i.e., the singular locus of $\mathcal{G}_{(\mathcal{I}(H),b)} = \mathcal{O}_V[\mathcal{I}(H)W^b](\subset \mathcal{O}_V[W])$, is the closed set of points of multiplicity at least $b$ of $H$ (which may be empty). The order of $\mathcal{G}_{(\mathcal{I}(H),b)}$ at a point in the singular locus is the multiplicity of $H$ divided by $b$.

EXAMPLE 9.10. In the same manner, if $J \subset \mathcal{O}_V$ is an arbitrary non-zero sheaf of ideals, and $b$ is a non-negative integer, then $\mathrm{Sing}\ \mathcal{G}_{(J,b)}$ consists of the points of $V$ where the order of $J$ is at least $b$, and $\mathrm{ord}_{\mathcal{G}_{(J,b)}}(\xi) = \frac{\nu_\xi(J)}{b}$ for all $\xi \in \mathrm{Sing}\ \mathcal{G}_{(J,b)}$.

**9.11. Singular locus, order, and integral closure.** Two Rees algebras with the same integral closure have the same singular locus and the same order at a point [**EV3**, Proposition 6.4 (2)].

**9.12. Transforms of Rees algebras by blow ups.** A smooth closed subscheme $Y \subset V$ is said to be *permissible* for $\mathcal{G} = \bigoplus_n J_nW^n \subset \mathcal{O}_V[W]$ if $Y \subset \mathrm{Sing}\ \mathcal{G}$. A *permissible monoidal transformation* is the blow up at a permissible center, $V \leftarrow V_1$. If $H_1 \subset V_1$ denotes the exceptional divisor, then for each $n \in \mathbb{N}$,

$$J_n\mathcal{O}_{V_1} = I(H_1)^nJ_{n,1}$$

for some sheaf of ideals $J_{n,1} \subset \mathcal{O}_{V_1}$. The *transform of* $\mathcal{G}$ in $V_1$ is then defined as:

$$\mathcal{G}_1 := \bigoplus_n J_{n,1}W^n;$$

and for a given homogeneous element $fW^m \in \mathcal{G}$, a *weighted transform*, $f_1W^m \in \mathcal{G}_1$, is defined by choosing any generator of the principal ideal

$$I(H_1)^{-m}\cdot\langle f\rangle\mathcal{O}_{V_1}.$$

The next proposition gives a local description of the transform of a Rees algebra after a permissible monoidal transformation.

PROPOSITION 9.13. [**EV3**, Proposition 1.6] *Let $\mathcal{G} = \bigoplus_n I_nW^n$ be a Rees algebra on a smooth scheme $V$ over a field $k$, and let $V \leftarrow V_1$ be a permissible monoidal transformation. Assume, for simplicity, that $V$ is affine. If $\mathcal{G}$ is generated by $\{f_1W^{n_1}, \dots, f_sW^{n_s}\}$, then its transform $\mathcal{G}_1$ is generated by $\{f_{1,1}W^{n_1}, \dots, f_{s,1}W^{n_s}\}$, where $f_{i,1}$ denotes a weighted transform of $f_i$ in $V_1$ for $i = 1, \dots, s$.*

REMARK 9.14. The transforms of two Rees algebras with the same integral closure also have the same integral closure (see for instance [**BrG-EV**, Section 2.3].

**9.15. Resolution of Rees algebras.** A *resolution* of a Rees algebra $\mathcal{G}$ on a smooth scheme $V$ is a finite sequence of blowing ups,

$$\begin{array}{ccccccc} V = V_0 & \xleftarrow{\rho_0} & V_1 & \xleftarrow{\rho_1} & \dots & \xleftarrow{\rho_{n-1}} & V_n \\ \mathcal{G} = \mathcal{G}_0 & & \mathcal{G}_1 & & \dots & & \mathcal{G}_n \end{array} \tag{9.5}$$

at permissible centers $Y_i \subset \operatorname{Sing} \mathcal{G}_i$ (here $\mathcal{G}_i$ denotes the transform of $\mathcal{G}_{i-1}$) for $i = 0, 1, \dots, n-1$, so that

(1) Sing $\mathcal{G}_n = \emptyset$;
(2) The exceptional locus of the composition $V \leftarrow V_n$ is a union of smooth hypersurfaces having only normal crossings in $V_n$.

## 10. Weak equivalence

As it was indicated before, from the point of view of resolution, it seems quite natural not to distinguish between a Rees algebra and its integral closure. More generally, given a Rees algebra $\mathcal{G}$ on a smooth scheme $V$, we can ask which $\mathcal{O}_V$-Rees algebras are indistinguishable from $\mathcal{G}$ from the point of view of resolution. This leads us to the notion of *weak equivalence* introduced by Hironaka.

Weak equivalence is an equivalence relation among all Rees algebras on a given smooth scheme $V$, and it is established by taking into account a *tree of closed sets* determined by the singular locus of a Rees algebra, $\mathcal{G}$, and the singular loci of transforms of $\mathcal{G}$ under suitable morphisms (see Definitions 10.2 and 10.3 below).

Definition 10.1. Let $V$ be a smooth scheme over a perfect field $k$. A *local sequence over* $V$ is a sequence of the form

$$V = V_0 \xleftarrow{\pi_0} V_1 \xleftarrow{\pi_1} \dots \xleftarrow{\pi_{m-1}} V_m$$

where for $i = 0, 1, \dots, m-1$, each $\pi_i$ is either the blow up at a smooth closed subscheme, or a smooth morphism.

Definition 10.2. If $\mathcal{G}$ is an $\mathcal{O}_V$-Rees algebra, a *$\mathcal{G}$-local sequence over* $V$ is a local sequence over $V$,

$$(V = V_0, \mathcal{G} = \mathcal{G}_0) \xleftarrow{\pi_0} (V_1, \mathcal{G}_1) \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_{m-1}} (V_m, \mathcal{G}_m), \tag{10.1}$$

where for $i = 0, 1, \dots, m-1$ each $\pi_i$ is either a permissible monoidal transformation for $\mathcal{G}_i \subset \mathcal{O}_{V_i}[W]$ (and then $\mathcal{G}_{i+1}$ is the transform of $\mathcal{G}_i$ in the sense of 9.12), or a smooth morphism (and then $\mathcal{G}_{i+1} = \pi_i^*(\mathcal{G}_i)$, i.e., the pull-back of $\mathcal{G}_i$ in $V_{i+1}$).

Definition 10.3. Let $\mathcal{G}$ be an $\mathcal{O}_V$-Rees algebra, and let

$$(V = V_0, \mathcal{G} = \mathcal{G}_0) \xleftarrow{\pi_0} (V_1, \mathcal{G}_1) \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_{m-1}} (V_m, \mathcal{G}_m), \tag{10.2}$$

be a $\mathcal{G}$-local sequence over $V$. Then the collection of closed subsets

$$\operatorname{Sing} \mathcal{G}_0 \subset V_0, \quad \operatorname{Sing} \mathcal{G}_1 \subset V_1, \quad \dots, \quad \operatorname{Sing} \mathcal{G}_N \subset V_N$$

determined by the $\mathcal{G}$-local sequence (10.2) is a *branch of closed subsets over $V$ determined by $\mathcal{G}$* (notice that the branch of closed subsets consists of the closed subsets Sing $\mathcal{G}_i$ and the maps among them). The union of all branches of closed subsets, obtained by considering all $\mathcal{G}$-local sequences over $V$, is the *tree of closed subsets over $V$ determined by $\mathcal{G}$*. We will denote it by $\mathscr{F}_V(\mathcal{G})$.

DEFINITION 10.4. If $\mathcal{G}$ and $\mathcal{K}$ are two $\mathcal{O}_V$-Rees algebras, then a *$\mathcal{G}$-$\mathcal{K}$-local sequence over* $V$ is a local sequence over $V$ that is both $\mathcal{G}$-local and $\mathcal{K}$-local.

DEFINITION 10.5. Two algebras, $\mathcal{G}$ and $\mathcal{K}$, are said to be *weakly equivalent* if:

(1) Sing $\mathcal{G}$ = Sing $\mathcal{K}$;
(2) Any local $\mathcal{G}$-local sequence over $V$ induces a $\mathcal{K}$-local sequence over $V$, and any $\mathcal{K}$-local sequence over $V$ induces a $\mathcal{G}$-local sequence over $V$;
(3) For any $\mathcal{G}$-local sequence over $V$,

$$(V,\mathcal{G}) = (V_0,\mathcal{G}_0) \xleftarrow{\pi_0} (V_1,\mathcal{G}_1) \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_{m-1}} (V_m,\mathcal{G}_m),$$

and the corresponding induced $\mathcal{K}$-local over $V$,

$$(V,\mathcal{G}) = (V_0,\mathcal{K}_0) \xleftarrow{\pi_0} (V_1,\mathcal{K}_1) \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_{m-1}} (V_m,\mathcal{K}_m),$$

there is an equality of closed sets, Sing $(\mathcal{G}_j)$ = Sing $(\mathcal{K}_j)$ for $0 \leq j \leq m$; and vice versa.

REMARK 10.6. Weak equivalence is an equivalence relation within the class of Rees algebras defined over $V$. We denote by $\mathscr{C}_V(\mathcal{G})$ the equivalence class of given Rees algebra $\mathcal{G}$. By definition two Rees algebras are weakly equivalent when they determine the same tree of closed subsets over $V$; i.e., two Rees algebras over $V$, say $\mathcal{G}$ and $\mathcal{K}$, are weakly equivalent when $\mathscr{F}_V(\mathcal{G}) = \mathscr{F}_V(\mathcal{K})$.

EXAMPLE 10.7. If two $\mathcal{O}_V$-Rees algebras $\mathcal{G}$ and $\mathcal{K}$ have the same integral closure, then by 9.11, Sing $\mathcal{G}$ = Sing $\mathcal{K}$. If $\varphi : V_1 \to V$ is a smooth morphism, then the pull backs of $\mathcal{G}$ and $\mathcal{K}$ in $V_1$, i.e., $\varphi^*(\mathcal{G})$ and $\varphi^*(\mathcal{K})$, also have the same integral closure in $V_1$, and therefore Sing $\varphi^*(\mathcal{G})$ = Sing $\varphi^*(\mathcal{K})$. Moreover, if $V \leftarrow V_1$ is a permissible transformation with center $Y \subset$ Sing $\mathcal{G}$ = Sing $\mathcal{K}$, then the transforms of $\mathcal{G}$ and $\mathcal{K}$ in $V_1$, say $\mathcal{G}_1$ and $\mathcal{K}_1$, also have same integral closure (see [**BrG-EV**, Section 2.3]) and therefore Sing $\mathcal{G}_1$ = Sing $\mathcal{K}_1$. Thus algebras with the same integral closure are equivalent.

Given two Rees algebras $\mathcal{G}$ and $\mathcal{K}$ we can make sense of the expression $\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{K})$ in a natural way:

DEFINITION 10.8. Let $\mathcal{K}$ and $\mathcal{G}$ be two Rees algebras on $V$. We will say that

$$\mathscr{F}_V(\mathcal{K}) \subset \mathscr{F}_V(\mathcal{G})$$

if Sing $\mathcal{K} \subset$ Sing $\mathcal{G}$, and any $\mathcal{K}$-local sequence over $V$,

$$(V,\mathcal{K}) = (V_0,\mathcal{K}_0) \xleftarrow{\pi_0} (V_1,\mathcal{K}_1) \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_{m-1}} (V_m,\mathcal{K}_m),$$

induces a $\mathcal{G}$-local sequence over $V$,

$$(V,\mathcal{G}) = (V_0,\mathcal{G}_0) \xleftarrow{\pi_0} (V_1,\mathcal{G}_1) \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_{m-1}} (V_m,\mathcal{G}_m),$$

with Sing $\mathcal{K}_i \subset$ Sing $\mathcal{G}_i$ for $i = 0, \dots, m$. We will say that

$$\mathscr{F}_V(\mathcal{K}) = \mathscr{F}_V(\mathcal{G})$$

if $\mathscr{F}_V(\mathcal{K}) \subset \mathscr{F}_V(\mathcal{G})$ and $\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{K})$.

REMARK 10.9. Observe that if $\mathcal{G} \subset \mathcal{K}$ is an inclusion of graded rings, then $\mathscr{F}_V(\mathcal{K}) \subset \mathscr{F}_V(\mathcal{G})$. Moreover, $\mathscr{F}_V(\mathcal{G}) = \mathscr{F}_V(\mathcal{K})$ if and only if $\mathscr{C}_V(\mathcal{G}) = \mathscr{C}_V(\mathcal{K})$ for any two $\mathcal{O}_V$-Rees algebras $\mathcal{G}$ and $\mathcal{K}$.

There are two different (natural) ways in which a Rees algebra $\mathcal{G}$ can be extended and still remain within the same equivalence class: The first consists in taking its integral closure, say $\overline{\mathcal{G}}$ (see Example 10.7). A second form is obtained by extending $\mathcal{G}$ by the action of differential operators, say $\mathbb{D}iff(\mathcal{G})$, and this will be discussed in Section 11). Let us advance that Hironaka shows that a combination of both operations leads to a complete characterization of each equivalence class:

Theorem 10.10. [**BrG-EV**, Theorem 3.10] **Duality.** *Let $V$ be a smooth scheme over a perfect field $k$, and let $\mathcal{G}$ and $\mathcal{K}$ be Rees algebras. Then $\mathscr{F}_V(\mathcal{K}) \subset \mathscr{F}_V(\mathcal{G})$ if and only if $\overline{\mathbb{D}iff(\mathcal{G})} \subset \overline{\mathbb{D}iff(\mathcal{K})}$.*

Theorem 10.11. [**BrG-EV**, Theorem 3.11] **Canonicity.** *Let $V$ be a smooth scheme over a perfect field $k$, and let $\mathcal{G}$ be a Rees algebra. Then the differential Rees algebra $\overline{\mathbb{D}iff(\mathcal{G})}$ contains any other Rees algebra weakly equivalent to $\mathcal{G}$. In particular, $\overline{\mathbb{D}iff(\mathcal{G})}$ is the canonical representative of $\mathscr{C}_V(\mathcal{G})$.*

These statements derive from Hironaka's Finite Presentation Theorem in [**Hi7**]. In [**BrG-EV**] these theorems using techniques coming from commutative algebra.

Theorem 10.10 asserts that given two Rees algebras, $\mathcal{G}$ and $\mathcal{K}$, there are canonical representatives for both $\mathscr{C}_V(\mathcal{G})$ and $\mathscr{C}_V(\mathcal{K})$, namely $\overline{\mathbb{D}iff(\mathcal{G})}$ and $\overline{\mathbb{D}iff(\mathcal{K})}$, in such a way that $\mathscr{F}_V(\mathcal{K}) \subset \mathscr{F}_V(\mathcal{G})$ if and only if there is an inclusion between the canonical representatives, i.e., if and only if $\overline{\mathbb{D}iff(\mathcal{G})} \subset \overline{\mathbb{D}iff(\mathcal{K})}$.

## 11. Differential Rees algebras and Giraud's Lemma

We have already mentioned that a Rees algebra and its integral closure are weakly equivalent (see Example 10.7). In this section we will see that a Rees algebra and the *differential Rees algebra* expanded by it (see Definition 11.1 below) are also weakly equivalent. This is essentially a result of Giraud (see Lemma 11.6). See also [**W**] and [**K**] for other results in this line.

Let $V$ be a smooth scheme over a perfect field $k$. For any non-negative integer $r$, denote by $\mathrm{Diff}^r_{V|k}$ the (locally free) sheaf of $k$-linear differential operators of order at most $r$ (see [**Gr**] or [**K**] for references to the theory of Differential operators).

Definition 11.1. A Rees algebra $\mathcal{G} = \bigoplus_n I_n W^n$ is said to be a *differential Rees algebra*, if the following condition holds:

> There is an affine open covering of $V$, $\{U_i\}$, such that for any $D \in \mathrm{Diff}^r_{V|k}(U_i)$ and any $h \in I_n(U_i)$ we have that $D(h) \in I_{n-r}(U_i)$ provided that $n \geq r$.

In particular, $I_{n+1} \subset I_n$, since $\mathrm{Diff}^0_{V|k}(U_i) \subset \mathrm{Diff}^1_{V|k}(U_i)$.

Remark 11.2. Given any Rees algebra $\mathcal{G}$ on a smooth scheme $V$, there is a natural way to construct the smallest differential Rees algebra containing it: the *differential Rees algebra generated by* $\mathcal{G}$, $\mathbb{D}iff(\mathcal{G})$ (see [**Vi6**, Theorem 3.4]). More precisely, if $\mathcal{G}$ is locally generated by $\{f_1 W^{n_1}, \dots, f_s W^{n_s}\}$ on an affine open set $U$, then it can be shown that $\mathbb{D}iff(\mathcal{G}(U))$ is generated by the elements

$$\{D(f_i) W^{n_i - r} : D \in \mathrm{Diff}^r_{V|k}(U), 0 \leq r < n_i, i = 1, \dots, s\}. \tag{11.1}$$

Note that $\mathrm{Diff}^r_{V|k}(U) \subset \mathrm{Diff}^\ell_{V|k}(U)$ if $r \leq \ell$. Thus, if $D \in \mathrm{Diff}^r_{V|k}(U) \subset \mathrm{Diff}^\ell_{V|k}(U)$ is a differential operator, then $D(f_i) W^{n_i - r}$, and also $D(f_i) W^{n_i - \ell}$, are in (11.1), as long as $r \leq \ell < n_i$. Moreover, it suffices to take $D$ as part of a finite system of generators of $\mathrm{Diff}^{r}_{V|k}(U)$ with $r$ being strictly smaller than $n_i$, which in particular implies that the differential algebra generated by $\mathcal{G}$ is a (finitely generated) Rees algebra (cf. [**Vi6**, Proof of Theorem 3.4]).

**11.3. Local generators for the sheaf of differential operators.** Along these lines we give a local description of the generators of the locally free sheaf $\mathrm{Diff}^{r}_{V|k}$. Let $\xi \in V$ be a closed point, and let $\{z_1, z_2, \ldots, z_d\} \subset \mathcal{O}_{V,\xi}$ be a regular system of parameters. On $\widehat{\mathcal{O}_{V,\xi}} \simeq k'[[z_1, \ldots, z_d]]$, where $k'$ is the residue field at $\xi$, consider the Taylor expansion

(11.2)

$$\begin{array}{rrcl} \mathrm{Tay}: & k'[[z_1,\ldots,z_d]] & \longrightarrow & k'[[z_1,\ldots,z_d,T_1,\ldots,T_d]] \\ & f(z_1,z_2,\ldots,z_d) & \longmapsto & f(z_1+T_1,z_2+T_2,\ldots,z_d+T_d) = \sum_{\alpha\in\mathbb{N}^d} \Delta^{\alpha}(f)T^{\alpha} \end{array}$$

as in [**Vi4**, Definition 1.2] and in [**Vi6**, Theorem 3.4]. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d$,

$$\begin{array}{rrcl} \Delta^{(\alpha_1,\alpha_2,\ldots,\alpha_d)}: & \widehat{\mathcal{O}_{V,x}} & \longrightarrow & \widehat{\mathcal{O}_{V,x}} \\ & f & \longmapsto & \Delta^{(\alpha_1,\alpha_2,\ldots,\alpha_d)}(f) \end{array} \tag{11.3}$$

is a differential operator of order $|\alpha| = \alpha_1+\alpha_2+\ldots+\alpha_d$, which defines by restriction a differential operator $D^{\alpha} : \mathcal{O}_V(U) \to \mathcal{O}_V(U)$ in some neighborhood $U$ of $\xi$, since $V$ is smooth over the perfect field $k$ (see [**Cu1**]). Moreover, the sheaf of differential operators up to order $r$, say $\mathrm{Diff}^{r}_{V|k}$, is locally generated by the $D^{\alpha}$ with $|\alpha| \leq r$ (at $U$).

**11.4. Differential Rees algebras, order, and singular locus.** Let $J \subset \mathcal{O}_V$ be a non-zero sheaf of ideals. Then, for a non-negative integer $r$, define

$$\mathrm{Diff}^{r}_{V|k}(J) := \langle D(f) : D \in \mathrm{Diff}^{r}_{V|k} \text{ and } f \in J\rangle.$$

Let $\xi \in V$, and let $b$ be a non-negative integer. Since $V$ is smooth over a perfect field $k$, using Taylor expansions as in 11.3, note that

$$\nu_{\xi}(J) \geq b \Leftrightarrow x \in V(\mathrm{Diff}^{b-1}_{V|k}(J)).$$

Therefore, if $\mathcal{G} = \bigoplus_n I_n W^n$, then,

$$\mathrm{Sing}\ \mathcal{G} = \bigcap_{n\geq 1} V(\mathrm{Diff}^{n-1}_{V|k}(I_n)),$$

(see [**Vi6**, Definition 4.2 and Proposition 4.4]). In particular,

$$\mathrm{Sing}\ \mathcal{G} = \mathrm{Sing}\ \mathbb{D}iff(\mathcal{G});$$

and moreover, if $\xi \in \mathrm{Sing}\ \mathcal{G} = \mathrm{Sing}\ (\mathbb{D}iff(\mathcal{G}))$ then

$$\mathrm{ord}_{\mathcal{G}}(\xi) = \mathrm{ord}_{\mathbb{D}iff(\mathcal{G})}(\xi)$$

(cf. [**EV3**, Proposition 6.4 (3)]). Furthermore, if $\mathcal{G}$ is a differential Rees algebra, then $\mathrm{Sing}\ \mathcal{G} = V(I_n)$ for any positive integer $n$ (see [**Vi6**, Proposition 4.4 (5)]).

**11.5. Local sequences and differential extensions.** The pull-back of a Differential Rees algebra by a smooth morphism is a Differential Rees algebra again. The following main result states that the similarities between $\mathcal{G}$ and $\mathbb{D}iff(\mathcal{G})$, studied in 11.4, also hold after applying a monoidal transformation.

LEMMA 11.6. **Giraud's Lemma** [**EV3**, Theorem 4.1]. *Let $\mathcal{G} \subset \mathcal{K} \subset \mathcal{R}$ be an inclusion of Rees algebras, such that $\mathcal{R} = \mathbb{D}iff(\mathcal{G})$, and let $V \leftarrow V_1$ be a permissible monoidal transformation with center $Y \subset Sing\ \mathcal{R}(= Sing\ \mathcal{G} = Sing\ \mathcal{K})$. Then:*

(1) *There is an inclusion of transforms*

$$\mathcal{G}_1 \subset \mathcal{K}_1 \subset \mathcal{R}_1.$$

(2) *Even if $\mathcal{R}_1$ is not a differential Rees algebra over $V_1$, the three algebras $\mathcal{G}_1 \subset \mathcal{K}_1 \subset \mathcal{R}_1$ span the same differential Rees algebra, and therefore*

$$Sing\ \mathcal{G}_1 = Sing\ \mathcal{K}_1 = Sing\ \mathcal{R}_1.$$

**Summarizing: who is in $\mathscr{C}_V(\mathcal{G})$?** One can check from the previous discussion that $\mathcal{G}$ and $\mathbb{D}iff(\mathcal{G})$ are weakly equivalent.

So far we have seen that a Rees algebra $\mathcal{G}$ is weakly equivalent to both $\overline{\mathcal{G}}$ and $\mathbb{D}iff(\mathcal{G})$. Theorem 10.11 asserts that combining both operators we get a characterization all the Rees algebras in $\mathscr{C}_V(\mathcal{G})$: namely, a Rees algebra $\mathcal{K} \in \mathscr{C}_V(\mathcal{G})$ if and only if $\overline{\mathbb{D}iff(\mathcal{K})} = \overline{\mathbb{D}iff(\mathcal{G})}$.

REMARK 11.7. In Definitions 10.1 and 10.2 all smooth morphisms are allowed. In particular, étale morphisms are permitted as well. As we will see, some of our arguments are built in étale topology, so it is quite natural to consider this type of morphisms. Now, assume that in Definitions 10.1 and 10.2 we only consider smooth maps of the following type:

1) Projection on the first coordinate, $V_1 = V \times \mathbb{A}^n_k \xrightarrow{\varphi} V$.
2) Restriction to a Zariski's open subset $V_1$ of $V$, $V_1 \xrightarrow{\varphi} V$.

In the same manner as we did in Definition 10.5, one can define a new equivalence relation on $\mathcal{O}_V$-Rees algebras, by considering local sequences in which only smooth morphisms of type (1) or (2) are considered. It can be shown that this apparently new equivalence relation leads to the same partition as that given by the weak equivalence that we considered before (see [**BrG-EV**, §8]). For this reason, we will restrict to smooth morphisms as in (1) and (2) whenever possible if this simplifies the discussion.

REMARK 11.8. It follows from the definition of weak equivalence that if two Rees algebras are weakly equivalent, then a resolution of one of them induces a resolution of the other. As we will see the *resolution functions* that are used to address the resolution of Rees algebras take the same value on weakly equivalent Rees algebras. The purpose of the next section is to introduce the main tools needed to define so called resolution functions.

# FIRST STEPS TOWARDS RESOLUTION OF REES ALGEBRAS

## 12. Exceptional divisors enter in scene: basic objects

When addressing the resolution of a Rees algebra, see 9.15, some exceptional hypersurfaces are introduced. In a constructive resolution we will require that they have normal crossings. This motivates the notion of *basic object*, a new structure which is well adapted to this information.

Definition 12.1. A *basic object* is a triple $(V, \mathcal{G}, E)$ where

(1) V is a smooth scheme,
(2) $\mathcal{G}$ is a $\mathcal{O}_V$-Rees algebra, and
(3) $E = \{H_1, H_2, \ldots, H_r\}$ is a set of smooth hypersurfaces in $V$ so that their union has normal crossings.

If the dimension of $V$ is $n$ we say that $(V, \mathcal{G}, E)$ is an $n$-dimensional basic object.

Definition 12.2. We say that a smooth closed subscheme $Y \subset \operatorname{Sing} \mathcal{G}$ is *permissible center* for $(V, \mathcal{G}, E)$ if, in addition, it has normal crossings with the union of the hypersurfaces in $E$. A *permissible transformation* for a basic object is the blow up, $V \leftarrow V_1$, at a permissible center. The *transform* of $(V, \mathcal{G}, E)$ is:

$$(V_1, \mathcal{G}_1, E_1)$$

where $\mathcal{G}_1$ is the transform of $\mathcal{G}$, as in 9.12, and $E_1 = \{H_1, \ldots, H_r, H_{r+1}\}$. Here $H_i \in E_1$ denotes the strict transform of $H_i \in E$ for $i = 1, \ldots, r$, and $H_{r+1}$ is the exceptional hypersurface of the blow up.

The notion of resolution of a basic object generalizes that of resolution of a Rees algebra in the obvious manner:

Definition 12.3. A *resolution of a a basic object* $(V, \mathcal{G}, E)$ is a finite sequence of permissible transformations:

$$(V, \mathcal{G}, E) = (V_0, \mathcal{G}_0, E_0) \xleftarrow{\pi_0} (V_1, \mathcal{G}_1, E_1) \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_{m-1}} (V_m, \mathcal{G}_m, E_m) \tag{12.1}$$

so that $\operatorname{Sing} \mathcal{G}_m = \emptyset$.

Before stating the main theorem of this section, we will establish an equivalence relation among basic objects in the same spirit as we did for Rees algebras (see Definition 10.5).

**12.4. Local sequences for basic objects.** In the same way as we did in Definition 10.2, we will define *local sequences for basic objects*. Fix a basic object $(V, \mathcal{G}, E)$. We say that a sequence

$$(V, \mathcal{G}, E) = (V_0, \mathcal{G}_0, E_0) \xleftarrow{\pi_0} (V_1, \mathcal{G}_1, E_1) \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_{m-1}} (V_m, \mathcal{G}_m, E_m) \tag{12.2}$$

is a $(V, \mathcal{G}, E)$-*local sequence* if, for $i = 0, 1, \ldots, m-1$, each $\pi_i$ is either a permissible monoidal transformation for $(V_i, \mathcal{G}_i, E_i)$ (and then $(V_{i+1}, \mathcal{G}_{i+1}, E_{i+1})$ is the transform of $(V_i, \mathcal{G}_i, E_i)$ in the sense of 12.2), or a smooth morphism as in Remark 11.7 (and then $\mathcal{G}_{i+1}$ and $E_{i+1}$ are, respectively, the pull-backs of $\mathcal{G}_i$ and $E_i$ in $V_{i+1}$).

Therefore, in this new context, the condition of normal crossings is imposed in the notion of local sequence. And we can define a new equivalence relation, now on basic objects in the obvious way: we will say that $B = (V, \mathcal{G}, E)$ and $B' = (V, \mathcal{K}, E)$ are *equivalent basic objects* if $\mathcal{G}$ and $\mathcal{K}$ are equivalent with this new (more restrictive) definition of local sequence (since we take into account the hypersurfaces in $E$ in the notion of permissible center).

However, it can be shown that two basic objects, $B = (V, \mathcal{G}, E)$ and $B' = (V, \mathcal{K}, E)$, are equivalent if and only if $\mathcal{G}$ and $\mathcal{K}$ are weakly equivalent as in Definition 10.5 (see [**BrG-EV**, §8]). This leads to the following definition:

DEFINITION 12.5. Two basic objects $(V, \mathcal{G}, E)$ and $(V', \mathcal{G}', E')$ are said to be *weakly equivalent* if $V = V'$, $E = E$ and $\mathcal{G}$ is weakly equivalent to $\mathcal{G}'$.

Now we can state the main theorem for *constructive resolution* in characteristic zero. This theorem, which is well known over fields of characteristic zero (see [**BiMi**], [**BrEV**], [**Cu1**] [**EH**], [**EV1**], [**EV2**], [**Ko**], [**Vi1**], [**Vi2**], [**W**]), will be formulated now so as to encompass some results in positive characteristic, and will be discussed in Section 18.

THEOREM 12.6. [**BrV1**] *Let $(V, \mathcal{G}, E)$ be a basic object. Then a finite sequence of blow ups at permissible centers can be* constructed*,*

$$(V,\mathcal{G},E) = (V_0,\mathcal{G}_0,E_0) \xleftarrow{\pi_0} (V_1,\mathcal{G}_1,E_1) \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_{m-1}} (V_m,\mathcal{G}_m,E_m) \tag{12.3}$$

*so that:*

(1) *When the characteristic of the base field is zero, Sing $\mathcal{G}_m = \emptyset$;*
(2) *When the characteristic of the base field is positive, either Sing $\mathcal{G}_m = \emptyset$, or else, Sing $\mathcal{G}_m$ is "simpler" than Sing $\mathcal{G}$ (see Theorem* 18.7*).*
(3) *Weakly equivalent basic objects share the same constructive resolution (in characteristic zero) or simplification (in positive characteristic).*

The claim in (2) is that the algebra $\mathcal{G}_m$ acquires a particular form, expressed in terms of *elimination*. This is a notion to be discussed in forthcoming sections.

**12.7. Resolution functions.** A proof of Theorem 12.6, and a precise formulation of (2), is sketched in Section 18 (see also 13.5 for an idea of the strategy of the proof, and Theorem 18.9 for a more precise statement). But we anticipate that this *constructive resolution* (or simplification) of basic objects is obtained via the definition of the so called *resolution functions*. These are upper-semicontinuous functions defined on the singular locus of a given Rees algebra which tell us where to blow up. In fact such functions have the following main properties:

- Their maximum locus is smooth and determines the center to blow up;
- Their maximum value drops after blowing up.

To address the definition of these functions first we need to introduce suitable *resolution invariants* (see Section 13 below). The main resolution invariants are *Hironaka's order function* and *Hironaka's $\tau$-invariant*.

Using Hironaka's order function, the *satellite functions* are defined (see Section 14). Satellite functions are the building blocks used to obtain the resolution functions.

The point is that the satellite functions can be naturally refined. This refinement will be done by using *elimination and elimination algebras* (see Sections 16 and 17). Finally, the Resolution Function, namely that which will lead us to the construction of the sequences (12.3), is nothing but the function obtained from the satellite functions by successive refinements.

On the other hand, Hironaka's $\tau$-invariant will appear in this process of refinements. It will lead to the notion of the *codimensional type of a basic object*, which will play a key role on the inductive argument of the proof of Theorem 12.6.

We should mention here that the definition of the resolution invariants, and hence the construction of the resolution functions, is of local nature. However, the Canonicity Principle for Rees algebras (Theorem 10.11) will ensure that they globalize and lead to a global resolution (or simplification if the characteristic is positive). This settles the local-global problem in constructive resolution (see 18.10 for more details).

## 13. Resolution invariants

The purpose of this section is to present the main *invariants* (see Definition 13.1 below) that we use in order to address the constructive resolution of a Rees algebra, or more generally, of a basic object, in characteristic zero. If the characteristic of the base field is positive, we will see how to construct a form of simplification, in which the singular locus of the given basic object is included in a union of smooth hypersurfaces with normal crossings.

DEFINITION 13.1. Let $V$ be a smooth scheme over a perfect field $k$, and let $\mathscr{R}$ be the set of all (finitely generated) $\mathcal{O}_V$-Rees algebras. Recall that we can assign a closed set to each basic object $(V, \mathcal{G}, E)$ with $\mathcal{G} \in \mathscr{R}$, namely Sing $\mathcal{G}$. Suppose we assign a value to each $(V, \mathcal{G}, E)$, at each point $x \in$ Sing $\mathcal{G}$, and denote it by $\mu_{\mathcal{G}}(x)$. We will say that $\mu_{\mathcal{G}}(x)$ is an *invariant* if for any Rees algebra $\mathcal{K}$ weakly equivalent to $\mathcal{G}$, one has that $\mu_{\mathcal{G}}(x) = \mu_{\mathcal{K}}(x)$.

**13.2. Hironaka's main invariants [Hi4], [G], [Hi2], [Hi3], [Hi5], [O].**

13.2.1. *Hironaka's order function.* Let $\mathcal{G} = \bigoplus_n I_n W^n$ be a Rees algebra on a smooth scheme $V$ over a perfect field $k$. Recall that Hironaka's order function is defined as:

$$\begin{array}{rccl} \operatorname{ord}_{\mathcal{G}}: & \operatorname{Sing} \mathcal{G} & \longrightarrow & \mathbb{Q} \geq 1 \\ & \xi & \longmapsto & \operatorname{ord}_{\mathcal{G}}(\xi) = \inf_{n \geq 1} \left\{ \frac{\nu_{\xi}(I_n)}{n} \right\}, \end{array} \tag{13.1}$$

(see 9.8). The statements in 9.11, 11.4 and Theorem 10.11 guarantee that Hironaka's order is an invariant (see [**Hi4**, Theorem 10.10]).

13.2.2. *Hironaka's $\tau$-invariant for Rees algebras.* Let $\mathcal{G} = \bigoplus_n I_n W^n$ be as before, and let $\xi \in$ Sing $\mathcal{G}$ be a closed point with residue field $k'$. Fix a regular system of parameters, $\{z_1, \ldots, z_d\} \subset \mathcal{O}_{V,\xi}$, and consider the graded $k'$-algebra associated to its maximal ideal $m_\xi$, $\operatorname{Gr}_{m_\xi} \mathcal{O}_{V,\xi}$. This graded ring is isomorphic to a polynomial ring in $d$-variables with coefficients in $k'$, i.e., $k'[Z_1, \ldots, Z_d]$, where $Z_i$ denotes the initial form of $z_i$ in $m_\xi / m_\xi^2$.

Note that $\operatorname{Gr}_{m_\xi} \mathcal{O}_{V,\xi}$ is the coordinate ring associated to the tangent space of $V$ at $\xi$, namely Spec $(\operatorname{Gr}_{m_\xi} \mathcal{O}_{V,\xi}) = \mathbb{T}_{V,\xi}$. The *initial ideal* or *tangent ideal* of $\mathcal{G}$ at $\xi$, $\operatorname{In}_\xi(\mathcal{G})$, is the ideal of $\operatorname{Gr}_{m_\xi} \mathcal{O}_{V,\xi}$ generated by the homogeneous elements

$$\operatorname{In}_\xi(I_n) := \frac{I_n + m_\xi^{n+1}}{m_\xi^{n+1}}$$

for all $n \geq 1$. Observe that $\operatorname{In}_\xi(\mathcal{G})$ is zero unless $\operatorname{ord}_{\mathcal{G}}(\xi) = 1$. The zero set of the tangent ideal in Spec $(\operatorname{Gr}_{m_\xi} \mathcal{O}_{V,\xi})$ is the *tangent cone* of $\mathcal{G}$ at $\xi$, $\mathcal{C}_{\mathcal{G},\xi}$.

The *$\tau$-invariant of $\mathcal{G}$ at $\xi$* is the minimum number of variables needed to generate $\operatorname{In}_\xi(\mathcal{G})$. This in turn is the codimension of the largest linear subspace $\mathcal{L}_{\mathcal{G},\xi} \subset \mathcal{C}_{\mathcal{G},\xi}$ such that $u + v \in \mathcal{C}_{\mathcal{G},\xi}$ for all $u \in \mathcal{C}_{\mathcal{G},\xi}$ and $v \in \mathcal{L}_{\mathcal{G},\xi}$.

The $\tau$-invariant of $\mathcal{G}$ at $\xi$ is denoted by $\tau_{\mathcal{G},\xi}$. The inclusion $\mathcal{G} \subset \mathbb{D}iff(\mathcal{G})$ defines an inclusion $\mathcal{C}_{\mathbb{D}iff(\mathcal{G}),\xi} \subset \mathcal{C}_{\mathcal{G},\xi}$, and in fact,

$$\mathcal{C}_{\mathbb{D}iff(\mathcal{G}),\xi} = \mathcal{L}_{\mathbb{D}iff(\mathcal{G}),\xi} = \mathcal{L}_{\mathcal{G},\xi}.$$

Note, in particular, that $\mathcal{G}$, $\overline{\mathcal{G}}$, and $\mathbb{D}iff(\mathcal{G})$ have the same $\tau$-invariant at all singular point (see for instance [**B**, Remark 4.5, Theorem 5.2]), so it is an invariant.

Some of these invariants have also been studied by H. Kawanue and K. Matsuki in the frame of their "Idealistic filtration program" (see [**K**] and [**KM**]).

**13.3. Some properties of the $\tau$-invariant.** In the following lines we list some of the relevant properties of this invariant for algebras over perfect fields.

(1) If $\mathcal{G}$ is a Differential Rees algebra over a field of characteristic zero, the $\tau$-invariant at a point $\xi \in \operatorname{Sing} \mathcal{G}$ has the following simple interpretation: it is the biggest integer $r$ so that there is a regular system of parameters, $\{x_1, \dots, x_d\} \in \mathcal{O}_{V,\xi}$, for which the following inclusion holds

$$\mathcal{O}_{V,\xi}[x_1 W, \dots, x_r W] \subset \mathcal{G} \tag{13.2}$$

in a neighborhood of $\xi$. In fact, it easy to check that if $\mathcal{G}$ is a Differential Rees in characteristic zero, $\tau_{\mathcal{G},\xi} \geq r$ if and only if there is a regular system of parameters such that the inclusion (13.2) holds.

(2) Let $\xi \in \operatorname{Sing} \mathcal{G}$ be a closed point. If $Y \subset \operatorname{Sing} \mathcal{G}$ is a permissible center, then $\operatorname{codim}_\xi Y \geq \tau_{\mathcal{G},\xi}$: note that $\mathbb{T}_{Y,\xi} \subset \mathbb{T}_{V,\xi}$, is a linear subspace. Moreover, $\mathbb{T}_{Y,\xi} \subset \mathcal{L}_{\mathcal{G},\xi}$ for all $\xi \in Y \subset \operatorname{Sing} \mathcal{G}$, thus $\operatorname{codim}_\xi Y \geq \tau_{\mathcal{G},\xi}$ (cf. [**BrV1**, Theorem 6.5]).

(3) The $\tau$-invariant bounds the local codimension of the singular locus of a Rees algebra. Let $\xi \in \operatorname{Sing} \mathcal{G}$ be a closed point. We say that $\mathcal{G}$ *is of* codimensional type $e$ at $\xi \in \operatorname{Sing} \mathcal{G}$, or that $\mathcal{G}$ *is e-simple at* $\xi$ if $\tau_{\mathcal{G},\xi} \geq e$. We say that $\mathcal{G}$ *is of codimensional type* $\geq e$ or *e-simple* if $\tau_{\mathcal{G},\xi} \geq e$ for all $\xi \in \operatorname{Sing} \mathcal{G}$. In this case we will write $\tau_{\mathcal{G}} \geq e$. We will say that a *basic object* $(V, \mathcal{G}, E)$ *is of codimensional type* $\geq e$ or *e-simple* if $\mathcal{G}$ is of codimensional type $\geq e$.

(4) Rees algebras that are weakly equivalent share the same $\tau$-invariant. Hence the same statement holds for weakly equivalent basic objects (see [**B**, Remark 4.5, Theorem 5.2]).

(5) If $\mathcal{G}$ is of codimensional type $\geq e$, then, if the underlying field is perfect, all the codimension $e$-components of $\operatorname{Sing} \mathcal{G}$ are smooth and disconnected. Thus they are natural centers to blow up. Moreover, if the codimension at a closed point $\xi \in \operatorname{Sing} \mathcal{G}$ is $\tau_{\mathcal{G},\xi} = e$, then, up to integral closure, the restriction to an open neighborhood of $\xi$ is

$$\mathcal{G} = \mathcal{O}_V[x_1 W, \dots, x_e W] \tag{13.3}$$

for some regular system of parameters $\{x_1, \dots x_e, \dots, x_d\} \in \mathcal{O}_{V,\xi}$ (see [**BrV1**, Lemma 13.2, Theorem 13.1 and 13.4], and (5) in 16.7). In such case we will say that $\mathcal{G}$ is *e-trivial at* $\xi$.

Remark 13.4. Let $\mathcal{G}$ be a differential $\mathcal{O}_V$-Rees algebra and let $\xi \in \operatorname{Sing} \mathcal{G}$. If

$$(V, \mathcal{G}) = (V_0, \mathcal{G}_0) \xleftarrow{\pi_0} (V_1, \mathcal{G}_1) \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_{m-1}} (V_m, \mathcal{G}_m),$$

is a $\mathcal{G}$-local sequence with $\pi_i(\xi_i) = \xi_{i-1}$ for $i = 1, \dots, m$, then

$$\tau_{\mathcal{G}_0,\xi_0} \leq \tau_{\mathcal{G}_1,\xi_1} \leq \dots \leq \tau_{\mathcal{G}_m,\xi_m}$$

(see Remarks 16.10, and 16.11). Since the $\tau$-invariant is the same for Rees algebras that are weakly equivalent, it follows that the claim holds for arbitrary Rees algebras (differentially saturated or not).

**13.5. A resolution strategy.** Let $(V, \mathcal{G}, E)$ be a $d$-dimensional basic object defined over a perfect field $k$. Note that $d$ is an upper bound for the codimensional type, and the point is that in characteristic zero, a resolution of $(V, \mathcal{G}, E)$ can be constructed by decreasing induction on the codimensional type. First observe that if $(V, \mathcal{G}, E)$ is of codimensional type $d$, then Sing $\mathcal{G}$ consists of finitely many points. In such case a resolution can be obtained by simply blowing up these points (see 13.3 (5) for a local description of $\mathcal{G}$ in this case). The inductive step is that if $(V, \mathcal{G}, E)$ is of codimensional type $\geq e$ at a given point $\xi \in \operatorname{Sing} \mathcal{G}$, then, either there is a smooth natural center of codimension $e$ (see (5) in 13.3), or else it is possible to *associate* to $(V, \mathcal{G}, E)$ another basic object, say $(V, \widehat{\mathcal{G}}, E')$, of codimensional type larger than $e$, so that

$$\mathcal{G} \subset \widehat{\mathcal{G}}. \tag{13.4}$$

and whose resolution leads to an *improvement* of the singular locus of $\mathcal{G}$ (see Proposition 18.2). This motivates the use of induction on the codimensional type of a basic object as a strategy for resolution. On the other hand, when the characteristic is positive, the same inductive strategy leads to some form of simplification of the singularities of a given basic object (see Proposition 18.2 and Theorem 18.7).

As was previously indicated the construction of the resolution functions involved in the previous strategy makes use of the so called *satellite functions*, which are described in Section 14 below, and of refinement of satellite functions. This refinement makes use of a form of elimination presented in Sections 16 and 17.

## 14. Satellite functions

In this section we present the so called *satellite functions* used in constructive resolution (see [**Cu1**], [**Vi1**], [**EV1**, 4.11, 4.15]). As indicated before, satellite functions are the building blocks used to defined the resolution functions.

We will see that satellite functions derive from Hironaka's order function, and hence the value of a satellite function at each point is an invariant in the sense of Definition 13.1. To describe them we find it convenient to use the language of *pairs* instead of that of Rees algebras, since we think that it clarifies the exposition.

**14.1. The language of pairs.** Let $V$ be a smooth scheme over a perfect field $k$. A *pair* is given by a couple $(J, b)$ where $J$ is a non-zero sheaf of ideals, and $b$ is a non-negative integer. The *singular locus of a pair* is the closed set

$$\operatorname{Sing}(J, b) := \{\xi \in V : \nu_\xi(J) \geq b\}, \tag{14.1a}$$

where $\nu_\xi(J)$ denotes the order of $J$ in the regular local ring $\mathcal{O}_{V,\xi}$. With this notation, *Hironaka's order function* is defined as

$$\begin{array}{rccl} \operatorname{ord}_{(J,b)} : & \operatorname{Sing}(J, b) & \longrightarrow & \mathbb{Q}_{\geq 0} \\ & \xi & \longmapsto & \operatorname{ord}_{(J,b)}(\xi) := \frac{\nu_\xi(J)}{b}. \end{array} \tag{14.1b}$$

A *permissible center* for a pair $(J, b)$ is a smooth closed subscheme $Y \subset \operatorname{Sing}(J, b)$. The *transform of* $(J, b)$ after blowing up at a permissible center $Y$,

$$V \xleftarrow{\rho} V_1$$

is defined as a pair $(J_1, b)$ where

$$J_1 := I(H_1)^{-b} J \mathcal{O}_{V_1} \tag{14.1c}$$

and $I(H_1)$ is the defining ideal of the exceptional divisor $\rho^{-1}(Y)$. A *resolution of a pair* is a finite sequence of blow ups at permissible centers

$$\begin{array}{ccccccc} V=V_0 & \xleftarrow{\rho_0} & V_1 & \xleftarrow{\rho_1} & \dots & \xleftarrow{\rho_{n-1}} & V_n \\ (J,b)=(J_0,b) & & (J_1,b) & & \dots & & (J_n,b), \end{array}$$

such that:

(1) Sing $(J_n, b) = \emptyset$; and
(2) The exceptional locus of the composition $V \leftarrow V_n$ is a union of smooth hypersurfaces having only normal crossings in $V_n$.

**14.2. Rees algebras reformulated as pairs.** Observe that we can attach an almost-Rees ring to a given pair $(J, b)$, say $\mathcal{G}_{(J,b)} = \mathcal{O}_V[JW^b]$. Also, as indicated in 9.5, see also 9.6, up to integral closure, any Rees algebra is an almost-Rees ring. In other words, for every Rees algebra $\mathcal{G} = \bigoplus_{n\geq 0} I_n W^n$ there is some $N$ such that $\mathbb{V}_N(\mathcal{G}) = \mathcal{O}_V[I_N W^N] = \mathcal{G}_{(I_N,N)}$ is an almost-Rees ring, and hence, it can be interpreted as the Rees algebra associated to the pair $(I_N, N)$.

Moreover, by 9.11 and the definition of the singular locus of a pair (see (14.1a), and Example 9.10), one has that

$$\text{Sing } \mathcal{G} = \text{Sing } \mathbb{V}_N(\mathcal{G}) = \text{Sing } (I_N, N).$$

In addition, by 9.11 and by the definition of Hironaka's order function for a pair (see (14.1b)), one has that for any $\xi \in \text{Sing } \mathcal{G}$

$$\text{ord}_{\mathcal{G}}(\xi) = \text{ord}_{\mathbb{V}_N(\mathcal{G})}(\xi) = \text{ord}_{(I_N,N)}(\xi).$$

The same integer, $N$, that links Rees algebras and pairs passing through an almost-Rees ring, is preserved by permissible monoidal transformations. More precisely, let $\mathcal{G} = \bigoplus_n J_n W^n$ be a Rees algebra. If $\mathbb{V}_N(\mathcal{G})$ is an almost-Rees ring, i.e., if $\mathbb{V}_N(\mathcal{G}) = \mathcal{G}_{(J_N,N)}$ for some pair $(J_N, N)$, then observe that

$$\mathbb{V}_N(\mathcal{G})_1 = \mathbb{V}_N(\mathcal{G}_1). \tag{14.2}$$

Hence $\mathbb{V}_N(\mathcal{G}_1)$ is also an almost-Rees ring, and, moreover, $\mathbb{V}_N(\mathcal{G}_1) = \mathcal{G}_{(J_{N,1},N)}$ where $(J_{N,1}, N)$ is the transform of the pair $(J_N, N)$ by the permissible transformation $V \leftarrow V_1$. This shows that, if $\mathbb{V}_N(\mathcal{G})$ is an almost Rees ring, then so is $\mathbb{V}_N(\mathcal{G}_1)$ for the same $N$.

Thus the transformation law under permissible transformations is compatible for both, Rees algebras and pairs (see 9.12). Hence, if $\mathbb{V}_N(\mathcal{G}) = \mathcal{G}_{(J,b)}$ for some pair $(J, b)$, then a resolution of $\mathcal{G}$ as in (9.5) gives a resolution of $(J, b)$.

So far we have considered permissible transformations which are blow ups. However the link $\mathbb{V}_N(\mathcal{G}) = \mathcal{G}_{(J,b)}$ is also preserved when taking a pull-back by a smooth morphism of smooth schemes. Note that pull-backs also apply to pairs. Recall the definition of a $\mathcal{G}$-local sequence over $V$ from Definition 10.2,

$$(V = V_0, \mathcal{G} = \mathcal{G}_0) \xleftarrow{\pi_0} (V_1, \mathcal{G}_1) \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_{m-1}} (V_m, \mathcal{G}_m), \tag{14.3}$$

and note, in addition, that the definition of any such sequence of transforms over $\mathcal{G}$ is equivalent to a sequence of transforms of the pair $(J, b)$,

$$\begin{array}{ccccccc} V = V_0 & \xleftarrow{\pi_0} & V_1 & \xleftarrow{\pi_1} & \dots & \xleftarrow{\pi_{n-1}} & V_n \\ \mathcal{G} = \mathcal{G}_0 & & \mathcal{G}_1 & & \dots & & \mathcal{G}_n \\ (J, b) = (J_0, b) & & (J_1, b) & & \dots & & (J_n, b). \end{array} \tag{14.4}$$

and vice versa. This is due to the fact that $\mathbb{V}_N(\mathcal{G}_i) = \mathbb{V}_N(\mathcal{G})_i = \mathcal{G}_{(J_i,b)}$ for $i = 0, 1, \dots, n$. Therefore, from the point of view of resolution, the information encoded by both, pairs and Rees algebras, is equivalent. The concept of weak equivalence, formulated for algebras in Definition 10.5, extends naturally to the class of pairs $(J, b)$. In addition, the notion of invariant introduced in Definition 13.1 also applies for pairs, replacing Sing $\mathcal{G}$ by Sing $(J, b)$.

Observe that the previous discussion together with the last lines in 13.2.1 indicate that the rational number $\text{ord}_{(J,b)}(\xi) := \frac{\nu_\xi(J)}{b}$ in 14.1b is an invariant. See [**BrG-EV**, §2] for a more detailed exposition of these ideas.

**Notation.** In what follows, given an upper-semicontinuous function

$$F : \text{Sing } \mathcal{G} \longrightarrow \Lambda$$

max $F$ will be the maximum value of $F$, and denote by $\underline{\text{Max}}$ $F$ the closed set of points where this maximum is achieved.

**14.3. The first satellite function [Cu1], [Vi1], [EV1, 4.11].** Let $(V, (J, b), E)$ be an $n$-dimensional basic object with $E = \{H_1, \dots, H_r\}$, and consider *any* local sequence as defined in the section 12.4,

$$(V, (J, b), E) = (V_0, (J_0, b), E_0) \longleftarrow (V_1, (J_1, b), E_1) \longleftarrow \cdots \longleftarrow (V_m, (J_m, b), E_m). \tag{14.5}$$

Let $\{H_{r+1}, \dots, H_{r+m'}\} (\subset E_m)$ with $m' \leq m$ denote the exceptional hypersurfaces introduced by the steps that are permissible monoidal transformations (i. e., by the steps not given by smooth morphisms). We may assume, for simplicity, that these hypersurfaces are irreducible. Then for $i = 1, \dots, m$ there is a well defined factorization of the sheaves of ideals $J_i \subset \mathcal{O}_{V_i}$, say:

$$J_i = I(H_{r_1})^{b_1} I(H_{r_2})^{b_2} \cdots I(H_{r_{i'}})^{b_{i'}} \cdot \widetilde{J}_i \tag{14.6}$$

so that $\widetilde{J}_i$ does not vanish along $H_{r_j}$ for $j = 1, \dots i'$. Define $\text{w-ord}^{(n)}_{(J_i,b)}$ (or simply $\text{w-ord}^n_i$):

$$\begin{array}{rccl} \text{w-ord}^{(n)}_i : & \text{Sing } (J_i, b) & \longrightarrow & \mathbb{Q} \\ & \xi & \longmapsto & \text{w-ord}^{(n)}_i(\xi) = \frac{\nu_\xi(\widetilde{J}_i)}{b} \ (= \text{ord}_{(\widetilde{J}_i, b)}(\xi)), \end{array} \tag{14.7}$$

where $\nu_\xi(\widetilde{J}_i)$ denotes the order of $\widetilde{J}_i$ at $\mathcal{O}_{V_i,\xi}$. As we will show below, these functions derive from Hironaka's order function, and hence are invariants (i.e., they take the same value on weakly equivalent basic objects). The w-ord-function is introduced because its maximum value does not increase by blow ups at permissible centers contained in $\underline{\text{Max}}$ w-ord while the ord-function does not have this property. To see this it suffices to consider the pair $(\langle x^2 - y^3 z^2 \rangle, 1)$ in Spec$[x, y, z]$ and blow up at the origin.

**14.4. The second satellite function: the inductive function t [Cu1], [Vi1], [EV1, 4.15].** Let $(V,(J,b),E)$ be an $n$-dimensional basic object and consider *any* local sequence as in the section 12.4, where now if $V_i \leftarrow V_{i+1}$ is a blow up, it is assumed that the center $Y_i \subset \underline{\mathrm{Max}}\ \text{w-ord}_i^{(n)}$,

$$(14.8)\qquad (V,(J,b),E) \xleftarrow{\rho_0} (V_1,(J_1,b),E_1) \xleftarrow{\rho_1} \cdots \xleftarrow{\rho_{m-1}} (V_m,(J_m,b),E_m),$$

Then,

$$(14.9)\qquad \max \text{w-ord}^{(n)} \geq \max \text{w-ord}_1^{(n)} \geq \cdots \geq \max \text{w-ord}_m^{(n)}.$$

We now define a function $t_m$, only under the assumption that $\max \text{w-ord}_m > 0$. Set $l \leq m$ such that

(14.10)

$$\max \text{w-ord}^{(n)} \geq \ldots \geq \max \text{w-ord}_{l-1}^{(n)} > \max \text{w-ord}_l^{(n)} = \max \text{w-ord}_{l+1}^{(n)} \cdots = \max \text{w-ord}_m^{(n)},$$

and write:

$$(14.11)\qquad E_m = E_m^+ \sqcup E_m^- \text{ (disjoint union)},$$

where $E_m^-$ are the strict transforms of hypersurfaces in $E_l$. Define

(14.12)

$$\begin{array}{rccl} t_m^{(n)}: & \text{Sing }(J_m,b) & \longrightarrow & \mathbb{Q}\times\mathbb{N} \\ & \xi & \longmapsto & t_m^{(n)}(\xi) = (\text{w-ord}_m^{(n)}(\xi), \sharp\{H_i \in E_m^- : \xi \in H_i\}) \end{array}$$

where $\mathbb{Q}\times\mathbb{N}$ is a set ordered lexicographically, and $\sharp S$ denotes the total number of elements of a set $S$. We underline that:

(1) If each step $(V_i,(J_i,b),E_i) \leftarrow (V_{i+1},(J_{i+1},b),E_{i+1})$ in (14.8) is defined with center $Y_i \subset \underline{\mathrm{Max}}\ t_i$, then

$$(14.13)\qquad \max t^{(n)} \geq \max t_1^{(n)} \geq \cdots \geq \max t_m^{(n)}.$$

(2) If $\max t_m^{(n)} = (\frac{d}{b}, a)$, then $\max \text{w-ord}_m^{(n)} = \frac{d}{b}$. Clearly

$$\underline{\mathrm{Max}}\ t_m^{(n)} \subset \underline{\mathrm{Max}}\ \text{w-ord}_m^{(n)}.$$

Recall that the functions $t_i^{(n)}$ are defined only if $\max \text{w-ord}_i > 0$. We say that a sequence of transformations is $t^{(n)}$*-permissible* when $Y_i \subset \underline{\mathrm{Max}}\ t_i^{(n)}$ for all $i$. Similarly, a sequence is *w-permissible* when $Y_i \subset \underline{\mathrm{Max}}\ \text{w-ord}_i$ for all $i$.

We will show next that these functions derive from Hironaka's order function, and hence are invariants (i.e. they take the same value on weakly equivalent basic objects).

**14.5. Satellite functions derive from Hironaka's order function.** Let us draw attention here on the fact that the function $\text{w-ord}^{(n)}$ from (14.7), and the factorization in (14.6), grow from Hironaka's order function. Fix $H_{r+i}$ as in (14.6). Assume, for simplicity, that all steps in sequence (14.5) are permissible monoidal transformations with centers $Y_{i-1} \subset \text{Sing }(J_{i-1},b)$, for $i = 0,\ldots,m-1$. Then define the function $\exp_i$ along the points in Sing $(J_i,b)$ by setting

$$(14.14)\qquad \exp_i(x) = \begin{cases} \frac{b_i}{b} = \text{ord}_{Y_{i-1}} J_{i-1} - 1 & \text{if } x \in H_{r+i} \cap \text{Sing }(J_i,b); \\ \\ 0 & \text{otherwise}. \end{cases}$$

Since $Y_{i-1} \subset \text{Sing }(J_{i-1},b)$, one has that $b_i \geq 0$. So, we can express each rational number $\exp_i(x)$ in terms of the functions $\text{ord}_{(J_j,b)}$, for $j < i$. More precisely, in terms of the functions $\mathrm{ord}_{(J_j,b)}$ evaluated at the generic points, say $y_j$, of the centers $Y_j(\subset V_j)$ of the monoidal transformation. Finally note that, by induction $i$,

$$\text{w-ord}^{(n)}_{(J_i,b)}(x) = \mathrm{ord}_{(J_i,b)}(x) - \exp_1(x) - \exp_2(x) - \cdots - \exp_i(x).$$

Thus the satellite functions derive from Hironaka's order functions and hence they are invariants (they take the same value for any basic object $(V,(I,c),E)$ weakly equivalent to $(V,(J,b),E)$) (see 14.2).

**14.6. The role of satellite functions.** An $n$-dimensional basic object

$$(V,(J,b),E = \{H_1,\dots,H_r\})$$

is said *to be within the monomial case* if

$$J = \mathcal{I}(H_1)^{b_1}\cdots\mathcal{I}(H_r)^{b_r}, \tag{14.15}$$

for some $b_1,\dots,b_r \in \mathbb{N}$ (a Rees algebra is said to be within the monomial case if, up to integral closure, is of the form $\mathcal{O}_V[JW^b]$, with $J$ as in (14.15), see 14.2).

A resolution of a basic object that is within the monomial case can be achieved by means of a combinatorial argument (see [**EV1**]). Observe that the first coordinate of the $t^{(n)}$-function defined above measures how far a basic object is from being within the monomial case; while the second coordinate will ultimately ensure that once this monomial case is achieved all exceptional hypersurfaces in $E = \{H_1,\dots,H_r\}$ will end up having normal crossings. To illustrate this fact, we suggest the reader to see the example in Section 19. There it is clear from the beginning that the curve $C$ has to be the center of a blow up at some step of the resolution process (see expression (19.1)). However, to ensure that it is a permissible center first it has to be desingularized (here the w-ord-function plays a role), and then it needs to have normal crossings with the exceptional divisors (and here the second coordinate of the $t$-function plays a role).

Thus the goal is to successively lower the maximum value of $t^{(n)}$ by constructing a suitable finite sequence of transformations with centers contained in $\underline{\mathrm{Max}}\ t^{(n)}$. This will be shown to follow from an inductive argument on the codimensional type of the given basic object (this is the key point in Proposition 18.2).

However, this strategy deserves some clarification. Sometimes we will not reach the monomial case because this case shows up in lower dimensions. For instance, suppose $K$ is a field of characteristic zero. If $V = \mathrm{Spec}\ K[x,y,z]$ and $\mathcal{G}$ is the Rees algebra generated by $x$ and $(y^3-z^2)$ in degree one, then Sing $\mathcal{G}$ is contained in the codimension one smooth scheme $\{x=0\}$. In this case, the algorithm of resolution will produce a finite sequence of permissible transformations so that Sing $\mathcal{G}$ is supported on the zero set of a monomial ideal *within the strict transform of* $\{x=0\}$ (see Theorem 18.7 for the general statement). In fact, after three blow ups at closed points, $V \longleftarrow V_3$, it can be checked that the transform of $\mathcal{G}$ in $V_3$ is

$$\mathcal{G}_3 = \mathcal{O}_{V_3}[\mathcal{I}(X_3)W, \mathcal{I}(H_1)\mathcal{I}(H_2)\mathcal{I}(H_3)^2\mathcal{I}(C_3)W]$$

where $\mathcal{I}(X_3)$ denotes the ideal of the strict transform of $\{x=0\}$ in $V_3$, $C_3$ denotes the ideal of the strict transform of $\{y^3-z^2=0\}$ in $V_3$, and $H_1,H_2,H_3$ denote the exceptional divisors. A further blow up at $X_3\cap C_3$ leads to the monomial case.

This example also illustrates that $\underline{\mathrm{Max}}\ t^{(n)}$ may not be smooth (in the example, $\underline{\mathrm{Max}}\ t^{(3)} = \{x=0, y^3-z^2=0\}$). Thus, in general the $t^{(n)}$-function needs to be refined in order to define a resolution function (see 12.7). This refinement can be obtained using the theory of elimination which will be exposed in the following sections (see 18.6 for a hint on how the $t^{(n)}$-functions are refined via elimination).

## Elimination in the class of Rees Algebras

As was previously indicated, for a given $n$-dimensional basic object $(V, \mathcal{G}, E)$, the definition of the satellite function $t^{(n)}$ is a first step towards the construction of a resolution function (see 12.7 and 14.6). In this part of the notes we will see how the theory of *elimination* can be used to refine the function $t^{(n)}$. In what follows, all base fields are assumed to be perfect. Also, we will use the notation $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ for an $n$-dimensional basic object.

### 15. Elimination algebras: a motivation

Suppose we are given an $n$-dimensional pair $(V^{(n)}, \mathcal{G}^{(n)})$. We would like to, somehow, assign a new pair $(V^{(d)}, \mathcal{G}^{(d)})$, with $d \leq n$ (most hopefully with $d < n$), and to find a smooth surjective morphism $\beta : V^{(n)} \to V^{(d)}$ so that the following conditions hold:

(1) The morphism $\beta$ induces a homeomorphism between $\operatorname{Sing} \mathcal{G}^{(n)}$ and $\operatorname{Sing} \mathcal{G}^{(d)}$;

(2) Any $\mathcal{G}^{(d)}$-local sequence (see Remark 11.7):

$$(V^{(d)}, \mathcal{G}^{(d)}) = (V_0^{(d)}, \mathcal{G}_0^{(d)}) \leftarrow (V_1^{(d)}, \mathcal{G}_1^{(d)}) \leftarrow \dots (V_r^{(d)}, \mathcal{G}_m^{(d)})$$

induces a $(V^{(n)}, \mathcal{G}^{(n)})$-local sequence

$$(V^{(n)}, \mathcal{G}^{(n)}) = (V_0^{(n)}, \mathcal{G}_0^{(n)}) \leftarrow (V_1^{(n)}, \mathcal{G}_1^{(n)}) \leftarrow \dots (V_r^{(n)}, \mathcal{G}_r^{(n)})$$

and smooth surjective morphisms together with commutative diagrams

$$\begin{array}{ccccccc}
\mathcal{G}^{(n)} = \mathcal{G}_0^{(n)} & & \mathcal{G}_1^{(n)} & & & & \mathcal{G}_m^{(n)} \\
V^{(n)} = V_0^{(n)} & \xleftarrow{\rho_0} & V_1^{(n)} & \xleftarrow{\rho_1} & \cdots & \xleftarrow{\rho_{m-1}} & V_m^{(n)} \\
\downarrow \beta & & \downarrow \beta_1 & & & & \downarrow \beta_m \\
V^{(d)} = V_0^{(d)} & \xleftarrow{\overline{\rho}_0} & V_1^{(d)} & \xleftarrow{\overline{\rho}_1} & \cdots & \xleftarrow{\overline{\rho}_{m-1}} & V_m^{(d)} \\
\mathcal{G}^{(d)} = \mathcal{G}_0^{(d)} & & \mathcal{G}_1^{(d)} & & & & \mathcal{G}_m^{(d)}
\end{array} \tag{15.1}$$

and so that for $i = 1, \dots, m$, the smooth surjective morphisms $\beta_i$ induce homeomorphisms between $\operatorname{Sing} \mathcal{G}_i^{(n)}$ and $\operatorname{Sing} \mathcal{G}_i^{(d)}$.

(3) Any $\mathcal{G}^{(n)}$-local sequence

$$(V^{(n)}, \mathcal{G}^{(n)}) = (V_0^{(n)}, \mathcal{G}_0^{(n)}) \leftarrow (V_1^{(n)}, \mathcal{G}_1^{(n)}) \leftarrow \dots (V_r^{(n)}, \mathcal{G}_m^{(n)})$$

induces a $\mathcal{G}^{(d)}$-local sequence

$$(V^{(d)}, \mathcal{G}^{(d)}) = (V_0^{(d)}, \mathcal{G}_0^{(d)}) \leftarrow (V_1^{(d)}, \mathcal{G}_1^{(d)}) \leftarrow \dots (V_r^{(d)}, \mathcal{G}_m^{(d)})$$

and commutative diagrams

$$(15.2)\qquad \begin{array}{ccccccc}
\mathcal{G}^{(n)} = \mathcal{G}_0^{(n)} & & \mathcal{G}_1^{(n)} & & & & \mathcal{G}_m^{(n)} \\
V^{(n)} = V_0^{(n)} & \xleftarrow{\rho_0} & V_1^{(n)} & \xleftarrow{\rho_1} & \cdots & \xleftarrow{\rho_{m-1}} & V_m^{(n)} \\
\downarrow \beta & & \downarrow \beta_1 & & & & \downarrow \beta_m \\
V^{(d)} = V_0^{(d)} & \xleftarrow{\overline{\rho}_0} & V_1^{(d)} & \xleftarrow{\overline{\rho}_1} & \cdots & \xleftarrow{\overline{\rho}_{m-1}} & V_m^{(d)} \\
\mathcal{G}^{(d)} = \mathcal{G}_0^{(d)} & & \mathcal{G}_1^{(d)} & & & & \mathcal{G}_m^{(d)}
\end{array}$$

so that for $i = 1, \dots, m$, the morphisms $\beta_i$ are smooth and surjective, and moreover, induce homeomorphisms between Sing $\mathcal{G}_i^{(n)}$ and Sing $\mathcal{G}_i^{(d)}$.

It follows from conditions (1)-(3) that a resolution of $(V^{(n)}, \mathcal{G}^{(n)})$ induces a resolution of $(V^{(d)}, \mathcal{G}^{(d)})$ and vice versa. If $d$ happens to be strictly smaller than $n$, we can think that we pass from $(V^{(n)}, \mathcal{G}^{(n)})$ to $(V^{(d)}, \mathcal{G}^{(d)})$ by *eliminating variables*.

If given $(V^{(n)}, \mathcal{G}^{(n)})$ we can find some $(V^{(d)}, \mathcal{G}^{(d)})$ as above, Hironaka's order function $\operatorname{ord}_{\mathcal{G}^{(n)}}$ can be refined with the function $\operatorname{ord}_{\mathcal{G}^{(d)}}$ via the identification of their singular loci. This in turn can be used to refine Hironaka's order function (and thus the satellite functions defined on $(V^{(n)}, \mathcal{G}^{(n)})$).

On the other hand notice that if we are able to assign a pair $(V^{(d)}, \mathcal{G}^{(d)})$ to a given pair $(V^{(n)}, \mathcal{G}^{(n)})$ as above, then $(V^{(d)}, \mathcal{G}^{(d)})$ can also be assigned to any other pair of the form $(V^{(n)}, \mathcal{G}'^{(n)})$ so far as $\mathcal{G}^{(n)}$ is weakly equivalent to $\mathcal{G}'^{(n)}$.

Example 15.1. Suppose $\varphi : V^{(n)} \longrightarrow V^{(d)}$ is a smooth surjective morphism of smooth spaces that has a section $s : V^{(d)} \longrightarrow V^{(n)}$, and let $\mathcal{G}^{(d)}$ be an $\mathcal{O}_{V^{(d)}}$-Rees algebra. Then the image of $V^{(d)}$ by $s$ is a smooth closed subscheme of $V^{(n)}$, defined by a sheaf of ideals say $I \subset \mathcal{O}_{V^{(n)}}$. Now define

$$\mathcal{G}^{(n)} := \mathcal{O}_{V^{(n)}}[IW] \odot \alpha^*(\mathcal{G}^{(d)}).$$

Then it can be checked that the pairs $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)})$ satisfy conditions (1)-(3).

Example 15.2. Suppose $\varphi : V^{(n)} \longrightarrow V^{(d)}$ is a smooth surjective morphism of smooth spaces, and let $\mathcal{G}^{(d)}$ be an $\mathcal{O}_{V^{(d)}}$-Rees algebra. Now let $\xi \in$ Sing $\mathcal{G}^{(d)}$ be a closed point, choose a regular system of parameters $\{x_1, \dots, x_d\} \subset \mathcal{O}_{V^{(d)},\xi}$, and select some closed point $\xi'$ in the fiber of $\xi$. Since $\varphi^* : \mathcal{O}_{V^{(d)},\xi} \to \mathcal{O}_{V^{(n)},\xi'}$ is smooth, $\{x_1, \dots, x_d\}$ can be extended to some regular system of parameters $\{x_1, \dots, x_d, x_{d+1}, \dots, x_n\} \subset \mathcal{O}_{V^{(n)},\xi'}$. The ideal $\langle x_{d+1}, \dots, x_n \rangle \subset \mathcal{O}_{V^{(n)},\xi'}$ defines a regular closed subscheme $C$ in some neighborhood $U$ of $\xi'$, and there is a diagram

$$\begin{array}{ccc}
 & & U \subset V^{(n)} \\
 & \nearrow & \downarrow \varphi \\
C & \longrightarrow & \varphi(U) \subset V^{(d)}
\end{array}$$

where the horizontal map is étale. Thus taking the fiber product,

$$\begin{array}{ccc}
C \times_{\varphi(U)} U & \longrightarrow & U \\
\downarrow & \nearrow & \downarrow \varphi \\
C & \longrightarrow & \varphi(U)
\end{array}$$

we get vertical smooth maps, horizontal étale maps and a section $s : C \to C \times_{\varphi(U)} V^{(n)}$. Now using the section and the pull back of $\mathcal{G}^{(d)}$ to $C$, the argument of the previous example can be repeated, and a new Rees algebra $\mathcal{G}^{(n)}$ can be defined in étale neighborhood of $\xi'$. Now, setting $U^{(n)} = C \times_{\varphi(U)} U$ it can be checked that the pairs $(U^{(n)}, \mathcal{G}^{(n)})$ and $(C, \varphi^*(\mathcal{G}^{(d)})$ satisfy properties (1)-(3).

Despite the examples, our problem is rather the opposite: our data will be $(V^{(n)}, \mathcal{G}^{(n)})$ and we will want to find $(V^{(d)}, \mathcal{G}^{(d)})$ with $d < n$ satisfying properties (1)-(3) from above. This can be accomplished when the characteristic is zero using elimination algebras (see Section 16). We will see in Remark 16.10, that, at least when the characteristic of the base field is zero, the link between $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)})$ will be that described in Example 15.2.

## 16. Elimination algebras: first properties

Let $V^{(n)}$ be a smooth $n$-dimensional scheme over a perfect field, and let $\mathcal{G}^{(n)}$ be an $\mathcal{O}_{V^{(n)}}$-Rees algebra. Suppose that $\tau_{\mathcal{G}^{(n)}} \geq e \geq 1$. Then Sing $\mathcal{G}^{(n)}$ can be expressed as a disjoint union

$$\text{Sing } \mathcal{G}^{(n)} = F_e \sqcup Z$$

where $F_e$ denotes the union of the codimension-$e$ components of Sing $\mathcal{G}^{(n)}$. As indicated in 13.3 (5), the set $F_e$ is a disjoint union of smooth closed subschemes. Thus $F_e$ is the canonical center to blow-up. Recall that $\mathcal{G}^{(n)}$ is said to be $e$-trivial at $\xi \in$ Sing $\mathcal{G}^{(n)}$ if $\xi \in F_e$. From the point of view of resolution we need to focus on points of Sing $\mathcal{G}^{(n)}$ that are non-$e$-trivial. This is the purpose of the following discussion.

**16.1. Transversality.** Let $\beta : V^{(n)} \to V^{(n-e)}$ be a smooth morphism of smooth schemes of dimensions $n$ and $(n-e)$ respectively, with $1 \leq e \leq n$. For each closed point $\xi \in V^{(n)}$ denote by $d\beta_\xi : \mathbb{T}_{V^{(n)},\xi} \to \mathbb{T}_{V^{(n-e)},\beta(\xi)}$ the linear (surjective) map induced on the corresponding tangent spaces. Let $\mathcal{G}^{(n)} = \bigoplus_n I_n W^n$ be an $\mathcal{O}_{V^{(n)}}$-Rees algebra, and assume that $\xi \in$ Sing $\mathcal{G}^{(n)}$ is a closed point with $\tau_{\mathcal{G}^{(n)},\xi} \geq e \geq 1$. We say that $\beta : V^{(n)} \to V^{(n-e)}$ is *transversal to* $\mathcal{G}^{(n)}$ *at* $\xi$ if the subspaces $\mathcal{L}_{\mathcal{G}^{(n)},\xi}$ from 13.2, and $\ker(d\beta_\xi)$ intersect at $\mathbb{O}$ in the vector space $\mathbb{T}_{V^{(n)},\xi}$ (recall that the codimension of $\mathcal{L}_{\mathcal{G}^{(n)},\xi}$ in $\mathbb{T}_{V^{(n)},\xi}$ coincides with $\tau_{\mathcal{G}^{(n)},\xi}$; see 13.2.2). We will say that $\beta : V^{(n)} \to V^{(n-e)}$ is *transversal to* $\mathcal{G}^{(n)}$ if it is transversal at every point of $\xi \in$ Sing $\mathcal{G}^{(n)}$.

For a given $\mathcal{O}_{V^{(n)}}$-Rees algebra $\mathcal{G}^{(n)}$ and a point $\xi \in$ Sing $\mathcal{G}^{(n)}$ with $\tau_{\mathcal{G}^{(n)},\xi} \geq e \geq 1$, it is possible to construct a smooth morphism $\beta : V^{(n)} \to V^{(n-e)}$ transversal to $\mathcal{G}^{(n)}$ at every point in some neighborhood $\xi$ (here we may use étale topology, see [**BrV1**, §8]).

If $\beta : V^{(n)} \to V^{(n-e)}$ is transversal to $\mathcal{G}^{(n)}$, then it can be shown that:

1. Sing $\mathcal{G}^{(n)}$ and $\beta$(Sing $\mathcal{G}^{(n)}$) are homeomorphic;
2. If a closed subset $Y \subset$ Sing $\mathcal{G}^{(n)}$ is smooth then $\beta(Y) \subset \beta$(Sing $\mathcal{G}^{(n)}$) is smooth.

See [**BrV1**, 8.4].

**16.2. Transversality, local transformations, and commutative diagrams.** Let $\mathcal{G}^{(n)}$ be an $\mathcal{O}_{V^{(n)}}$-Rees algebra and suppose that $\beta : V^{(n)} \longrightarrow V_1^{(n)}$ is a smooth surjective morphism transversal to $\mathcal{G}^{(n)}$. We would like to show that any $\mathcal{G}^{(n)}$-local transformation, say $V^{(n)} \longleftarrow V_1^{(n)}$ induces a local transformation $V^{(n-e)} \longleftarrow V_1^{(n-e)}$ and a commutative diagram of vertical smooth surjective morphisms:

$$\begin{array}{ccc} V^{(n)} & \xleftarrow{\rho^{(n)}} & V_1^{(n)} \\ \downarrow \beta & \circlearrowleft & \downarrow \beta_1 \\ V^{(n-e)} & \xleftarrow{\rho^{(n-e)}} & V_1^{(n-e)}, \end{array} \tag{16.1}$$

where $\beta_1 : V_1^{(n)} \longrightarrow V_1^{(n-e)}$ transversal to the transform $\mathcal{G}_1^{(n)}$ of $\mathcal{G}^{(n)}$.

According to Remark 11.7, it can be assumed that $V^{(n)} \longleftarrow V_1^{(n)}$ is one of the following transformations:

(1) The multiplication of $V^{(n)}$ by an affine line $\mathbb{A}_k^1$;
(2) The restriction to some open subset of $V^{(n)}$;
(3) The blow up at a permissible center $Y \subset \mathcal{G}^{(n)}$.

The existence of a commutative diagram like (16.1) is clear for the transformations in (1). Regarding to the transformations in (2), in which case $\rho^{(n)}(V_1^{(n)}) \subset V^{(n)}$ is open, it suffices to define $V_1^{(n-e)}$ as $\beta(V_1^{(n)})$ which will be open in $V^{(n-e)}$ since $\beta$ is smooth, and hence flat.

Now suppose that $V^{(n)} \longleftarrow V_1^{(n)}$ is the blow up at a smooth closed $Y \subset \operatorname{Sing} \mathcal{G}^{(n)}$. Then $\beta(Y)$ is also smooth and closed (see 16.1 (2) above), and thus $V^{(n-e)} \longleftarrow V_1^{(n-e)}$ is defined as the blow up of $V^{(n-e)}$ at $\beta(Y)$. However, in general it is not possible to obtain a commutative diagram with vertical transversal morphisms as (16.1) unless we restrict to a suitable open subset $U_1^{(n)} \subset V_1^{(n)}$ containing the singular locus of $\operatorname{Sing} \mathcal{G}_1^{(n)}$. Then, $V_1^{(n-e)}$ can be interpreted as the (open subset) obtained from the image of $U_1^{(n)}$ by $\beta_1$. Under these assumptions, it can be checked that $\beta_1$ is transversal to $\mathcal{G}_1^{(n)}$ (see [**BrV1**, §9]). In particular, this implies that $\operatorname{Sing} \mathcal{G}_1^{(n)}$ and $\beta_1(\operatorname{Sing} \mathcal{G}_1^{(n)})$ are homeomorphic.

From this discussion it follows that any $\mathcal{G}^{(n)}$-local sequence,

$$\begin{array}{ccccccc} V^{(n)} = V_0^{(n)} & \longleftarrow & V_1^{(n)} & \longleftarrow & \dots & \longleftarrow & V_r^{(n)} \\ \mathcal{G}^{(n)} = \mathcal{G}_0^{(n)} & & \mathcal{G}_1^{(n)} & & \dots & & \mathcal{G}_r^{(n)}. \end{array} \tag{16.2}$$

induces a local sequence

$$V^{(n-e)} = V_0^{(n-e)} \longleftarrow V_1^{(n-e)} \longleftarrow \dots \longleftarrow V_r^{(n-e)}. \tag{16.3}$$

and commutative diagrams of horizontal local sequences and vertical transversal smooth surjective projections, say,

$$\begin{array}{ccccccc}
\mathcal{G}^{(n)} = \mathcal{G}_0^{(n)} & & \mathcal{G}_1^{(n)} & & & & \mathcal{G}_m^{(n)} \\
V^{(n)} = V_0^{(n)} & \xleftarrow{\rho_0} & V_1^{(n)} & \xleftarrow{\rho_1} & \cdots & \xleftarrow{\rho_{m-1}} & V_m^{(n)} \\
\downarrow \beta & & \downarrow \beta_1 & & & & \downarrow \beta_m \\
V^{(d)} = V_0^{(d)} & \xleftarrow{\overline{\rho}_0} & V_1^{(d)} & \xleftarrow{\overline{\rho}_1} & \cdots & \xleftarrow{\overline{\rho}_{m-1}} & V_m^{(d)},
\end{array}$$

where it is understood that if $V_i^{(n)} \longleftarrow V_{i+1}^{(n)}$ is the blow up at a permissible $Y_i \subset \operatorname{Sing} \mathcal{G}_i^{(n)}$, then $V_{i+1}^{(n)}$ is in fact a suitable open subset of the blow up of $V_i^{(n)}$ at $Y_i$ that contains $\operatorname{Sing} \mathcal{G}_{i+1}^{(n)}$.

**16.3. Absolute (and relative) differential Rees algebras.** Suppose that a smooth morphism between smooth schemes as above, $\beta : V^{(n)} \to V^{(n-e)}$ is transversal to a given $\mathcal{O}_{V^{(n)}}$-Rees algebra $\mathcal{G}^{(n)}$. We will say that $\mathcal{G}^{(n)}$ is a *$\beta$-relative differential Rees algebra* if $\mathcal{G}^{(n)}$ is closed under the action of the sheaf of relative differential operators $\operatorname{Diff}_{V^{(n)}/V^{(n-e)}}$. The requirement in this latter condition is similar to that formulated in Definition 11.1, where for any

$$D \in Diff^r_{V^{(n)}/V^{(n-e)}}(U_i)$$

and any $h \in I_n(U_i)$ we have that $D(h) \in I_{n-r}(U_i)$ provided that $n \geq r$. Obviously, any absolute differential Rees algebra has this property.

Suppose that $\mathcal{G}^{(n)}$ is a $\beta$-relative differential Rees algebra and consider a local transformation as in Remark 11.7, $V^{(n)} \leftarrow V_1^{(n)}$. As before, there is a commutative diagram:

$$\begin{array}{ccc}
V^{(n)} & \xleftarrow{\rho^{(n)}} & V_1^{(n)} \\
\downarrow \beta & \circlearrowleft & \downarrow \beta_1 \\
V^{(n-e)} & \xleftarrow{\rho^{(n-e)}} & V_1^{(n-e)},
\end{array} \tag{16.4}$$

with smooth vertical arrows and horizontal local transformations (see 16.2). Denote by $\mathcal{G}_1^{(n)}$ the weak transform of $\mathcal{G}^{(n)}$ in $V_1^{(n)}$. Then it can be proved that, not only $\beta_1$ is transversal to $\mathcal{G}_1^{(n)}$, but also $\mathcal{G}_1^{(n)}$ is $\beta_1$-relative differential (see [**BrV1**, §9]).

To define an elimination algebra for a given Rees algebra $\mathcal{G}^{(n)}$ we will make use of suitable transversal smooth projections, and the structure of relative differential Rees algebra of $\mathcal{G}^{(n)}$. On the other hand, typically our starting point will be a differential Rees algebra $\mathcal{G}^{(n)}$ which will later undergo a local sequence of transformations. The permissible transform of an absolute differential algebra may not be an absolute differential algebra any more. However, as we have indicated, the relative differential structure of $\mathcal{G}^{(n)}$ is preserved by local transformations. This will allow us to define an elimination algebra at each step of a given local sequence.

**16.4. Admissible projections.** Let $\mathcal{G}^{(n)}$ be an $\mathcal{O}_{V^{(n)}}$-Rees algebra, and let $\xi \in \operatorname{Sing} \mathcal{G}^{(n)}$ be a closed point with $\tau_{\mathcal{G}^{(n)},\xi} \geq e \geq 1$ (i.e., $\xi$ is an $e$-simple point). A smooth projection to some $(n-e)$ dimensional smooth scheme,

$$\beta : V^{(n)} \to V^{(n-e)},$$

is said to be $\mathcal{G}^{(n)}$-*admissible at* $\xi$ if the following conditions hold:

(1) The point $\xi$ is not contained in a codimension $e$-component of Sing $\mathcal{G}^{(n)}$ (i.e., $\mathcal{G}^{(n)}$ is non-$e$-trivial at $\xi$);
(2) The morphism $\beta$ is transversal to $\mathcal{G}^{(n)}$;
(3) The Rees algebra $\mathcal{G}^{(n)}$ is closed by the action of the sheaf of relative differential operators $\mathrm{Diff}_{V^{(n)}/V^{(n-e)}}$, i.e., $\mathcal{G}^{(n)}$ is a $\beta$-relative differential algebra.

It can be shown that if a morphism $\beta$ is $\mathcal{G}^{(n)}$-admissible at a point $\xi$, then it is admissible in a neighborhood $U$ of $\xi$ (see [**BrV1**, Remark 8.5]).

As previously indicated, if $\beta : V^{(n)} \to V^{(n-e)}$ is admissible for $\mathcal{G}^{(n)}$, then it is also admissible for the absolute saturation $Diff(\mathcal{G}^{(n)})$, typically our starting point.

Given a differential Rees algebra $\mathcal{G}^{(n)}$ and a closed point $\xi \in$ Sing $\mathcal{G}^{(n)}$ with $\tau_{\mathcal{G}^{(n)},\xi} \geq e \geq 1$, it is not hard to construct a $\mathcal{G}^{(n)}$-admissible projection to some smooth $(n-e)$ dimensional smooth scheme in an étale neighborhood of $\xi$ (see [**BrV1**, §8]). Such an admissible projection will be preserved by permissible blow ups; and this is the context in which elimination theory for Rees algebras holds.

**16.5. Elimination algebras.** Let $\mathcal{G}^{(n)}$ be a Rees algebra on $V^{(n)}$ and consider a closed point $\xi \in$ Sing $\mathcal{G}^{(n)}$ with $\tau_{\mathcal{G}^{(n)},\xi} \geq e \geq 1$. Given a $\mathcal{G}^{(n)}$-admissible $\beta$ in a neighborhood of $\xi$,

$$\beta : V^{(n)} \to V^{(n-e)},$$

one can define an *elimination algebra* $\mathcal{G}^{(n-e)} \subset \mathcal{O}_{V^{(n-e)}}[W]$ (see [**Vi4**, 1.25, Definitions 1.42 and 4.10], [**BrV1**, 8.11], for more details; related ideas in the case of finite morphisms are discussed in Sections 6, 7, and 8 of this manuscript). If $\xi_e = \beta(\xi)$, then it can be shown that the inclusion $\beta^* : \mathcal{O}_{V^{(n-e)},\xi_e} \to \mathcal{O}_{V^{(n)},\xi}$ induces an inclusion of Rees algebras $\mathcal{G}^{(n-e)} \subset \mathcal{G}^{(n)}$, and $\mathcal{G}^{(n-e)}$ can be defined as the largest $\mathcal{O}_{V^{(n-e)}}$-Rees algebra contained in $\mathcal{G}^{(n)}$ ([**Vi4**, Theorem 4.13]). The reader can find further results and applications of elimination algebras over perfect fields in [**B**], [**BV1**], and [**BV2**].

EXAMPLE 16.6. Let $V^{(2)} = \mathrm{Spec}\ K[x,y]$, and let $\mathcal{G}^{(2)}$ be the differential Rees algebra generated by $x^2-y^3$ in degree two. Then it can be checked that projection induced by the natural inclusion $K[y] \to K[x,y]$ is $\mathcal{G}^{(2)}$-admissible. Now, if char $K \neq 2$, then $\mathcal{G}^{(2)} = K[x,y][xW, y^2W, y^3W^2]$ and the corresponding elimination algebra is $\mathcal{G}^{(1)} = K[y][y^2W, y^3W^2]$. If char $K = 2$ then $\mathcal{G}^{(2)} = K[x,y][y^2W, (x^2-y^3)W^2]$ and $\mathcal{G}^{(1)} = K[y][y^2W]$.

**16.7. First properties of elimination algebras.** The elimination algebra depends on the projection $\beta$, but still it satisfies important properties. With the same notation as in 16.5, one has that:

(1) Sing $\mathcal{G}^{(n)}$ maps injectively into Sing $\mathcal{G}^{(n-e)}$, in particular
$$\beta(\mathrm{Sing}\ \mathcal{G}^{(n)}) \subset \mathrm{Sing}\ \mathcal{G}^{(n-e)}$$
with equality if the characteristic is zero, or if $\mathcal{G}^{(n)}$ is a differential Rees algebra, and in this case the sets are homeomorphic (cf. [**Vi4**, §4]). Moreover, Sing $\mathcal{G}^{(n)}$ and $\beta(\mathrm{Sing}\ \mathcal{G}^{(n)})$ are homeomorphic (see 16.1).
(2) The homeomorphism from Sing $\mathcal{G}^{(n)}$ to $\beta(\mathrm{Sing}\ \mathcal{G}^{(n)})$ has the following properties:

If $Z \subset \operatorname{Sing} \mathcal{G}^{(n-e)}$ is a smooth closed subscheme, then $\beta^{-1}(Z)_{\mathrm{red}} \cap \operatorname{Sing} \mathcal{G}^{(n)}$ is smooth; and if $Y \subset \operatorname{Sing} \mathcal{G}^{(n)}$ is a smooth closed subscheme, then so is $\beta(Y) \subset \operatorname{Sing} \mathcal{G}^{(n-e)}$ ([**BrV1**, 8.4], [**Vi3**, Lemma 1.7]).

(3) If $\mathcal{G}^{(n)}$ is a differential Rees algebra, then so is $\mathcal{G}^{(n-e)}$ (see [**Vi4**, Corollary 4.14]).

(4) If $\mathcal{G}^{(n)} \subset \mathcal{G}'^{(n)}$ is a finite extension, then $\mathcal{G}^{(n-e)} \subset \mathcal{G}'^{(n-e)}$ is a finite extension (see [**Vi4**, Theorem 4.11]).

(5) If $\mathcal{G}^{(n)}$ is a differential algebra, then $\tau_{\mathcal{G}^{(n-e)},\beta(\xi)} = \tau_{\mathcal{G}^{(n)},\xi} - e$; in particular, if $e \geq 2$, then $\tau_{\mathcal{G}^{(n-1)},\beta(\xi)} \geq 1$ (cf. [**B**, Theorem 6.4]). In particular if $\mathcal{G}^{(n)}$ is $e$-simple, with $e \geq 2$, then $\mathcal{G}^{(n-1)}, \dots, \mathcal{G}^{(n-e+1)}$ are simple. The claim follows from Remark 16.10 in characteristic zero.

(6) Using (3) one readily checks that for any $\mathcal{G}^{(n)}$-local sequence with smooth morphisms as in Remark 11.7, there are commutative diagrams

$$\begin{array}{ccccccc}
\mathcal{G}^{(n)} = \mathcal{G}_0^{(n)} & & \mathcal{G}_1^{(n)} & & & & \mathcal{G}_m^{(n)} \\
V^{(n)} = V_0^{(n)} & \xleftarrow{\rho_0} & V_1^{(n)} & \xleftarrow{\rho_1} & \cdots & \xleftarrow{\rho_{m-1}} & V_m^{(n)} \\
\downarrow \beta & & \downarrow \beta_1 & & & & \downarrow \beta_m \\
V^{(n-e)} = V_0^{(n-e)} & \xleftarrow{\overline{\rho}_0} & V_1^{(n-e)} & \xleftarrow{\overline{\rho}_1} & \cdots & \xleftarrow{\overline{\rho}_{m-1}} & V_m^{(n-e)} \\
\mathcal{G}^{(n-e)} = \mathcal{G}_0^{(n-e)} & & \mathcal{G}_1^{(n-e)} & & & & \mathcal{G}_m^{(n-e)}
\end{array} \tag{16.5}$$

of transversal projections and transforms, such that for $i = 1, \dots, m$:

(a) If $V_{i-1}^{(n)} \overset{\rho_{i-1}}{\longleftarrow} V_i^{(n)}$ is a permissible transformation with center

$$Y_{i-1} \subset \operatorname{Sing} \mathcal{G}_{i-1}^{(n)},$$

then $V_{i-1}^{(n-e)} \overset{\overline{\rho}_{i-1}}{\longleftarrow} V_i^{(n-e)}$ is the permissible blow up at $\beta_{i-1}(Y_{i-1})$ and $\beta_i : V_i^{(n)} \longrightarrow V_i^{(n-e)}$ is $\mathcal{G}_i^{(n)}$-admissible in an open subset $U_i \subset V_i^{(n)}$ containing $\operatorname{Sing} \mathcal{G}_i^{(n)}$.

(b) The Rees algebra $\mathcal{G}_i^{(n-e)}$ is an elimination algebra of $\mathcal{G}_i^{(n)}$ (i.e., the transform of an elimination algebra of a given Rees algebra $\mathcal{G}^{(n)}$ is the elimination algebra of the transform of $\mathcal{G}^{(n)}$);

(c) There is an inclusion of closed sets:

$$\beta_i(\operatorname{Sing} \mathcal{G}_i^{(n)}) \subseteq \operatorname{Sing} \mathcal{G}_i^{(n-e)}, \tag{16.6}$$

and $\operatorname{Sing} \mathcal{G}_i^{(n)}$ and $\beta_i(\operatorname{Sing} \mathcal{G}_i^{(n)})$ are homeomorphic. If the characteristic is zero then the inclusion (16.6) is an equality (see Example 16.9 to see that in positive characteristic the inclusion may be strict).

See [**BrV1**, Theorem 9.1]. See also 16.1.

(7) Conversely, if the characteristic is zero, any $\mathcal{G}^{(n-e)}$-local sequence with smooth morphisms as in Remark 11.7 induces a $\mathcal{G}^{(n)}$-local sequence and commutative diagrams of transversal projections and transforms of Rees algebras as in (16.5) satisfying properties (a), (b) and (c) as above.

**16.8. Conclusion.** From 16.7 if follows that, at least when the characteristic is zero, elimination algebras fulfill properties (1)-(3) from Section 15. In fact, given an $n$-dimensional pair $(V^{(n)}, \mathcal{G}^{(n)})$ and once a $\mathcal{G}^{(n)}$-admissible projection is fixed in a neighborhood of a point $\xi \in \operatorname{Sing} \mathcal{G}^{(n)}$, say $\beta : V^{(n)} \longrightarrow V^{(d)}$, $d < n$, the elimination algebra is the unique $\mathcal{O}_{V^{(d)}}$-Rees algebra (up to weak equivalence) that fullfils properties (1)-(3) from Section 15.

EXAMPLE 16.9. The inclusion in 16.7 (6.c) may be strict if the characteristic of the base field is positive. For instance, suppose that $K$ is a field of characteristic 3, let $V^{(2)} = \text{Spec } K[x, z]$ and let $\mathcal{G}^{(2)}$ be the differential Rees algebra generated by $x^3 + z^5$ in degree 3, i.e., up to integral closure, $\mathcal{G}^{(2)} = K[x, z][z^2W, (x^3 + z^5)W^3]$. Then it can be checked that the projection $\beta$ induced by the natural inclusion $K[z] \to K[x, z]$ is $\mathcal{G}^{(2)}$-admissible, and that $\mathcal{G}^{(1)} = K[x, z][z^2W]$. Then $\beta(\text{Sing } (\mathcal{G}^{(2)})) = \text{Sing } (\mathcal{G}^{(1)})$. Now let $V^{(2)} \longleftarrow V_1^{(2)}$, respectively $V^{(1)} \longleftarrow V_1^{(1)}$, be the blow up at the origin, and let $\mathcal{G}_1^{(2)}$, respectively, $\mathcal{G}_1^{(1)}$ be the transforms of $\mathcal{G}^{(2)}$ and $\mathcal{G}^{(1)}$. Then $\text{Sing } (\mathcal{G}_1^{(2)}) = \emptyset$ while $\text{Sing } (\mathcal{G}_1^{(1)}) \neq \emptyset$.

REMARK 16.10. (*Elimination algebras over fields of characteristic zero.*) The elimination algebra introduced in 16.5 has the following characterization over fields of characteristic zero (compare to Example 15.2). Let $\mathcal{G}^{(n)}$ be an $\mathcal{O}_{V^{(n)}}$-differential Rees algebra, let $\beta : V^{(n)} \to V^{(n-e)}$ be a $\mathcal{G}^{(n)}$-admissible morphism in a neighborhood of a point $\xi \in \text{Sing } \mathcal{G}^{(n)}$ with $\tau_{\mathcal{G}^{(n-e)},\xi} \geq e \geq 1$, and let $\mathcal{G}^{(n-e)}$ be the corresponding elimination algebra.

**Claim 1.** *The pairs* $(V^{(n)}, \mathcal{G}^{(n)})$ *and* $(V^{(n-e)}, \mathcal{G}^{(n-e)})$ *are linked as follows: a regular system of parameters* $\{x_1, \dots, x_{n-e}\} \in \mathcal{O}_{V^{(n-e)},\beta(\xi)}$, *can be extended to a regular system of parameters,* $\{x_1, \dots, x_{n-e}, x_{n-e+1}, \dots, x_n\} \in \mathcal{O}_{V^{(n)},\xi}$ *so that, up to weak equivalence*

$$\mathcal{G}^{(n)} = \mathcal{O}_{V^{(n)},\xi}[x_{n-e+1}W, \dots, x_nW] \odot \beta^*(\mathcal{G}^{(n-e)}) \tag{16.7}$$

*in a neighborhood of* $\xi$ *(see (9.1)).*

Taking the claim for granted it can be checked that the zero set of the ideal spanned by $\{x_{n-e+1}, \dots, x_n\}$ is the image of a section of $\beta : V^{(n)} \to V^{(n-e)}$ in some étale neighborhood of $(V^{(n-e)}, \beta(\xi))$.

To show that a presentation like (16.7) can be found we proceed as follows. Let $\beta : V^{(n)} \to V^{(n-e)}$ be a $\mathcal{G}^{(n)}$-admissible morphism in a neighborhood of a point $\xi \in \text{Sing } \mathcal{G}^{(n)}$, and let $\mathcal{G}^{(n-e)}$ be the corresponding elimination algebra.

Since $\beta$ is $\mathcal{G}^{(n)}$-admissible in a neighborhood of $\xi \in \text{Sing } \mathcal{G}^{(n)}$ one has that

$$\mathcal{L}_{\mathcal{G}^{(n)},\xi} \cap \ker(d\beta_\xi) = \mathbb{O} \in \mathbb{T}_{V^{(n)},\xi}.$$

Given a subspace $S$ in a vector space $V$, let $S^0$ denote the annihilator of S in V*. Thus, at the dual space,

$$\mathcal{L}^o_{\mathcal{G}^{(n)},\xi} + \ker(d\beta_\xi)^o = \mathbb{O}^o = \mathbb{T}^*_{V^{(n)},\xi}, \tag{16.8}$$

from where it follows that a regular system of parameters $x_1 \dots, x_{n-e} \in \mathcal{O}_{V^{(n-e)},\beta(\xi)}$ can be extended to a regular system of parameters , say

$$x_1 \dots, x_{n-e}, x_{n-e+1}, \dots, x_n \in \mathcal{O}_{V^{(n)},\xi} \tag{16.9}$$

so that

$$\mathcal{O}_{V^{(n)}}[x_{n-e+1}W, \dots, x_nW] \subset \mathcal{G}^{(n)}, \tag{16.10}$$

since $\mathcal{G}^{(n)}$ is a differential Rees algebra. Observe that the restriction of $\beta$ to $X := \mathbb{V}(\langle x_{n-e+1}, \dots, x_n \rangle)$ is étale over $V^{(n-e)}$, which in turn is equivalent to saying that $X$ is the image of a section of $\beta$ (here we may have to replace $(V^{(n-e)}, \beta(\xi))$ by an étale neighborhood). Thus, after an étale extension at $\beta(\xi)$, we may assume that $V^{(n-e)}$ is isomorphic to $X$.

By 16.5, there is an inclusion of Rees algebras,

$$\mathcal{G}' := \beta^*(\mathcal{G}^{(n-e)}) \odot \mathcal{O}_{V^{(n)}}[x_{n-e+1}W, \dots, x_nW] \subset \mathcal{G}^{(n)},$$

and we claim that the two Rees algebras are equal up to weak equivalence. To proof the claim, we have to show that

$$\mathscr{F}_{V^{(n)}}(\mathcal{G}') = \mathscr{F}_{V^{(n)}}(\mathcal{G}^{(n)}) \tag{16.11}$$

(see Definition 10.3).

Now, the following result will be used (this is essentially the outcome of Hironaka's Restriction Property [**Hi7**]):

**Proposition** [**BrG-EV**, Lemma 6.3, and Proposition 6.6] *Let $V$ be a smooth scheme, and let $X \subset V$ be a smooth closed subscheme. Set $\mathcal{X} = \mathcal{O}_V[I(X)W]$ and let $\mathcal{G}$ be an arbitrary Rees algebra. Then*

$$\mathscr{F}_X((\mathbb{D}iff(\mathcal{G}))|_X) = \mathscr{F}_V(\mathcal{G}) \cap \mathscr{F}_V(\mathcal{X}) = \mathscr{F}_V(\mathcal{G} \odot \mathcal{X}). \tag{16.12}$$

*Moreover, $(\mathbb{D}iff(\mathcal{G}))|_X$ is an $\mathcal{O}_X$-differential Rees algebra.*

Set $\mathcal{X} = \mathcal{O}_{V^{(n)}}[x_{n-e+1}W, \dots, x_nW]$. Notice that:

$$\mathscr{F}_{V^{(n)}}(\mathcal{G}^{(n)}) = \mathscr{F}_{V^{(n)}}(\mathcal{G}^{(n)} \odot \mathcal{X}) = \mathscr{F}_X(\mathcal{G}^{(n)}|_X)$$

where the last equality follows from the equality (16.12). Similarly,

$$\mathscr{F}_{V^{(n)}}(\mathcal{G}') = \mathscr{F}_{V^{(n)}}(\mathcal{G}' \odot \mathcal{X}) = \mathscr{F}_X(\mathcal{G}'|_X).$$

Thus, to show the equality (16.11), it suffices to prove that

$$\mathscr{F}_X(\mathcal{G}^{(n)}|_X) = \mathscr{F}_X(\mathcal{G}'|_X).$$

But

$$\mathscr{F}_X(\mathcal{G}'|_X) = \mathscr{F}_X(\mathcal{G}^{(n-e)}|_X)$$

and since $X$ is isomorphic to $V^{(n-e)}$,

$$\mathscr{F}_X(\mathcal{G}^{(n-e)}|_X) = \mathscr{F}_{V^{(n-e)}}(\mathcal{G}^{(n-e)}).$$

Use now 16.7 (6) and (7) to conclude that up to weak equivalence there is an expression like (16.7). See Remark 16.11 for a counterexample to Claim 1 when the characteristic is positive.

The following claim also holds in positive characteristic:

**Claim 2.** *Suppose $\beta : V^{(n)} \to V^{(n-e)}$ is a smooth morphism of smooth schemes of dimensions $n$ and $(n-e)$, respectively, with $e \geq 1$, and assume that $z_1, \dots, z_e$ defines a section of $\beta$ (in an étale neighborhood of $V^{(n-e)}$). Then if $\mathcal{H}$ is an $\mathcal{O}_{V^{(n-e)}}$-Rees algebra, the $\mathcal{O}_{V^{(n)}}$- algebra*

$$\mathcal{G}^{(n)} := \langle z_1, \dots, z_e \rangle W \odot \beta^*(\mathcal{H})$$

*satisfies that:*

1. *The morphism $\beta$ is $\mathcal{G}^{(n)}$-admissible;*
2. *The elimination algebra of $\mathcal{G}^{(n)}$ via $\beta$, say $\mathcal{G}^{(n-e)}$, is equal to $\mathcal{H}$ up to weak equivalence.*

To prove (1) notice that, since $z_1, \dots, z_e$ defines a section of $\beta$, the morphism is necessarily transversal to $\{z_1 = 0, \dots, z_e = 0\}$, and hence it is transversal to $\mathcal{G}^{(n)}$. On the other hand, it can be checked that $\mathcal{G}^{(n)}$ is closed under the action of the sheaf of relative differential operators $\mathrm{Diff}_{V^{(n)}/V^{(n-e)}}$ (see 16.4). Thus an elimination algebra $\mathcal{G}^{(n-e)} \subset \mathcal{O}_{V^{(n-e)}}[W]$ of $\mathcal{G}^{(n)}$ can be defined, and by the argument exhibited above,

$$\mathcal{G}^{(n)} = \langle z_1, \dots, z_e \rangle \odot \mathcal{G}^{(n-e)}. \tag{16.13}$$

Set $Z = \mathbb{V}(\langle z_1, \dots, z_e \rangle)$. Replacing $V^{(n-e)}$ by an étale neighborhood if needed, we may assume that $V^{(n-e)}$ is isomorphic to $Z$. To show that (2) holds we have to prove that

$$\mathscr{F}_{V^{(n-e)}}(\mathcal{G}^{(n-e)}) = \mathscr{F}_{V^{(n-e)}}(\mathcal{H})$$

but, since $Z$ is isomorphic to $V^{(n-e)}$, it is enough to show that

$$\mathscr{F}_Z(\mathcal{G}^{(n-e)}|_Z) = \mathscr{F}_Z(\mathcal{H}|_Z).$$

Now notice that by construction, $\mathcal{G}^{(n)}|_Z = \mathcal{H}|_Z$, and by (16.13)

$$\mathcal{G}^{(n)}|_Z = \mathcal{G}^{(n-e)}|_Z.$$

Remark 16.11. When the characteristic is positive, it can be shown that there is also a presentation in the spirit of (16.7) in which $x_{n-e+1}, \dots, x_n \in \mathcal{O}_{V^{(n)},\xi}$ are replaced by suitably defined elements $f_{n-e+1}, \dots, f_n \in \mathcal{O}_{V^{(n)},\xi}$ with the property that $\langle f_{n-e+1}, \dots, f_n \rangle$ defines a complete intersection variety (see [**BV2**, Proposition 2.11] and [**BrV2**, Section 12]). Observe that in this case, from equality (16.8) one can only conclude that there is a set of linearly independent linear forms $l_{n-e+1}, \dots, l_n \in \mathbb{T}_{V^{(n)},\xi}$ so that $l_{n-e+1}^{p^{r_{n-e+1}}}, \dots, l_n^{p^{r_n}} \in \mathcal{L}_{\mathcal{G}^{(n)},\xi}$. In fact, suppose that $k$ is a perfect field of characteristic two, and let $\mathcal{G}^{(2)}$ be the $k[x, y]$-differential Rees algebra generated by $x^2 - y^3$ in degree two. Then the arguments exhibited in [**BrG-EV**, §11] indicate that it is not possible to find a $\mathcal{G}^{(2)}$-admissible projection so that $\mathcal{G}^{(2)}$ can be expressed like in (16.7). This general pattern is crucial and specific of the positive characteristic, we refer to [**K**] and [**KM**] for an alternative treatment, and to [**B**], [**BV1**], and [**BV2**] for further results on elimination and resolution of singularities in low dimension.

## 17. New invariants that can be defined via elimination

Given a differential Rees algebra $\mathcal{G}^{(n)}$ of codimensional type $\geq e \geq 1$, and an elimination algebra $\mathcal{G}^{(n-e)}$ on some $(n-e)$-dimensional smooth scheme, the function $\mathrm{ord}^{(n-e)}_{\mathcal{G}^{(n)}}$ is defined as:

$$\begin{array}{rccl} \mathrm{ord}^{(n-e)}_{\mathcal{G}^{(n)}}: & \mathrm{Sing}\,\mathcal{G}^{(n)} & \longrightarrow & \mathbb{Q}_{\geq 0} \\ & \xi & \longmapsto & \mathrm{ord}_{\mathcal{G}^{(n-e)}}(\beta(\xi)), \end{array} \tag{17.1}$$

where $\mathrm{ord}_{\mathcal{G}^{(n-e)}}$ is the usual Hironaka's order function for a Rees algebra as in 9.8. Note that the elimination algebra provides information of local nature, and that the choice of the local projection for its construction is not unique. A fundamental theorem for elimination algebras is that this information does not depend on the choice of projection, and that, in fact, it is an invariant (see Definition 13.1).

THEOREM 17.1. [**BrV1**, Theorem 10.1] *Let $V^{(n)}$ be a $n$-dimensional scheme smooth over a perfect field $k$, let $\mathcal{G}^{(n)} \subset \mathcal{O}_{V^{(n)}}[W]$ be a differential algebra, let $\xi \in Sing\ \mathcal{G}^{(n)}$ be a simple closed point, and let $m \leq \tau_{\mathcal{G}^{(n)},\xi}$. Consider two different $\mathcal{G}^{(n)}$-admissible local projections to some $(n-m)$-dimensional smooth schemes with their corresponding elimination algebras:*

$$\begin{array}{rcl} \beta_{1_{n,n-m}}: \ (V^{(n)},\xi) & \longrightarrow & (V_1^{(n-m)},\xi_{m,1}) \\ \mathcal{G}^{(n)} & \rightarrow & \mathcal{G}_1^{(n-m)} \\ \textit{and} & & \\ \beta_{2_{n,n-m}}: \ (V^{(n)},\xi) & \longrightarrow & (V_2^{(n-m)},\xi_{m,2}) \\ \mathcal{G}^{(n)} & \rightarrow & \mathcal{G}_2^{(n-m)}. \end{array} \tag{17.2}$$

*Then:*

$$ord_{\xi_{m,1}}\mathcal{G}_1^{(n-m)} = ord_{\xi_{m,2}}\mathcal{G}_2^{(n-m)}.$$

*Moreover, if $V^{(n)} \leftarrow V^{(n)'}$ is a composition of permissible monoidal transformations, $\xi' \in Sing\ \mathcal{G}^{(n)'}$ a closed point dominating $\xi$, and*

$$\begin{array}{ccc} (V^{(n)},\xi) & \longleftarrow & (U \subset V^{(n)'},\xi') \\ \mathcal{G}^{(n)} & & \mathcal{G}^{(n)'} \\ \downarrow & \circlearrowleft & \downarrow \\ (V_j^{(n-m)},\xi_{m,j}) & \longleftarrow & (V_j^{(n-m)'},\xi'_{m,j}) \\ \mathcal{G}_j^{(n-m)} & & \mathcal{G}_j^{(n-m)'} \end{array}$$

*is the corresponding commutative diagram of elimination algebras and admissible projections for $j = 1, 2$, then*

$$ord_{\xi'_{m,1}}\mathcal{G}_1^{(n-m)'} = ord_{\xi'_{m,2}}\mathcal{G}_2^{(n-m)'}.$$

REMARK 17.2. Now we will check that the function $\mathrm{ord}^{(n-e)}$ is an invariant. Suppose that $\mathcal{G}^{(n)}$ and $\mathcal{G}'^{(n)}$ are weakly equivalent. Then for each $\xi \in \mathrm{Sing}\ \mathcal{G}^{(n)} = \mathrm{Sing}\ \mathcal{G}'^{(n)}$ one has that $\tau_{\mathcal{G}^{(n)},\xi} = \tau_{\mathcal{G}'^{(n)},\xi}$ (see 13.2.2). Thus if $\tau_{\mathcal{G}^{(n)},\xi} \geq e$, we can find a $\mathcal{G}^{(n)}$-admissible projection to some $(n-e)$-dimensional smooth scheme $V^{(n)} \longrightarrow V^{(n-e)}$ which can be assumed to be $\mathcal{G}'^{(n)}$-admissible too. Now we saturate $\mathcal{G}^{(n)}$ and $\mathcal{G}'^{(n)}$ by the action of the differential operators, say $\mathbb{D}iff(\mathcal{G}^{(n)})$ and $\mathbb{D}iff(\mathcal{G}'^{(n)})$, and then we compute the elimination algebras $\mathcal{G}^{(n-e)}$ and $\mathcal{G}'^{(n-e)}$. Since $\mathbb{D}iff(\mathcal{G}^{(n)})$ and $\mathbb{D}iff(\mathcal{G}'^{(n)})$ are equal up to integral closure, $\mathcal{G}^{(n-e)}$ and $\mathcal{G}'^{(n-e)}$ are equal up to integral closure by 16.7 (4). Thus, $\mathrm{Sing}\ \mathcal{G}^{(n-e)} = \mathrm{Sing}\ \mathcal{G}'^{(n-e)}$ and $\mathrm{ord}_{\mathcal{G}^{(n-e)}}(\xi) = \mathrm{ord}_{\mathcal{G}'^{(n-e)}}(\xi)$ for all $\xi \in \mathrm{Sing}\ \mathcal{G}^{(n-e)} = \mathrm{Sing}\ \mathcal{G}'^{(n-e)}$ (see 13.2.1). Therefore

$$\mathrm{ord}^{(n-e)}_{\mathcal{G}^{(n)}}(\xi) = \mathrm{ord}^{(n-e)}_{\mathcal{G}'^{(n)}}(\xi),$$

for all $\xi \in \mathrm{Sing}\ \mathcal{G}^{(n)} = \mathrm{Sing}\ \mathcal{G}'^{(n)}$ and hence the upper-semi continuous function $\mathrm{ord}^{(n-e)}$ is an invariant.

**17.3. More satellite functions.** Let $\mathcal{G}^{(n)}$ be an $\mathcal{O}_{V^{(n)}}$ differential Rees algebra, together with a $\mathcal{G}^{(n)}$-admissible projection

$$\beta : V^{(n)} \rightarrow V^{(n-e)},$$

and an elimination algebra $\mathcal{G}^{(n-e)} \subset \mathcal{O}_{\mathcal{O}^{(n-e)}}[W]$ in a neighborhood of some point $\xi \in \operatorname{Sing} \mathcal{G}^{(n)}$. As indicated in 16.7 the blow up at a permissible $Y \subset \operatorname{Sing} \mathcal{G}^{(n)}$ induces a blow up at a permissible $\beta(Y) \subset \operatorname{Sing} \mathcal{G}^{(n-e)}$ and a commutative diagram of transversal projections and elimination algebras:

$$\begin{array}{ccc} \mathcal{G}^{(n)} & & \mathcal{G}_1^{(n)} \\ V^{(n)} & \xleftarrow{\rho} & V_1^{(n)} \\ \downarrow \beta & & \downarrow \beta_1 \\ V^{(n-e)} & \xleftarrow{\overline{\rho}} & V_1^{(n-e)} \\ \mathcal{G}^{(n-e)} & & \mathcal{G}_1^{(n-e)} \end{array} \tag{17.3}$$

In addition, if $H$ denotes the exceptional divisor of $\rho$ and if $\overline{H}$ denotes the exceptional divisor of $\overline{\rho}$, then $\beta^{-1}(\overline{H}) = H$. Thus, in the same fashion as in Section 14, satellite functions of $\operatorname{ord}^{(n-e)}_{\mathcal{G}^{(n)}}$ can be defined, namely, w-$\operatorname{ord}^{(n-e)}_{\mathcal{G}^{(n)}}$ and $t^{(n-e)}_{\mathcal{G}^{(n)}}$, and by 14.5 and Remark 17.2 they are also invariants (although these satellite functions were originally defined for pairs, they can also be defined for Rees algebras, see 14.2 for the dictionary between pairs and Rees algebras).

The following result illustrates a relevant application of the invariants introduced above.

Theorem 17.4. [**BrV1**, Theorem 13.1] *Let $\mathcal{G}^{(n)}$ be a differential algebra on a smooth n-dimensional scheme $V^{(n)}$ over a field k. Let $\mathbb{Q}^* = \mathbb{Q} \cup \{\infty\}$ and let*

$$I_n = \underbrace{\mathbb{Q}^* \times \mathbb{Q}^* \times \ldots \times \mathbb{Q}^*}_{n-times}$$

*ordered lexicographically. Then there is an upper semi-continuous function,*

$$\gamma_{\mathcal{G}^{(n)}} : Sing\ \mathcal{G}^{(n)} \to I_n$$

*such that:*

(1) *The level sets of $\gamma_{\mathcal{G}^{(n)}}$ stratify Sing $\mathcal{G}^{(n)}$ in smooth locally closed strata;*
(2) *If $\mathcal{G}^{(n)}$ and $\mathcal{K}^{(n)}$ are weakly equivalent, then the level sets of both $\gamma_{\mathcal{G}^{(n)}}$ and $\gamma_{\mathcal{K}^{(n)}}$ are the same on Sing $\mathcal{G}^{(n)}$ = Sing $\mathcal{K}^{(n)}$;*
(3) *If k is a field of characteristic zero then $\gamma_{\mathcal{G}^{(n)}}$ coincides with the resolution function used for resolution of singularities in characteristic zero.*

Remark 17.5. The function $\gamma_{\mathcal{G}^{(n)}}$ of Theorem 17.4 is built up using Hironaka's order function and Hironaka's $\tau$-invariant. Thus, if two Rees algebras are weakly equivalent then, since they share the same resolution invariants, the stratification of their singular loci is the same for both of them. See Appendix B, for more details.

## Constructive Resolution of Rees algebras

### 18. Sketch of proof of Theorem 12.6

The purpose of this section is to give son ideas about the proof of Theorem 12.6 on the resolution of basic objects in characteristic zero (or simplification in positive characteristic). As indicated in 13.5, the constructive resolution (or simplification) of a basic object is achieved by induction on the codimensional type. Here we exhibit the explicit results on which this argument is supported, and conclude stating a stronger result: Theorem 18.9.

**18.1. The key to the induction on the codimensional type.** Recall that a basic object $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ is said to be simple if $\mathrm{ord}_{\mathcal{G}^{(n)}}(\xi) = 1$ for all $\xi \in \mathrm{Sing}\, \mathcal{G}^{(n)}$. Observe that if $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ is simple, then $\tau_{\mathcal{G}^{(n)},\xi} \geq 1$ for all $\xi \in \mathrm{Sing}\, \mathcal{G}^{(n)}$, and the same will hold for any transform by a permissible transformation.

**Why simple basic objects?** Simple basic objects play a central role because, in characteristic zero, their resolution can be addressed in an inductive manner. Traditionally the approach to resolve simple basic objects by induction makes use of restriction to smooth hypersurfaces (of maximal contact). Here we describe an alternative form of induction for resolution of simple basic objects making use of the notion of the codimensional type of a Rees algebra. We can summarize this strategy as follows:

- **Step 1:** Input: a basic object, $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$;
- **Step 2:** Attach to $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ some invariants;
- **Step 3:** Attach to the invariants a new basic object $(V^{(n)}, \widehat{\mathcal{G}}^{(n)}, \widehat{E}^{(n)})$ *"simpler"* or easier to deal with than $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$; moreover, Theorem 10.11 will be used to show that the new basic object $(V^{(n)}, \widehat{\mathcal{G}}^{(n)}, \widehat{E}^{(n)})$ is unique up to weak equivalence (see 12.4 and Definition 12.5).

Proposition 18.2 below asserts that the strategy described in steps 1-3 can be pursued for any basic object.

PROPOSITION 18.2. [**BrV1**, Theorems 12.7 and 12.9] *Let $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ be a basic object with $V^{(n)}$ smooth over a perfect field $k$. Consider a $t^{(n)}_{\mathcal{G}^{(n)}}$-permissible local sequence,*

(18.1)
$$(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)}) \longleftarrow (V^{(n)}_1, \mathcal{G}^{(n)}_1, E^{(n)}_1) \longleftarrow \cdots \longleftarrow (V^{(n)}_m, \mathcal{G}^{(n)}_m, E^{(n)}_m),$$

*and assume that* $\max w\text{-}ord^{(n)}_{\mathcal{G}^{(n)}_m} \neq 0$. *Let $l$ be the smallest index so that* $\max t^{(n)}_{\mathcal{G}^{(n)}_l} = \max t^{(n)}_{\mathcal{G}^{(n)}_m}$. *Then:*

(1) *There is a* **simple** *$\mathcal{O}_{V^{(n)}_l}$-Rees algebra $\widehat{\mathcal{G}}^{(n)}$ (or say there is a simple basic object $(V^{(n)}_l, \widehat{\mathcal{G}}^{(n)}, \widehat{E}^{(n)})$), with the following property: Any* **local sequence** *starting on $(V^{(n)}_l, \widehat{\mathcal{G}}^{(n)}, \widehat{E}^{(n)})$, say*

$$(V^{(n)}_l, \widehat{\mathcal{G}}^{(n)}, \widehat{E}^{(n)}) \longleftarrow (\widehat{V}^{(n)}_{l+1}, \widehat{\mathcal{G}}^{(n)}_1, \widehat{E}^{(n)}_1) \longleftarrow \cdots \longleftarrow (\widehat{V}^{(n)}_{l+S}, \widehat{\mathcal{G}}^{(n)}_S, \widehat{E}^{(n)}_S),$$

*induces a $t^{(n)}_{\mathcal{G}^{(n)}_l}$-permissible local sequence starting on $(V^{(n)}_l, \mathcal{G}^{(n)}_l, E^{(n)}_l)$ (also enlarging the first $l$-steps of sequence (*18.1*)), say:*

(18.2)
$$(V^{(n)}_l, \mathcal{G}^{(n)}_l, E^{(n)}_l) \longleftarrow (\widetilde{V}^{(n)}_{l+1}, \widetilde{\mathcal{G}}^{(n)}_{l+1}, \widetilde{E}^{(n)}_{l+1}) \longleftarrow \cdots \longleftarrow (\widetilde{V}^{(n)}_{l+S}, \widetilde{\mathcal{G}}^{(n)}_{l+S}, \widetilde{E}^{(n)}_{l+S}),$$

*with the following condition on the functions $t^{(n)}_{\mathcal{G}^{(n)}_j}$ defined for this last sequence (*18.2*):*

(a) $\underline{Max}\, t^{(n)}_{\mathcal{G}^{(n)}_{l+k}} = Sing\ (\widehat{\mathcal{G}}^{(n)}_k)$ *for* $k = 0, 1, \dots, S-1$*;*

(b) $\max t^{(n)}_{\mathcal{G}^{(n)}_l} = \max t^{(n)}_{\widetilde{\mathcal{G}}^{(n)}_{l+1}} = \cdots = \max t^{(n)}_{\widetilde{\mathcal{G}}^{(n)}_{l+S-1}} \geq \max t^{(n)}_{\widetilde{\mathcal{G}}^{(n)}_{l+S}}$*;*

(c) $\max t^{(n)}_{\widetilde{\mathcal{G}}^{(n)}_{l+S-1}} = \max t^{(n)}_{\widetilde{\mathcal{G}}^{(n)}_{l+S}}$ *if and only if* $Sing\ (\widehat{\mathcal{G}}^{(n)}_S) \neq \emptyset$*, in which case* $\underline{Max}\, t^{(n)}_{\widetilde{\mathcal{G}}^{(n)}_{l+S}} = Sing\ (\widehat{\mathcal{G}}^{(n)}_S)$*;*

(2) *(Canonicity) If an* $\mathcal{O}_{V^{(n)}_l}$*-Rees algebra* $\mathcal{K}^{(n)}$ *also fulfills the conditions listed in (1), then* $\mathcal{K}^{(n)}$ *and* $\widehat{\mathcal{G}}^{(n)}$ *are weakly equivalent.*

REMARK 18.3. Several observations:

i) The assertion in (1) is that a lowering of $\max t^{(n)}_{\mathcal{G}^{(l)}}$ can be achieved by resolving a simple basic object (i.e., a basic object of codimensional type $\geq 1$).

ii) The claim in (2) already follows from the conditions imposed in (1) and the observation in step 3 above.

iii) Proposition 18.2 is valid for basic objects $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ with $V^{(n)}$ smooth over a perfect field $k$.

See Section 37 in Appendix B for some hints about the construction of the simple basic object $(V^{(n)}_l, \widehat{\mathcal{G}}^{(n)}, \widehat{E}^{(n)})$.

REMARK 18.4. The starting point in the previous proposition is an arbitrary $n$-dimensional basic object with a function $t^{(n)}$. The proposition asserts that we can associate to max $t^{(n)}$ another basic object of codimensional type $\geq 1$, which is easier to resolve. Hence, by using this fact, Remark 18.3 (i) says that we can lower the maximum value of the function $t^{(n)}$ by successive monoidal transformations. Another feature of this procedure is that by successive iteration of this method we come to the case in which the maximum value of w-ord is zero, which corresponds to the monomial case (see 14.6).

EXAMPLE 18.5. Let $K$ be a field of characteristic zero. Consider the Rees algebra $\mathcal{G}^{(3)}$ generated by $z^2 + (x^2 - y^3)^2$ in degree one. Then Sing $\mathcal{G}^{(2)} = \{z^2 + (x^2 - y^3)^2 = 0\} \subset$ Spec $K[x, y, z] = V^{(3)}$. Let $C := \{z = 0, x^2 - y^3 = 0\}$. Notice that $t^{(3)}_{\mathcal{G}^{(3)}}(\xi) = (1, 0)$ for $\xi \in$ Sing $\mathcal{G}^{(2)} \setminus C$ and that $t^{(3)}_{\mathcal{G}^{(3)}}(\xi) = (2, 0)$ for $\xi \in C$. Thus max $t^{(3)}_{\mathcal{G}^{(3)}} = (2, 0)$ and the basic object $(V^{(3)}, \mathcal{G}^{(3)}, E^{(3)} = \emptyset)$ is non-simple. Now we attach to max $t^{(3)}_{\mathcal{G}^{(3)}}$ the differential Rees algebra $\hat{\mathcal{G}}^{(3)}$ generated by $z^2 + (x^2 - y^3)^2$ in degree two, i.e., up to integral closure, $\hat{\mathcal{G}}^{(3)} = K[x, y][zW, (x^2 - y^3)W]$. Then it can be checked that:

(1) $\hat{\mathcal{G}}^{(3)}$ is simple;

(2) $\underline{\text{Max}}\ t^{(3)}_{\mathcal{G}^{(3)}} =$ Sing $\hat{\mathcal{G}}^{(3)}$;

(3) Finding a resolution of $\hat{\mathcal{G}}^{(3)}$ is equivalent to finding sequence of permissible transformations hat lowers max $t^{(3)}_{\mathcal{G}^{(3)}}$;

(4) Up to weak equivalence, $\hat{\mathcal{G}}^{(3)}$ is the unique Rees algebra fulfilling properties (1)-(3) (see [**BrV1**, Theorems 12.7 and 12.9]).

Thus an improvement of the singularities of (Spec $K[x, y, z], \mathcal{G}^{(3)}, E^{(3)} = \{\emptyset\}$) is achieved by resolving (Spec $K[x, y, z], \hat{\mathcal{G}}^{(3)}, E^{(3)} = \{\emptyset\}$) which has larger codimensional type. Notice also that since $\hat{\mathcal{G}}^{(3)}$ is simple, a $\hat{\mathcal{G}}^{(3)}$-admissible projection can be constructed, and the corresponding elimination algebra $\hat{\mathcal{G}}^{(2)}$ can be defined. In fact, in this case a resolution of $\hat{\mathcal{G}}^{(2)}$ induces a resolution of $\hat{\mathcal{G}}^{(3)}$ and therefore an improvement of a singularities of the original basic object (Spec $K[x,y,z], \mathcal{G}^{(3)}, E^{(3)} = \{\emptyset\}$).

The previous strategy is applied several times in the discussion of the example of Section 19.

**18.6. On the construction of the Resolution Functions.** Fix a basic object $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ of codimensional type $\geq 0$. Proposition 18.2 asserts that we can attach to $t^{(n)}$ another basic object of codimensional type $\geq 1$. This will be the first step in our inductive argument in which we will attach to a function defined on a basic object of codimensional type $\geq e$, another basic object of codimensional type $\geq e+1$.

Here we list some examples to illustrate this philosophy. Our starting point is a basic object $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ of codimensional type $\geq e$ where we assume $e \geq 0$, and we proceed as follows. Let $\xi \in$ Sing $\mathcal{G}^{(n)}$ and suppose first that $\mathrm{codim}_\xi(\mathrm{Sing}\ \mathcal{G}^{(n)}) > e$. Construct a sequence of admissible smooth projections and elimination algebras in a neighborhood of $\xi$:

$$\begin{array}{l} V^{(n)}, \mathcal{G}^{(n)} \\ \downarrow \\ V^{(n-1)}, \mathcal{G}^{(n-1)} \\ \downarrow \\ \vdots \\ \downarrow \\ V^{(n-e)}, \mathcal{G}^{(n-e)} \end{array} \tag{18.3}$$

Then we can write the first $e$-coordinates of the resolution functions:

- If $e \geq 1$, and $E^{(n)} = \emptyset$, then the first coordinates of the resolution function at $\xi$ would be:
$$(\mathrm{ord}^{(n)}_{\mathcal{G}^{(n)}}(\xi), 0), \ldots, (\mathrm{ord}^{(n-e+1)}_{\mathcal{G}^{(n)}}(\xi), 0), (\mathrm{ord}_{\mathcal{G}^{(n)}}(\xi), 0), \ldots) =$$
$$= (\underbrace{(1,0), \ldots, (1,0)}_{e-1}, t^{(n-e)}_{\mathcal{G}^{(n)}}(\xi), \ldots).$$
This follows from the fact that if $\mathcal{G}^{(n)}$ is $e$-simple, then $\mathcal{G}^{(n-1)}$ is $(e-1)$-simple (see 16.7 (5)).
- More generally, for arbitrary $e$, and $E^{(n)}$ non-necessarily empty, the string of invariants at $\xi$ will have the following first $e$-components:
$$(t^{(n)}_{\mathcal{G}^{(n)}}(\xi), \ldots, t^{(n-e)}_{\mathcal{G}^{(n)}}(\xi), \ldots).$$
For instance, if $e \geq 1$, the string will look like:
$$(\underbrace{(1,*), \ldots, (1,*)}_{e-1}, t^{(n-e)}_{\mathcal{G}^{(n)}}(\xi), \ldots),$$
and recall that $t^{(n-e)}_{\mathcal{G}^{(n)}}$ is defined via the function $t^{(n-e)}_{\mathcal{G}^{(n-e)}}$ for some $(n-e)$-dimensional elimination algebra of $\mathcal{G}^{(n)}$.

Now by applying successively Proposition 18.2 (see 37.7) and elimination, we can attach to max $t^{(n-e)}_{\mathcal{G}^{(n-e)}}$ a simple $(n-e)$-dimensional basic object

$$(V^{(n-e)}, \widehat{\mathcal{G}}^{(n-e)}, \widehat{E}^{(n-e)}).$$

This in turn is used to defined another $n$-dimensional basic object

$$(V^{(n)}, \widehat{\mathcal{G}}^{(n)}, \widehat{E}^{(n)})$$

with larger codimensional type than $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$. This allows us to fill in the other coordinates of the resolution function for $\mathcal{G}^{(n)}$:

$$(t^{(n)}_{\mathcal{G}^{(n)}}(\xi), , \ldots, t^{(n-e)}_{\mathcal{G}^{(n)}}(\xi), t^{(n-e-1)}_{\widehat{\mathcal{G}}^{(n)}}(\xi), \ldots).$$

See Section 37 in Appendix B for more details, and the example in Section 19 where the basic object $(V^{(n)}, \widehat{\mathcal{G}}^{(n)}, \widehat{E}^{(n)})$ is constructed in different stages of the resolution process. The previous steps are repeated until we find a natural center to blow up.

On the other hand, if $\tau_{\mathcal{G}^{(n)}} \geq e \geq 1$ then all codimension-$e$ components of Sing $\mathcal{G}^{(n)}$ are smooth and disconnected (see 13.3 (5)). Hence if $E^{(n)} = \emptyset$, they are the natural centers to blow up. The value of the resolution function would be:

$$(\mathrm{ord}^{(n)}_{\mathcal{G}^{(n)}}(\xi), 0), \ldots, (\mathrm{ord}^{(n-e+1)}_{\mathcal{G}^{(n)}}(\xi), 0), \infty, \ldots) = (\underbrace{(1,0), \ldots, (1,0)}_{e}, \infty, \ldots).$$

Observe that $\mathcal{G}^{(n)}$ could be $e$-trivial for different values of $e$. In such case, according to the definition of the resolution function, those components with smaller codimension are blown up first.

Using Proposition 18.2 and an inductive argument on the codimensional type, we can assume, that, after a finite sequence of blowing ups at permissible centers, the function $t$ is exhausted (i.e., its value decreases to $(0,0)$), and hence we have achieved the monomial case (see 14.6). Thus the following theorem holds:

Theorem 18.7. [**BrV1**, Part 5] [**Vi5**, Corollary 6.15] *Let $V^{(n)}$ be a smooth scheme over a perfect field $k$, and let $\mathcal{G}^{(n)}$ be a differential $\mathcal{O}_{V^{(n)}}$-Rees algebra of codimensional type $\geq e$. Assume that we know how to construct the resolution of basic objects of codimensional type $\geq (e+1)$. Let $\beta : V^{(n)} \to V^{(n-e)}$ be a transversal projection in a neighborhood of a point $\xi \in$ Sing $\mathcal{G}^{(n)}$, and let $\mathcal{G}^{(n-e)} \subset \mathcal{O}_{V^{(n-e)}}[W]$ be an elimination algebra. Then a sequence of permissible transformations can be defined,*

(18.4)
$$\begin{array}{ccccccc}
\mathcal{G}^{(n)} = \mathcal{G}^{(n)}_0 & & \mathcal{G}^{(n)}_1 & & & & \mathcal{G}^{(n)}_m \\
V^{(n)} = V^{(n)}_0 & \xleftarrow{\rho_0} & V^{(n)}_1 & \xleftarrow{\rho_1} & \cdots & \xleftarrow{\rho_{m-1}} & V^{(n)}_m \\
\downarrow \beta & & \downarrow \beta_1 & & & & \downarrow \beta_m \\
V^{(n-e)} = V^{(n-e)}_0 & \xleftarrow{\overline{\rho}_0} & V^{(n-e)}_1 & \xleftarrow{\overline{\rho}_1} & \cdots & \xleftarrow{\overline{\rho}_{m-1}} & V^{(n-e)}_m \\
\mathcal{G}^{(n-e)} = \mathcal{G}^{(n-e)}_0 & & \mathcal{G}^{(n-e)}_1 & & & & \mathcal{G}^{(n-e)}_m
\end{array}$$

*so that either*

$$\mathit{Sing}\ \mathcal{G}^{(n-e)}_m = \emptyset$$

*in which case Sing $\mathcal{G}^{(n)}_m = \emptyset$, or else, up to integral closure,*

(18.5)
$$\mathcal{G}^{(n-e)}_m = \mathcal{O}_{V^{(n-e)}_m}[\mathcal{M}W^s],$$

*where $\mathcal{M}$ is a locally principal ideal supported on the exceptional divisor of*

$$V_0^{(n-e)} \longleftarrow V_m^{(n-e)}.$$

When $\mathrm{char}(k) = 0$, the inclusion (16.6) is an equality, and, as a consequence, sequence (18.4) can be enlarged so as to obtain a resolution of $\mathcal{G}^{(n)}$ using arguments of combinatorial nature. On the other hand, if $\mathrm{char}(k) = p > 0$, the inclusion in (16.6) may be strict. Consider, for instance, the differential Rees algebra $\mathcal{G}^{(3)}$ generated by $x^2 + y^2t^3$ in degree two, when the characteristic is two. The natural projection given by the inclusion $k[y,t] \subset k[x,y,z]$ induces a $\mathcal{G}^{(3)}$-admissible projection with an elimination algebra $\mathcal{G}^{(2)} = k[y,t][y^2t^2W]$. Notice that Sing $\mathcal{G}^{(3)}$ is the union of two curves, say $C := \{x = 0, y = 0\}$ and $S := \{x = 0, t = 0\}$. After two blow ups at closed points, the strict transforms of the curves can be blown up. At this point it can be checked that the transform of $\mathcal{G}^{(2)}$, $\mathcal{G}_3^{(2)}$, is within the monomial case, and that the projection of the singular locus of the transform of $\mathcal{G}^{(3)}$, $\mathcal{G}_3^{(3)}$, is strictly contained in Sing $\mathcal{G}_3^{(2)}$.

Still, formula (18.5) indicates that the singularities of $\mathcal{G}^{(n)}$ can be, somehow, simplified. In particular, in [**BV2**] it is shown how sequence (18.4) can be enlarged so as to obtain resolution of two-dimensional schemes in positive characteristic.

Remark 18.8. Using Theorem 18.7 and Remark 16.10, we can extract the following consequence when the characteristic is zero. If $\xi \in \mathrm{Sing}\,\mathcal{G}_m^{(n)} \neq \emptyset$, then there is a regular system of parameters $x_1, \dots, x_n \in \mathcal{O}_{V_m^{(n)}}$, so that, up to integral closure,

$$\mathcal{G}_m^{(n)} = \mathcal{O}_{V_m^{(n)}}[x_1W, \dots, x_eW] \odot \mathcal{M}W^s,$$

where $\mathcal{M}$ is locally principal and supported on the exceptional divisor of $V_0^{(n)} \leftarrow V_m^{(n)}$.

**Resolution and weak equivalence.** Since the resolution functions are defined via invariants, it is clear that weakly equivalent Rees algebras share the same constructive resolution.

With the theory presented up to now, the following theorem (which is stronger than Theorem 12.6) can be stated:

Theorem 18.9. *There is a totally ordered set $(\Gamma, \leq)$, and for each basic object $\mathscr{B} = (V, \mathcal{G}, E)$, an upper-semi continuous function*

$$f_{\mathscr{B}} : \mathit{Sing}\ \mathcal{G} \longrightarrow \Gamma$$

*with the following properties:*

(1) *$\underline{Max}\ f_{\mathscr{B}}$ is smooth;*
(2) *The sequence,*

(18.6)
$$\mathscr{B} = (V, \mathcal{G}, E) = (V_0, \mathcal{G}_0, E_0) = \mathscr{B}_0 \leftarrow \mathscr{B}_1 = (V_1, \mathcal{G}_1, E_1) \leftarrow \cdots \leftarrow \mathscr{B}_m = (V_m, \mathcal{G}_m, E_m)$$

*defined by blowing up successively at $\underline{Max}\ f_{\mathscr{B}_i}$ is so that:*
- (a) *When the characteristic of the base field is zero, Sing $\mathcal{G}_m = \emptyset$;*
- (b) *When the characteristic of the base field is positive, either Sing $\mathcal{G}_m = \emptyset$, or else, Sing $\mathcal{G}_m$ is "simpler" than Sing $\mathcal{G}$ (see Theorem* 18.7*).*
- (c) *Weakly equivalent basic objects share the same constructive resolution (in characteristic zero) or simplification (in positive characteristic).*

(d) *When the characteristic is zero, if $E = \emptyset$ and if $\tau_{\mathcal{G}} \geq e \geq 1$, then, up to weak equivalence $\mathcal{G} = \mathcal{I}(X)W \odot \mathcal{H}$, where $X$ is some smooth scheme of codimension $e$, and $\mathcal{H}$ is an elimination algebra in some smooth scheme of dimension $\dim V - e$ (see Remark* 16.10*); in such case sequence (*18.6*) induces a resolution of $\mathcal{H}$ (see* 18.6*).*

**18.10. Final remarks.** The resolution function $f_{\mathscr{B}}$ is defined with the philosophy of the exposition of 18.6. The construction of the resolution function is of local nature because of our form of elimination, and induction on the dimension (see Section 17). However, the Canonicity Theorem 10.11 guarantees the globalization of the resolution function (e.g., given a basic object $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ as in Proposition 18.2, the assignment of a simple basic object $(V^{(n)}, \widehat{\mathcal{G}}^{(n)}, \widehat{E}^{(n)})$ to the maximum value of the $t^{(n)}_{\mathcal{G}^{(n)}}$ is done locally, in a neighborhood of each point in $\underline{\mathrm{Max}}\ \mathrm{t}^{(n)}_{\mathcal{G}^{(n)}}$; the Canonicity Theorem ensures that $\widehat{\mathcal{G}}^{(n)}$, is unique up to weak equivalence, and thus, this local construction globalizes on $\underline{\mathrm{Max}}\ \mathrm{t}^{(n)}_{\mathcal{G}^{(n)}}$). More details on these and related matters can be found in Part III of this manuscript. Finally, note that Theorem 12.6 follows from Theorem 18.9.

## 19. Example

Let $k$ be a field of characteristic zero. To find a resolution of singularities of

$$X := \{z^2 + (x^2 - y^3)^2 = 0\} \subset \mathbb{A}^3_k,$$

we start by stratifying $X$ using the multiplicity function on $X$, $\mathrm{Mult}_X$. The highest multiplicity, $\max \mathrm{Mult}_X$, is 2, and

$$\underline{\mathrm{Max}}\ \mathrm{Mult}_X = \{z = 0, x^2 - y^3 = 0\} = C. \tag{19.1}$$

We attach to $\max \mathrm{Mult}_X$ the differential Rees algebra generated by $\mathcal{I}(X)$ in weight 2,

$$\mathcal{G}^{(3)} = k[x,y,z][zW, (x(x^2 - y^3))W, (y^2(x^2 - y^3))W, (z^2 + (x^2 - y^3)^2)W^2].$$

Up to integral closure, we can assume that

$$\mathcal{G}^{(3)} = k[x,y,z][zW, (x^2 - y^3)W].$$

Recall that a resolution of the basic object

$$(\mathbb{A}^3_k, \mathcal{G}^{(3)}, E^{(3)} = \{\emptyset\})$$

say

$$(\mathbb{A}^3_k, \mathcal{G}^{(3)}, E^{(3)}) \longleftarrow (V^{(3)}_1, \mathcal{G}^{(3)}_1, E^{(3)}_1) \longleftarrow \ldots \longleftarrow (V^{(3)}_s, \mathcal{G}^{(3)}_s, E^{(3)}_s), \tag{19.2}$$

induces a finite sequence of blow ups at smooth centers,

$$X \longleftarrow X_1 \longleftarrow \ldots \longleftarrow X_s,$$

so that $X_s$ does not have points of multiplicity 2, namely that all points in $X_s$ have multiplicity equal to one. So the construction of the sequence (19.2) is natural step towards resolving the singularities of $X$.

**Step 0.** We start by constructing the resolution function for $(\mathbb{A}^3_k, \mathcal{G}^{(3)}, E^{(3)} = \{\emptyset\})$. Recall that the building blocks are the satellite functions. First we use $t^{(3)}_{\mathcal{G}^{(3)}}$, and notice that

$$\Gamma_X(\xi) = ((1,0), *)$$

for all $\xi \in \operatorname{Sing} \mathcal{G}^{(3)}$. As we see, $t^{(3)}_{\mathcal{G}^{(3)}}$ is too coarse to stratify $\operatorname{Sing} \mathcal{G}^{(3)}$ in smooth strata. Thus we refine it using elimination algebras. Consider the natural projection $\beta : \mathbb{A}^3_k \longrightarrow \mathbb{A}^2_k$ induced by the inclusion

$$k[x,y] \longrightarrow k[x,y,z]$$

and the corresponding elimination algebra

$$\mathcal{G}^{(2)} = k[x,y][(x^2 - y^3)W].$$

Now, with the information provided by $t^{(2)}_{\mathcal{G}^{(2)}}$, we can add a few more coordinates to $\Gamma_X$,

$$\Gamma_X(\xi) = \begin{cases} ((1,0),(2,0),*) & \text{if} \quad \xi = (0,0,0) \\ ((1,0),(1,0),\infty) & \text{if} \quad \xi \in C \setminus (0,0,0), \end{cases}$$

where $C$ is as in (19.1). Notice the "$\infty$"-coordinate of $\Gamma_X$ at the points $\xi \in C \setminus (0,0,0)$. This corresponds to the fact at $C \setminus (0,0,0)$, $\mathcal{G}^{(3)}$ has codimensional type 2, which in turn coincides with the local codimension of $\operatorname{Sing} \mathcal{G}^{(3)}$. This indicates that, locally, at any point of $\mathbb{A}^3 \setminus (0,0,0)$, the curve $C \setminus (0,0,0)$ is a natural center to blow up. This is not the situation at $(0,0,0)$, where the basic object $(\mathbb{A}^2_k, \mathcal{G}^{(2)}, E^{(2)} = \{\emptyset\})$ is non-simple (the codimensional type of $\mathcal{G}^{(3)}$ at the origin is 1). In fact, our procedure will lead to blow up at this point first.

We attach to

$$\max\, t^{(2)}_{\mathcal{G}^{(2)}} = (\max \text{ w-ord}^{(2)}_{\mathcal{G}^{(2)}}, 0) = (2,0)$$

the simple basic object (s.b.o.),

$$(\mathbb{A}^2_k, \widehat{\mathcal{G}}^{(2)}, \{\emptyset\})$$

where $\widehat{\mathcal{G}}^{(2)}$ is the differential Rees algebra

$$\widehat{\mathcal{G}}^{(2)} = k[x,y][(x^2 - y^3)W^2, xW, y^2W] = k[x,y][xW, y^2W, y^3W^2],$$

(see Theorem 37.5).

Now, consider the natural projection

$$k[y] \longrightarrow k[x,y]$$

and the elimination algebra of $\widehat{\mathcal{G}}^{(2)}$,

$$\widehat{\mathcal{G}}^{(1)} = k[y][y^2W, y^3W^2],$$

together with the corresponding basic object

$$(\mathbb{A}^1_k, \widehat{\mathcal{G}}^{(1)}, \{\emptyset\}).$$

Now $t^{(1)}_{\widehat{\mathcal{G}}^{(1)}}$ is defined on $\underline{\operatorname{Max}}\ t^{(2)}_{\mathcal{G}^{(2)}} = \underline{\operatorname{Max}}$ w-ord$^{(2)}_{\mathcal{G}^{(2)}}$, and this provides the third coordinate of $\Gamma_X$:

$$\Gamma_X(\xi) = \begin{cases} ((1,0),(2,0),(\frac{3}{2},0)) & \text{if} \quad \xi = (0,0,0) \\ ((1,0),(1,0),\infty) & \text{if} \quad \xi \in C \setminus (0,0,0). \end{cases}$$

Still, since $\widehat{\mathcal{G}}^{(1)}$ is non-simple at the origin, say (0), we attach to

$$\max t^{(1)}_{\widehat{\mathcal{G}}^{(1)}} = (\max \text{w-ord}^{(1)}_{\widehat{\mathcal{G}}^{(1)}}, 0) = \left(\frac{3}{2}, 0\right)$$

the simple basic object (s.b.o.):

$$(\mathbb{A}^1_k, \widehat{\widehat{\mathcal{G}}^{(1)}}, \{\emptyset\})$$

with $\widehat{\widehat{\mathcal{G}}^{(1)}} = k[y][yW]$ (see Theorem 37.5). Observe that the codimensional type of $\widehat{\widehat{\mathcal{G}}^{(1)}}$ is 1, which in turn coincides with the codimension of its singular locus. So we have found the first center to blow up.

Summarizing: Step 0:

$$\begin{array}{ccccc}
(\mathbb{A}^3_k, \mathcal{G}^{(3)}, \{\emptyset\}) & & & & \\
\text{Projection} \downarrow & \text{s. b. o.} & & & \\
(\mathbb{A}^2_k, \mathcal{G}^{(2)}, \{\emptyset\}) & \leftrightarrow & (\mathbb{A}^2_k, \widehat{\mathcal{G}}^{(2)}, \{\emptyset\}) & & \\
& & \downarrow \text{Projection} & \text{s. b. o.} & \\
& & (\mathbb{A}^1_k, \widehat{\mathcal{G}}^{(1)}, \{\emptyset\}) & \leftrightarrow & (\mathbb{A}^1_k, \widehat{\widehat{\mathcal{G}}^{(1)}}, \{\emptyset\})
\end{array} \tag{19.3}$$

Diagram (19.3) can be read as follows:

- A resolution of the simple basic object $(\mathbb{A}^3_k, \mathcal{G}^{(3)}, \{\emptyset\})$ can be constructed by finding a resolution of $(\mathbb{A}^2_k, \mathcal{G}^{(2)}, \{\emptyset\})$.

- Lowering the maximum order of $\mathcal{G}^{(2)}$ in $\mathbb{A}^2_k$ (reached at $(0,0)$) is equivalent to finding a resolution of the simple basic object $(\mathbb{A}^2_k, \widehat{\mathcal{G}}^{(2)}, \{\emptyset\})$. Observe that the codimensional type of $\widehat{\mathcal{G}}^{(2)}$ is 1, whereas the codimensional type of $\mathcal{G}^{(2)}$ is zero. As we will see, a resolution of the simple basic object $(\mathbb{A}^2_k, \widehat{\mathcal{G}}^{(2)}, \{\emptyset\})$ will lead to lowering of max w-ord$_{\mathcal{G}^{(2)}}$. Thus the improvement is achieved by resolving a basic object of larger codimensional type. This fact can also be read as follows: define

$$\widehat{\mathcal{G}}^{(3)} := \mathcal{G}^{(3)} \odot \beta^*(\widehat{\mathcal{G}}^{(2)}).$$

Then the codimensional type of $\widehat{\mathcal{G}}^{(3)}$ at the origin is two while the codimensional type of $\mathcal{G}^{(3)}$ at the origin is one. Now,

$$\text{Sing } \widehat{\mathcal{G}}^{(3)} = \underline{\text{Max}}\ \mathrm{t}^{(2)}_{\mathcal{G}^{(3)}} \cap \underline{\text{Max}}\ \mathrm{t}^{(3)}_{\mathcal{G}^{(3)}}$$

and moreover a resolution of $\widehat{\mathcal{G}}^{(3)}$ induces a lowering of $(\max \mathrm{t}^{(3)}_{\mathcal{G}^{(3)}}, \max \mathrm{t}^{(2)}_{\mathcal{G}^{(3)}})$.

- A resolution of $(\mathbb{A}^2_k, \widehat{\mathcal{G}}^{(2)}, \{\emptyset\})$ can be constructed by resolving $(\mathbb{A}^1_k, \widehat{\mathcal{G}}^{(1)}, \{\emptyset\})$.

- The maximum order of $\widehat{\mathcal{G}}^{(1)}$ is forced to drop by resolving the simple basic object $(\mathbb{A}^1_k, \widehat{\widehat{\mathcal{G}}^{(1)}}, \{\emptyset\})$: an improvement of $(\mathbb{A}^1_k, \widehat{\mathcal{G}}^{(1)}, \{\emptyset\})$ is obtained by resolving a basic object of larger codimensional type.

**Step 1. The first blow up.** Now, the maximum value of $\Gamma_X(\xi)$ indicates that $(0,0,0)$ is the first center to blow up. Thus there is a commutative diagram of permissible transformations, and transformations of basic objects,

(19.4)
$$\begin{array}{ccc} (V^{(3)}, \mathcal{G}^{(3)}, \{\emptyset\}) & \longleftarrow & (V_1^{(3)}, \mathcal{G}_1^{(3)}, H_1^{(3)}) \\ \downarrow & & \downarrow \\ (V^{(2)}, \mathcal{G}^{(2)}, \{\emptyset\}) & \longleftarrow & (V_1^{(2)}, \mathcal{G}_1^{(2)}, H_1^{(2)}). \end{array}$$

And diagram (19.3) transforms as follows:

(19.5)
$$\begin{array}{ccccc} (V_1^{(3)}, \mathcal{G}_1^{(3)}, H_1^{(3)}) & & & & \\ \text{Projection}\downarrow & \text{s. b. o.} & & & \\ (V_1^{(2)}, \mathcal{G}_1^{(2)}, H_1^{(2)}) & \leftrightarrow & (V_1^{(2)}, \widehat{\mathcal{G}}_1^{(2)}, H_1^{(2)}) & & \\ & & \downarrow\text{Projection} & \text{s. b. o.} & \\ & & (V_1^{(1)}, \widehat{\mathcal{G}}_1^{(1)}, H_1^{(1)}) & \leftrightarrow & (V_1^{(1)}, \widehat{\widehat{\mathcal{G}}_1^{(1)}}, H_1^{(1)}) \end{array}$$

Let us analyze locally the effect of the blow up at each level (according to the dimension of the ambient space) of the previous diagram. Consider the affine chart that arises after dividing by $y$. Set $z_1 := \frac{z}{y}$, $x_1 := \frac{x}{y}$, $y_1 := y$.

3-dimensional basic object

$$(\mathbb{A}_k^3, \{\emptyset\}) \qquad \leftarrow \qquad (\operatorname{Spec} k[x_1, y_1, z_1], \{H_1^{(3)}\})$$

$$\mathcal{G}^{(3)} = \mathcal{O}_{V^{(3)}}[zW, (x^2 - y^3)W] \quad \leftarrow \quad \mathcal{G}_1^{(3)} = k[x_1, y_1, z_1][z_1 W, y_1(x_1^2 - y_1)W]$$

Projecting: 2-dimensional basic object

$$(\mathbb{A}_k^2, \{\emptyset\}) \qquad \leftarrow \qquad (k[x_1, y_1], \{H_1^{(2)}\})$$

$$\mathcal{G}^{(2)} = \mathcal{O}_{V^{(2)}}[(x^2 - y^3)W] \quad \leftarrow \quad \mathcal{G}_1^{(2)} = k[x_1, y_1][y_1(x_1^2 - y_1)W]$$

And, moreover,

Transform for $\widehat{\mathcal{G}}^{(2)}$

$$(k[x, y], \{\emptyset\}) \qquad \leftarrow \qquad (k[x_1, y_1], \{H_2^{(1)}\})$$

$$\widehat{\mathcal{G}}^{(2)} = k[x, y][xW, y^2W, y^3W^2] \quad \leftarrow \quad \widehat{\mathcal{G}}_1^{(2)} = k[x_1, y_1][x_1W, y_1W, y_1W^2]$$

Projecting: 1-dimensional basic object

$$(k[x], \{\emptyset\}) \qquad \leftarrow \qquad (k[x_1], \{H_1^{(1)}\})$$

$$\widehat{\mathcal{G}}^{(1)} = k[y][y^2W, y^3W^2] \quad \leftarrow \quad \widehat{\mathcal{G}}_1^{(1)} = k[y_1][y_1W, y_1W^2]$$
$$\widehat{\widehat{\mathcal{G}}^{(1)}} = k[y][yW] \quad \leftarrow \quad \widehat{\widehat{\mathcal{G}}_1^{(1)}} = k[y_1][W]$$

Thus, $\operatorname{Sing} \widehat{\widehat{\mathcal{G}}_1^{(1)}} = \emptyset$. Moreover, in this case, $\operatorname{Sing} \widehat{\mathcal{G}}_1^{(1)} = \emptyset$, hence $\operatorname{Sing} \widehat{\mathcal{G}}_1^{(2)} = \emptyset$, and

$$2 = \max \text{w-ord}^{(2)}_{\mathcal{G}^{(2)}} > \max \text{w-ord}^{(2)}_{\mathcal{G}_1^{(2)}} = 1,$$

so the second coordinate of the $t^{(2)}_{\mathcal{G}_1^{(2)}}$-function enters in scene by counting *old* exceptional divisors.

Denote by $C_1$ the strict transform of $C$. Then:

$$\Gamma_{X_1}(\xi) = \begin{cases} ((1,0),(1,1),*) & \text{if} \quad \xi_1 = C_1 \cap H_1^{(3)} \\ ((1,0),(1,0),\infty) & \text{if} \quad \xi \in C_1 \setminus (0,0,0). \end{cases}$$

Notice that points at $\{z_1 = 0\} \cap H_1^{(3)}$ different from $(0,0,0)$ (the origin of the affine chart Spec $k[x_1, y_1, z_1]$), are already within the monomial case (see Remark 18.8).

Now we determine the third pair of coordinates of $\Gamma_{X_1}$ at $\xi_1$. To this end, we attach to max $t^{(2)}_{\mathcal{G}_1^{(2)}} = (1,1)$ the simple basic object

$$(V_1^{(2)}, \widetilde{\mathcal{G}}^{(2)}, \{\emptyset\}),$$

where

$$\widetilde{\mathcal{G}}^{(2)} = k[x_1, y_1][(x_1^2 - y_1)W, y_1W] = k[x_1, y_1][x_1^2W, y_1W],$$

(see 37.7). Consider the natural projection

$$k[x_1] \longrightarrow k[x_1, y_1]$$

and define the corresponding elimination algebra of $\widetilde{\mathcal{G}}^{(2)}$,

$$\widetilde{\mathcal{G}}^{(1)} = k[x_1][x_1^2W],$$

and the basic object

$$(\text{Spec } k[x_1], \widetilde{\mathcal{G}}^{(1)}, \{\emptyset\}).$$

This defines the function $t^{(1)}_{\widetilde{\mathcal{G}}^{(1)}}$ on $\underline{\text{Max}}\ t^{(2)}_{\mathcal{G}_1^{(2)}}$. Thus,

$$\Gamma_{X_1}(\xi) = \begin{cases} ((1,0),(1,1),(2,0) & \text{if} \quad \xi_1 = C_1 \cap H_1 \\ ((1,0),(1,0),\infty) & \text{if} \quad \xi \in C_1 \setminus (0,0,0). \end{cases}$$

Since $\widetilde{\mathcal{G}}^{(1)} = k[x_1][x_1^2W]$ is non-simple, we attach to $2 = \max \text{w-ord}^{(1)}_{\widetilde{\mathcal{G}}^{(1)}}$ the simple basic object (s.b.o.):

$$(\text{Spec } k[x_1], \widetilde{\widetilde{\mathcal{G}}^{(1)}}, \{\emptyset\})$$

with $\widetilde{\widetilde{\mathcal{G}}^{(1)}} = k[x_1][x_1W]$.

Summarizing:

(19.6)

$$\begin{array}{ccccc} (V_1^{(3)}, \mathcal{G}_1^{(3)}, \{H_1^{(3)}\}) & & & & \\ \text{Projection} \downarrow & \text{s. b. o.} & & & \\ (V_1^{(2)}, \mathcal{G}_1^{(2)}, \{H_1^{(2)}\}) & \leftrightarrow & ((V_1^{(2)}, \widetilde{\mathcal{G}}^{(2)}, \{\emptyset\}) & & \\ & & \downarrow \text{Projection} & \text{s. b. o.} & \\ & & (V'^{(1)}, \widetilde{\mathcal{G}}^{(1)}, \{\emptyset\}) & \leftrightarrow & (V'^{(1)}, \widetilde{\widetilde{\mathcal{G}}^{(1)}}, \{\emptyset\}). \end{array}$$

Notice that a lowering of max $\text{t}^{(2)}_{\mathcal{G}^{(2)}}$ is obtained by resolving $(V_1^{(2)}, \widetilde{\mathcal{G}}^{(2)}, \{\emptyset\})$ whose codimensional type is larger than that of $(V_1^{(2)}, \mathcal{G}_1^{(2)}, \{H_1^{(2)}\})$; similarly, a lowering of max $\text{w-ord}^{(1)}_{\widetilde{\mathcal{G}}^{(1)}}$ is achieved by resolving $(V'^{(1)}, \widetilde{\widetilde{\mathcal{G}}^{(1)}}, \{\emptyset\})$ which has a larger codimensional type.

**Step 2. The second blow up.** The maximum value of $\Gamma_{X_1}(\xi)$ indicates the second center to be blow up: the origin of the affine chart Spec $k[x_1, y_1, z_1]$. So we enlarge the sequence from diagram (19.4),

$$\begin{array}{ccc} (V_1^{(3)}, \mathcal{G}_1^{(3)}, \{H_1^{(3)}\}) & \longleftarrow & (V_2^{(3)}, \mathcal{G}_2^{(3)}, \{H_1^{(3)}, H_2^{(3)}\}) \\ \downarrow & & \downarrow \\ (V_1^{(2)}, \mathcal{G}_1^{(2)}, \{H_1^{(2)}\}) & \longleftarrow & (V_2^{(2)}, \mathcal{G}_2^{(2)}, \{H_1^{(3)}, H_2^{(3)}\}). \end{array} \tag{19.7}$$

And diagram (19.6) transforms as follows:

(19.8)

$$\begin{array}{ccccc} (V_2^{(3)}, \mathcal{G}_2^{(3)}, \{H_1^{(3)}, H_2^{(3)}\}) & & & & \\ \text{Projection} \downarrow & \text{s. b. o.} & & & \\ (V_2^{(2)}, \mathcal{G}_2^{(2)}, \{H_1^{(2)}, H_2^{(2)}\}) & \leftrightarrow & ((V_2^{(2)}, \widetilde{\mathcal{G}}_1^{(2)}, \{H_2^{(2)}\}) & & \\ & & \downarrow \text{Projection} & \text{s. b. o.} & \\ & & (V'^{(1)}_1, \widetilde{\mathcal{G}}_1^{(1)}, \{H_2^{(1)}\}) & \leftrightarrow & (V'^{(1)}_1, \widetilde{\widetilde{\mathcal{G}_1}}^{(1)}, \{H_2^{(1)}\}) \end{array}$$

We analyze locally the effect of the blow up at each level (according to the dimension of the ambient space) of the previous diagram. Consider, the affine chart after dividing by $x_1$. Set $z_2 := \frac{z_1}{x_1}$, $x_2 := x_1$ and $y_2 = \frac{y_1}{x_1}$. Then

3-dimensional basic object

$$(\text{Spec } k[z_1, x_1, y_1], \{H_1^{(3)}\}) \quad \leftarrow \quad (\text{Spec } k[z_2, x_2, y_2], \{H_1^{(3)}, H_2^{(3)}\})$$

$$\mathcal{G}_1^{(3)} = k[z_1, x_1, y_1][z_1 W, y_1(x_1^2 - y_1)W] \quad \leftarrow \quad \mathcal{G}_2^{(3)} = k[z_2, x_2, y_2][z_2 W, y_2 x_2(x_2 - y_2)W]$$

Projecting: 2-dimensional basic object

$$(k[x_1, y_1], \{H_1^{(2)}\}) \quad \leftarrow \quad (k[x_2, y_2], \{H_1^{(2)}, H_2^{(2)}\})$$

$$\mathcal{G}_1^{(2)} = k[x_1, y_1][(x_1^2 - y_1)W] \quad \leftarrow \quad \mathcal{G}_2^{(2)} = k[x_2, y_2][y_2 x_2(x_2 - y_2)W]$$

And, moreover,

Transform for $\widetilde{\mathcal{G}}^{(2)}$

$$(k[x_1, y_1], \{\emptyset\}) \quad \leftarrow \quad (k[x_2, y_2], \{H_2^{(2)}\})$$

$$\widetilde{\mathcal{G}}^{(2)} = k[x_1, y_1][x_1^2 W, y_1 W] \quad \leftarrow \quad \widetilde{\mathcal{G}}_1^{(2)} = k[x_2, y_2][x_2 W, y_2 W]$$

Projecting: 1-dimensional basic object

$$(k[x_1], \{\emptyset\}) \quad \leftarrow \quad (k[x_2], \{H_2^{(1)}\})$$

$$\widetilde{\mathcal{G}}^{(1)} = k[x_1][x_1^2 W] \quad \leftarrow \quad \widetilde{\mathcal{G}}_1^{(1)} = k[x_2][x_2 W]$$
$$\widetilde{\widetilde{\mathcal{G}}}^{(1)} = k[x_1][x_1 W] \quad \leftarrow \quad \widetilde{\widetilde{\mathcal{G}}}_1^{(1)} = k[x_2][W]$$

Now, Sing $\widetilde{\widetilde{\mathcal{G}}}_1^{(1)} = \emptyset$, hence,

$$2 = \max \text{ w-ord}^{(1)}_{\widetilde{\mathcal{G}}^{(1)}} > \max \text{ w-ord}^{(1)}_{\widetilde{\mathcal{G}}_1^{(1)}} = 1$$

and the second coordinate of the $t^{(1)}_{\widetilde{\mathcal{G}}^{(1)}_1}$-function plays a role in counting *old* exceptional divisors.

Denoting by $C_2$ the strict transform of $C$, we start defining the resolution function:

$$\Gamma_{X_2}(\xi) = \begin{cases} ((1,0),(1,1),(1,1)) & \text{if} \quad \xi_2 = C_2 \cap H_2 \cap H_1 \\ ((1,0),(1,0),\infty) & \text{if} \quad \xi \in C_2 \setminus (0,0,0). \end{cases}$$

**Step 3. The third blow up.** Blow up the origin of the affine chart Spec $k[x_2, y_2, z_2]$ and enlarge sequence (19.7):

$$\begin{array}{ccc} (V_2^{(3)}, \mathcal{G}_2^{(3)}, \{H_1^{(3)}, H_2^{(3)}\}) & \longleftarrow & (V_3^{(3)}, \mathcal{G}_3^{(3)}, \{H_1^{(3)}, H_2^{(3)}, H_3^{(3)}\}) \\ \downarrow & & \downarrow \\ (V_2^{(2)}, \mathcal{G}_2^{(2)}, \{H_1^{(2)}, H_2^{(2)}\}) & \longleftarrow & (V_3^{(2)}, \mathcal{G}_3^{(2)}, \{H_1^{(3)}, H_2^{(3)}, H_3^{(2)}\}). \end{array} \tag{19.9}$$

Diagram (19.8) transforms:

$$\begin{array}{ccc} (V_3^{(3)}, \mathcal{G}_3^{(3)}, \{H_1^{(3)}, H_2^{(3)}, H_3^{(3)}\}) & & \\ \text{Projection} \downarrow & \text{s. b. o.} & \\ (V_3^{(2)}, \mathcal{G}_3^{(2)}, \{H_1^{(2)}, H_2^{(2)}, H_3^{(2)}\}) & \leftrightarrow & ((V_3^{(2)}, \widetilde{\mathcal{G}}_3^{(2)}, \{H_2^{(2)}, H_3^{(2)}\}) \\ & & \downarrow \text{Projection} \\ & & (V'^{(1)}_2, \widetilde{\mathcal{G}}_3^{(1)}, \{H_2^{(1)}, H_3^{(1)}\}) \end{array} \tag{19.10}$$

One can check that Sing $\widetilde{\mathcal{G}}_3^{(1)} = \emptyset$, thus, Sing $\widetilde{\mathcal{G}}_3^{(2)} = \emptyset$, so,

$$(1,1) = \max\ t^{(2)}_{\mathcal{G}_1^{(2)}} = \max\ t^{(2)}_{\mathcal{G}_2^{(2)}} > \max\ t^{(2)}_{\mathcal{G}_3^{(2)}} = (1,0).$$

Now, if $C_3$ denotes the strict transform of $C_2$ in $V_3^{(3)}$, we have that for all $\xi \in C_3$,

$$\Gamma_{X_3}(\xi) = ((1,0),(1,0),\infty).$$

The rest of the points in Sing $\mathcal{G}_3^{(3)}$ are within the monomial case (see Remark 18.8). At this point, $\mathcal{G}_3^{(3)}$ is of codimensional type 2 at any point of $C_3$, which in addition has normal crossings with all the exceptional divisors.

**Step 4. The fourth blow up.** Therefore, $C_3$ is the next center that we blow up:

$$\begin{array}{ccc} (V_3^{(3)}, \mathcal{G}_3^{(3)}, \{H_1^{(3)}, H_2^{(3)}, H_3^{(3)}\}) & \longleftarrow & (V_4^{(3)}, \mathcal{G}_4^{(3)}, \{H_1^{(3)}, H_2^{(3)}, H_3^{(3)}, H_4^{(3)}\}) \\ \downarrow & & \downarrow \\ (V_3^{(2)}, \mathcal{G}_3^{(2)}, \{H_1^{(2)}, H_2^{(2)}, H_3^{(2)}\}) & \longleftarrow & (V_4^{(2)}, \mathcal{G}_4^{(2)}, \{H_1^{(3)}, H_2^{(3)}, H_3^{(2)}, H_4^{(4)}\}). \end{array}$$

Notice that $\mathcal{G}_4^{(2)}$ is generated in degree one by a locally principal ideal supported on exceptional divisors:

$$\mathcal{G}_4^{(2)} = \mathcal{O}_{V^{(4)}}[\mathcal{I}(H_1^{(2)}) \cdot \mathcal{I}(H_2^{(2)}) \cdot \mathcal{I}(H_3^{(2)})^2 \cdot \mathcal{I}(H_4^{(2)})W].$$

Therefore, we have achieved the monomial case in the sense of Remark 18.8, and a resolution of

$$(V_4^{(3)}, \mathcal{G}_4^{(3)}, \{H_1^{(3)}, H_2^{(3)}, H_3^{(3)}, H_4^{(3)}\})$$

follows from a combinatorial argument: a new resolution function can be defined to treat this case (see [**EV1**]).

## 20. Some applications of constructive resolution of basic objects

The main application of the theorem on constructive resolution of basic objects from Section 12 (see Theorem 12.6) is the Theorem of Constructive Resolution of Singularities of Algebraic Varieties over fields of characteristic zero. This will be addressed in Section 30.

In the present section we exhibit other applications with a twofold goal: on the one hand they are interesting by themselves; on the other, they will be used in Section 30 to obtain different versions of the Theorem of Resolution of Singularities of Algebraic Varieties.

In the following lines all schemes are assumed to be defined over a field of characteristic zero. We will use the language of pairs (instead of that of Rees algebras) since we that think the notation is simpler in this case (see 14.1 and 14.2 for the dictionary between Rees algebras and pairs).

**20.1. Application 1.** Let $V$ be a smooth scheme, let $E$ be a collection of hypersurfaces with normal crossing support, and let $\mathcal{L}$ be an invertible sheaf of ideals on $V$. Recall that, since $V$ is smooth, for all $\xi \in V$, the local ring $\mathcal{O}_{V,\xi}$ is regular, and therefore a unique factorization domain. Denote by $\widetilde{\mathcal{L}}$ the invertible sheaf of ideals that results after factoring out from $\mathcal{L}$ those components supported on $E$.

Consider the constructive resolution of $(V, (\widetilde{\mathcal{L}}, 1), E)$, say

(20.1)
$$(V, (\widetilde{\mathcal{L}}, 1), E) = (V_0, (\widetilde{\mathcal{L}}_0, 1), E_0) \longleftarrow (V_1, (\widetilde{\mathcal{L}}_1, 1), E_1) \longleftarrow \dots \longleftarrow (V_r, (\widetilde{\mathcal{L}}_r, 1), E_r),$$

and denote by $\pi$ the composite morphism $V \longleftarrow V_r$. Then:

(1) The invertible sheaf $\pi^*\mathcal{L} = \mathcal{L}\mathcal{O}_{V_r}$ is supported on $E_r$. So, in particular, it defines a normal crossing divisor supported on the exceptional divisor of the morphism $\pi$.

(2) The morphism $\pi$ induces an isomorphism on $V \setminus \mathbb{V}(\mathcal{L})$, where $\mathbb{V}(\mathcal{L})$ denotes the closed subscheme defined by $\mathcal{L}$ in $V$.

Observe that at some steps in sequence (20.1) we may be blowing up at a (smooth) hypersurface. Even though this induces the identity morphism, the transform of a pair after such blow up has a non-trivial law of transformation.

**20.2. Application 2.** Suppose that $V$ is a smooth scheme, $\mathcal{L}$ is an invertible sheaf of ideals on $V$, and let $X \subset V$ be a closed smooth subscheme such that the restriction of $\mathcal{L}$ to each component of $X$, say $\mathcal{L}_{|X}$, is non-zero, and different from $\mathcal{O}_X$. Then, a property of the constructive resolution of basic objects is that the constructive resolution of $(V, ((\mathcal{I}(X) + \mathcal{L}), 1), E = \emptyset)$ induces the constructive resolution of $(X, (\mathcal{L}_{|X}, 1), E = \emptyset)$ and vice versa. Moreover if

$$(V, ((\mathcal{I}(X) + \mathcal{L}), 1), E = \emptyset) \overset{\pi}{\longleftarrow} (V_r, ((\mathcal{I}(X) + \mathcal{L})_r, 1), E_r)$$

is such resolution, then the strict transform of $X$ in $V_r$, say $X_r$, has normal crossings with $E_r \cup \pi^*\mathcal{L}$, and $\pi$ induces an isomorphism on $V \setminus \mathbb{V}(\mathcal{L})$, and hence on a dense open set of $X$. Moreover, $X_r$ is smooth since all centers are chosen to be smooth. This fact will be used later for the proof of Theorem 30.7.

To prove the assertion, first notice that if $X \subset V$ is a closed subscheme, and if $Y \subset X \subset V$ is our choice of center, then the blow up at $Y$ induces a commutative diagram of blow ups and closed immersions:

$$\begin{array}{ccc} V & \xleftarrow{\rho} & V_1 \\ \uparrow & & \uparrow \\ X & \xleftarrow[\overline{\rho}]{} & X_1 \end{array}$$

Moreover, if $H_1 \subset V_1$ denotes the exceptional divisor, then

$$\mathcal{I}(H_1)_{|X_1} = \overline{\rho}^*(\mathcal{I}(Y)).$$

Now set $\mathcal{N} = \mathcal{I}(X) + \mathcal{L}$ and consider the basic objects

$$(V, (\mathcal{N}, 1), E = \emptyset) \text{ and } (X, (\mathcal{N}_{|X}, 1), F = \emptyset),$$

and

$$(V, (\mathcal{L}, 1), E = \emptyset) \text{ and } (X, (\mathcal{L}_{|X}, 1), F = \emptyset).$$

Since $X$ is smooth, it can be shown that a finite sequence of permissible transformations of $(V, (\mathcal{N}, 1), E = \emptyset)$ induces a finite sequence of permissible transformations of $(X, (\mathcal{N}_{|X}, 1), F = \emptyset)$ and vice versa (see [**BrG-EV**, §6]). Moreover, setting:

$$\begin{array}{c} (V_0, (\mathcal{N}_0, 1), E_0) := (V, (\mathcal{N}, 1), E = \emptyset); \\ (X_0, (\mathcal{N}_{|X,0}, 1), F_0) := (X, (\mathcal{N}_{|X}, 1), F = \emptyset); \end{array} \tag{20.2}$$

and

$$(V_0, (\mathcal{L}_0, 1), E_0) := (V, (\mathcal{L}, 1), E = \emptyset); \quad (X_0, (\mathcal{L}_{|X,0}, 1), F_0) := (X, (\mathcal{L}_{|X}, 1), F = \emptyset),$$

a finite sequence of permissible transformations of one of the basic objects in (20.2), induces commutative diagrams of transformations and restrictions:

(20.3)

$$\begin{array}{ccccccc} (V_0, (\mathcal{N}_0, 1), E_0) & \longleftarrow & (V_1, (\mathcal{N}_1, 1), E_1) & \longleftarrow & \dots & \longleftarrow & (V_r, (\mathcal{N}_r, 1), E_r) \\ \downarrow & & \downarrow & & & & \downarrow \\ (X_0, (\mathcal{N}_{|X,0}, 1), F_0) & \longleftarrow & (X_1, (\mathcal{N}_{|X,1}, 1), F_1) & \longleftarrow & \dots & \longleftarrow & (X_r, (\mathcal{N}_{|X,r}, 1), F_r) \end{array}$$

and at the same time induce permissible transformations:

$$(V_0, (\mathcal{L}_0, 1), E_0) \longleftarrow (V_1, (\mathcal{L}_1, 1), E_1) \longleftarrow \dots \longleftarrow (V_r, (\mathcal{L}_r, 1), E_r)$$

and

$$(X_0, (\mathcal{L}_{|X,0}, 1), F_0) \longleftarrow (X_1, (\mathcal{L}_{|X,1}, 1), F_1) \longleftarrow \dots \longleftarrow (X_r, (\mathcal{L}_{|X,r}, 1), F_r),$$

with the following properties:

(1) Sing $(\mathcal{N}_0, 1) \cap X_0 =$ Sing $(\mathcal{N}_{|X,0}, 1) =$ Sing $(\mathcal{L}_{|X,0}, 1)$, i.e.,

$$\text{Sing } (\mathcal{I}(X) + \mathcal{L}, 1) \cap X = \text{Sing } (\mathcal{L}_{|X}, 1);$$

And for $i = 1, \dots, r$,

(2) $(\mathcal{N}_i, 1) = ((\mathcal{I}(X) + \mathcal{L})_i, 1) = ((\mathcal{I}(X_i) + \mathcal{L}_i), 1)$;

(3) Furthermore:

$$\text{Sing } (\mathcal{N}_i, 1) = \text{Sing } ((\mathcal{I}(X) + \mathcal{L})_i, 1) = \text{Sing } ((\mathcal{I}(X_i) + \mathcal{L}_i), 1) =$$
$$= \text{Sing } ((\mathcal{I}(X_i) + \mathcal{L}_i), 1) \cap X_i = \text{Sing } (\mathcal{L}_i, 1) \cap X_i = \text{Sing } (\mathcal{L}_{i|X_i}, 1) =$$
$$= \text{Sing } (\mathcal{L}_{|X,i}, 1).$$

Now let $\pi$ be the composition of the $r$-permissible transformations $V \longleftarrow V_r$, and assume any of the two sequences in (20.3) is a resolution of the corresponding basic object. Then the strict transforms of the components of $\mathcal{L}$ in $V_r$ do not intersect $X_r$. Since all the permissible centers where contained in $X$ it can be concluded that $X_r$ has normal crossings with $\pi^*\mathcal{L} \cup E_r$. In addition, $\pi$ defines an isomorphism on $V \setminus \mathbb{V}(\mathcal{L})$ and hence it induces an isomorphism on an open dense set of $X$.

# Part III: The Identification Theorem for Rees Algebras and compatibility of constructive resolution.

Let $X$ be a non-smooth variety defined over a perfect field $k$. Part I of this manuscript has been devoted to showing that the set of points of maximum multiplicity $X$ can be described as the singular locus of a suitably defined Rees algebra $\mathcal{G}$ defined on some smooth scheme $V$. Moreover, as we have seen in Part II, if $V$ is defined over a field of characteristic zero, a resolution of any Rees algebra can be constructed. Thus a lowering of the maximum multiplicity is achieved via a resolution of $\mathcal{G}$. However, the initial choice of $\mathcal{G}$ and the smooth scheme $V$ may not be unique. In addition, the association of the Rees algebra $\mathcal{G}$ is made locally, in an (étale) neighborhood of each point in the maximum multiplicity locus of $X$. These observations raise some natural questions:

(1) Can we compare the (constructive) resolution of two different Rees algebras (defined over different smooth schemes) that describe the maximum multiplicity locus in a neighborhood of a point of $X$?
(2) Does this local assignation lead to a global sequence of blow ups at smooth centers that lower the maximum multiplicity of $X$?

The purpose of this part is to establish a criterion to compare pairs $(V, \mathcal{G})$, and more generally, basic objects $(V, \mathcal{G}, E)$, defined on different ambient spaces. First we formulate an equivalence relation among pairs defined on different ambient spaces (see Definition 21.2 for the notion of *identifiable or equivalent pairs*), which will be extended to basic objects accordingly (see Definition 27.1 for the notion of *identifiable or equivalent basic basic objects*). As we will see, this new equivalence relation generalizes that of weakly equivalent Rees algebras introduced in Section 10: observe that weak equivalence is established among Rees algebras that are defined on the same smooth scheme.

We will study upper-semi continuous functions that are compatible with this new equivalence relation (see Sections 22, 24, 25 and 26). Finally we state the main result of this part: the Resolution Theorem for Rees algebras 27.5, which asserts that given two identifiable basic objects, the constructive resolution of one of them given by Theorem 18.9 is naturally compatible with that of the other.

The results obtained in the next sections will be used in Part IV to answer affirmatively to the questions raised in (1) and (2). In the present Part III, all fields are assumed to be of characteristic zero.

## 21. Identifiable pairs

A couple of the form $(V^{(n)}, \mathcal{G}^{(n)})$ is said to be an *n-dimensional pair*, or simply a *pair*, if $V^{(n)}$ is an $n$-dimensional smooth scheme of finite type over a field $k$ (which we assume to be of characteristic zero), and $\mathcal{G}^{(n)}$ is an $\mathcal{O}_{V^{(n)}}$-Rees algebra. Two pairs $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V'^{(n)}, \mathcal{G}'^{(n)})$ will be said to be *weakly equivalent* if $V^{(n)} = V'^{(n)}$ and $\mathcal{G}^{(n)}$ is weakly equivalent to $\mathcal{G}'^{(n)}$ (see Definition 10.5). In what follows we will not distinguish between two weakly equivalent pairs, and this will enable us to assume that the Rees algebras are differential, unless otherwise indicated. However, even if our starting point is a differential Rees algebra, recall that its transform after a permissible transformation will not be differentially saturated any more (see 16.3).

Definition 21.1. We say that $(V^{(d)}, \mathcal{G}^{(d)})$ is an *elimination pair* for $(V^{(n)}, \mathcal{G}^{(n)})$ if there is a $\mathcal{G}^{(n)}$-admissible (smooth) projection $\beta_{n,d} : V^{(n)} \to V^{(d)}$, so that $\mathcal{G}^{(d)}$ is an elimination algebra for $\mathcal{G}^{(n)}$ (see Section 16). Here necessarily $d < n$.

Definition 21.2. Two pairs $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)})$ are said to be *identifiable* if there is a third pair, $(V^{(m)}, \mathcal{G}^{(m)})$, for which both $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)})$ are elimination pairs.

Remark 21.3. Definition 21.2 is valid over perfect fields, and so is Lemma 21.9 below. The treatment in such context requires different arguments and the results are still partial. So we present here results only in characteristic zero, such as Lemma 21.10.

Remark 21.4. The pair $(V^{(m)}, \mathcal{G}^{(m)})$ from Definition 21.2 may not be unique. So, whenever we say that $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)})$ are identifiable, we will be assuming that we have fixed a pair $(V^{(m)}, \mathcal{G}^{(m)})$ for which the conditions of the definition hold. We will refer to

$$\begin{array}{ccc} & (V^{(m)}, \mathcal{G}^{(m)}) & \\ {}^{\lambda}\swarrow & & \searrow^{\lambda'} \\ (V^{(n)}, \mathcal{G}^{(n)}) & & (V^{(d)}, \mathcal{G}^{(d)}) \end{array}$$

as a *diagram of identifications*.

**How do we search for identifiable pairs?** Let $(V^{(n)}, \mathcal{G}^{(n)})$ be a pair. The purpose of the following examples is to exhibit some pairs which are identifiable with $(V^{(n)}, \mathcal{G}^{(n)})$.

Example 21.5. If $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V'^{(n)}, \mathcal{G}'^{(n)})$ are isomorphic pairs, meaning that there is an isomorphism, $\phi : V^{(n)} \longrightarrow V'^{(n)}$ with $\phi^*(\mathcal{G}'^{(n)}) = \mathcal{G}^{(n)}$, then $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V'^{(n)}, \mathcal{G}'^{(n)})$ are identifiable. To see this, consider the diagram

$$\begin{array}{ccc} & & V^{(n+1)} := V^{(n)} \times_{\mathrm{Spec}\ k} \mathbb{A}^1_k \\ & {}^{\alpha}\swarrow & \downarrow{\scriptstyle \alpha'} \\ V^{(n)} & \xrightarrow{\ \phi\ } & V'^{(n)}, \end{array}$$

where $\alpha' = \phi \circ \alpha$. Now if $\{x = 0\}$ is the image of a section of $\alpha$, set $\mathcal{G}^{(n+1)} := \mathcal{O}_{V^{(n+1)}}[xW] \odot \alpha^*(\mathcal{G}^{(n)})$. Notice that by the hypothesis, also $\mathcal{G}^{(n+1)} = \mathcal{O}_{V^{(n+1)}}[xW] \odot \alpha^*(\phi^*(\mathcal{G}'^{(n)})) = \mathcal{O}_{V^{(n+1)}}[xW] \odot \alpha'^*(\mathcal{G}'^{(n)})$. Now, by Remark 16.10, $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V'^{(n)}, \mathcal{G}'^{(n)})$ are elimination pairs for $(V^{(n+1)}, \mathcal{G}^{(n+1)})$.

Remark 21.6. Given a pair $(V^{(d)}, \mathcal{G}^{(d)})$ we can easily construct another pair $(V^{(n)}, \mathcal{G}^{(n)})$ with $n > d$, for which $(V^{(d)}, \mathcal{G}^{(d)})$ is an elimination pair. To do so, it suffices to consider the natural extension $\mathcal{O}_{V^{(d)}} \to \mathcal{O}_{V^{(d)}}[X_1, \ldots, X_{n-d}]$ which induces a smooth morphism $\beta_{n,d} : V^{(d)} \times \mathbb{A}^{n-d}_k \to V^{(d)}$, and then set

$$\mathcal{G}^{(n)} := \beta^*_{n,d}(\mathcal{G}^{(d)}) \odot \mathcal{O}_{V^{(n)}}[\langle X_1, \ldots, X_{d-n} \rangle W].$$

It can be checked that $(V^{(d)}, \mathcal{G}^{(d)})$ is an elimination pair of $(V^{(n)}, \mathcal{G}^{(n)})$ (see Remark 16.10). More generally, given a pair $(V^{(d)}, \mathcal{G}^{(d)})$ and a smooth morphism $\beta : V^{(n)} \to V^{(d)}$ for some $V^{(n)}$ with $n > d$, we can perform a construction (similar to that in

Remark 21.6) whenever there is a section of $\beta$. This is possible, locally at any point of $V^{(d)}$ using étale topology (see Remark 16.10).

EXAMPLE 21.7. If $(V^{(d)}, \mathcal{G}^{(d)})$ is an elimination pair for $(V^{(n)}, \mathcal{G}^{(n)})$ then the pairs $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)})$ are identifiable. To see this consider the diagram:

$$\begin{array}{ccc} & & V^{(n+1)} := V^{(n)} \times_{\mathrm{Spec}(k)} \mathbb{A}^1_k \\ & \overset{\alpha}{\swarrow} & \downarrow {\scriptstyle \beta\circ\alpha} \\ V^{(n)} & & \\ & \underset{\beta}{\searrow} & \\ & & V^{(d)}. \end{array}$$

Suppose that $\{x = 0\}$ is the image of a section of $\alpha$, and set $\mathcal{G}^{(n+1)} := \mathcal{O}_{V^{(n+1)}}[xW] \odot \alpha^*(\mathcal{G}^{(n)})$. By Remark 21.6, it follows that $(V^{(n)}, \mathcal{G}^{(n)})$ is an elimination pair for $(V^{(n+1)}, \mathcal{G}^{(n+1)})$. To check that $(V^{(d)}, \mathcal{G}^{(d)})$ is an elimination pair for $(V^{(n+1)}, \mathcal{G}^{(n+1)})$ we use Remark 16.10 to write $\mathcal{G}^{(n)}$ as $\mathcal{O}_{V^{(n)}}[z_1 W, \dots, z_{n-d} W] \odot \beta^*(\mathcal{G}^{(d)})$ where $\{z_1 = 0, \dots, z_{n-d} = 0\}$ defines the image of a section of $\beta$. Now observe that $\{x = 0, \alpha^*(z_1) = 0, \dots, \alpha^*(z_{n-d} = 0)\}$ is the image of a section of $\beta \circ \alpha$, and that

$$\mathcal{G}^{(n+1)} = \mathcal{O}_{V^{(n+1)}}[xW] \odot \alpha^*(\mathcal{G}^{(n)}) =$$

$$= \mathcal{O}_{V^{(n+1)}}[xW, \alpha^*(z_1)W, \dots, \alpha^*(z_{n-d})W] \odot \alpha^*(\beta^*(\mathcal{G}^{(d)})),$$

and the claim follows from Remark 16.10 once more.

REMARK 21.8. *Being identifiable* is an equivalence relation among pairs. Reflexivity follows from Example 21.5, symmetry is obvious, and transitivity follows from Lemma 21.9 below.

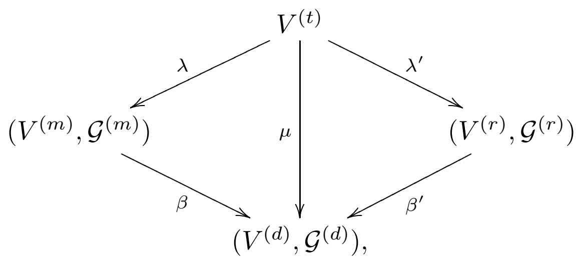
LEMMA 21.9. *If $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)})$ are identifiable, and if $(V^{(d)}, \mathcal{G}^{(d)})$ and $(V^{(l)}, \mathcal{G}^{(l)})$ are identifiable, then $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(l)}, \mathcal{G}^{(l)})$ are identifiable.*

PROOF. By the hypotheses, $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)})$ are elimination pairs for some pair $(V^{(m)}, \mathcal{G}^{(m)})$ with $m > n, d$, and $(V^{(d)}, \mathcal{G}^{(d)})$ and $(V^{(l)}, \mathcal{G}^{(l)})$ are elimination pairs for some pair $(V^{(r)}, \mathcal{G}^{(r)})$ with $r > d, l$. Thus we have the following diagrams of identifications:

$$\begin{array}{ccccccc} & (V^{(m)}, \mathcal{G}^{(m)}) & & & & (V^{(r)}, \mathcal{G}^{(r)}) & \\ \swarrow & & \overset{\beta}{\searrow} & & \overset{\beta'}{\swarrow} & & \searrow \\ (V^{(n)}, \mathcal{G}^{(n)}) & & & (V^{(d)}, \mathcal{G}^{(d)}) & & & (V^{(l)}, \mathcal{G}^{(l)})). \end{array}$$

Now it will be enough to show that $(V^{(m)}, \mathcal{G}^{(m)})$ and $(V^{(r)}, \mathcal{G}^{(r)})$ can be seen as elimination pairs of a third pair. To do so, set $V^{(t)} := V^{(m)} \times_{V^{(d)}} V^{(r)}$, thus

$t = m + r - d$. Consider the commutative diagram




and define

$$\mathcal{G}^{(t)} := \lambda^*(\mathcal{G}^{(m)}) \odot \lambda'^*(\mathcal{G}^{(r)}).$$

We claim that $(V^{(m)}, \mathcal{G}^{(m)})$ and $(V^{(r)}, \mathcal{G}^{(r)})$ are elimination pairs of $(V^{(t)}, \mathcal{G}^{(t)})$. To prove the claim it is enough to argue locally.

Let $\xi \in \operatorname{Sing} \mathcal{G}^{(d)}$, and set:

$$\xi_\beta = \beta^{-1}(\xi) \cap \operatorname{Sing} \mathcal{G}^{(m)}, \text{ and } \xi_{\beta'} = \beta'^{-1}(\xi) \cap \operatorname{Sing} \mathcal{G}^{(r)}.$$

Now since $\tau_{\mathcal{G}^{(m)}, \xi_\beta} \geq (m - d)$ (see Remark 16.10), locally, at $\mathcal{O}_{V^{(m)}, \xi_\beta}$, there is a regular system of parameters $x_1, \ldots, x_m$, so that

$$\mathcal{G}^{(m)} = \mathcal{O}_{V^{(m)}}[x_1 W, \ldots, x_{m-d} W] \odot \beta^*(\mathcal{G}^{(d)}).$$

In fact, this equality holds in some open set $U^{(m)}$ containing $\xi_\beta$. Similarly, since $\tau_{\mathcal{G}^{(r)}, \xi_{\beta'}} \geq (r - d)$, locally at $\mathcal{O}_{V^{(r)}, \xi_{\beta'}}$, there is a regular system of parameters $y_1, \ldots, y_r$, so that

$$\mathcal{G}^{(r)} = \mathcal{O}_{V^{(m)}}[y_1 W, \ldots, y_{r-d} W] \odot \beta'^*(\mathcal{G}^{(d)}),$$

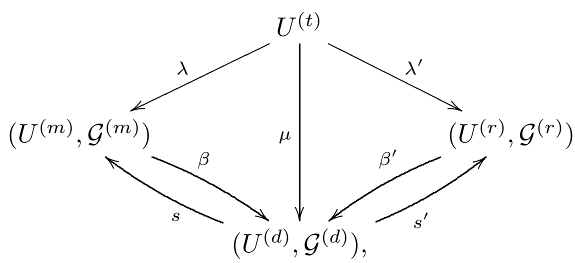
and in fact this equality holds in a neighborhood $U^{(r)}$ of $\xi_{\beta'}$. Replace $V^{(d)}$ by some open neighborhood $U^{(d)}$ of $\xi$, so that

$$\beta : U^{(m)} \to U^{(d)}, \text{ and } \beta' : U^{(r)} \to U^{(d)}.$$

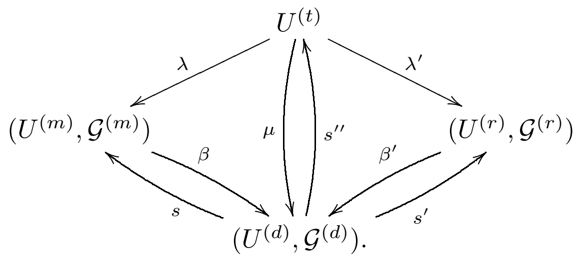
Now consider:

$$U^{(t)} := U^{(m)} \times_{U^{(d)}} U^{(r)} \subset V^{(t)},$$

and the commutative diagram:




with sections $s : U^{(d)} \to U^{(m)}$ and $s' : U^{(d)} \to U^{(r)}$ whose images are determined by the zeroes of $x_1, \ldots, x_{m-d}$ and $y_1, \ldots, y_{r-d}$ respectively (here we may need to replace $U^{(d)}$ by some étale neighborhood, which we denote by $U^{(d)}$ again to simplify the notation, see Remark 16.10). By the universal property of the fiber product, there is a section $s'' : U^{(d)} \to U^{(t)}$ so that the following diagram commutes:




Notice that

$$\mathcal{G}^{(t)} = \lambda^*(\mathcal{G}^{(m)}) \odot \lambda'^*(\mathcal{G}^{(r)}) = \mathcal{O}_{V^{(t)}}[\lambda^*(x_1)W, \dots, \lambda^*(x_{m-d})W, \lambda'^*(y_1)W, \dots, \lambda'^*(y_{r-d})W] \odot \mu^*(\mathcal{G}^{(d)}), \tag{21.1}$$

and that $\lambda^*(x_1), \dots, \lambda^*(x_{m-d}), \lambda'^*(y_1), \dots, \lambda'^*(y_{r-d})$ define the image of

$$s'' : U^{(d)} \to U^{(t)}.$$

Therefore, $\lambda^*(x_1), \dots, \lambda^*(x_{m-d}), \lambda'^*(y_1), \dots, \lambda'^*(y_{r-d})$ can be extended to some regular system of parameters in an open subset of $U^{(t)}$. From the expression (21.1) it follows that $\mathcal{G}^{(d)}$ is an elimination algebra for $\mathcal{G}^{(t)}$ (see Remark 16.10).

Now, since $\mu$ is transversal to $\mathcal{G}^{(t)}$ so is $\lambda$. Thus we can construct an elimination algebra of $\mathcal{G}^{(t)}$ in $V^{(m)}$, say $\widetilde{\mathcal{G}}^{(m)}$. Notice that by construction,

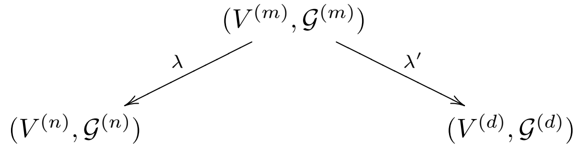
$$\mathcal{G}^{(m)} \subset \widetilde{\mathcal{G}}^{(m)},$$

and hence by 16.5,

$$\operatorname{Sing} \mathcal{G}^{(m)} \supset \operatorname{Sing} \widetilde{\mathcal{G}}^{(m)}.$$

By 16.7 (6) and (7) we have that: any $\mathcal{G}^{(d)}$-local sequence induces a $\mathcal{G}^{(m)}$-local sequence and a $\mathcal{G}^{(t)}$-local sequence and identification of singular loci. Any $\mathcal{G}^{(t)}$-local sequence in turn induces a $\widetilde{\mathcal{G}}^{(m)}$-local sequence with identification of singular loci. Iterating this argument it can be checked that $\widetilde{\mathcal{G}}^{(m)}$ and $\mathcal{G}^{(m)}$ are weakly equivalent. Therefore, $(V^{(m)}, \mathcal{G}^{(m)})$ is an elimination pair for $(V^{(t)}, \mathcal{G}^{(t)})$. The claim on $(V^{(r)}, \mathcal{G}^{(r)})$ follows from a similar argument. □

LEMMA 21.10. **Being identifiable is preserved by considering local sequences.** *Suppose that* $(V^{(n)}, \mathcal{G}^{(n)})$ *and* $(V^{(d)}, \mathcal{G}^{(d)})$ *are identifiable via a diagram of identifications:*




*Then any local* $\mathcal{G}^{(n)}$*-sequence as the ones considered in Remark* 11.7*,*

$$(V^{(n)}, \mathcal{G}^{(n)}) = (V_0^{(n)}, \mathcal{G}_0^{(n)}) \longleftarrow (V_1^{(n)}, \mathcal{G}_1^{(n)}) \longleftarrow \dots \longleftarrow (V_s^{(n)}, \mathcal{G}_s^{(n)}), \tag{21.2}$$

*induces a* $\mathcal{G}^{(d)}$*-local sequence*

$$(V^{(d)}, \mathcal{G}^{(d)}) = (V_0^{(d)}, \mathcal{G}_0^{(d)}) \longleftarrow (V_1^{(d)}, \mathcal{G}_1^{(d)}) \longleftarrow \dots \longleftarrow (V_s^{(d)}, \mathcal{G}_s^{(d)}), \tag{21.3}$$

*and a* $\mathcal{G}^{(m)}$*-local sequence*

$$(V^{(m)},\mathcal{G}^{(m)}) = (V_0^{(m)},\mathcal{G}_0^{(m)}) \longleftarrow (V_1^{(m)},\mathcal{G}_1^{(m)}) \longleftarrow \ldots \longleftarrow (V_s^{(m)},\mathcal{G}_s^{(m)}), \tag{21.4}$$

*and vice versa. Moreover, the pairs* $(V_i^{(n)},\mathcal{G}_i^{(n)})$ *and* $(V_i^{(d)},\mathcal{G}_i^{(d)})$ *from sequences (*21.2*) and (*21.3*) are identifiable for each* $i = 0, 1, \ldots, s$ *via diagrams of identification*

$$\begin{array}{ccc} & (V_i^{(m)},\mathcal{G}_i^{(m)}) & \\ {}^{\lambda_i}\swarrow & & \searrow^{\lambda'_i} \\ (V_i^{(n)},\mathcal{G}_i^{(n)}) & & (V_i^{(d)},\mathcal{G}_i^{(d)}), \end{array}$$

*and for* $i = 0, 1, \ldots, s$*:*

1. *Sing* $\mathcal{G}_i^{(n)}$ *is homeomorphic to Sing* $\mathcal{G}_i^{(d)}$*;*
2. *A smooth center* $Y_i \subset$ *Sing* $\mathcal{G}_i^{(n)}$ *corresponds to a smooth center in Sing* $\mathcal{G}_i^{(d)}$ *and vice versa.*

Proof. From the hypotheses, $(V^{(n)},\mathcal{G}^{(n)})$ and $(V^{(d)},\mathcal{G}^{(d)})$ are elimination pairs for the pair $(V^{(m)},\mathcal{G}^{(m)})$. Now the assertion follows from 16.7 (1), (2), (6) and (7). □

## Identifiable pairs and invariants

In the following sections we will be interested in studying what we call *invariant functions on identifiable pairs* (see Definition 22.1 below). Examples of these functions will be the *dimensional type of a pair*, *Hironaka's order function,* and the corresponding *satellite functions* (see Section 14).

## 22. Invariant functions on pairs

Suppose that $(V^{(n)},\mathcal{G}^{(n)})$ and $(V^{(d)},\mathcal{G}^{(d)})$ are identifiable and fix a diagram of identifications as in Remark 21.4,

$$\begin{array}{ccc} & (V^{(m)},\mathcal{G}^{(m)}) & \\ {}^{\lambda}\swarrow & & \searrow^{\lambda'} \\ (V^{(n)},\mathcal{G}^{(n)}) & & (V^{(d)},\mathcal{G}^{(d)}). \end{array} \tag{22.1}$$

For each $\xi \in$ Sing $\mathcal{G}^{(n)}$, let $\xi_\lambda \in$ Sing $\mathcal{G}^{(m)}$ be the unique point dominating $\xi$, and set $\xi' := \lambda'(\xi_\lambda)$. In other words, since $(V^{(n)},\mathcal{G}^{(n)})$ and $(V^{(d)},\mathcal{G}^{(d)})$ are identifiable, the closed sets Sing $\mathcal{G}^{(n)}$ and Sing $\mathcal{G}^{(d)}$ are homeomorphic, and $\xi'$ is the image of $\xi$ via the homeomorphism induced by the diagram of identifications (22.1).

Definition 22.1. With the same notation as above, an *invariant function for identifiable pairs* is an assignment of a function $\gamma_{(V^{(n)},\mathcal{G}^{(n)})}$ to each pair $(V^{(n)},\mathcal{G}^{(n)})$ so that the following conditions hold:

1. Each $\gamma_{(V^{(n)},\mathcal{G}^{(n)})}$ is defined on Sing $\mathcal{G}^{(n)}$ and takes values in some well ordered set $\Lambda$:

$$\gamma_{(V^{(n)},\mathcal{G}^{(n)})} : \text{Sing } \mathcal{G}^{(n)} \longrightarrow \Lambda;$$

(2) If $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)})$ are identifiable with a diagram of identifications like (22.1) then, for all $\xi \in \operatorname{Sing} \mathcal{G}^{(n)}$

$$\gamma_{(V^{(n)},\mathcal{G}^{(n)})}(\xi) = \gamma_{(V^{(d)},\mathcal{G}^{(d)})}(\xi').$$

## 23. Trivial pairs

**23.1. Simple pairs, trivial pairs.** A pair $(V^{(n)}, \mathcal{G}^{(n)})$ is said to be *simple* (resp. *e-simple*) at $\xi \in \operatorname{Sing} \mathcal{G}^{(n)}$ if $\mathcal{G}^{(n)}$ is simple (resp. $e$-simple) at $\xi$ (see 13.3 (3)). Recall that if $\tau_{\mathcal{G}^{(n)},\xi} \geq e \geq 1$ then there is a regular system of parameters $x_1, \dots, x_n \in \mathcal{O}_{V^{(n)},\xi}$, so that, up to weak equivalence, $x_1W, \dots, x_eW \in \mathcal{G}^{(n)}$. Thus, if $(V^{(n)}, \mathcal{G}^{(n)})$ is $e$-simple at $\xi$, then it is $e$-simple in a neighborhood of $\xi$. We will say that $(V^{(n)}, \mathcal{G}^{(n)})$ is *simple (resp. e-simple)* if it is simple (resp. $e$-simple) at any point $\xi \in \operatorname{Sing} \mathcal{G}^{(n)}$.

If $(V^{(n)}, \mathcal{G}^{(n)})$ is $e$-simple pair for some $e \geq 1$ then the $e$-codimensional components of $\operatorname{Sing} \mathcal{G}^{(n)}$ are smooth and disconnected in $\operatorname{Sing} \mathcal{G}^{(n)}$, and $\operatorname{Sing} \mathcal{G}^{(n)}$ is a disjoint union

$$\operatorname{Sing} \mathcal{G}^{(n)} = F_e \sqcup Z$$

where $F_e$ denotes the (disjoint) union of the components of codimension $e$. We will say that $(V^{(n)}, \mathcal{G}^{(n)})$ is *e-trivial at each point* $\xi \in F_e$ (see 13.3 (5)).

PROPOSITION 23.2. *Assume $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)})$ are identifiable with a diagram of identifications as in (*22.1*). Let $\xi \in$ Sing $\mathcal{G}^{(n)}$, and let $\xi' \in$ Sing $\mathcal{G}^{(d)}$ be the point identified with $\xi$ via the homeomorphism induced by $\lambda$ and $\lambda'$. Then, $(V^{(n)}, \mathcal{G}^{(n)})$ is e-trivial at $\xi$ for some $e \geq 1$ if and only if $(V^{(d)}, \mathcal{G}^{(d)})$ is $e'$-trivial at $\xi$ for some $e' \geq 1$.*

PROOF. Assume that $(V^{(n)}, \mathcal{G}^{(n)})$ is $e$-trivial at $\xi$ for some $e \geq 1$ and let $\xi'' \in \operatorname{Sing} \mathcal{G}^{(m)}$ be its preimage via $\lambda$. Then by the Claim 1 in Remark 16.10, $\mathcal{G}^{(m)}$ is $(m-n)+e$ trivial. Observe that here necessarily $(m-n)+e > (m-d)$ (since $(V^{(d)}, \mathcal{G}^{(d)})$ is an elimination pair for $(V^{(m)}, \mathcal{G}^{(m)})$). Again by Remark 16.10, $\mathcal{G}^{(d)}$ one has that $(m-n)+e-(m-d) = (d+e-n)$-trivial. □

**Conclusion.** Being trivial is an *invariant quality* of identifiable pairs.

## 24. The dimensional type of a pair

DEFINITION 24.1. A pair $(V^{(n)}, \mathcal{G}^{(n)})$ is said to be *of dimensional type d* if it is identifiable with some $d$-dimensional pair. A pair $(V^{(n)}, \mathcal{G}^{(n)})$ is said to be *locally of dimensional type d at a point $\xi \in$ Sing $\mathcal{G}^{(n)}$* if it is identifiable with some $d$-dimensional pair in a neighborhood of $\xi$.

REMARK 24.2. If $(V^{(n)}, \mathcal{G}^{(n)})$ is of dimensional type $d$, then, in particular, we can see a homeomorphic copy of $\operatorname{Sing} \mathcal{G}^{(n)}$ embedded in some $d$-dimensional smooth scheme.

REMARK 24.3. Since being identifiable is an equivalence relation among pairs, it is clear that if a pair $(V^{(n)}, \mathcal{G}^{(n)})$ is of dimensional type $d$, then so is any other pair identifiable with $(V^{(n)}, \mathcal{G}^{(n)})$. Thus the dimensional type is an invariant on identifiable pairs according to Definition 22.1 (as opposed to the codimensional type of a pair, see 13.3 (3), which is an invariant only for weakly equivalent pairs).

*What are the possible dimensional types of a given pair?* Let $(V^{(n)},\mathcal{G}^{(n)})$ be an $n$-dimensional pair. By Remark 21.6, $(V^{(n)},\mathcal{G}^{(n)})$ has dimensional type $m$ for any $m \geq n$. What about smaller dimensions? This question requires a more careful analysis, and will be discussed in the following lines.

**24.4. Trivial pairs.** If $(V^{(n)},\mathcal{G}^{(n)})$ is $e$-trivial for some $e \geq 1$ then it can be represented in dimensions $n,\dots,n-(e-1)$ in a neighborhood of $\xi$, but, of course, it cannot be represented in dimension $(n-e)$ (recall that in our setting, the singular locus of a Rees algebra $\mathcal{G}$ in $V$ is a proper subset of $V$). This settles the question about the representability of trivial pairs in lower dimensions. Proposition 23.2 states that if two pairs are identifiable, then the homeomorphism between their singular loci maps the trivial components of one of them to the trivial components of the other.

Definition 24.5. Let $(V^{(n)},\mathcal{G}^{(n)})$ be an $e$-simple pair. We will say that $(V^{(n)},\mathcal{G}^{(n)})$ is *non-e-trivial* if $F_e = \emptyset$ (see the notation in 23.1).

Proposition 24.6. *An $n$-dimensional pair $(V^{(n)},\mathcal{G}^{(n)})$ has dimensional types $(n-1) \geq \ldots \geq (n-e)$ in a neighborhood of a point $\xi \in$ Sing $\mathcal{G}^{(n)}$ if and only if $\tau_{\mathcal{G}^{(n)}\xi} \geq e \geq 1$ and $(V^{(n)},\mathcal{G}^{(n)})$ is non-e-trivial.*

Proof. If $\tau_{\mathcal{G}^{(n)}} \geq e > 1$ in a neighborhood $U$ of $\xi \in \operatorname{Sing} \mathcal{G}^{(n)}$ and $\mathcal{G}^{(n)}$ is non-$e$-trivial at $\xi$ then $\mathcal{G}^{(n)}$-admissible projections can be constructed to smooth $(n-i)$-dimensional schemes $V^{(n-i)}$ for all $i = 1,\dots,e$,

$$\beta_{n,n-i} : U \subset V^{(n)} \to V^{(n-i)},$$

and elimination algebras $\mathcal{G}^{(n-i)} \subset \mathcal{O}_{V^{(n-i)}}[W]$ can be defined (see 16.7).

To show the converse, assume, to get a contradiction, that $(V^{(n)},\mathcal{G}^{(n)})$ is representable in dimensions $(n-1) \geq \ldots \geq (n-e)$ in a neighborhood of $\xi$. Then $(V^{(n)},\mathcal{G}^{(n)})$ is identifiable with some $(n-e)$-dimensional pair say $(V^{(n-e)},\mathcal{G}^{(n-e)})$. Fix a diagram of identifications:

$$\begin{array}{ccccc} & & (V^{(m)},\mathcal{G}^{(m)}) & & \\ & \swarrow^{\lambda} & & \searrow^{\lambda'} & \\ (V^{(n)},\mathcal{G}^{(n)}) & & & & (V^{(n-e)},\mathcal{G}^{(n-e)}), \end{array} \tag{24.1}$$

and let $\xi' \in \operatorname{Sing} \mathcal{G}^{(m)}$ be such that $\lambda(\xi') = \xi$. By Remark 16.10, there is a regular system of parameters $x_1,\dots,x_m \in \mathcal{O}_{V^{(m)},\xi'}$ such that

$$\mathcal{O}_{V^{(m)},\xi'}[x_1W,\dots,x_{m-(n-e)}W] \subset \mathcal{G}^{(m)}.$$

Hence $\tau_{\mathcal{G}^{(m)},\xi'} \geq m-(n-e)$ and by 16.7 (5), $\tau_{\mathcal{G}^{(n)},\xi} \geq (m-(n-e))-(m-n) = e$. Now, if $\mathcal{G}^{(n)}$ were $e$-trivial at $\xi$, then, again by Remark 16.10, $\mathcal{G}^{(m)}$ would be $(m-n)+e$-trivial at $\xi'$ (see also Proposition 23.2) and by the discussion in 24.4 it only be would be representable up to dimension $m-((m-n)+e)+1 = n-e+1$ which is a contradiction. □

**24.7. Summarizing: bounds on the dimensional type of a pair.** Let $(V^{(n)},\mathcal{G}^{(n)})$ be an $e$-simple pair. We distinguish two cases:

**The trivial case.** If $(V^{(n)}, \mathcal{G}^{(n)})$ is $e$-trivial for some $e \geq 1$, then it is easy to see that its dimensional types are $\dots, (n+1), n, \dots, (n-e+1)$. If a pair is trivial then so is any other pair identifiable with it (see Proposition 23.2).

**The non-trivial case.** If $(V^{(n)}, \mathcal{G}^{(n)})$ is non-$e$-trivial and $\tau_{\mathcal{G}^{(n)}} = e \geq 0$ then the dimensional types of $(V^{(n)}, \mathcal{G}^{(n)})$ are at least $\dots, (n+1), n, \dots, (n-e)$.

## 25. Other invariant functions on identifiable pairs: order functions

Let $(V^{(n)}, \mathcal{G}^{(n)})$ be a pair. As indicated before, our purpose is to define invariant functions on pairs. In this section, we will present upper-semi continuous functions with values in some well ordered set $(\Lambda, \geq)$,

$$F_{\mathcal{G}^{(n)}} : \quad \operatorname{Sing} \mathcal{G}^{(n)} \quad \longrightarrow \quad (\Lambda, \geq)$$

that are invariants on identifiable pairs (see Section 22) and that are defined at each step of a $\mathcal{G}^{(n)}$-local sequence. So if

(25.1)
$$\begin{array}{ccccccc} V_0^{(n)} = V^{(n)} & \leftarrow & V_1^{(n)} & \leftarrow & \dots & \leftarrow & V_m^{(n)} \\ \mathcal{G}_0^{(n)} = \mathcal{G}^{(n)} & & \mathcal{G}_1^{(n)} & & \dots & & \mathcal{G}_m^{(n)} \end{array}$$

is a local sequence, then the functions

$$F_{\mathcal{G}_i^{(n)}} : \quad \operatorname{Sing} \mathcal{G}_i^{(n)} \quad \longrightarrow \quad (\Lambda, \geq)$$

are natural extensions of $F_{\mathcal{G}_0^{(n)}}$ for $i = 1, \dots, m$ in some sense to be discussed.

REMARK 25.1. Let $(V^{(n)}, \mathcal{G}^{(n)})$ be a pair. An interesting example of an upper-semi continuous function on pairs is Hironaka's order function (see 9.8):

(25.2)
$$\begin{array}{cccc} \operatorname{ord}^{(n)}_{\mathcal{G}^{(n)}} : & \operatorname{Sing} \mathcal{G}^{(n)} & \longrightarrow & \mathbb{Q} \\ & \eta & \mapsto & \operatorname{ord}_{\mathcal{G}^{(n)}}(\eta). \end{array}$$

Notice that Hironaka's order function is defined at any step of a $\mathcal{G}^{(n)}$-local sequence like (25.1):

(25.3)
$$\begin{array}{cccc} \operatorname{ord}^{(n)}_{\mathcal{G}_i^{(n)}} : & \operatorname{Sing} \mathcal{G}_i^{(n)} & \longrightarrow & \mathbb{Q} \\ & \eta & \mapsto & \operatorname{ord}_{\mathcal{G}_i^{(n)}}(\eta), \end{array}$$

and takes the same value on weakly equivalent pairs. The purpose of this section is to study these order functions and to show that in fact, these are invariant functions on identifiable pairs (see 25.4)).

**25.2. Order functions defined for a given pair.** Assume that $(V^{(n)}, \mathcal{G}^{(n)})$ is of dimensional type $d$, and fix a diagram of identifications:

(25.4)
$$\begin{array}{ccccc} & & (V^{(m)}, \mathcal{G}^{(m)}) & & \\ & \overset{\lambda}{\swarrow} & & \overset{\lambda'}{\searrow} & \\ (V^{(n)}, \mathcal{G}^{(n)}) & & & & (V^{(d)}, \mathcal{G}^{(d)}). \end{array}$$

For each $\eta \in \operatorname{Sing} \mathcal{G}^{(n)}$, let $\eta' \in \operatorname{Sing} \mathcal{G}^{(d)}$ be the corresponding point via the homeomorphism induced by $\lambda$ and $\lambda'$. Then define:

(25.5)
$$\begin{array}{cccc} \operatorname{ord}^{(d)}_{\mathcal{G}^{(n)}} : & \operatorname{Sing} \mathcal{G}^{(n)} & \longrightarrow & \mathbb{Q} \\ & \eta & \mapsto & \operatorname{ord}_{\mathcal{G}^{(d)}}(\eta'). \end{array}$$

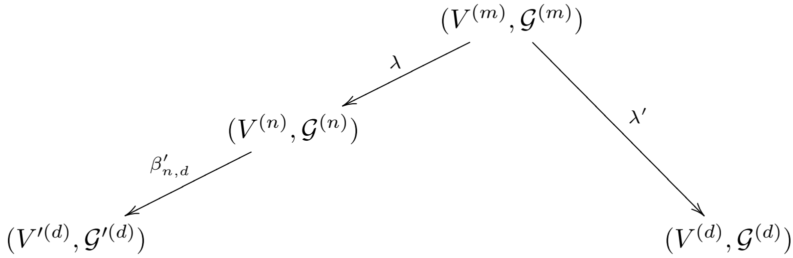
To see that this function is well defined, we consider two cases:

**Case $d > n$.** Notice that $\mathrm{ord}^{(d)}_{\mathcal{G}^{(n)}}$ is constantly equal to 1 on Sing $\mathcal{G}^{(n)}$ for any $d$-dimensional pair identifiable with $(V^{(n)}, \mathcal{G}^{(n)})$ with $d > n$. This follows from the fact that $\mathcal{G}^{(d)}$ is necessarily $(d-n)$-simple, see Proposition 24.6.

**Case $d \leq n$.** If $n = d$ then the assertion follows from Theorem 17.1. If $n > d$, then, since the assumption is that $(V^{(n)}, \mathcal{G}^{(n)})$ is of dimensional type $d$, necessarily $\tau_{\mathcal{G}^{(n)}} \geq (n-d)$ (see Proposition 24.6), and $\mathcal{G}^{(n)}$ is not $(n-d)$-trivial, so a $\mathcal{G}^{(n)}$-admissible projection can be constructed to some $d$-dimensional smooth scheme,

$$\beta'_{n,d} : V^{(n)} \longrightarrow V'^{(d)}$$

with a corresponding elimination algebra $\mathcal{G}'^{(d)}$.
(25.6)




But, via $\lambda$, the pair $(V'^{(d)}, \mathcal{G}'^{(d)})$ is also an elimination pair for $(V^{(m)}, \mathcal{G}^{(m)})$. Thus, by Theorem 17.1 again, the function $\mathrm{ord}^{(d)}_{\mathcal{G}^{(n)}}$ coincides with that defined by any $d$-dimensional elimination pair for $(V^{(n)}, \mathcal{G}^{(n)})$, and hence it is well defined.

Definition 25.3. Assume that $(V^{(n)}, \mathcal{G}^{(n)})$ is of dimensional type $d$. Then we will refer to the *$d$-th order function of* $\mathcal{G}^{(n)}$ as the function

$$\mathrm{ord}^{(d)}_{\mathcal{G}^{(n)}} : \mathrm{Sing}\ \mathcal{G}^{(n)} \to \mathbb{Q},$$

from 25.2.

Remark 25.4. From the discussion in 25.2, it follows that the functions of Definition 25.3 are invariant functions on identifiable pairs.

Remark 25.5. If $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V'^{(n)}, \mathcal{G}'^{(n)})$ are isomorphic via some isomorphism $\phi : V^{(n)} \longrightarrow V'^{(n)}$ as in Example 21.5, then the pairs $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V'^{(n)}, \mathcal{G}'^{(n)})$ are identifiable, and by Remark 25.4,

$$\mathrm{ord}^{(n)}_{\mathcal{G}^{(n)}}(\xi) = \mathrm{ord}^{(n)}_{\mathcal{G}'^{(n)}}(\phi(\xi)).$$

Moreover, if $(V^{(n)}, \mathcal{G}^{(n)})$ is of dimensional type $d$, then so is $(V'^{(n)}, \mathcal{G}'^{(n)})$, and

$$\mathrm{ord}^{(d)}_{\mathcal{G}^{(n)}}(\xi) = \mathrm{ord}^{(d)}_{\mathcal{G}'^{(n)}}(\phi(\xi)).$$

## 26. More invariant functions on pairs: satellite functions

Let $(V^{(n)}, \mathcal{G}^{(n)})$ be a pair. Recall that we will focus our attention on upper-semi continuous functions $F_{\mathcal{G}^{(n)}}$ defined on Sing $\mathcal{G}^{(n)}$ that can be naturally extended along a $\mathcal{G}^{(n)}$-local sequence, and that are invariants on identifiable pairs in the sense of Definition 22.1.

The maximum value of an upper-semi continuous function $F_{\mathcal{G}^{(n)}}$ will be denoted by max $F_{\mathcal{G}^{(n)}}$, and we associate to this value the closed set:

$$\underline{\mathrm{Max}}\ F_{\mathcal{G}^{(n)}} := \left\{\xi \in \mathrm{Sing}\ \mathcal{G}^{(n)} : F_{\mathcal{G}^{(n)}}(\xi) = \max\ F_{\mathcal{G}^{(n)}}\right\}.$$

Definition 26.1. Let $F_{\mathcal{G}^{(n)}}$ be an upper-semi continuous function defined on a pair $(V^{(n)}, \mathcal{G}^{(n)})$. A $\mathcal{G}^{(n)}$-local sequence

$$\begin{array}{ccccccc} V_0^{(n)} = V^{(n)} & \leftarrow & V_1^{(n)} & \leftarrow & \dots & \leftarrow & V_m^{(n)} \\ \mathcal{G}_0^{(n)} = \mathcal{G}^{(n)} & & \mathcal{G}_1^{(n)} & & \dots & & \mathcal{G}_m^{(n)} \end{array}$$

is said to be *max*$-F_{\mathcal{G}^{(n)}}$*-local*, if whenever $V_i \leftarrow V_{i+1}$ is the blow up at a permissible center $Y_i \subset V_i^{(n)}$, one has that

$$Y_i \subset \underline{\mathrm{Max}}\ F_{\mathcal{G}_i^{(n)}} \subset \mathrm{Sing}\ \mathcal{G}_i^{(n)}.$$

Remark 26.2. We stress here that some of the functions that will be considered may also depend on the particular choice of the $\mathcal{G}^{(n)}$-local sequence above. An example of such a function would be the second coordinate of the $t^{(n)}$-function introduced in 14.4.

Definition 26.3. An upper-semi continuos function $F_{\mathcal{G}^{(n)}}$ defined on a pair $(V^{(n)}, \mathcal{G}^{(n)})$ is said to be *strongly upper-semi continuous* if given any max $- F_{\mathcal{G}^{(n)}}$-local sequence

$$\begin{array}{ccccccc} V_0^{(n)} = V^{(n)} & \leftarrow & V_1^{(n)} & \leftarrow & \dots & \leftarrow & V_m^{(n)} \\ \mathcal{G}_0^{(n)} = \mathcal{G}^{(n)} & & \mathcal{G}_1^{(n)} & & \dots & & \mathcal{G}_m^{(n)} \end{array}$$

one has that:

$$\max\ F_{\mathcal{G}_0^{(n)}} \geq \max\ F_{\mathcal{G}_1^{(n)}} \geq \dots \geq \max\ F_{\mathcal{G}_m^{(n)}}.$$

Let $\mathcal{G}^{(n)}$ be an $\mathcal{O}_{V^{(n)}}$-Rees algebra, and suppose that max $\mathrm{ord}_{\mathcal{G}^{(n)}}^{(n)} > 1$ (this is the case of the points $\xi \in \mathrm{Sing}\ \mathcal{G}^{(n)}$ where $\tau_{\mathcal{G}^{(n)},\xi} = 0$). Then, if

$$\begin{array}{ccc} V^{(n)} & \longleftarrow & V_1^{(n)} \\ \mathcal{G}^{(n)} & & \mathcal{G}_1^{(n)} \end{array}$$

is the blow up at a permissible $Y \subset \underline{\mathrm{Max}}\ \mathrm{ord}_{\mathcal{G}^{(n)}}^{(n)}$ it is no longer true that

$$\max\ \mathrm{ord}_{\mathcal{G}^{(n)}}^{(n)} \geq \max\ \mathrm{ord}_{\mathcal{G}_1^{(n)}}^{(n)}.$$

In the same manner, if $d < n$ and $\mathrm{ord}_{\mathcal{G}^{(n)}}^{(d)}$ is not constantly equal to 1 (this is the case of the points $\xi \in \mathrm{Sing}\ \mathcal{G}^{(n)}$ where $\tau_{\mathcal{G}^{(n)},\xi} = n - d$), and if

$$\begin{array}{ccc} V^{(n)} & \longleftarrow & V_1^{(n)} \\ \mathcal{G}^{(n)} & & \mathcal{G}_1^{(n)} \end{array}$$

is the blow up at a permissible $Y \subset \max\ \mathrm{ord}_{\mathcal{G}^{(n)}}^{(d)}$ it is no longer true that

$$\max\ \mathrm{ord}_{\mathcal{G}^{(n)}}^{(d)} \geq \max\ \mathrm{ord}_{\mathcal{G}_1^{(n)}}^{(d)}.$$

**26.4. Satellite functions are invariants on identifiable pairs.** The discussion above motivates the use of the so called *w-ord*$^{(d)}_{\mathcal{G}^{(n)}}$*-functions* (see Section 14), which are slight modifications of Hironaka's order function that in addition have a good behavior under blow ups at permissible centers. In other words, they all are strongly upper-semi continuos functions. The fact that they are well defined at any step of a $\mathcal{G}^{(n)}$-local sequence, assuming that $(V^{(n)}, \mathcal{G}^{(n)})$ is of dimensional type $d$, follows from the discussion in 25.2 and from 14.5. The same argument applies to the $t^{(d)}_{\mathcal{G}^{(n)}}$-functions from Section 14. These are all examples of (strongly) invariant upper-semicontinuous functions on (identifiable) pairs (see Remark 25.4 and 14.5). In particular, if two pairs are isomorphic as in Example 21.5, then their satellite functions take the same value on identified points (see Remark 25.5).

Now the following theorem can be proven:

THEOREM 26.5. *Let* $(V^{(n)}, \mathcal{G}^{(n)})$ *and* $(V^{(d)}, \mathcal{G}^{(d)})$ *be identifiable pairs, and fix a diagram of identifications*

(26.1)
$$\begin{array}{ccc} & (V^{(m)}, \mathcal{G}^{(m)}) & \\ {\scriptstyle\lambda}\swarrow & & \searrow{\scriptstyle\lambda'} \\ (V^{(n)}, \mathcal{G}^{(n)}) & & (V^{(d)}, \mathcal{G}^{(d)}). \end{array}$$

*Then the stratification of Sing* $\mathcal{G}^{(n)}$ *into smooth strata given by Theorem* 17.4*, induces via the diagram (*26.1*) the stratification of Sing* $\mathcal{G}^{(d)}$ *given by Theorem* 17.4.

PROOF. The theorem follows from the fact that the stratification is built taking into account the $t^{(i)}$-functions and the dimensional type of a pair (see Theorem 17.4 and Remark 17.5), and both are invariant functions on pairs. More details can be found in Appendix B. □

REMARK 26.6. If two pairs $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V'^{(n)}, \mathcal{G}'^{(n)})$ are isomorphic in the sense of Example 21.5, via an isomorphism $\phi : V^{(n)} \longrightarrow V'^{(n)}$, then the stratification of Sing $\mathcal{G}^{(n)}$ given by Theorem 17.4, induces, via $\phi$, a stratification of Sing $\mathcal{G}'^{(n)}$. This stratification of Sing $\mathcal{G}'^{(n)}$ coincides with that given by Theorem 17.4 when applied to $(V'^{(n)}, \mathcal{G}'^{(n)})$. This follows from the fact that isomorphic pairs are identifiable.

## CONSTRUCTIVE RESOLUTION OF IDENTIFIABLE BASIC OBJECTS

### 27. Resolution

As indicated in Section 14, when $V^{(n)}$ is a smooth scheme over a field of characteristic zero, a resolution of a $\mathcal{O}_{V^{(n)}}$-Rees algebra $\mathcal{G}^{(n)}$ can be constructed (see Theorem 18.9).

> **Fundamental fact.** *To face the constructive resolution of a Rees algebra two main invariants are used: Hironaka's order function and Hironaka's* $\tau$*-invariant (see Section* 13*). All the other invariants that are involved in the resolution process (the so called satellite functions) derive from these two main invariants (see Sections* 14 *and* 18*).*

Hironaka's order function and Hironaka's $\tau$-invariant are usually referred to as *resolution invariants*. Thus, taking this fact for granted, it immediately follows that a resolution of a given $\mathcal{O}_{V^{(n)}}$-Rees algebra $\mathcal{G}^{(n)}$ leads to the resolution of any other $\mathcal{O}_{V^{(n)}}$-Rees algebra weakly equivalent to $\mathcal{G}^{(n)}$, since they share the same resolution invariants (see Section 18).

In this section we will see that if $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)})$ are identifiable, then the constructive resolution of $(V^{(n)}, \mathcal{G}^{(n)})$ given by Theorem 18.9 induces, via the identification, the constructive resolution of $(V^{(d)}, \mathcal{G}^{(d)})$ given by the same theorem, and vice versa.

**Exceptional divisors.** Recall that once we start a resolution process, exceptional divisors appear after each blow up, and that we will require that the collection of all of them have normal crossing support at the end of the resolution process. As indicated in Section 12 we collect this information in terms of the so called basic objects, and that there is accordingly a notion of permissible transformation. Now we introduce the notion of *identifiable basic objects*.

Definition 27.1. We will say that two basic objects $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)}, E^{(d)})$ are *identifiable* if there is an $m$-dimensional basic object

$$(V^{(m)}, \mathcal{G}^{(m)}, E^{(m)})$$

so that:

(1) Both $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)})$ are elimination pairs of $(V^{(m)}, \mathcal{G}^{(m)})$,

$$\begin{array}{ccccc} & & (V^{(m)}, \mathcal{G}^{(m)}) & & \\ & \swarrow^{\beta} & & \searrow^{\beta'} & \\ (V^{(n)}, \mathcal{G}^{(n)}) & & & & (V^{(d)}, \mathcal{G}^{(d)}); \end{array} \tag{27.1}$$

thus in particular $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)})$ are identifiable;

(2) There are equalities of sets:

$$\beta^{-1}(E^{(n)}) = E^{(m)} = \beta'^{-1}(E^{(d)}).$$

We will say that $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)}, E^{(d)})$ *are identifiable locally, in a neighborhood $U$ of a point $\xi \in$ Sing* $\mathcal{G}^{(n)}$, if conditions (1) and (2) are satisfied locally in $U$.

Remark 27.2. Suppose that $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ and $(V'^{(n)}, \mathcal{G}'^{(n)}, E'^{(n)})$ are isomorphic meaning that there is an isomorphism $\phi : V^{(n)} \longrightarrow V'^{(n)}$ with $\phi^*(\mathcal{G}^{(n)}) = \mathcal{G}'^{(n)}$ as in Example 21.5, which in addition maps $E^{(n)}$ to $E'^{(n)}$, i.e., $\phi(E^{(n)}) = E'^{(n)}$. Then it can be checked that $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ and $(V'^{(n)}, \mathcal{G}'^{(n)}, E'^{(n)})$ are identifiable.

Definition 27.3. We will say that $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ is *of dimensional type $d$* if $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ is identifiable with some basic object of dimensional type $d$.

Remark 27.4. In characteristic zero, if $(V^{(n)}, \mathcal{G}^{(n)})$ is an elimination pair for some $m$-dimensional pair $(V^{(m)}, \mathcal{G}^{(m)})$ via a morphism

$$\beta_{m,n} : V^{(m)} \longrightarrow V^{(n)},$$

then any sequence of permissible transformations for $(V^{(m)}, \mathcal{G}^{(m)})$, say

$$(27.2)\quad (V^{(m)}, \mathcal{G}^{(m)}, E^{(m)} = \emptyset) \leftarrow (V_1^{(m)}, \mathcal{G}_1^{(m)}, E_1^{(m)}) \leftarrow \dots \leftarrow (V_s^{(m)}, \mathcal{G}_1^{(m)}, E_s^{(m)})$$

induces, via $\beta_{m,n}$, a permissible sequence of transformations for $(V^{(n)}, \mathcal{G}^{(n)})$

$$(27.3)\qquad (V^{(n)}, \mathcal{G}^{(n)}, E^{(n)} = \emptyset) \leftarrow (V_1^{(n)}, \mathcal{G}_1^{(n)}, E_1^{(n)}) \leftarrow \dots \leftarrow (V_s^{(n)}, \mathcal{G}_1^{(n)}, E_s^{(n)})$$

and for each $i = 1, \dots, s$, the basic objects $(V_i^{(m)}, \mathcal{G}_i^{(m)}, E_i^{(m)})$ and $(V_i^{(n)}, \mathcal{G}_i^{(n)}, E_i^{(n)})$ are identifiable. And conversely, any permissible sequence like (27.3) induces a permissible sequence like (27.2) (see 16.7 (6), (7)). As a consequence if $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)})$ are identifiable then the corresponding basic objects that arise after considering a sequence of permissible transformations of one of them are identifiable with the ones induced by the other.

THEOREM 27.5. **Resolution Theorem for Rees algebras revisited.** *Suppose* $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ *and* $(V^{(d)}, \mathcal{G}^{(d)}, E^{(d)})$ *are identifiable via some diagram like (*27.1*). Then the constructive resolution of* $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ *given by Theorem* 18.9 *induces, via the identification (*27.1*), the constructive resolution of* $(V^{(d)}, \mathcal{G}^{(d)}, E^{(d)})$ *given by Theorem* 18.9 *and vice versa.*

PROOF. The string of relevant invariants that leads to the constructive resolution of both basic objects is the same (see Theorem 18.9): identifiable pairs share the same satellite functions (see 26.4 and Section 24). The proof of resolution of basic objects follows by (increasing) induction on the dimensional type (see Remark 27.6 below). □

REMARK 27.6. Theorem 18.9 was proved by induction on the codimensional type of a given basic object, namely by assuming that one can resolve basic object of higher codimensional type. Since the codimensional type is not an invariant for identifiable basic objects, we have therefore introduced the notion of dimensional type. We can now change the form of induction in the resolution of basic objects, by arguing by induction on the dimensional type. We assume that we can resolve basic objects of smaller dimensional type. The theorem clearly holds for basic objects of dimensional type 1. Now suppose that the theorem holds for basic objects of dimensional type smaller than $n$, and let $(V, \mathcal{G}, E)$ be a basic object. If the dimensional type of $(V, \mathcal{G}, E)$ is smaller than $n$, then the result follows from the inductive hypothesis. Otherwise, if the dimensional type of $(V, \mathcal{G}, E)$ is $n$, then by Proposition 18.2, we associate to the maximum value of the function $t_{\mathcal{G}}^{(n)}$ a basic object $(V, \widehat{\mathcal{G}}, \widehat{E})$ of smaller dimensional type.

REMARK 27.7. If $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ and $(V'^{(n)}, \mathcal{G}'^{(n)}, E'^{(n)})$ are isomorphic basic objects as in Remark 27.7, then it follows that the constructive resolution of $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)})$ given by Theorem 18.9 induces, via the isomorphism, the constructive resolution of $(V'^{(n)}, \mathcal{G}'^{(n)}, E'^{(n)})$ given by the same theorem.

## Part IV: Stratification of the singular locus and resolution of singularities

In this part of the notes we will see how the Theorem of resolution of singularities of algebraic varieties follows from the Theorem of constructive resolution of basic objects, when the characteristic is zero. As we will see, this will be a consequence of the notion of local presentations introduced at the end of Part I, and the Theorem of resolution of basic objects stated in Parts II and III. There are several ways to approach resolution of singularities using the Theorem of Resolution of Basic Objects. Here we will address the resolution of singularities in a way which differs from that in previous literature (see [**Cu1**], [**EV1**], [**Vi1**], [**Vi2**]).

### 28. Strongly upper-semi continuous functions on varieties

Let $X$ be a variety. A *local sequence on* $X$ will be defined as a sequence of morphisms:

$$X = X_0 \overset{\varphi_1}{\longleftarrow} X_1 \overset{\varphi_2}{\longleftarrow} \dots \overset{\varphi_m}{\longleftarrow} X_m \tag{28.1}$$

where each $X_i \longleftarrow X_{i+1}$ is either blow up at a smooth center $Y_i \subset X_i$, or a smooth morphism of one of the following forms (see Remark 11.7):

(1) The restriction to an open Zariski subset of $X_i$;
(2) $X_{i+1}$ is of the form $X_i \times \mathbb{A}^n_k$, and then $X_i \longleftarrow X_{i+1}$ is the projection on the first coordinates.

We will be interested in studying certain upper-semi continuous functions on $X$ that naturally extend at each step of a local sequence on $X$.

Now fix a well ordered set $(\Lambda, \geq)$. Given an upper-semi continuous function $F$ defined on each variety,

$$\mathrm{F}_X : X \mapsto (\Lambda, \geq)$$

we will denote by $\max \mathrm{F}_X$ the maximum value of $\mathrm{F}_X$, and define the closed subset of $X$,

$$\underline{\mathrm{Max}}\ \mathrm{F}_X := \{\xi \in X : \mathrm{F}_X(\xi) = \max \mathrm{F}_X\}.$$

A local sequence on $X$ like (28.1) is said to be $F_X$*-local*, if whenever $\varphi_i : X_{i+1} \longrightarrow X_i$ is the blow up at a smooth center $Y_i \subset X_i$, one has that $Y_i \subset \underline{\mathrm{Max}}\ \mathrm{F}_{X_i}$.

DEFINITION 28.1. We will say that $F_X$ is a *strongly upper-semi continuous* if :

(1) Given any $F_X$-local sequence,

$$X = X_0 \overset{\varphi_1}{\longleftarrow} X_1 \overset{\varphi_2}{\longleftarrow} \dots \overset{\varphi_m}{\longleftarrow} X_m \;\text{ one has that} \tag{28.2}$$

$$\max \mathrm{F}_{X_0} \geq \max \mathrm{F}_{X_1} \geq \dots \geq \max \mathrm{F}_{X_m}.$$

(2) If $\alpha : X' \to X$ is étale, then $F_{X'} = F_X \circ \alpha$.

EXAMPLE 28.2. For a given variety $X$, both, the Hilbert-Samuel function, $\mathrm{HS}_X$, and the multiplicity, $\mathrm{M}_X$, are examples of strongly upper-semi continuous functions.

As we will be studying upper-semi continuous functions defined for every variety, we will refer to them as *upper-semi continuous functions defined on varieties.* If $F$ is an upper-semi continuous function defined on varieties, we denote by $F_X$ the function defined on a concrete $X$.

DEFINITION 28.3. Let $F$ be an upper-semi continuous function defined on varieties. We will say that $F$ *is globally representable for a variety* $X$, if whenever $X$ admits an embedding in some smooth scheme $X \subset V^{(n)}$ there is an $\mathcal{O}_{V^{(n)}}$-Rees algebra $\mathcal{G}^{(n)}$ so that the following conditions hold:

(1) There is an equality of closed sets:

$$\underline{\mathrm{Max}}\ F_X = \mathrm{Sing}\ \mathcal{G}^{(n)};$$

(2) Any $F_X$-local sequence $X = X_0 \leftarrow X_1 \leftarrow \dots \leftarrow X_m$ with

$$\max \mathrm{F}_X = \max \mathrm{F}_{X_0} \cdots = \max \mathrm{F}_{X_{m-1}} \geq \max \mathrm{F}_{X_m}$$

induces a sequence

$$\begin{array}{lllll} V_0^{(n)} = V^{(n)} & \leftarrow & V_1^{(n)} & \leftarrow \dots & \leftarrow V_m^{(n)} \\ \mathcal{G}_0^{(n)} = \mathcal{G}^{(n)} & & \mathcal{G}_1^{(n)} & \dots & \mathcal{G}_m^{(n)} \end{array}$$

and:

$$\underline{\mathrm{Max}}\ F_{X_i} = \mathrm{Sing}\ \mathcal{G}_i^{(n)} \text{ for } i = 1, \dots, m-1,$$

$$\underline{\mathrm{Max}}\ F_{X_m} = \mathrm{Sing}\ \mathcal{G}_m^{(n)}, \text{ if } \max\ F_{X_{m-1}} = \max\ F_{X_m},$$

$$\mathrm{Sing}\ \mathcal{G}_m^{(n)} = \emptyset, \text{ if } \max\ F_{X_{m-1}} > \max\ F_{X_m}.$$

(3) Conversely, any $\mathcal{G}^{(n)}$-local sequence

$$\begin{array}{lllll} V_0^{(n)} = V^{(n)} & \leftarrow & V_1^{(n)} & \leftarrow \dots & \leftarrow V_m^{(n)} \\ \mathcal{G}_0^{(n)} = \mathcal{G}^{(n)} & & \mathcal{G}_1^{(n)} & \dots & \mathcal{G}_m^{(n)} \end{array}$$

$$\text{with Sing } \mathcal{G}_i \neq \emptyset \text{ for } i = 0, \dots, m-1,$$

induces an $F_X$-local sequence

$$X = X_0 \leftarrow X_1 \leftarrow \dots \leftarrow X_m \text{ with}$$

$$\max \mathrm{F}_X = \max \mathrm{F}_{X_0} \cdots = \max \mathrm{F}_{X_{m-1}} \geq \max \mathrm{F}_{X_m},$$

and:

$$\underline{\mathrm{Max}}\ F_{X_i} = \mathrm{Sing}\ \mathcal{G}_i^{(n)} \text{ for } i = 1, \dots, m-1,$$

$$\max\ F_{X_{m-1}} = \max\ F_{X_m} \text{ if } \underline{\mathrm{Max}}\ F_{X_m} = \mathrm{Sing}\ \mathcal{G}_m^{(n)} \neq \emptyset,$$

$$\max\ F_{X_{m-1}} > \max\ F_{X_m} \text{ if Sing } \mathcal{G}_m^{(n)} = \emptyset.$$

DEFINITION 28.4. We will say that a strongly upper-semi-continuous function defined on varieties $F$ is *representable via local embeddings*, if for each variety $X$ and every point $\xi \in X$, the previous definition holds at some étale neighborhood.

EXAMPLE 28.5. A Theorem of Aroca (see [**Hi5**, §4]) asserts that the Hilbert-Samuel function, $\mathrm{HS}_X$, is representable via local embeddings. Moreover, if $X$ is globally embedded in a smooth ambient space $V^{(n)}$, then after taking a suitable étale cover of $V^{(n)}$, we may assume that there is an $\mathcal{O}_{V^{(n)}}$-Rees algebra $\mathcal{G}^{(n)}$ unique up to weak equivalence, such that $\mathrm{Sing}\ \mathcal{G}^{(n)} = \underline{\mathrm{Max}}\ \mathrm{HS}_X$ (see [**Hi7**] or [**BrG-EV**, §9]).

EXAMPLE 28.6. Let $X$ be a variety over a perfect field. In Appendix A it is shown that there is a finite covering in étale topology so that at each restriction, say $X'$, a finite morphism $\delta : X' \to V$ is defined with the conditions of Proposition 8.8. In particular, the expression in (8.10) in Remark 8.9 is a local presentation (a Rees algebra) attached to the multiplicity. Therefore if $X$ is a variety defined over a perfect field field then the multiplicity is a strongly upper-semicontinuous function that is representable via local embeddings.

REMARK 28.7. From the definition it follows that if $F$ is is globally representable for a variety $X$, and if $F_X$ is represented by a pair $(V^{(n)}, \mathcal{G}^{(n)})$, then a resolution of $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)} = \{\emptyset\})$ induces a sequence of blowing ups on $X$,

$$X = X_0 \longleftarrow X_1 \longleftarrow \dots \longleftarrow X_m \tag{28.3}$$

so that

$$\max \mathrm{F}_X = \max \mathrm{F}_{X_0} \cdots = \max \mathrm{F}_{X_{m-1}} > \max \mathrm{F}_{X_m}.$$

This observation leads naturally to the following questions:

(1) Here the data are $F$ and $X$. Does sequence (28.3) depend on the particular choice of the pair $(V^{(n)}, \mathcal{G}^{(n)})$?
(2) Some of the strongly upper-semi continuous functions that we will be interested in are the Hilbert Samuel function and the Multiplicity. However, as indicated above, they are only representable via local embeddings. Does this (étale-local) information lead to the construction of a globally defined sequence like (28.3)?

The following theorem will be the key to answer the previous questions.

THEOREM 28.8. **Identification Theorem for Varieties.** *Suppose that $F$ is a strongly upper-semi continuous function defined on varieties that is representable via local embeddings, and let $X$ be a variety defined over a perfect field $k$. Let $\xi \in \underline{Max}\ F_X$, and suppose that there is a neighborhood of $\xi$ so that $F_X$ is globally represented via two different local embeddings. Then the two corresponding pairs are identifiable (see Section* 21*).*

The proof of the previous theorem will be addressed in Section 29 (see 29.3). Using this result, and the theorem of resolution of basic objects in characteristic zero (see Theorems 18.9 and 27.5) we will answer affirmatively to questions (1) and (2) from Remark 28.7 (see Theorem 30.2).

As another application of the output of Theorem 28.8 the next definition can be established:

DEFINITION 28.9. Assume that $F$ is a strongly upper-semicontinuous function defined on varieties, and let $X$ be a variety. We will say that $F_X$ *is locally representable in dimension* $d$ if locally, in some étale neighborhood of each point $\xi \in \underline{\mathrm{Max}}\ X$ there is an pair, say $(V, \mathcal{G})$, of dimensional type $d$, (see Section 24) so that the conditions of Definition 28.3.

The following theorem illustrates a feature of representable functions:

THEOREM 28.10. *Let $F$ be a strongly upper-semi continuous function defined on varieties that is representable via local embeddings. Then, if the characteristic is zero, $F_X$ is locally representable in dimension $d = dim\ X$.*

*Proof:* The proof of [**BrV2**, Proposition 11.4] can be adapted to show that any upper-semi continuous function $F_X$ that is representable via a local embedding under the hypotheses of the theorem is of dimensional type is less than or equal to the dimension of $X$. $\square$

Remark 28.11. It follows from Theorem 28.10 that both, the Hilbert-Samuel function and the multiplicity of a $d$-dimensional variety are representable in dimension $d$, if the characteristic is zero. See also Theorem 7.2, Proposition 7.5, Proposition 8.4 in this manuscript, which also indicate that the Multiplicity is locally representable in dimension $d$ when the characteristic is zero. When the characteristic is positive Theorem 28.10 may not hold: it suffices to consider the example of the multiplicity for the curve $\{z^2 + x^3 = 0\}$ in the affine plane when the characteristic is two. As it is shown in [**BrG-EV**, §11] the multiplicity cannot be represented in dimension one.

**28.12. Equivariance.** A variety is a scheme obtained by patching $k$-algebras for some field $k$, together with some additional conditions. We say that a variety induces an abstract scheme simply by neglecting the structure over the field $k$. Suppose that $F$ is an upper-semi continuous function defined on varieties. We say that $F$ is *equivariant* if whenever $\Theta : X' \longrightarrow X$ is an isomorphism of the underlying abstract schemes, one has that

$$F_{X'}(\xi) = F_X(\Theta(\xi))$$

for all $\xi \in X'$. Notice that condition (2) in Definition 28.1 above already says that if $F$ is strongly upper semicontinuous, then, in particular, it is equivariant.

Both, the Hilbert-Samuel function and the multiplicity are examples of strongly upper-semi continuous (hence equivariant) functions defined on varieties.

If $F$ is locally representable, and if $F_X$ is represented by a pair $(V^{(n)}, \mathcal{G}^{(n)})$, then $(V^{(n)}, \mathcal{G}^{(n)})$ also represents $F_{X'}$ via $\Theta$. Thus, a resolution of $(V^{(n)}, \mathcal{G}^{(n)}, E^{(n)} = \{\emptyset\})$ induces a sequence of blowing ups at smooth centers for both $X$ and $X'$, by considering pull backs,

$$\begin{array}{ccccccc} X = X_0 & \longleftarrow & X_1 & \longleftarrow & \dots & \longleftarrow & X_m \\ \uparrow{\scriptstyle \theta=\theta_0} & & \uparrow{\scriptstyle \theta_1} & & & & \uparrow{\scriptstyle \theta_m} \\ X' = X'_0 & \longleftarrow & X'_1 & \longleftarrow & \dots & \longleftarrow & X'_m \end{array}$$

so that

$$\max \mathrm{F}_{X'} = \max \mathrm{F}_{X'_0} \cdots = \max \mathrm{F}_{X'_{m-1}} > \max \mathrm{F}_{X'_m}$$

and

$$\max \mathrm{F}_{X} = \max \mathrm{F}_{X_0} \cdots = \max \mathrm{F}_{X_{m-1}} > \max \mathrm{F}_{X_m}.$$

One readily checks from this fact that a sequence like (28.3) in Remark 28.7 is compatible with isomorphisms. This is a deep observation due to Hironaka, and the key for the property of equivariance in constructive resolution which we discuss in our proofs below (see also [**Vi2**]).

## 29. Proof of Theorem 28.8

The proof of Theorem 28.8 (which is given in 29.3) will make use of Lemma 29.1 and Remark 29.2.

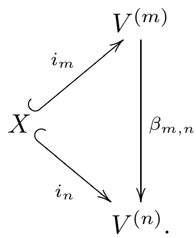
Lemma 29.1. *Let $F$ be a strongly upper-semi continuous functions defined on varieties, and let $X$ be a variety defined over a perfect field $k$. Assume that:*

(1) *There is an embedding of $X$ in some $n$-dimensional smooth scheme $V^{(n)}$,*
$$i_n : X \hookrightarrow V^{(n)}$$
*and a pair $(V^{(n)}, \mathcal{G}^{(n)})$ representing $F_X$;*

(2) *There is a smooth morphism of smooth schemes*
$$\beta_{m,n} : V^{(m)} \longrightarrow V^{(n)}$$
*for some $m \geq n$;*

(3) *There is an embedding of $i_m : X \hookrightarrow V^{(m)}$ inducing a commutative diagram of embeddings and smooth morphisms:*

(29.1)




*Then, locally at any point $\xi \in \underline{Max}\ F_X$, $F_X$ is also representable via the embedding $i_m : X \hookrightarrow V^{(m)}$, i.e., there is a pair of the form $(V^{(m)}, \mathcal{G}^{(m)})$ that represents $F_X$. Moreover, $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(m)}, \mathcal{G}^{(m)})$ are identifiable (see Section 21).*

Proof. By the Canonicity Principle (see Theorem 10.11) it suffices to exhibit the algebra $\mathcal{G}^{(m)}$ locally. Let $\xi \in V^{(m)}$ be a point and let $\{x_1, \ldots, x_n\}$ be a regular system of parameters at $\mathcal{O}_{V^{(n)}, \beta_{m,n}(\xi)}$. Via the inclusion of local rings,
$$\beta^*_{m,n} : \mathcal{O}_{V^{(n)}, \beta_{m,n}(\xi)} \hookrightarrow \mathcal{O}_{V^{(m)}, \xi}$$
extend $\{x_1, \ldots, x_n\}$ to a regular system of parameters in $\mathcal{O}_{V^{(m)}, \xi}$, say
$$\{x_1, \ldots, x_n, y_1, \ldots, y_{m-n}\}.$$
Since diagram (29.1) is commutative, there are $(m-n)$ functions $g_1, \ldots, g_{(m-n)} \in \mathcal{O}_{V^{(n)}, \beta_{m,n}(\xi)}$ so that:
$$y_1 - g_1, \ldots, y_{(m-n)} - g_{(m-n)} \in \mathcal{I}_m(X)_\xi \subset \mathcal{O}_{V^{(m)}, \xi},$$
where $\mathcal{I}_m(X)_\xi$ denotes the defining ideal of $X$ at $\mathcal{O}_{V^{(m)}, \xi}$. Now set
$$\mathcal{G}^{(m)} := \mathcal{O}_{V^{(m)}, \xi}[(y_1 - g_1)W, \ldots, (y_{(m-n)} - g_{(m-n)})W] \odot \beta^*_{m,n}(\mathcal{G}^{(n)}).$$
Observe that $\mathcal{G}^{(n)}$ is an elimination algebra of $\mathcal{G}^{(m)}$ (see Remark 16.10), so it remains to show that
$$\text{Sing}\ \mathcal{G}^{(m)} = \underline{\text{Max}}\ \text{F}_X \tag{29.2}$$
via the embedding $i_m : X \hookrightarrow V^{(m)}$. Replacing $V^{(n)}$ by an étale neighborhood if needed, it can be assumed that $V^{(n)}$ is isomorphic to $\mathbb{V}(\langle (y_1 - g_1), \ldots, (y_{(m-n)} - g_{(m-n)}) \rangle)$. Thus
$$\text{Sing}\ \mathcal{G}^{(m)} (\subset V^{(n)}) = \text{Sing}\ \mathcal{G}^{(n)},$$
where the equality follows because $\mathcal{G}^{(n)}$ is an elimination algebra of $\mathcal{G}^{(m)}$. $\square$

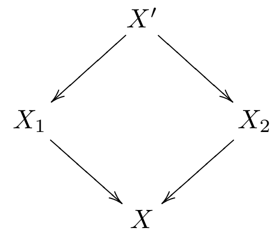
REMARK 29.2. **Some properties of étale topology.** In what follows we will list some properties of étale topology that will be used in the proof of Theorem 28.8. For more details and definitions we refer to Section 32 in Appendix A.

**(1)** Let $\xi \in X$ be a point and let $X' \to X$ be an étale morphism mapping $\xi' \in X'$ to $\xi \in X$. Then $\mathcal{O}_{X',\xi'}$ is isomorphic to an étale $\mathcal{O}_{X,\xi}$- algebra of the form $(\mathcal{O}_{X,\xi}[X]/\langle P(X)\rangle)_{G(X)}$, with $P(X)$ and $G(X)$ in $\mathcal{O}_{X,\xi}[X]$, $P(X)$ is monic and $G(X)$ is chosen so that $P'(X)$ is a unit in $(\mathcal{O}_{X,\xi}[X]/\langle P(X)\rangle)_{G(X)}$ (see [**R**, Théorème 1, p. 51]).

**(2)** If, in addition, there is a closed immersion $X \subset Y$ in a neighborhood of $\xi \in X$, then there is an étale neighborhood of $\xi \in Y$, say $\xi' \in Y'$, and a closed immersion $X' \subset Y'$ locally at $\xi'$. To check this, use (1) and the local surjection $\mathcal{O}_{Y,\xi} \to \mathcal{O}_{X,\xi}$ to lift the polynomial $P(X)$ and the element $G(X)$ to $\mathcal{O}_{Y,\xi}[X]$.

**(3)** Suppose given a commutative diagram of étale morphisms

(29.3)



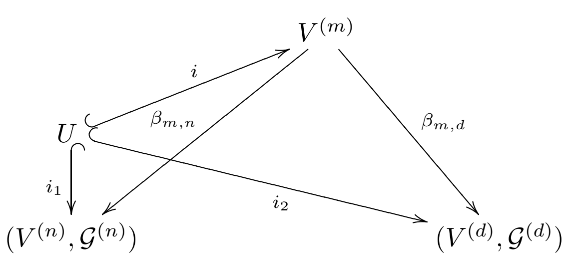

together with points $\xi \in X$, $\xi_1 \in X_1$, $\xi_2 \in X_2$, $\xi' \in X'$ that are in correspondence. Suppose, in addition, that there are local embeddings, say $X_1 \subset V_1$ in a neighborhood of $\xi_1$ and $X_2 \subset V_2$ in a neighborhood of $\xi_2$ , with $V_1$ and $V_2$ smooth. Then, by (2) we get two different embeddings, say

$$X' \subset V_1' \text{ and } X' \subset V_2'$$

in a neighborhood of $\xi'$, so that each $V_i'$ is a (smooth) étale neighborhood of $V_i$, for $i = 1, 2$. Namely, there is an étale neighborhood of $X$ that admits two different embeddings.

**29.3. Proof of Theorem 28.8.** By the arguments exhibited in Remark 29.2 (3), the proof of the Theorem is reduced to to the following setting. Let $\xi \in \underline{\operatorname{Max}}\, \mathrm{F}_X$, and suppose that there are two pairs $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)})$ representing $F_X$ in a neighborhood $U$ of $\xi$. Consider the fiber product $V^{(m)} := V^{(n)} \times_{\operatorname{Spec} k} V^{(d)}$ with $m = n + d$, and the commutative diagram of embeddings and smooth morphisms:

(29.4)




that follow from the universal property of products. Applying Lemma 29.1 to the embeddings of $U$ in $V^{(m)}$ and $V^{(n)}$ we conclude that there is an $\mathcal{O}_{V^{(m)}}$-Rees algebra $\mathcal{G}^{(m)}$ representing $F_X$.

The same lemma now applied to $(V^{(d)}, \mathcal{G}^{(d)})$ and $V^{(m)}$ ensures that there is an $\mathcal{O}_{V^{(m)}}$-Rees algebra $\mathcal{G}'^{(m)}$ representing $F_X$. Hence $\mathcal{G}^{(m)}$ and $\mathcal{G}'^{(m)}$ are weakly equivalent, and moreover, from the proof of Lemma 29.1 it follows that both $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)})$ are elimination pairs of $(V^{(m)}, \mathcal{G}^{(m)})$ and therefore both are identifiable. $\square$

## 30. Constructive resolution, equivariance, and stratifications into smooth strata

The purpose of this section is to state different versions of the Theorem of Resolution of Singularities of Algebraic varieties in characteristic zero: Theorems 30.1, 30.6, 30.7.

THEOREM 30.1. *Let $X$ be a non-smooth variety defined over a field of characteristic zero. Then there is a finite sequence of blow ups at smooth centers:*

$$X = X_0 \longleftarrow X_1 \longleftarrow \ldots \longleftarrow X_s \tag{30.1}$$

*so that:*

(1) *$X_s$ is smooth;*
(2) *The morphism $X \leftarrow X_s$ induces an isomorphism on $X \setminus Sing\ X$.*

*Moreover, the process is constructive and equivariant (see* 28.12*).*

The proof of the theorem (to be addressed in 30.5) follows from Theorem 30.2, Corollary 30.3 (to be discussed below), and an inductive argument.

THEOREM 30.2. *Let $F$ be a strongly upper-semi continuous function defined on varieties, and let $X$ be a variety defined over a perfect field $k$. Assume that $F$ is representable via local embeddings. Then:*

(1) *$\underline{Max}\ F_X$ can be stratified in smooth strata in a natural manner (independently of the particular choice of the local representation of $F_X$);*
(2) *If the characteristic is zero, a finite sequence of blow ups at smooth centers can be constructed,*

$$X = X_0 \longleftarrow X_1 \longleftarrow \ldots \longleftarrow X_s \tag{30.2}$$

*so that*

$$Max\ F_X = Max\ F_{X_0} = Max\ F_{X_1} = \ldots > \ Max\ F_{X_s}, \tag{30.3}$$

*and the sequence (*30.2*) does not depend on the particular choice of the (local) representation of $F_X$.*

(3) *(Equivariance) If $\Theta : X' \to X$ is an isomorphism then the smooth stratification of $\underline{Max}\ F_{X'}$ from (1) coincides with that induced by the smooth stratification of $\underline{Max}\ F_X$ from (1) via pull back; moreover, if the characteristic is zero, the sequence*

$$X' = X'_0 \longleftarrow X'_1 \longleftarrow \ldots \longleftarrow X'_{s'} \tag{30.4}$$

*with*

$$Max\ F_{X'} = Max\ F_{X'_0} = Max\ F_{X'_1} = \ldots > \ Max\ F_{X'_{s'}}, \tag{30.5}$$

*from (2) coincides with that induced by the sequence (*30.2*) for $X$ via pull backs. Thus in particular, $s' = s$.*

PROOF. Consider an open covering of $X$, $\{U_i\}_{\in I}$ so that for each $i \in I$ there is a local embedding $U_i \hookrightarrow V^{(n_i)}$ in some smooth $V^{(n_i)}$ together with an $\mathcal{O}_{V^{(n_i)}}$-Rees algebra $\mathcal{G}^{(n_i)}$ so that the pair $(V^{(n_i)}, \mathcal{G}^{(n_i)})$ represents $\underline{\text{Max}}\ \text{F}_X \cap U_i$.

(1) By Theorem 17.4, Sing $\mathcal{G}^{(n_i)}$ can be stratified in smooth strata. This stratification induces a smooth stratification of $\underline{\text{Max}}\ \text{F}_X \cap U_i$. By Theorems 28.8 and 26.5, this stratification is independent of the choice of the pair $(V^{(n_i)}, \mathcal{G}^{(n_i)})$ chosen for the representation of $\underline{\text{Max}}\ \text{F}_X \cap U_i$. And by the same reason, the stratification of $\underline{\text{Max}}\ \text{F}_X \cap U_i \cap U_j$ induced by $(V^{(n_i)}, \mathcal{G}^{(n_i)})$ and $(V^{(n_j)}, \mathcal{G}^{(n_j)})$ coincides for each $i, j \in I$. This gives a (global) smooth stratification of $\underline{\text{Max}}\ \text{F}_X$.

(2) When the characteristic is zero, a resolution of $(V^{(n_i)}, \mathcal{G}^{(n_i)})$ can be constructed for all $i \in I$ (see Theorem 12.6). For each $i \in I$, a resolution of $(V^{(n_i)}, \mathcal{G}^{(n_i)})$,

$$\begin{array}{ccccccc} V_0^{(n_i)} = V^{(n_i)} & \longleftarrow & V_1^{(n_i)} & \longleftarrow & \dots & \longleftarrow & V_{s_i}^{(n_i)} \\ \mathcal{G}_0^{(n_i)} = \mathcal{G}^{(n_i)} & & \mathcal{G}_1^{(n_i)} & & \dots & & \mathcal{G}_{s_i}^{(n_i)} \end{array} \tag{30.6}$$

induces a sequence of blow ups at smooth centers,

$$\begin{array}{ccccccc} U_{i,0} = U_i & \longleftarrow & U_{i,1} & \longleftarrow & \dots & \longleftarrow & U_{i,s_i} \\ \cap & & \cap & & \dots & & \cap \\ X_0 = X & & X_1 & & \dots & & X_{s_i}, \end{array} \tag{30.7}$$

so that

$$\max \text{F}_{X_0} \cap U_{i,0} = \max \text{F}_{X_1} \cap U_{i,1} = \dots = \max \text{F}_{X_{s-1}} \cap U_{i,s-1} > \max \text{F}_{X_s} \cap U_{i,s_i}.$$

By Theorem 28.8 and Theorem 27.5, sequence (30.7) is independent on the choice of the pair representing $\underline{\text{Max}}\ \text{F}_X$ on $U_i$. And by a similar argument, for each $i, j \in I$, the resolution of $(V^{(n_i)}, \mathcal{G}^{(n_i)})$ and $(V^{(n_j)}, \mathcal{G}^{(n_j)})$ induce the same sequence of blow ups at smooth centers on $U_i \cap U_j$. Therefore, the sequences (30.7) patch so as to define a sequence of blow ups at smooth centers

$$X_0 = X \quad \longleftarrow \quad X_1 \quad \longleftarrow \quad \dots \quad \longleftarrow \quad X_s, \tag{30.8}$$

so that

$$\max \text{F}_{X_0} = \max \text{F}_{X_1} = \dots = \max \text{F}_{X_{s-1}} > \max \text{F}_{X_s}.$$

Finally, part (3) follows from the fact that local representations for $\text{F}_X$ induce, via $\Theta$, local representations for $\text{F}_{X'}$ (see 28.12). □

COROLLARY 30.3. *Let $X$ be a non-smooth variety. Then both, the maximum stratum of the Hilbert-Samuel Function of $X$, $\underline{\text{Max}}\ HS_X$, and the maximum stratum of the multiplicity of $X$, $\underline{\text{Max}}\ M_X$, can be stratified (in a natural way). When the characteristic of the base field is zero, the maximum value of any of those functions can be lowered via a finite sequence of blow up at smooth centers. Moreover, the process is constructive and equivariant.*

PROOF. Both, the Hilbert-Samuel Function and the Multiplicity of $X$ are representable via local embeddings (see Examples 28.5 and 28.6; and Propositions 8.4 and 8.8 in this manuscript) and equivariant. Thus the assertions follow from Theorem 30.2. □

REMARK 30.4. In [**Hi1**], Hironaka showed that the singularities of a non smooth algebraic variety can be resolved by successively lowering the maximum value of the Hilbert-Samuel function. On the other hand, an equidimensional dimensional variety is smooth if an only if its maximum multiplicity is one (see 2.4 and 2.10).

**30.5. Proof of Theorem 30.1.** Using either the Hilbert-Samuel function or the multiplicity (see Remark 30.4), a resolution can be constructed by Corollary 30.3 and an inductive argument. □

Next we state other versions of the Theorem of Resolution of Singularities.

THEOREM 30.6. *Let $X$ be a non-smooth variety defined over a field of characteristic zero. Then a finite sequence of blow ups at smooth centers can be constructed:*

$$X = X_0 \longleftarrow X_1 \longleftarrow \dots \longleftarrow X_r \tag{30.9}$$

*so that:*

1. *$X_r$ is smooth;*
2. *The composition $X \leftarrow X_r$ induces an isomorphism on $X \setminus \mathrm{Sing}\, X$;*
3. *The exceptional divisor of $X \leftarrow X_r$ has normal crossing support.*

*Moreover, the process is constructive and equivariant.*

PROOF. Use Theorem 30.1 to construct a resolution of singularities of $X$,

$$X = X_0 \longleftarrow X_1 \longleftarrow \dots \longleftarrow X_s. \tag{30.10}$$

Thus $X \xleftarrow{\pi} X_s$ satisfies conditions (1) and (2).

Since each step $X_i \longleftarrow X_{i+1}$ in the sequence (30.10) is the blow at a smooth center $Y_i \subset \mathrm{Sing}\, X_i$, we can attach to it a well defined invertible sheaf $\mathcal{I}(Y_i)\mathcal{O}_{X_{i+1}}$. Now define $\mathcal{K}$ as $\mathcal{I}(Y_1)\cdots\mathcal{I}(Y_{s-1})\mathcal{O}_{X_s}$. Observe that $\mathcal{K}$ is a locally invertible ideal supported on $\pi^{-1}(\mathrm{Sing}\, X) \subset X_r$.

Set $\mathcal{K}_s := \mathcal{K}$, $F_s := \emptyset$, and use Theorem 12.6 to construct a resolution of the the basic object expressed in terms of pairs $(X_s, (\mathcal{K}_s, 1), F_s)$, say

$$(X_s, (\mathcal{K}_s, 1), F_s) \longleftarrow (X_{s+1}, (\mathcal{K}_{s+1}, 1), F_{s+1}) \longleftarrow \dots \longleftarrow (X_r, (\mathcal{K}_r, 1), F_r). \tag{30.11}$$

See 14.1 and 14.2 for the dictionary between Rees algebras and pairs. Notice that:

1. Since sequence (30.11) is a composition of blow ups at smooth centers, $X_r$ is smooth.
2. Since for $i = s, \dots, r-1$ the center of each blow up
$$(X_i, (\mathcal{K}_i, 1), F_i) \longleftarrow (X_{i+1}, (\mathcal{K}_{i+1}, 1), F_{i+1})$$
is contained in $\mathrm{Sing}\, (\mathcal{K}_i, 1)$, the composition
$$X \longleftarrow X_s \longleftarrow X_r$$
induces an isomorphism on $X \setminus \mathrm{Sing}\, X$;
3. Since $\mathrm{Sing}\, (\mathcal{K}_r, 1) = \emptyset$, the total transform of $\mathcal{K}$ in $X_r$ is supported on $F_r$. Thus the exceptional divisor of $X \longleftarrow X_r$ has normal crossings support.

Finally, to see that the process is equivariant, observe that if $\Theta : X' \to X$ is an isomorphism, then the equivariant resolution of $X$ (30.10) given by Theorem 30.1, induces the resolution of $X'$ given by the same theorem,

$$\begin{array}{ccccccc} X = X_0 & \longleftarrow & X_1 & \longleftarrow & \dots & \longleftarrow & X_s \\ \uparrow{\scriptstyle \theta=\theta_0} & & \uparrow{\scriptstyle \theta_1} & & & & \uparrow{\scriptstyle \theta_s} \\ X' = X'_0 & \longleftarrow & X'_1 & \longleftarrow & \dots & \longleftarrow & X'_m. \end{array}$$

The latter is therefore obtained by blowing up smooth centers $Y'_i \subset \operatorname{Sing} X'_i$ with $\theta_i(Y'_i) = Y_i$. Thus, if $\mathcal{K}' = \mathcal{I}(Y'_1)\cdots\mathcal{I}(Y'_{s-1})\mathcal{O}_{X'_s}$, then $\theta_s$ induces an isomorphism between $\mathcal{K}$ and $\mathcal{K}'$. Therefore, the basic objects, $(X_s, (\mathcal{K}_s, 1), F_s = \emptyset)$ and $(X'_s, (\mathcal{K}'_s, 1), F'_s = \emptyset)$ are identifiable (see Remark 27.2), and the resolution of one of them given Theorem 18.9 induces the resolution of the other given by the same theorem (see Theorem 27.5 and Remark 27.7). □

THEOREM 30.7. **Embedded resolution of singularities.** *Let $X$ be a non-smooth variety defined over a field of characteristic zero, embedded in a smooth $V$. Then a finite sequence of blow ups at smooth centers can be constructed:*

$$\begin{array}{ccccccc} V = V_0 & \longleftarrow & V_1 & \longleftarrow & \dots & \longleftarrow & V_t \\ \cup & & \cup & & \dots & & \cup \\ X = X_0 & \longleftarrow & X_1 & \longleftarrow & \dots & \longleftarrow & X_t \end{array} \tag{30.12}$$

*so that:*

1. *$X_t$ is smooth;*
2. *The composition $X \leftarrow X_t$ induces an isomorphism on $X \setminus \operatorname{Sing} X$;*
3. *$X_t$ has normal crossings with the exceptional divisor of $V \leftarrow V_t$;*
4. *The exceptional divisor of $V \leftarrow V_t$ has normal crossing support.*

*Moreover, the process is constructive and equivariant.*

PROOF. Use Theorem 30.1 to construct a resolution of singularities of $X$,

$$X = X_0 \longleftarrow X_1 \longleftarrow \dots \longleftarrow X_s, \tag{30.13}$$

so that:

1. $X_s$ is smooth;
2. The morphism $X \leftarrow X_s$ induces an isomorphism on $X \setminus \operatorname{Sing} X$.

Sequence (30.13) induces a finite sequence of blow ups at smooth centers and commutative diagrams of blow ups and closed immersions:

$$\begin{array}{ccccccc} V = V_0 & \xleftarrow{\pi_0} & V_1 & \xleftarrow{\pi_1} & \cdots & \xleftarrow{\pi_{s-1}} & V_s \\ \uparrow & & \uparrow & & & & \uparrow \\ X = X_0 & \xleftarrow{\overline{\pi}_0} & X_1 & \xleftarrow{\overline{\pi}_1} & \cdots & \xleftarrow{\overline{\pi}_{s-1}} & X_s. \end{array} \tag{30.14}$$

This follows from the fact that if $Y_i \subset X_i \subset V_i$ is our choice of center, then the blow up at $Y_i$ induces a commutative diagram of blow ups and closed immersions:

$$\begin{array}{ccc} V_i & \xleftarrow{\pi_i} & V_{i+1} \\ \uparrow & & \uparrow \\ X_i & \xleftarrow[\overline{\pi}_i]{} & X_{i+1}. \end{array}$$

Also, if $H_{i+1} \subset V_{i+1}$ denotes the exceptional divisor, then

$$\mathcal{I}(H_{i+1})_{|X_{i+1}} = \overline{\pi}_i{}^*(\mathcal{I}(Y_i)).$$

Now let $\mathcal{L}$ be the exceptional divisor of $V \longleftarrow V_s$ (i.e., define $\mathcal{L} := \mathcal{I}(Y_1)\cdots\mathcal{I}(Y_s)\mathcal{O}_{V_s}$), and set:

- $\mathcal{L}_s := \mathcal{L}$;
- $E_s = \emptyset$;
- $F_s = \emptyset$.

Then, $\mathcal{L}_{s|X_s}$ is a locally invertible ideal, and coincides with the invertible ideal $\mathcal{K}$ from the proof of Theorem 30.6.

The constructive resolution of $(X_s, (\mathcal{L}_{s|X_s}, 1), F_s)$ that we considered in Theorem 30.6, induces the constructive resolution of the basic object

$$(V_s, (\mathcal{I}(X_s) + \mathcal{L}_s), 1), E_s)$$

(see 20.2). This induces an enlargement of sequence (30.14),

$$\begin{array}{ccccccc} V_s & \xleftarrow{\pi_s} & V_{s+1} & \xleftarrow{\pi_{s+1}} & \cdots & \xleftarrow{\pi_{r-1}} & V_r \\ \uparrow & & \uparrow & & & & \uparrow \\ X_s & \xleftarrow{\overline{\pi}_s} & X_{s+1} & \xleftarrow{\overline{\pi}_{s+1}} & \cdots & \xleftarrow{\overline{\pi}_{r-1}} & X_r, \end{array} \tag{30.15}$$

and we have that:

- Since all the centers are smooth, both $X_r$ and $V_r$ are smooth;
- For $i = s, \ldots, r-1$ the centers are contained in Sing $(\mathcal{L}_i, 1) \cap X_i$, therefore $X \longleftarrow X_r$ induces an isomorphism on $X \setminus \text{Sing } X$;
- $X_r$ does not intersect the strict transforms of the components of $\mathcal{L}$ in $V_r$; and since for $i = s, \ldots, r-1$ all centers were strictly contained in $X_i$ it follows that $X_r$ has normal crossings with the exceptional divisor of $V \longleftarrow V_r$.

Finally, to address (4), define $\widetilde{\mathcal{L}}_r$ as the locally invertible sheaf of ideals that results after factoring out from $\mathcal{L}\mathcal{O}_{V_r}$ those components supported on $E_r$. Then the constructive resolution of

$$(V_r, (\widetilde{\mathcal{L}}_r, 1), E_r)$$

induces an enlargement of sequence (30.15),

$$\begin{array}{ccccccc} V_r & \xleftarrow{\pi_r} & V_{r+1} & \xleftarrow{\pi_{r+1}} & \cdots & \xleftarrow{\pi_{tr-1}} & V_t \\ \uparrow & & \uparrow & & & & \uparrow \\ X_r & \xleftarrow{\overline{\pi}_r} & X_{r+1} & \xleftarrow{\overline{\pi}_{r+1}} & \cdots & \xleftarrow{\overline{\pi}_{t-1}} & X_t, \end{array} \tag{30.16}$$

and has the following properties (see 20.1):

- Since for $i = r, \ldots, t-1$ all centers are contained in Sing $(\widetilde{\mathcal{L}}_i, 1)$, the composition $X_r \longleftarrow X_t$ is the identity map; thus $X_t$ is smooth; $X \longleftarrow X_t$ induces an isomorphism on $X \setminus X_t$ and $X_t$ has normal crossings with $E_t$;
- The locally invertible sheaf of ideals $\mathcal{L}\mathcal{O}_{V_t}$ is supported on $E_t$, therefore the exceptional divisor of $V \longleftarrow V_t$ has normal crossings support.

□

## Appendix A: Transversality

In Part I of these notes, we studied the multiplicity of varieties. More precisely, suppose given an affine variety $X$ over a field $k$, with coordinate ring $B$. Then, the multiplicity at points of $X$ was studied under the additional assumption that there is a **finite** extension of $k$-algebras,

$$S \subset B$$

where $S$ is smooth, satisfying some *extra conditions* (which we will recall below). One purpose of this appendix is to show that, at a given point $\xi \in X$, one can define, locally (in and étale neighborhood), a finite morphism on a smooth variety, say,

$$X \to V = \text{Spec } S$$

which satisfies the previous *extra conditions.*

In the discussion above, we do not view $X$ as embedded in a smooth scheme, whereas in Proposition 36.1, we study $X$ together with an immersion (see 36). More precisely, it is proved that if $X \subset W^{(d)}$ is a hypersurface, where $W^{(d)}$ is a smooth scheme of dimension $d$, then one can construct a smooth morphism of smooth schemes,

$$W^{(d)} \to W^{(d-1)}$$

so that, at least locally, in an (étale) neighborhood of a point, the hypersurface $X \subset W^{(d)}$ can be expressed as the zero set of a monic polynomial with coefficients in $\mathcal{O}_{W^{(d-1)}}$.

### 31. The initial setting and main results

Let $k$ be a perfect field, let $X'$ be a scheme of finite type over $k$, and let $\xi' \in X'$ be a closed point. In this part of the notes we show how to construct an *étale morphism of $k$-schemes* in a neighborhood of $(X', \xi)$, say,

$$(X, \xi) \longrightarrow (X', \xi')$$

and a morphism of $k$-algebras in a neighborhood of $\xi'$,

$$\delta : X \to V$$

so that:

(A) The local rings $\mathcal{O}_{V,\delta(\xi)}$ and $\mathcal{O}_{X,\xi}$ have the same residue field;
(B) $m_{\delta(\xi)}\mathcal{O}_{X,\xi}$ is a reduction of $m_\xi$, where $m_\xi$ and $m_{\delta(\xi)}$ denote the maximal ideals of $\mathcal{O}_{X,\xi}$ and $\mathcal{O}_{V,\delta(\xi)}$ respectively;
(C) $\xi$ is the only point of $X$ mapping to $\delta(\xi) \in V$;
(D) $V$ is smooth and $\delta : X \to V$ is finite.

In this way we answer the question posed in 1.8. Recall that, under conditions (A)-(D), the Multiplicity Formula of Zariski ensures that if $\xi \in X$ is a point of multiplicity $n$, then the finite morphism $\delta : X \to V$ has generic rank $n$, i.e., $[L : K] = e(\mathcal{O}_{X,\xi})$, where $K$ denotes the ring of rational functions of the smooth scheme $V$, and $L = \mathcal{O}_{X,\xi} \otimes_{\mathcal{O}_V} K$.

As indicated, the construction of $\delta : X \to V$ will be done in a neighborhood of a closed $\xi'$ of $X'$. We must qualify here that the notion of neighborhood is meant in the sense of *étale topology.* This will lead us to the definition of étale morphisms (see 32.1), and, in particular, to explain why we require the field $k$ to be perfect. In fact, finite extensions of the base field $k$ will be required in our construction of

$\delta : X \to V$, and these extensions induce étale morphisms over $X'$ when the field $k$ is perfect.

The precise formulation of the previous construction is given in Proposition 31.1 below, whose proof will be addressed in Section 35.

PROPOSITION 31.1. *Let $k$ be a perfect field, let $D$ be $k$-algebra, and let $M \subset D$ be a maximal ideal. Then, after replacing $D$ by a suitable étale extension, if needed, a morphism*

$$\alpha : \mathrm{Spec}(D) \to \mathbb{A}^d$$

*can be constructed, so that:*

(a) *The local rings $\mathcal{O}_{\mathbb{A}^d,\alpha(M)}$ and $D_M$ have the same residue field;*
(b) *The maximal ideal of $\mathcal{O}_{\mathbb{A}^d,\alpha(M)}$ generates a reduction of $M$ in $D_M$.*

*Moreover, there is a commutative diagram of affine schemes of finite type over $k$, say,*

$$\begin{array}{ccc} \mathrm{Spec}(D) & \longleftarrow & X \\ \downarrow & & \downarrow \delta \\ \mathbb{A}^d = \mathrm{Spec}(k[X_1, \dots, X_d]) & \longleftarrow & V \end{array} \tag{31.1}$$

*so that $V$ is smooth and:*

(1) *Both horizontal maps are étale.*
(2) *There is a (unique) point $\xi \in X$ mapping to $M \in \mathrm{Spec}(D)$.*
(3) *There is a (unique) point $\eta \in V$ mapping to the origin in $\mathbb{A}^d$.*
(4) *Both local rings $\mathcal{O}_{X,\xi}$ and $\mathcal{O}_{V,\eta}$ have the same residue field.*
(5) *$\delta$ is finite, and the maximal ideal of $\mathcal{O}_{V,\eta}$ induces a reduction of the maximal ideal of $\mathcal{O}_{X,\xi}$.*

*In addition $\delta : X \to V$ can be constructed so that $\xi \in X$ is the unique point dominating $\eta \in V$.*

Finally, and starting in Section 36, we discuss an embedded version of Proposition 31.1 for the case of a hypersurface embedded in a smooth scheme (see Proposition 36.1).

## 32. Étale morphisms

Along these notes we have considered the notions of varieties and of schemes. An affine variety $Y$ over a field $k$ is a structure defined in correlation with a domain, say $D$, which is a $k$-algebra of finite type. In this correspondence $Y$ is the set of maximal ideals of $D$, and given $\xi \in Y$ the local ring $\mathcal{O}_{Y,\xi}$ is a localization at the corresponding maximal ideal, say $D_M$.

It is more convenient, at least for our further discussion, to replace varieties by schemes. Namely, if $Y$ is defined in terms of the domain $D$, it is convenient to replace $Y$ by $\mathrm{Spec}(D)$. The latter is a scheme of finite type over $k$.

There are many problems in local algebraic geometry which lead us, in a natural manner, to consider a *change of the base field $k$*. Let $k \subset K$ be a field extension, then one can replace $D$ by $D \otimes_k K$. For example, if $k \subset K$ is a finite extension, $D \subset D' = D \otimes_k K$ is a finite extension of rings. So if $M$ is a maximal ideal in $D$, there is at least one maximal ideal, say $M'$ in the $K$-algebra $D \otimes_k K$, dominating $M$, so

$$D_M \subset D'_{M'}, \tag{32.1}$$

and we would like to replace $D_M$ by $D'_{M'}$. For example, if the residue field of the first is $K$, we would like to replace $D_M$ by a local ring $D'_{M'}$ with residue field $K$. The point here is to make sure that the information concerning $D_M$ is preserved when replacing it by $D'_{M'}$. For instance, if one of them is regular, then we want to make sure that so is the other one.

In taking changes of the base field $k$, the first observation is that it is too rigid to consider only $k$-algebras $D$ which are domains. In fact this last property is not preserved by these changes. The first advantage of taking into account perfect fields is that if we relax this condition by letting $D$ be a reduced algebra, then at least $D \otimes_k K$ is also reduced, for any field extension $k \subset K$.

On the other hand, most of the important information of a local ring $D_M$ will be encoded in the graded ring $\mathrm{Gr}_M(D)$. Such is the case of the Hilbert-Samuel function, or the multiplicity. Another major advantage of working over perfect fields, at least for our purpose, is that it ensures a peculiar condition on the morphism

$$\mathrm{Gr}_M(D) \to \mathrm{Gr}_{M'}(D') \tag{32.2}$$

induced by (32.1): that it is also defined by a change of base field in degree zero. Namely that

$$\mathrm{Gr}_{M'}(D') = \mathrm{Gr}_M(D) \otimes_{D/M} D'/M'. \tag{32.3}$$

This property ensures, for example, that both local rings, $D_M$ and $D'_{M'}$, have the same Hibert-Samuel function, and hence that one is regular if and only the other is regular. Property which fails when $k$ is not perfect.

**32.1. Flat and étale morphisms.** Let $f : Y' \to Y$, be a morphism of affine $k$-varieties, induced by a morphism of $k$-algebras, $D \to D'$ (or consider $f : \mathrm{Spec}(D') \to \mathrm{Spec}(D)$). Suppose that $\xi' \in Y'$ maps to $\xi \in Y$, so if $M'$ and $M$ are the corresponding maximal ideals, there is an homomorphism of local rings $D_M \to D'_{M'}$.

(1) The morphism $f : Y' \to Y$ is said to be *flat at* $\xi$ when $D_M \to D'_{M'}$ is flat. In such a case $D_M \subset D'_{M'}$ is an inclusion.

When $D \to D'$ is given by a finite extension of the base field, the morphism $f : Y' \to Y$ is flat (i.e., is flat at every point). Another example is given by $D \to D' = D[X]/\langle f(X)\rangle$ where $f(X)$ is a monic polynomial in $D[X]$. In this case $D'$ is a (finite and) free extension of $D$, in particular it is flat, and therefore the induced morphism, say $g : \mathrm{Spec}(D') \to \mathrm{Spec}(D)$, is flat.

(2) The morphism $f : Y' \to Y$ (or say $f : \mathrm{Spec}(D') \to \mathrm{Spec}(D)$) is said to be *étale at* $\xi' \in Y'$, or say at $M'$, if $D_M \to D'_{M'}$ fulfills the following three conditions:
   (a) It is flat (in particular $D_M \subset D'_{M'}$);
   (b) $MD'_{M'} = M'D'_{M'}$.
   (c) $D/M \subset D'/M'$ is a finite separable extension.

When these conditions hold we say that $(Y', \xi')$*, or say* $f : (Y', \xi') \to (Y, \xi)$*, is an étale neighborhood of* $(Y, \xi)$. We abuse the notation and also say that $D_M \subset D'_{M'}$ is an étale neighborhood. Observe that, by definition, an étale morphism is of finite type.

Example 32.2. Given an algebra $D$ there is a simple manner to construct non trivial homomorphisms $D \to D'$ that are étale. Let $f(X) \in D[X]$ be a monic polynomial, and recall that the composition of flat homomorphisms is flat, so

$$D \to D_1 = D[X]/\langle f(X)\rangle \to D' = (D_1)_{\frac{\partial f}{\partial X}} \tag{32.4}$$

is flat. This ensures that $D \to D'$ is flat at every point (every time we fix a maximal ideal $M'$ in $D'$ dominating at a maximal ideal $M$ in $D$, $D_M \to D'_{M'}$ is flat). We claim now that this homomorphism of local rings, say $D_M \subset D'_{M'}$ is always étale. In fact, given a maximal ideal $M$ in $D$, and setting $K = D/M$, the fiber over $M$ is given by

$$D/M \otimes_D D' = (K[X]/\overline{f}(X))_{\frac{\partial \overline{f}}{\partial X}}$$

and one readily checks that this is a reduced ring, and a finite direct sum of separable extensions of $K$. Therefore conditions (b) and (c) in the definition of étale morphisms are fullfilled, and hence the morphism is étale. In fact, a theorem concerning étale maps states that any étale homomorphism $D_M \subset D'_{M'}$ arises from a construction as in (32.4) (see [**R**, Remark 2, pg. 19]).

Note here, in addition, that conditions (b) and (c) of 32.1 guarantee that

$$\mathrm{Gr}_{M'}(D') = \mathrm{Gr}_M(D) \otimes_{D/M} D'/M',$$

and therefore, when the inclusion of local rings is étale, both rings have the same Hilbert-Samuel function. The similarity with (32.3) is not casual: if $k$ is a perfect field, any finite field extension defines an étale morphism.

Remark 32.3. The notions of flat and étale morphism were introduced in 32.1 for a morphism $f : Y \to Y'$ between two affine varieties, and the definitions extend naturally to morphisms of schemes of finite type over $k$.

It is convenient to extend the notion of flat morphisms to the more general class of schemes. At least for a more general class of affine schemes. A morphism of affine schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$, defined in terms of a homomorphism $R \to B$, is said to be flat when $R \to B$ is flat. One example of this arises when $R$ is local, and $B$ denotes the completion of $R$ at the maximal ideal. Note that even if $R$ is the localization of a $k$-algebra of finite type, the completion is no longer of this form.

**32.4. Flatness and étalness are open conditions.** A) Let $E_1$ and $E_2$ be two algebras of finite type over $k$, and let $P_1$ and $P_2$ be prime ideals in $E_1$ and $E_2$ respectively. Suppose given a $k$-homomorphism of local rings,

$$\phi : (E_1)_{P_1} \to (E_2)_{P_2}. \tag{32.5}$$

In such case there is a natural homomorphism $E_1 \to (E_2)_{P_2}$. Moreover, since $E_1$ is finitely generated over $k$, there is an element $h \in E_2 \setminus P_2$ and a homomorphism, say

$$\phi' : E_1 \to E_2\left[\frac{1}{h}\right],$$

with the property that $P_2$ contracts to $P_1$, and that $\phi' : E_1 \to E_2\left[\frac{1}{h}\right]$ induces (32.5) by localization. In other words, a morphism of local rings as that in (32.5) can always be lifted to a morphism between algebras of finite type.

B) Let now $\delta : E_1 \to E_2$ be a homomorphism between $k$-algebras of finite type. Let $P_1$ and $P_2$ be prime ideals in $E_1$ and $E_2$ respectively, and assume that $P_2$ contracts to $P_1$. This defines a morphism of $k$-schemes, say

$$\mathrm{Spec}(E_2) \to \mathrm{Spec}(E_1)$$

mapping $P_2$ to $P_1$.

We define a *localization of* $\delta : E_1 \to E_2$ *locally at* $P_1$ *and* $P_2$, as a morphism say:

$$\delta_{f,g} : E_1\left[\frac{1}{f}\right] \to E_2\left[\frac{1}{g}\right]$$

for some $f \in E_1 \setminus P_1$ and $g \in E_2 \setminus P_2$. In this definition we are assuming that the image of $f$ in $E_2\left[\frac{1}{g}\right]$ is invertible so that $\delta_{f,g} : E_1\left[\frac{1}{f}\right] \to E_2\left[\frac{1}{g}\right]$ is defined by the universal property of localization.

Localization of morphisms allows us to formulate the two following properties of homomorphisms between algebras of finite type. Set $\delta : E_1 \to E_2$ and $P_1$ and $P_2$ as above:

**Property 1:** If the induced homomorphism $(E_1)_{P_1} \to (E_2)_{P_2}$ is flat homomorphism of local rings, then there is a localization $\delta_{f,g} : E_1\left[\frac{1}{f}\right] \to E_2\left[\frac{1}{g}\right]$ which is flat.

Similarly for étale homomorphisms:

**Property 2:** If $(E_1)_{P_1} \to (E_2)_{P_2}$ is étale homomorphism of local rings, then there is a localization $\delta_{f,g} : E_1\left[\frac{1}{f}\right] \to E_2\left[\frac{1}{g}\right]$ which is étale.

**32.5. Flat and étale morphisms are preserved by base change.** Let $\alpha : R \to B$ and $\beta : R \to C$ be two ring homomorphisms. Then there is a natural (commutative) diagram:

(32.6)
$$\begin{array}{ccc} B & \xrightarrow{\beta'} & B \otimes_R C \\ \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \alpha'} \\ R & \xrightarrow{\beta} & C \end{array}$$

which induces the diagram of affine schemes:

(32.7)
$$\begin{array}{ccc} \mathrm{Spec}(B) & \xleftarrow{\beta'_1} & Y = \mathrm{Spec}(B \otimes_R C) \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha'_1} \\ \mathrm{Spec}(R) & \xleftarrow[\beta_1]{} & \mathrm{Spec}(C) \end{array}$$

where $Y$ is the fiber product of $\mathrm{Spec}(B)$ and $\mathrm{Spec}(C)$ over $\mathrm{Spec}(R)$. Then:

(1) If $\beta_1 : \mathrm{Spec}(C) \to \mathrm{Spec}(R)$ is flat (i.e., if $\beta : R \to C$ is flat), then $\beta'_1 : X \to \mathrm{Spec}(B)$ is flat.

(2) If $\beta_1 : \mathrm{Spec}(C) \to \mathrm{Spec}(R)$ is étale (i.e., if $\beta : R \to C$ is étale), then $\beta_1' : X \to \mathrm{Spec}(B)$ is étale.

## 33. Henselian rings

In this section we will consider a class of local rings which has a property that is always fulfilled by fields: If $K$ is a field, and if $K \subset F$ is a finite extension, then $F$ is semilocal and it is the direct sum of the localizations at such local rings. Namely

$$F = F_1 \oplus \cdots \oplus F_r,$$

where each $F_i$ is a local artinian ring, and the localization of $F$ at a maximal ideal. This ensures, in particular, that each localization $F_i$ is also a finite extension of $K$.

Definition 33.1. A local ring $(R, m)$ is said to be *henselian* if for any finite extension, $R \subset C$, the semilocal ring $C$ is a direct sum of the localizations at the maximal ideals, i.e.,

$$C = C_{M_1} \oplus \cdots \oplus C_{M_r}$$

where $\{M_1, \ldots, M_r\}$ denote the maximal ideals in $C$. In particular, each localization $C_{M_i}$ is finite over $R$.

Example 33.2. Complete rings are examples of henselian rings, in particular if we replace the local ring $T = k[x_1, \ldots, x_d]_{\langle x_1,\ldots,x_d \rangle}$ by its completion $k[[x_1, \ldots, x_d]]$, the latter is henselian.

Proposition 33.3. *Let $(R, m)$ be a local henselian ring, let $B$ be an $R$-algebra of finite type and let $n$ be a prime ideal in $B$. Assume that:*

(i) *The prime ideal $n$ maps to the closed point $m$ of* $\mathrm{Spec}(R)$*;*
(ii) *The prime ideal $n$ is an isolated point of the fiber of* $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ *over the closed point $m$.*

*Then there is an element $g \in B$, $g \notin n$, such that:*

(1) *$B_g$ is a finite extension of $R$; and*
(2) *$B_g = B_n$.*

Proof. To proof the proposition we will make use of Zariski's Main Theorem:

**Zariski's Main Theorem as proved by Grothendieck.** [**R**, Thm 1, pg. 41, and Corollaries 1 and 2, pg. 42]. *Let $R$ be a ring, let $B$ be an $R$-algebra of finite type, and let $q \subset B$ be dominating $R$ at a prime $p$. Assume that $q$ is an isolated point of the fiber over $p$. Then, there is an $R$-algebra $C$,*

$$R \subset C \subset B$$

*such that:*

- *$C$ is finite over $R$;*
- *There is an element $f \in C \setminus q$, so that $B_f = C_f$.*

*In other words, locally, $B$ is the localization of a finite extension of $R$.*

According to Zariski's Main Theorem there is a finite extension $C$ of $R$, with $C \subset B$, and an element $f \in C$, $f \notin n$, so that $B_f = C_f$. Since $R$ is henselian,

$$C = C_{M_1} \oplus \cdots \oplus C_{M_r}$$

where $\{M_1, \dots, M_r\}$ denote the maximal ideals in $C$. Assume that $M_1 = n$. There is an index $s$, $1 < s \leq r$, so that

$$B_f = C_f = C_{M_1} \oplus \cdots \oplus C_{M_s}.$$

One can further localize at a multiple of $f$, say $g$ in $B$, so that $B_g = C_{M_1}$. Since each localization is finite over $R$, $B_g = C_{M_1}$ is also a finite extension of $R$. □

The previous result indicates that there is a neighborhood of the point $n$ in $\mathrm{Spec}(B)$, say $\mathrm{Spec}(B_f)$, which is finite over $\mathrm{Spec}(R)$. Moreover, $B_f$ is a local ring, so there is a unique prime dominating $R$ at $m$.

**33.4. The henselization of a local ring.** Given a local ring $(R, m)$ with residue field $K$, there is a local ring, say $(\tilde{R}, \tilde{m})$, and a homomorphism of local rings $(R, m) \to (\tilde{R}, \tilde{m})$, such that $(\tilde{R}, \tilde{m})$ is henselian. The local ring $(\tilde{R}, \tilde{m})$ is called the *strict henselization of* $(R, M)$ (see [**R**]). Roughly speaking, the local ring $(\tilde{R}, \tilde{m})$ is constructed by taking the direct limit of all local étale neighborhoods of $(R, m)$, and, in general, $(R, m) \to (\tilde{R}, \tilde{m})$ is not a morphism of finite type.

By construction, the homomorphism

$$(R, m) \to (\tilde{R}, \tilde{m})$$

complies properties which are similar to those in 32.1, (2) (a-c) of the definition of étale homomorphism. In fact, the residue field of $(\tilde{R}, \tilde{m})$ is the separable closure of $K$, the local morphism $(R, m) \to (\tilde{R}, \tilde{m})$ is flat, and $m\tilde{R} = \tilde{m}$. Moreover, the following properties hold:

- For any local étale homomorphism $(R, m) \subset (R', m')$ there is an inclusion $R \subset R' \subset \tilde{R}$.
- Given finitely many elements$g_1, \dots, g_r \in \tilde{R}$, there is a local étale homomorphism $(R, m) \subset (R', m')$ so that $g_1, \dots, g_r \in R'$.

Remark 33.5. Let $k[X_1, \dots, X_d]$ be a polynomial ring, let $N = \langle X_1, \dots, X_d \rangle$ denote the maximal ideal at the origin, let $\tilde{k}$ be the separable closure of $k$, and let $\tilde{k}\{\{X_1, \dots, X_d\}\}$ denote the strict henselization of $k[X_1, \dots, X_d]_N$.

Suppose given a $k$-algebra of finite type $D$ and a rational maximal ideal $M$ (i.e., $D/M = k$). Let $\{Y_1, \dots, Y_d\}$ be a system of parameters in $D_M$. Note that there is a natural $k$-homomorphism of local rings

$$k[X_1, \dots, X_d]_N \to D_M,$$

that maps $X_i$ to $Y_i$. It can be checked that this homomorphism is an inclusion, for example, by taking completions, in which case the setting is similar to that in (1.1) (it defines a finite extension of the complete rings of the same dimension).

## 34. Local-transversal morphisms

Let $S \subset B$ be an inclusion of $k$-algebras of finite type (not necessarily a finite extension), where $S$ is irreducible and smooth over a field $k$. Let $P \subset B$ be a prime ideal and let $\mathfrak{p} = P \cap S$. We say that the morphism

$$\mathrm{Spec}(B) \to \mathrm{Spec}(S)$$

is *local-transversal* at $P$, or that $S \subset B$ is *local-transversal at* $P$, if the following conditions hold:

(1) $B_P/PB_P = S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$;

(2) $\mathfrak{p}B_P$ is a reduction of $PB_P$;
(3) $P$ is an isolated point in the fiber over $\mathfrak{p}$.

The following proposition will be proven:

PROPOSITION 34.1. *Let $k$ be a perfect field, let $B$ be a $k$-algebra of finite type, let $\eta \in Spec(B)$ be a closed point and, let $n$ be its corresponding maximal ideal. Then, after considering a suitable étale extension $(B', n')$ of $(B, n)$ if needed, we can construct a local-transversal morphism at $n'$,*

$$S \longrightarrow B',$$

*from some smooth $k$-algebra of finite type $S$.*

The next theorem will be used in the proof of Proposition 34.1 since it gives a simple criterion to find parameter ideals which span a reduction of the maximal ideal (in *any* local noetherian ring). The proof of Proposition 34.1 will be detailed in 34.3.

THEOREM 34.2. *(see, e.g.* [**He**, Th 10.14]*) Let $(R, m)$ be a noetherian local ring, and let $\{x_1, \dots, x_d\}$ be elements in the maximal ideal $m$. Let $\overline{X_i}$ denote the class of $x_i$ in $m/m^2$. Then the following are equivalent:*

(1) *$\langle x_1, \dots, x_d\rangle$ is a reduction of $m$.*
(2) *$Gr_m(R)/\langle\overline{X_1}, \dots, \overline{X_d}\rangle$ is a graded ring of dimension zero.*

This result gives a nice geometrical condition for parameters to span a reduction of the maximal ideal. In fact, it asserts that a sufficiently general choice of parameters in $(R, m)$ will have this property. To be precise, we will use the fact that if a local ring $(R, m)$ has dimension $d$, then the corresponding graded ring $\mathrm{Gr}_m(R)$ is also $d$ dimensional. Thus:

- If $d = 0$, $R$ is an artinian local ring, and the ideal zero is a reduction of $m$;
- If $d \geq 1$, and using prime avoidance on the graded ring $\mathrm{Gr}_m(R)$, one can choose homogeneous elements in degree one, say $\{\overline{x}_1, \dots, \overline{x}_d\} \subset m/m^2$ so that the quotient $\mathrm{Gr}_m(R)/\langle\overline{x}_1, \dots, \overline{x}_d\rangle$ is zero dimensional.

One can also see this last observation as an application of Noether's Theorem in the context of graded algebras: if $k$ as the residue field of the local ring, then there is a finite extension

$$k[X_1, \dots, X_d] \subset \mathrm{Gr}_m(R)$$

which is also a morphism of graded rings.

**34.3. Proof of Proposition 34.1.** Observe that after extending the base field $k$ if needed, it can be assumed that the closed point $\eta$ is rational over $k$.Thus we can express $B$ as a quotient of a polynomial ring $T = k[x_1, \dots, x_n]$, i.e.,

$$B = k[\overline{x}_1, \dots, \overline{x}_n],$$

where the closed point $\eta$ corresponds to the origin.

Let $d = \dim B_m$. So $\mathrm{Gr}_{m_\eta}(B)$ is a $d$-dimensional ring, and a quotient of $\mathrm{Gr}_{(0,\dots,0)}(T)$ (the graded ring of $\mathbb{A}^n$ at the origin).

Again, after a finite extension of the base field if necessary, and making a linear change of variables, by Noether's Lemma (although $k$ may not be an infinite field, a suitable finite extension will enable us to fulfill this condition) it can be assumed that $\mathrm{Gr}_{m_\eta}(B)$ is a finite extension of $k[X_1,\dots,X_d]$, where $X_i$ denotes the class of $x_i$ in $m_\eta/m_\eta^2$ for $i=1,\dots,d$, and hence in $\mathrm{Gr}_{m_\eta}(B)$. Note that $\overline{x}_1,\dots,\overline{x}_d \in B_{m_\eta}$ are algebraically independent over $k$, and one can chose $g \in B \setminus m_\eta$ so that

$$k[\overline{x}_1,\dots,\overline{x}_d] = k[x_1,\dots,x_d] \subset B_g.$$

Observe that $B_g$ is also a $k$-algebra of finite type. This shows already that, after replacing $B$ by an open neighborhood of $m_\eta$, there is an inclusion $k[x_1,\dots,x_d] \subset B$ which induces a morphism that is local-transversal at $m_\eta$.

Therefore, after a finite change of the underlying field, and taking an open restriction at $m_\eta$, we can construct a morphism

$$\widetilde{\delta} : \mathrm{Spec}(\widetilde{B}) \to \mathbb{A}^n$$

which is local-transversal at $m_\eta$. □

**34.4. Fibers and fiber products.** Let $\alpha : A \to B$ and $\beta : A \to C$ be two ring homomorphisms. Then there is a natural (commutative) diagram:

(34.1)
$$\begin{array}{ccc} B & \xrightarrow{\beta'} & B \otimes_A C \\ \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\alpha'} \\ A & \xrightarrow{\beta} & C \end{array}$$

which induces the diagram of affine schemes:

(34.2)
$$\begin{array}{ccc} \mathrm{Spec}(B) & \xleftarrow{\beta_1'} & Y = \mathrm{Spec}(B \otimes_A C) \\ \downarrow{\scriptstyle\alpha_1} & & \downarrow{\scriptstyle\alpha_1'} \\ \mathrm{Spec}(A) & \xleftarrow[\beta_1]{} & \mathrm{Spec}(C) \end{array}$$

where $Y$ is the fiber product of $\mathrm{Spec}(B)$ and $\mathrm{Spec}(C)$ over $\mathrm{Spec}(A)$.

Let $P \subset A$ be a prime. Then the fiber of $\beta_1$ over $P$ is:

$$\beta_1^{-1}(P) = \mathrm{Spec}(k_A(P) \otimes_A C),$$

where $k_A(P)$ denotes the residue field of $A_P$. Now, if $Q$ is a prime in $B$ mapping to $P$, then the fiber (via $\beta_1'$) over $Q$ is

$$(\beta_1')^{-1}(Q) = \mathrm{Spec}(k_B(Q) \otimes_A C)$$

where $k_B(Q)$ denotes denotes the residue field of $B_Q$.

LEMMA 34.5. *Suppose that in the previous setting $k_A(P) = k_B(Q)$. Then:*

(1) *The fiber of $\beta_1'$ over $Q$ can be identified with the fiber of $\beta_1$ over $P$. In particular, for each point $P'$ of* Spec$(C)$ *mapping to $P$, there is a* unique point *$Q'$ in $Y$, that maps to $P'$ and to $Q$. Moreover,*

$$k_C(P') = k_B(Q) \otimes_A k_C(Q').$$

(2) *If the fiber of $\beta_1$ over $P$ is a unique point, say $P'$, then the fiber of $\beta_1'$ over $Q$ is a unique point, say $Q'$.*

(3) *If, in addition to (2), $A_P \to C_{P'}$ is flat, then $B_Q \to (B \otimes_A C)_{Q'}$ is flat.*

COROLLARY 34.6. *With the same notation as before, assume that:*

(1) *$A = A_P$ is a local regular ring, $Q$ is a prime in $B$ contacting to $P$, and $k_A(P) = k_B(Q)$.*
(2) *$C = C_{P'}$ is a local regular ring and $A_P \to C_{P'}$ is a flat homomorphism of local rings.*
(3) *$PC_{P'} = P'C_{P'}$.*

*Then setting $Q'$ as the unique prime in $B \otimes_A C$ mapping to $P'$ and to $Q$, there is a graded morphism of graded rings*

$$Gr_{P'}(C_{P'}) \to Gr_{Q'}(A \otimes B)_{Q'}$$

*which arises from $Gr_P(A_P) \to Gr_Q(B)$, by taking the change of base field $k_A(P) \to k_{C_{P'}}(P')$.*

Example 34.7. One example in which this situation occurs is that when we set $C$ to be the completion of the local regular ring $A$.

Example 34.8. **Local-transversal morphisms are preserved by étale base change.** Another example, in which Corollary 34.6 applies is that in which $A_P$, $B_Q$ and $C = C_{P'}$ are localizations of $k$-algebras of finite type at maximal ideals, where condition (1) of Corollary 34.6 holds, and where $C = C_{P'}$ is an étale neighborhood of $A_P$. If, in addition, we assume that $A_P$, $B_Q$ and $C = C_{P'}$ are localizations of $k$-algebras $A$, $B$, and $C$, and that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is local-transversal at $Q$, then $\mathrm{Spec}(B \otimes_A C) \to \mathrm{Spec}(C)$ is local-transversal at $Q'$.

## 35. Proof of Proposition 31.1

Let $k$ be a perfect field, let $D$ be $k$-algebra, and let $M \subset D$ be a maximal ideal as in the proposition. Then, by Proposition 34.1, after replacing $D$ by an étale extension if needed, it can be assumed that there is a local-transversal morphism at $M$:

$$\alpha : k[X_1, \ldots, X_d] \longrightarrow D,$$

mapping the closed point $M$ to the origin of $\mathbb{A}^n_k$. This means that:

- $D_M/MD_M = k[X_1, \ldots, X_d]/\langle X_1, \ldots, X_d\rangle = k$;
- $\langle X_1, \ldots, X_d\rangle D_M$ is a reduction of $MD_M$;
- $M$ is an isolated point in the fiber over $\langle X_1, \ldots, X_d\rangle$.

Under these assumptions, Zariski's Main Theorem asserts that there is a finite extension, $k[X_1, \ldots, X_d] \subset A(\subset D)$, and an element $f \in A \setminus M$, so that $A_f = D_f$.

To ease the notation let $M \subset A$ denote also the intersection $M \cap A$. Thus $D_M = A_M$. Let $\{M_1 = M, M_2, \ldots, M_s\}$ denote the maximal ideals of $A$ dominating $N = \langle X_1, \ldots, X_d\rangle(\subset k[X_1, \ldots, X_d])$. If $s > 1$ select $g \in A$ so that

$$g \in (M_2 \cap \cdots \cap M_s) \setminus M.$$

Therefore $\operatorname{Spec}(A_{fg})$ is also an open neighborhood of $M$ at $\operatorname{Spec}(D)$. Thus, there is a diagram like this:

$$\begin{array}{ccccc} D & \longrightarrow & D_f & \longrightarrow & D_{fg} = D_M \\ \uparrow & & \| & & \| \\ A & \longrightarrow & A_f & \longrightarrow & A_{fg}=A_M \\ \uparrow & & & \nearrow & \\ K[X_1,\dots,X_d] & & & & \end{array}$$

Notice that the morphism $K[X_1,\dots,X_d] \to D_{fg} = D_M$ fulfills properties (1)-(5) of Proposition 31.1 except for the fact that it may not be finite. In what follows we will show that, after considering suitable étale extension of both $D_{fg}$, and $K[X_1,\dots,X_d]$, all conditions (1)-(5) of Proposition 31.1 will be satisfied.

Let $\tilde{k}\{\{X_1,\dots,X_d\}$ be the strict henselization of $k[X_1,\dots,X_d]_{\langle X_1,\dots,X_d\rangle}$, and consider the diagram

$$\begin{array}{ccc} D & \xrightarrow{\beta'} & D\otimes\tilde{k}\{\{X_1,\dots,X_d\}\} \\ \alpha\uparrow & & \uparrow\alpha' \\ k[X_1,\dots,X_d] & \xrightarrow{\beta} & \tilde{k}\{\{X_1,\dots,X_d\}\}. \end{array} \tag{35.1}$$

To ease the notation set $D' = D\otimes_{k[X_1,\dots,X_d]}\tilde{k}\{\{X_1,\dots,X_d\}$. The previous diagram induces the fiber product of affine schemes,

$$\begin{array}{ccc} \operatorname{Spec}(D) & \xleftarrow{\beta_1'} & Z=\operatorname{Spec}(D') \\ \downarrow\alpha_1 & & \downarrow\alpha_1' \\ \mathbb{A}^n_k & \xleftarrow[\beta_1]{} & \operatorname{Spec}(\tilde{k}\{\{X_1,\dots,X_d\}\}) \end{array} \tag{35.2}$$

where $Z$ is the fiber product.

Since $D/M = k$, and the fiber of $\beta_1$ over the origin is $\operatorname{Spec}(\tilde{k})$, it follows that the fiber of $\beta_1'$ over $M$ is also isomorphic to $\operatorname{Spec}(\tilde{k})$. In particular, using Lemma 34.5 and 32.5 we conclude that:

(1) There is a unique maximal ideal, say $M'$, in $D'$ mapping to $M$ in $D$;
(2) $D_M \to D'_{M'}$ is flat; and
(3) $MD'_{M'} = M'D'_{M'}$.

Consider the diagram

$$\begin{array}{ccc} B & \xrightarrow{\beta'} & D' = D\otimes\tilde{k}\{\{X_1,\dots,X_d\}\} \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' = A\otimes\tilde{k}\{\{X_1,\dots,X_d\}\} \\ \alpha\uparrow & & \uparrow\alpha' \\ k[X_1,\dots,X_d] & \xrightarrow{\beta} & \tilde{k}\{\{X_1,\dots,X_d\}\} \end{array} \tag{35.3}$$

Let $\{M_1'', M_2'', \dots, M_S''\}$ be the maximal ideals of the semi-local ring $A' = A \otimes \tilde{k}\{\{X_1, \dots, X_d\}\}$. The same arguments used before show that:

(1) There is a unique maximal ideal, say $M''$, in $A'$ mapping to $M$ in $A$.
(2) $A_M \to A'_{M''}$ is flat, and
(3) $MA'_{M'} = M''A'_{M''}$.

Set $M_1'' = M''$.

By definition of henselian rings (see Definition 33.1) we get

$$A' = A'_{M''} \oplus A'_{M_2''} \oplus \cdots \oplus A'_{M'_S},$$

and each summand is a finite extension of $\tilde{k}\{\{X_1, \dots, X_d\}\}$.

Now recall that $f, g \in A$ were chosen so that $A_{fg} = A_M$. On the other hand, since $fg \notin M$, the image of $fg$ in $A'_{M''}$ is invertible. Therefore the element $\frac{1}{fg} \in A'_{M''}$.

On the other hand, since $A'_{M''}$ is finite over $\tilde{k}\{\{X_1, \dots, X_d\}\}$, there is an integer $n$ and there are elements $d_1, \dots, d_n \in \tilde{k}\{\{X_1, \dots, X_d\}\}$ such that

$$\left(\frac{1}{fg}\right)^n + d_1 \left(\frac{1}{fg}\right)^{n-1} + \cdots + d_n = 0. \tag{35.4}$$

Consider an étale neighborhood, of $k[X_1, \dots, X_d]_N$, say

$$k[X_1, \dots, X_d]_N \subset E'_{N'} (\subset \tilde{k}\{\{X_1, \dots, X_d\}\})$$

which contains all coefficients $d_i$, for $i = 1, \dots, n$ (see 33.4). Note that

$$E'_{N'} \subset E'_{N'} \otimes A$$

is a finite extension, and that there is a unique maximal ideal in $E'_{N'} \otimes A$, say $\mathcal{M}$, dominating $A$ at $M$. This shows that

$$(E'_{N'} \otimes A)_{fg} = (E'_{N'} \otimes A)_{\mathcal{M}}.$$

Moreover, since $E'_{N'} \subset \tilde{k}\{\{X_1, \dots, X_d\}\}$ is flat,

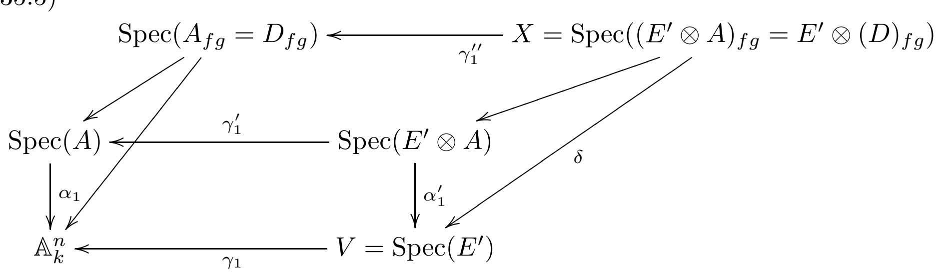
$$(E'_{N'} \otimes A)_{fg} \subset A'_{M'}$$

is a flat morphism of local rings. As flat morphisms of local rings are injective we conclude that (35.4) holds at $(E'_{N'} \otimes A)_{fg} = (E'_{N'} \otimes A)[\frac{1}{fg}]$, so this ring is a finite extension of $E'_{N'}$. Note also that

$$(E'_{N'} \otimes A)_{fg} = E'_{N'} \otimes (A)_{fg} = E' \otimes (D)_{fg}.$$

From the discussion in 32.4, it follows that $E'_{N'}$ can be assumed to be the localization of a $k$-algebra $E'$ of finite type over $k$ that is étale over $k[X_1, \dots, X_d]$. So, there is a diagram

(35.5)



where $\gamma_1$, $\gamma_1'$, and $\gamma_1''$ are étale, and $\delta : X \to V$ is finite. We conclude from this construction that the diagram

$$\begin{array}{ccc} \mathrm{Spec}(D) & \xleftarrow{\gamma_1'} & X = \mathrm{Spec}(E' \otimes (D)_{fg}) \\ \downarrow & & \downarrow \delta \\ \mathbb{A}^n_k & \xleftarrow[\gamma_1]{} & V = \mathrm{Spec}(E') \end{array} \tag{35.6}$$

fulfills all the conditions in Proposition 31.1. □

## 36. On the construction of finite-transversal morphisms for an embedded hypersurface

There is a small variation of the result in Proposition 31.1, which is worth pointing out as it is well adapted to the case in which the initial data are given by a singular variety together with an embedding in a smooth scheme. Therefore we discuss here an embedded version of the Proposition 31.1, which will be addressed only for the case of an embedded hypersurface.

In the formulation of Proposition 31.1 the starting point is a $k$-algebra $D$ together with a maximal ideal $M$. The result is the construction of a morphism at what is called *an étale neighborhood of $M$ in* Spec($D$), with prescribed properties. Now suppose we enlarge the initial data by considering a point $M$ in Spec($D$), together with an inclusion

$$\mathrm{Spec}(D) \subset W,$$

where $W$ is a smooth scheme. We will clarify and discuss this issue here for the case in which Spec($D$) is a hypersurface of dimension $d$ in a smooth affine scheme, say $W^{(d+1)} = \mathrm{Spec}(E)$, of dimension $d+1$. Within this frame we prove Proposition 36.1.

Proposition 36.1. *Let $X$ be a hypersurface embedded in some smooth $(d+1)$-dimensional scheme $W$ of finite type over a perfect field $k$. Let $\xi \in W$ be a rational point where the ideal $I(X)$ has order $n$ (thus the multiplicity of $X$ at $\xi$ is $n$). Then, at a suitable (étale) neighborhood of $\xi$, say $(W', \xi')$ there is a smooth morphism to some $d$-dimensional smooth scheme $\pi : W' \to V$, and an element $Z$ of order one at $\mathcal{O}_{W',\xi'}$ so that:*

1. *The smooth line $\pi^{-1}(\pi(\xi'))$ and the smooth hypersurface $\{Z = 0\}$ cut transversally at $\xi'$.*
2. *If $X'$ denotes the pull back of the hypersurface $X$ to $W'$, then $I(X')$ is spanned by a monic polynomial of the form*
$$f(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_1 Z + a_0 \in \mathcal{O}_V[Z],$$
*and the restriction of $W' \to V$, namely $\delta : X' \to V$, is finite (of generic rank $n$).*
3. *$\xi' \in X'$ is the unique point in $\delta^{-1}(\delta(\xi'))$, both points have residue field $k$, and the maximal ideal of $\mathcal{O}_{V,\delta(\xi')}$ spans a reduction of the maximal ideal of $\mathcal{O}_{X',\xi'}$.*

Proof. The basic tool for the proof of the proposition will be Weierstrass Preparation Theorem. This is a theorem which holds at henselian regular rings. These are local rings which are not localizations of $k$-algebras of finite type, however we shall extract from Weierstrass Preparation Theorem some consequences within the class of algebras of finite type.

Let $E$ be an open affine neighborhood of $\xi \in W$. Then $E$ is a finitely generated $k$-algebra which is regular. Suppose that the point $\xi$ corresponds to a maximal ideal $M \subset E$. From the hypothesis one has that the residue field of the local regular ring $E_M$ is $k$. Given a regular system of parameters, $\{X_1, \dots, X_d, X_{d+1}\} \subset E_M$ one has that the strict henselization of $E_M$, denoted by $\tilde{k}\{\{X_1, \dots, X_d, X_{d+1}\}\}$, is also a regular ring having $\{X_1, \dots, X_d, X_{d+1}\}$ as regular system of parameters. The residue field $\tilde{k}$ is the separable closure of $k$.

The graded ring of $E_M$, $\mathrm{Gr}_M(E_M)$, is a polynomial ring of dimension $d+1$ over $k$,

$$\mathrm{Gr}_M(E_M) = k[X_1, \dots, X_{d+1}]$$

where we identify $X_i$ with its initial form $\mathrm{In}_M(X_i)$, for $i = 1, \dots, d+1$. Similarly, the graded ring of $\tilde{k}\{\{X_1, \dots, X_d, X_{d+1}\}\}$ is the polynomial ring $\tilde{k}[X_1, \dots, X_{d+1}]$.

Now, let $f$ be a non-invertible element of $E_M$, defining the hypersurface $X$ in a neighborhood of the closed point $M$ in $\mathrm{Spec}(E)$, and let $n$ denote the multiplicity of the hypersurface at this point. Let $\mathrm{In}_M(f) \in k[X_1, \dots, X_{d+1}]$ be the initial form. We will assume that $\{X_1, \dots, X_d, X_{d+1}\}$ has been chosen so that the inclusion of graded rings

$$k[X_1, \dots, X_d] \subset k[X_1, \dots, X_{d+1}]/\langle \mathrm{In}_M(f) \rangle$$

is a finite extension.

Since Weierstrass Preparation Theorem holds at the henselization of $E_M$, i.e., at $\tilde{k}\{\{X_1, \dots, X_d, X_{d+1}\}\}$, we may assume that there is a unit $u$ so that

$$uf = (X_{d+1})^n + a_1(X_{d+1})^{n-1} + \cdots + a_n \in \tilde{k}\{\{X_1, \dots, X_d\}\}[X_{d+1}], \tag{36.1}$$

where $n$ is the multiplicity of $f$ at $E_M$.

Set

$$P(X_{d+1}) = (X_{d+1})^n + a_1(X_{d+1})^{n-1} + \cdots + a_n,$$

we claim that:

$$\tilde{k}\{\{X_1, \dots, X_d, X_{d+1}\}\}/\langle f \rangle = \tilde{k}\{\{X_1, \dots, X_d\}\}[X_{d+1}]/\langle P(X_{d+1}) \rangle. \tag{36.2}$$

To prove this claim we argue as for the case of complete rings. In fact the proof we now sketch is exactly the same which is used when considering complete rings instead of henselizations. Note first that as $f$ has multiplicity $n$ at the regular ring $\tilde{k}\{\{X_1, \dots, X_d, X_{d+1}\}\}$,

$$a_i \in \langle X_1, \dots, X_d \rangle^i \subset \tilde{k}\{\{X_1, \dots, X_d\}\},$$

for $i = 1, \dots, n$. This already implies that the ring at the right in (36.2) is local. Since it is also a finite extension of $\tilde{k}\{\{X_1, \dots, X_d\}\}$, it is also henselian, and therefore it coincides with its strict henselization. This, with the fact that henselization commutes with quotients, proves the equality in (36.2), since both rings are strictly henselian.

The regular system of parameters $\{X_1, \dots, X_d, X_{d+1}\}$ define an inclusion of regular local rings,

$$k[X_1, \dots, X_d]_{\langle X_1, \dots, X_d \rangle} \subset E_M.$$

This inclusion define a morphism

$$\mathrm{Spec}(E) \longrightarrow \mathrm{Spec}\, k[X_1, \dots, X_d]_{\langle X_1, \dots, X_d \rangle}$$

and $M$ is the unique point of the fiber over the the closed point of

$$\operatorname{Spec} k[X_1, \dots, X_d]_{\langle X_1,\dots,X_d\rangle}$$

where $X_{d+1} = 0$.

Any étale neighborhood $F'_{N'}$ of $k[X_1, \dots, X_d]_{\langle X_1,\dots,X_d\rangle}$ induces, by base change, an étale extension of $E_M$,

$$F'_{N'} \subset F'_{N'} \otimes_{k[X_1,\dots,X_d]_{\langle X_1,\dots,X_d\rangle}} E_M.$$

Notice that the ring on the right is also local. This holds because we assume that the residue field of $E_M$ is $k$, and this ensures, in addition, that both local rings have the same residue field.

Note also that $\{X_1, \dots, X_d\}$ is a regular system of parameters in $F'_{N'}$, and that $\{X_1, \dots, X_d, X_{d+1}\} \subset F'_{N'} \otimes_{k[X_1,\dots,X_d]} E$ defines a regular system of parameters at the local regular ring $F'_{N'} \otimes_{k[X_1,\dots,X_d]} E_M$. So, its strict henselization is again $\tilde{k}\{\{X_1, \dots, X_d, X_{d+1}\}\}$.

Let $P(X_{d+1}) = (X_{d+1})^n + a_1(X_{d+1})^{n-1} + \dots + a_n$ be as in (36.1), and let $F'_{N'}$ be an étale neighborhood of $k[T_1, \dots, T_d]_{\langle T_1,\dots,T_d\rangle}$ which contains all coefficients $a_i \in \tilde{k}\{\{X_1, \dots, X_d\}\}$ for $i = 1, \dots, n$.

Now $f$ and $P(X_{d+1})$ are two elements at the local ring $F'_{N'} \otimes_{k[X_1,\dots,X_d]} E_M$, and they both span the same ideal in this local ring since they span the same ideal at $\tilde{k}\{\{X_1, \dots, X_d, X_{d+1}\}\}$. This shows that

$$F'_{N'} \otimes_{k[X_1,\dots,X_d]} (E_M/\langle f\rangle) = (F'_{N'} \otimes_{k[X_1,\dots,X_d]} E_M)/\langle P(X_{d+1})\rangle.$$

Since $a_i \in N'$ for $i = 1, \dots, n$, it follows that the quotient ring

$$(F'_{N'} \otimes_{k[X_1,\dots,X_d]} E)/\langle P(X_{d+1})\rangle$$

is local. Moreover, the maximal ideal is that induced by $F'_{N'} \otimes_{k[X_1,\dots,X_d]} E_M$ and

$$(F'_{N'} \otimes_{k[X_1,\dots,X_d]} E_M)/\langle P(X_{d+1})\rangle = (F'_{N'} \otimes_{k[X_1,\dots,X_d]} E)/\langle P(X_{d+1})\rangle.$$

We claim now that $F''_{N''}$ can be chosen so that the finitely generated extension

$$F'_{N'} \subset (F'_{N'} \otimes_{k[X_1,\dots,X_d]} E)/\langle P(X_{d+1})\rangle$$

is, in addition, a finite extension.

Recall that $E$ is a finitely generated and smooth $k$-algebra, say $k[\alpha_1, \dots, \alpha_r]$. There is an inclusion of local rings

$$k[X_1, \dots, X_d]_{\langle X_1,\dots,X_d\rangle} \subset E_M/\langle f\rangle, \tag{36.3}$$

which induces, by change of the base ring, say $k[X_1, \dots, X_d]_{\langle X_1,\dots,X_d\rangle} \to \tilde{k}\{\{X_1, \dots, X_d\}\}$, the finite extension

$$\tilde{k}\{\{X_1, \dots, X_d\}\} \subset \tilde{k}\{\{X_1, \dots, X_d\}\}[X_{d+1}]/\langle (X_{d+1})^n + a_1(X_{d+1})^{n-1} + \dots + a_n\rangle$$

where the latter ring is the strict henselization of $E_M/\langle f\rangle$ (36.2).

Let $\overline{\alpha}_i$ denote that class of $\alpha_i$ in $E_M/\langle f\rangle$, for $i = 1, \dots, r$. As each $\overline{\alpha}_i$ can be identified with the image in the strict henselization of this local ring, each $\overline{\alpha}_i$ fulfills a monic polynomial relation with coefficients in $\tilde{k}\{\{X_1, \dots, X_d\}\}$. Each such monic polynomial involves only finitely many coefficients, and therefore one can choose $F'_{N'}$ so that each $\overline{\alpha}_i$ is an integral element over this ring. Therefore $F'_{N'}$ can be chosen so that the extension

$$F'_{N'} \subset (F'_{N'} \otimes_{k[X_1,\dots,X_d]} E)/\langle P(X_{d+1})\rangle$$

is finite. The previous discussion shows that the finitely generated $F'_{N'}$-algebra at the right is a local ring, which we view here as a subring in the strict henseization of $E_M/\langle f\rangle$, where

$$(F'_{N'} \otimes_{k[X_1,\dots,X_d]} E)/\langle P(X_{d+1})\rangle = F'_{N'}[\overline{\alpha}_1, \dots, \overline{\alpha}_r],$$

and each $\overline{\alpha_i}$, $i = 1, \dots, r$, is integral over $F'_{N'}$.

The proposition follows finally from this fact, where we now lift these constructions of morphisms to morphisms of finite type over a field by using general properties of localization of homomorphisms (see 32.4). This enables us to construct a square diagram of smooth schemes defined by a fiber product, say

(36.4)
$$\begin{array}{ccc} \operatorname{Spec}(E) & \longleftarrow & W' \\ \downarrow & & \downarrow \\ \mathbb{A}^n & \longleftarrow & V = \operatorname{Spec}(F') \end{array}$$

Moreover, the choice of coordinates $\{X_1, \dots, X_d, X_{d+1}\}$ in this construction, together with Theorem 3.1, show that the conditions (1), (2), and (3) hold.

$\square$

# Appendix B: An approach to Proposition 18.2

The purpose of this appendix is to give some hints about the proof of Proposition 18.2.

## 37. Describable strongly upper-semi continuous functions

DEFINITION 37.1. Let $(V^{(n)}, \mathcal{G}^{(n)})$ be a pair. A strongly upper-semi continuous function defined for $(V^{(n)}, \mathcal{G}^{(n)})$, $F_{\mathcal{G}^{(n)}}$, is said to be *describable* in $V^{(n)}$ if there is an $\mathcal{O}_{V^{(n)}}$-Rees algebra $\widetilde{\mathcal{G}}^{(n)}$ attached to $\max F_{\mathcal{G}^{(n)}}$ such that:

(1) There is an equality of closed sets:
$$\underline{\text{Max}}\ F_{\mathcal{G}^{(n)}} = \text{Sing}\ \widetilde{\mathcal{G}}^{(n)};$$
(2) Any $\max - F_{\mathcal{G}^{(n)}}$-local sequence
$$\begin{array}{cccccc} V_0^{(n)} = V^{(n)} & \leftarrow & V_1^{(n)} & \leftarrow & \dots & \leftarrow V_l^{(n)} \\ \mathcal{G}_0^{(n)} = \mathcal{G}^{(n)} & & \mathcal{G}_1^{(n)} & & \dots & \mathcal{G}_l^{(n)} \end{array}$$
with
$$\max F_{\mathcal{G}_0^{(n)}} = \max F_{\mathcal{G}_1^{(n)}} = \ldots = \max F_{\mathcal{G}_l^{(n)}}$$
induces a $\widetilde{\mathcal{G}}^{(n)}$-local sequence
$$\begin{array}{cccccc} V_0^{(n)} = V^{(n)} & \leftarrow & V_1^{(n)} & \leftarrow & \dots & \leftarrow V_l^{(n)} \\ \widetilde{\mathcal{G}}_0^{(n)} = \widetilde{\mathcal{G}}^{(n)} & & \widetilde{\mathcal{G}}_1^{(n)} & & \dots & \widetilde{\mathcal{G}}_l^{(n)} \end{array}$$
with
$$\underline{\text{Max}}\ F_{\mathcal{G}_i^{(n)}} = \text{Sing}\ \widetilde{\mathcal{G}}_i^{(n)}$$
for $i = 1, \ldots, l$;

(3) Any $\widetilde{\mathcal{G}}^{(n)}$-local sequence
$$\begin{array}{cccccc} V_0^{(n)} = V^{(n)} & \leftarrow & V_1^{(n)} & \leftarrow & \dots & \leftarrow V_l^{(n)} \\ \widetilde{\mathcal{G}}_0^{(n)} = \widetilde{\mathcal{G}}^{(n)} & & \widetilde{\mathcal{G}}_1^{(n)} & & \dots & \widetilde{\mathcal{G}}_l^{(n)} \end{array}$$
with
$$\text{Sing}\ \widetilde{\mathcal{G}}_i^{(n)} \neq \emptyset$$
for $i = 0, 1, \ldots, l$, induces a $\max - F_{\mathcal{G}^{(n)}}$-local sequence
$$\begin{array}{cccccc} V_0^{(n)} = V^{(n)} & \leftarrow & V_1^{(n)} & \leftarrow & \dots & \leftarrow V_l^{(n)} \\ \mathcal{G}_0^{(n)} = \mathcal{G}^{(n)} & & \mathcal{G}_1^{(n)} & & \dots & \mathcal{G}_l^{(n)} \end{array}$$
with
$$\max F_{\mathcal{G}_0^{(n)}} = \max F_{\mathcal{G}_1^{(n)}} = \ldots = \max F_{\mathcal{G}_l^{(n)}}$$
and
$$\underline{\text{Max}}\ F_{\mathcal{G}_i^{(n)}} = \text{Sing}\ \widetilde{\mathcal{G}}_i^{(n)}$$
for $i = 1, \ldots, l$.

(4) If a $\max - F_{\mathcal{G}^{(n)}}$-local sequence
$$\begin{array}{cccccc} V_0^{(n)} = V^{(n)} & \leftarrow & V_1^{(n)} & \leftarrow & \dots & \leftarrow V_l^{(n)} \\ \mathcal{G}_0^{(n)} = \mathcal{G}^{(n)} & & \mathcal{G}_1^{(n)} & & \dots & \mathcal{G}_l^{(n)} \end{array}$$
is so that
$$\max F_{\mathcal{G}_0^{(n)}} = \max F_{\mathcal{G}_1^{(n)}} = \ldots = \max F_{\mathcal{G}_{l-1}^{(n)}} > \max F_{\mathcal{G}_l^{(n)}}$$

then

$$\underline{\text{Max}}\ F_{\mathcal{G}_i^{(n)}} = \text{Sing}\ \widetilde{\mathcal{G}}_i^{(n)}$$

for $i = 1, \dots, l-1$ and Sing $\widetilde{\mathcal{G}}_l^{(n)} = \emptyset$;

(5) If a $\widetilde{\mathcal{G}}^{(n)}$-local sequence

$$\begin{array}{llll} V_0^{(n)} = V^{(n)} & \leftarrow \quad V_1^{(n)} & \leftarrow \quad \dots & \leftarrow V_l^{(n)} \\ \widetilde{\mathcal{G}}_0^{(n)} = \widetilde{\mathcal{G}}^{(n)} & \qquad \widetilde{\mathcal{G}}_1^{(n)} & \qquad \dots & \quad \widetilde{\mathcal{G}}_l^{(n)} \end{array}$$

is so that

$$\text{Sing}\ \widetilde{\mathcal{G}}_i^{(n)} \neq \emptyset$$

for $i = 1, \dots, l-1$ but Sing $\widetilde{\mathcal{G}}_l^{(n)} = \emptyset$, then the induced $\max - F_{\mathcal{G}^{(n)}}$-local sequence

$$\begin{array}{llll} V_0^{(n)} = V^{(n)} & \leftarrow \quad V_1^{(n)} & \leftarrow \quad \dots & \leftarrow V_m^{(n)} \\ \mathcal{G}_0^{(n)} = \mathcal{G}^{(n)} & \qquad \mathcal{G}_1^{(n)} & \qquad \dots & \quad \mathcal{G}_m^{(n)} \end{array}$$

is so that

$$\underline{\text{Max}}\ F_{\mathcal{G}_i^{(n)}} = \text{Sing}\ \widetilde{\mathcal{G}}_i^{(n)}$$

for $i = 1, \dots, l-1$, and

$$\max F_{\mathcal{G}_0^{(n)}} = \max F_{\mathcal{G}_1^{(n)}} = \dots = \max F_{\mathcal{G}_{l-1}^{(n)}} > \max F_{\mathcal{G}_l^{(n)}}.$$

In such case, we will also say that the pair $(V^{(n)}, \widetilde{\mathcal{G}}^{(n)})$ *describes* $F_{\mathcal{G}^{(n)}}$.

REMARK 37.2. Observe that the previous definition implies that a resolution of $\widetilde{\mathcal{G}}^{(n)}$ induces a $\max - F_{\mathcal{G}^{(n)}}$-local sequence which terminates with a lowering of the maximum value of $F_{\mathcal{G}^{(n)}}$.

EXAMPLE 37.3. Let $(V^{(n)}, \mathcal{G}^{(n)})$ be a pair and consider Hironaka's order function $\text{ord}_{\mathcal{G}^{(n)}}^{(n)}$ as in Example 25.1. If $\text{ord}_{\mathcal{G}^{(n)}}^{(n)}$ is constant and equal to 1 in a neighborhood $U$ of a point $\xi \in \text{Sing}\ \mathcal{G}^{(n)}$ (i.e., if $\tau_{\mathcal{G}^{(n)},\eta} \geq 1$ for all $\eta \in U$), then $\underline{\text{Max}}\ \text{ord}_{\mathcal{G}^{(n)}}^{(n)}$ is describable by $\mathcal{G}^{(n)}$ on $U$, and the integral closure of the differential saturation of $\mathcal{G}^{(n)}$ is a canonical representative within all $\mathcal{O}_{V^{(n)}}$-algebras that describe $\underline{\text{Max}}\ \text{ord}_{\mathcal{G}^{(n)}}^{(n)}$ on $U$.

EXAMPLE 37.4. More generally, let $\mathcal{G}^{(n)}$ be an $\mathcal{O}_{V^{(n)}}$-Rees algebra with $\tau_{\mathcal{G}^{(n)}} \geq e \geq 1$ in a neighborhood $U$ of some point $\xi \in \text{Sing}\ \mathcal{G}^{(n)}$. Then the upper-semi continuous functions $\text{ord}_{\mathcal{G}^{(n)}}^{(n-i)}$ of Example 25.2 are constantly equal to 1 on $U$ for $i = 0, 1, \dots, e-1$. Thus, in this case,

$$\underline{\text{Max}}\ \text{ord}_{\mathcal{G}^{(n)}}^{(d-i)} = \text{Sing}\ \mathcal{G}^{(n)} \cap U$$

and, moreover, the integral closure of the differential saturation of $\mathcal{G}^{(n)}$ is a canonical representative within all $\mathcal{O}_U$-Rees algebras describing $\underline{\text{Max}}\ \text{ord}^{(d-i)}$ on $U$ for $i = 0, 1, \dots, e-1$. Also, any other pair identifiable with $(U, \mathcal{G}^{(n)}|_U)$ describes $\text{ord}_{\mathcal{G}^{(d-i)}}^{(n)}$.

Now we face the question of the describability of the functions $\text{w-ord}_{\mathcal{G}^{(n)}}^{(d)}$ for a given pair $(V^{(n)}, \mathcal{G}^{(n)})$:

THEOREM 37.5. [**BrV1**, Theorems 12.7, 12.9] *Let $(V^{(n)},\mathcal{G}^{(n)})$ be a pair and let $\xi \in Sing\ \mathcal{G}^{(n)}$. Assume that $\tau_{\mathcal{G}^{(n)},\eta} \geq e$ for all $\eta \in Sing\ \mathcal{G}^{(n)}$ in a neighborhood $U$ of $\xi$. Then either $(V^{(n)},\mathcal{G}^{(n)})$ is e-trivial in $U$, or else, we can associate to max w-ord$^{(n-e)}_{\mathcal{G}^{(n-e)}}$ a pair $(V^{(n)},\widetilde{\mathcal{G}}^{(n)})$, such that*

$$\mathcal{G}^{(n)} \subset \widetilde{\mathcal{G}}^{(n)}; \quad \text{and} \quad \underline{Max}\ w\text{-}ord^{(n-e)}_{\mathcal{G}^{(n)}} = Sing\ \widetilde{\mathcal{G}}^{(n)}$$

*for all $\eta \in \underline{Max}$ w-ord$^{(n-e)}\mathcal{G}^{(n)}$ in $U$. Moreover:*

1. *The dimensional type of $(V^{(n)},\widetilde{\mathcal{G}}^{(n)})$ in $U$ is larger than that of $(V^{(n)},\mathcal{G}^{(n)})$ (i.e., $\tau_{\widetilde{\mathcal{G}}^{(n)},\eta} \geq e+1$ for all $\eta \in U$);*
2. *If $e = 0$, then $(V^{(n)},\widetilde{\mathcal{G}}^{(n)})$ describes the strongly upper-semi continuous function w-ord$^{(n)}_{\mathcal{G}^{(n)}}$ in $U$.*
3. *If $e \geq 1$ and if the characteristic of the base field is zero, then $(V^{(n)},\widetilde{\mathcal{G}}^{(n)})$ describes the strongly upper-semi continuous function w-ord$^{(n-e)}_{\mathcal{G}^{(n)}}$ in $U$.*

*Moreover, the pair $(V^{(n)},\widetilde{\mathcal{G}}^{(n)})$ is unique (up to weak equivalence).*

REMARK 37.6. Although the proof of this Theorem is detailed in [**BrV1**], we give here a brief outline of the idea of the argument. Suppose $\mathcal{G}^{(n)} = \oplus_l I^l W^l$. The hypotheses is that $\tau_{\mathcal{G}^{(n)},\eta} = e$ for all $\eta \in$ Sing $\mathcal{G}^{(n)}$ in a neighborhood $U$ of $\xi$. If $e = 0$ then set $\omega =$ max ord$^{(n)}_{\mathcal{G}^{(n)}}$ on $U$ and define $\widetilde{\mathcal{G}}^{(n)}$ as the smallest Rees algebra containing both $\mathcal{G}^{(n)}$ and the differential saturation of the *twisted algebra*

$$\mathcal{G}^{(n)}(\omega) := \oplus_{l\geq 0} I_{\frac{l}{\omega}} W^l$$

where it is assumed that $I_{\frac{l}{\omega}} = 0$ if $\frac{l}{\omega}$ is not an integer (see [**Vi5**]). This can be shown to be unique (up to integral closure) among all the Rees algebras verifying the conclusions of the theorem. If $e \geq 1$, then choose a $\mathcal{G}^{(n)}$-admissible projection to some $(n-e)$-dimensional smooth scheme:

$$\beta_{n,n-e} : V^{(n)} \longrightarrow V^{(n-e)}$$

and consider the corresponding elimination algebra $\mathcal{G}^{(n-e)}$. Then set

$$\omega = \text{max ord}^{(n-e)}_{\mathcal{G}^{(n-e)}}$$

on $\beta_{n,n-e}(U)$, and consider the differential saturation $\mathcal{H}^{(n-e)}$ of the twisted algebra $\mathcal{G}^{(n-e)}(\omega)$ in the same manner as above. Then

$$\widetilde{\mathcal{G}}^{(n)} := \mathcal{G}^{(n)} \odot \beta^*_{n,n-e}(\mathcal{H}^{(n-e)})$$

is shown to verify the conclusion of the theorem.

**37.7. Proposition 18.2 asserts that the strongly upper-semi continuous functions $t_{\mathcal{G}^{(n)}}$ are describable.** In this paragraph we give only a hint on why Proposition 18.2 holds. Let $(V^{(n)},\mathcal{G}^{(n)},E^{(n)})$ be an $n$-dimensional basic object. Consider a sequence of permissible transformations where each $V^{(n)}_i \leftarrow V^{(n)}_{i+1}$ is defined with center $Y_i \subset \underline{\text{Max}}$ w-ord$^{(n)}_i$,

$$(37.1) \quad (V^{(n)},\mathcal{G}^{(n)},E^{(n)}) \xleftarrow{\rho_0} (V^{(n)}_1,\mathcal{G}^{(n)}_1,E^{(n)}_1) \xleftarrow{\rho_1} \cdots \xleftarrow{\rho_{m-1}} (V^{(n)}_m,\mathcal{G}^{(n)}_m,E^{(n)}_m),$$

Then,

$$(37.2) \qquad \max \text{w-ord}^{(n)} \geq \max \text{w-ord}^{(n)}_1 \geq \cdots \geq \max \text{w-ord}^{(n)}_m.$$

Recall that the function $t_m^{(n)}$ is defined only under the assumption that max w-ord$_m > 0$. Set $l \leq m$ such that

(37.3)
$$\max \text{w-ord}^{(n)} \geq \ldots \geq \max \text{w-ord}_{l-1}^{(n)} > \max \text{w-ord}_l^{(n)} = \max \text{w-ord}_{l+1}^{(n)} \cdots = \max \text{w-ord}_m^{(n)},$$

and write:

$$E_m^{(n)} = E_m^+ \sqcup E_m^- \text{ (disjoint union)}, \tag{37.4}$$

where $E_m^-$ are the strict transforms of hypersurfaces in $E_l$. Then, recall that

$$\begin{array}{rccl} t_m^{(n)}: & \text{Sing}\,(J_m, b) & \longrightarrow & \mathbb{Q} \times \mathbb{N} \\ & \xi & \longmapsto & t_m^{(n)}(\xi) = (\text{w-ord}_m^{(n)}(\xi), \sharp\{H_i \in E_m^- : \xi \in H_i\}) \end{array} \tag{37.5}$$

where $\mathbb{Q} \times \mathbb{N}$ is a set ordered lexicographically, and $\sharp S$ denotes the total number of elements of a set $S$.

Now, suppose that $(V_l^{(n)}, \mathcal{G}_l^{(n)}, E_l^{(n)})$ is an arbitrary non necessarily simple basic object, i.e., of codimensional type $\geq 0$. Set $\widetilde{\mathcal{G}}_l^{(n)}$ as in Theorem 37.5, and define for $\xi \in \underline{\text{Max}}\ t_l^{(n)}$,

$$n(\xi) := \max\{r : \xi \in H_{i_1} \cap \ldots \cap H_{i_r}, \text{ with } H_i \in E_l^-\}$$

and define the Rees algebra

$$\widehat{\mathcal{G}}^{(n)} := \widetilde{\mathcal{G}}_l^{(n)} \odot \mathcal{O}_{V_l^{(n)}}[\mathcal{H}W]$$

with

$$\mathcal{H} = \prod_{H_i \in E_l^-} \left( \sum_{i_1 < \ldots < i_r} \mathcal{I}(H_i) \right).$$

Then the basic object

$$(V_l^{(n)}, \widehat{\mathcal{G}}^{(n)}, E_l^+)$$

describes the strongly upper-semi continuous function $t_{\mathcal{G}_l^{(n)}}^{(n)}$ and has codimensional type $\geq 1$. See [**EV1**, §9.5].

**37.8. Other strongly upper-semi continuous functions on pairs.** With the same notation as in Theorem 37.5, observe that new strongly upper-semi continuous functions can be defined on the closed set $\underline{\text{Max}}$ w-ord$^{(n-e)}\mathcal{G}^{(n)}$. Namely, if $(V^{(n)}, \widetilde{\mathcal{G}}^{(n)})$ is representable in dimensions $\ldots, (n+1), n, \ldots, n-e-1, \ldots, l$, then for $i = e+1, \ldots, l$, one can define:

$$\begin{array}{rccl} \text{w-ord}_{\mathcal{G}^{(n)}}^{(n-i)}: & \underline{\text{Max}}\text{-w-ord}_{\mathcal{G}^{(n)}}^{(n-e)} & \longrightarrow & \mathbb{Q} \\ & \eta & \mapsto & \text{w-ord}_{\widetilde{\mathcal{G}}^{(n-i)}}(\beta_{n,n-i}(\eta)). \end{array} \tag{37.6}$$

In the same fashion, $t_{\mathcal{G}^{(n)}}^{(n-i)}$-functions can be defined on $\underline{\text{Max}}$-w-ord$_{\mathcal{G}^{(n)}}^{(n-e)}$. These are strongly upper-semi continuous functions.

Now observe that if $(V^{(n)}, \mathcal{G}^{(n)})$ and $(V^{(d)}, \mathcal{G}^{(d)})$ are identifiable, then the closed sets $\underline{\text{Max}}$ w-ord$^{(n-e)}\mathcal{G}^{(n)}$ and $\underline{\text{Max}}$ w-ord$^{(n-e)}\mathcal{G}^{(d)}$ can be identified via the homeomorphisms from Lemma 21.10. Thus functions w-ord$_{\mathcal{G}^{(d)}}^{(n-i)}$ can be defined similarly on $\underline{\text{Max}}$ w-ord$_{\mathcal{G}^{(d)}}^{(n-e)}$, and it can be checked that w-ord$_{\mathcal{G}^{(n)}}^{(n-i)}$ and w-ord$_{\mathcal{G}^{(d)}}^{(n-i)}$ are invariants in the sense of Definition 22.1.

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Clay Mathematics Proceedings
Volume **20**, 2014

# A Simplified Game for Resolution of Singularities

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Abstract. We will describe a combinatorial game that models the problem of resolution of singularities of algebraic varieties over a field of characteristic zero. By giving a winning strategy for this game, we give another proof of the existence of resolution.

## Introduction

The proof of existence and construction of resolution of singularities of algebraic varieties in characteristic zero can be divided into two parts. First, there is an algebraic part, providing necessary constructions such as blowups, differential closure, transforms along blowup, descent in dimension, transversality conditions, and properties of these constructions. Second, there is a combinatorial part that consists in the setup of a tricky form of double induction taking various side conditions into account. The combinatorial part can be formulated as a game. The two parts of the proof can be cleanly separated: once the properties of the algebraic constructions are clear, it is no more necessary to do any algebra in the induction proof. In [**7**], the algebraic parts and the combinatoric parts of the proof are in separate sections that are logically independent of each other. The formulation is based on Villamayor's constructive proof [**15, 5, 16**], using ideas from other proofs [**4, 3, 18, 13, 2**]. It is needless to say that there are many more proofs that indirectly influenced our formulation. We just mention [**8, 1**]. For a more exhaustive account of other proofs, see [**7**] and the references cited therein.

In this paper, we give a simplified version of the game described in [**7**]. In contrast to the description there, the combinatorial part is not entirely independent of the algebraic part. In the game there, there are two players, one who tries to resolve and one who provides combinatorial data for the singularities according to a set of rules (and who is destined to lose). In this paper, the second player is replaced by an algebraic oracle that has complete information on the singularity, so that the rules are not needed. Also, the combinatorial data has been reduced: the stratification of the singular locus is not used any more.

The game described here has been introduced at the Clay Summer School in Obergurgl, 2012 on Resolution of Singularities. For me, this event was a unique experience full of intensive interactions with many highly qualified young researchers. Several simplifications are owed to the participants, for instance the precise notion

2010 *Mathematics Subject Classification.* Primary 14B05, 14E15; Secondary 12D10.

of the equivalence relation. The simplified game has actually been played at the summer school; see section 2 for a description of this play.

As a consequence of the winning strategy for the second player, we get another formulation of the resolution algorithm. A new aspect of this formulation is that it does not depend on local choices. All substeps in the algorithm can be done globally or locally, whichever is more useful. There is a single substep which requires an intermediate passing to a local cover, but the result of this substep is again global (see Remark 1.42).

Most proofs in Section 1 have been done as exercises in the Clay Summer School; we give them here (mostly through references) just for the sake of completeness. Apart from these, the existence proof of resolution in this paper is self-contained, and we hope that it serves as a gentle introduction/explanation of this classical result.

This version of the paper has been read by four reviewers, and I would like to thank them for their truly formidable reviews. They contained in total 116 suggestions, some of them independently by several reviewers, 9 additional references to the literature, on 17 pages in total. There was not a single comment which was not clear. I tried to follow most of their suggestions; many remarks and examples, for instance Example 1.29 of a singularity for which there is no global descent, are only here because of their persistence.

## 1. Habitats, Singularities, and Gallimaufries

In this section, we introduce the algebraic concepts which are needed for our setup of the resolution problem and algorithm: habitats, singularities, transform along blowups, differential closure, gallimaufries and descents. The terminology used in this paper is the same as in [**7**].

Let $K$ be a field of characteristic zero.

Definition 1.1. A *habitat* over $K$ is an equidimensional nonsingular algebraic variety $W$ over $K$ together with a finite sequence of nonsingular hypersurfaces $(E_1, \ldots, E_r)$ such that no two have a common component, and such that their sum is a normal crossing divisor. We denote this habitat by $(W, (E_1, \ldots, E_r))$, and when the hypersurface sequence is not important, we denote the habitat by $(W, *)$.

Examples of Habitats are the affine spaces $\mathbb{A}^n$, $n \geq 0$, with divisors defined by coordinates $x_i$, $1 \leq i \leq n$. In the analytic category, every habitat is locally isomorphic to such a habitat. It is also possible that some of the hypersurfaces are empty. This is a necessity because we consider habitats as local/global objects where we would like to restrict to open sets, or glue together habitats on an open cover when the restrictions to the intersections coincide. If we restrict to an open subset of the complement of $E_i$, $1 \leq i \leq r$, then the restricted habitat has an empty hypersurface at the $i$-th place.

Definition 1.2. A subvariety $Z$ of a habitat as above is called *straight* iff it is pure-dimensional, and for every point $p \in Z$, there is a regular system of parameters $u$ such that $Z$ is locally defined by a subset of $u$ and every hypersurface of the habitat sequence that contains $p$ is defined by an element of $u$. This concept also arises in [**7**], where it is called *transversal*, and in [**16**], where it is called *normal crossing*.

For instance, in the habitat $(\mathbb{A}^n, (x_1, \dots, x_n))$, the variety defined by $(x_1, \dots, x_m)$, $m \leq n$, is straight. If the habitat sequence is empty, then straightness is equivalent to smoothness.

Definition 1.3. Let $Z$ be a straight subvariety of a habitat $(W, (E_1, \dots, E_n))$. The *blowup* along $Z$ is the habitat $(W', (E_1', \dots, E_n', E_{n+1}))$, where $W'$ is the blowup of $W$ along $Z$, $E_i'$ is the strict transform of $E_i$ for $i = 1, \dots, n$, and $E_{n+1}$ is the exceptional divisor introduced by the blowup (the inverse image of the center $Z$).

We allow the following degenerate cases: if $Z = E_1$, then the blowup is $(W, (\emptyset, E_2, \dots, E_n, E_1))$. If $Z = W$, then the blowup is $(\emptyset, (\emptyset, \dots, \emptyset))$.

Definition 1.4. For any habitat $(W, *)$, we define an operator $\Delta$ from the set of ideal sheaves on $W$ to itself, as follows. For $I \subset \mathcal{O}_W$ and affine open subset $U \subset W$, $\Delta(I)|_U$ is the ideal sheaf generated by $I|_U$ and all first order partial derivatives of elements in $I|_U$.

The $i$-th iteration of the operator $\Delta$ is denoted by $\Delta^i$.

Definition 1.5. A *singularity* on a habitat $(W, *)$ is a finitely generated sheaf of Rees algebras $A = \oplus_{i=0}^{\infty} A_i$ over $A_0 = \mathcal{O}_W$, i.e. a sequence of ideal sheafs $A_i \subset \mathcal{O}_W$ such that $A_0 = \mathcal{O}_W$ and $A_i \cdot A_j \subseteq A_{i+j}$ and equality holds for sufficiently large indices $i, j$ (this is equivalent to finite generation).

We say a singularity $A$ is of *ideal-type* if there is an integer $b > 0$ and ideal sheaf $I$ such that $A_{nb} = I^n$ for all indices which are multiples of $b$, and $A_i = (0)$ otherwise. These singularities are denoted by $(I, b)$. (This is Hironaka's notion of pairs.)

The *singular locus* $\mathrm{Sing}(A)$ of a singularity $A = \oplus_{i=0}^{\infty} A_i$ is the intersection of the zero sets of $\Delta^{i-1}(A_i), i > 0$. We say that a singularity is *resolved* if its singular locus is empty.

Remark 1.6. The above concept of singularity is based on Hironaka's definition [**9**] of idealistic exponents (ideal-type singularities). I learned the description of singularities in terms by Rees algebras from [**6**], which is based on [**17**]. Similar description by algebras or filtration of rings have been used systematically in [**10, 11, 12**]. Note that our definition of Rees algebras slightly differs from the definition in [**14**].

Note that the intersection defining the singular locus is a finite intersection by Nötherianity. For computing the singular locus, it suffices to consider generating degrees.

The singular locus of an ideal-type singularity of the form $(I, 1)$ is just the zero set of $I$. The singular locus of an ideal-type singularity of the form $(I, b)$ with $b > 1$ is the zero set of points where the order of $I$ is at least $b$.

The trivial singularities are the zero singularity $A_i = 0$ for $i > 0$ and the unit singularity $A_i = \mathcal{O}_W$ for all $i \geq 0$. The singular locus of the zero singularity is $W$, and the singular locus of the unit singularity is the empty set (and so, the unit singularity is resolved).

Definition 1.7. Let $Z$ be a straight subvariety in the singular set of a singularity $A$. The *transform* of $A$ is the singularity $A' = \oplus_{i=0}^{\infty} A_i'$ on the blowup $(W, (\hat{A}, *, E_{n+1}))$, where $A_i'$ is such that $f^*(A_i) = A_i \mathcal{O}_{W'} = \mathrm{Ideal}(E_{n+1})^i \cdot A_i'$ for $i > 0$. Recall that $E_{n+1} = f^{-1}(Z)$ is the exceptional divisor.

Example 1.8. We consider the ideal-type singularity $(\langle x^2-y^3\rangle, 2)$ in the habitat $(\mathbb{A}^2, ())$. The singular locus of $A$ is the only point where $x^2-y^3$ has order 2, namely $(0,0)$. The blowup of $\mathbb{A}^2$ can be covered by two open affine charts, with coordinates $(x, \tilde{y}=\frac{y}{x})$ and $(\tilde{x}=\frac{x}{y}, y)$, respectively. In the first chart, the transform is the ideal-type singularity $(\langle 1-x\tilde{y}^3\rangle, 2)$; in the second chart, it is the ideal-type singularity $(\langle \tilde{x}^2-y\rangle, 2)$. Note that in both charts the singularity is resolved.

Definition 1.9. A *thread* is a sequence of singularity-habitat pairs, where the next is the transform of the previous under blowup of a straight subvariety in the singular locus. If the last singularity has empty singular locus, then we say the thread is a *resolution* of the first singularity of the thread.

For instance, the transform in the example above has empty singular locus. Therefore the thread consisting of the single blowup above is a resolution of the singularity $(\langle x^2-y^3\rangle, 2)$.

The objective in this paper is to show that every singularity admits a resolution. Desingularization of algebraic varieties over characteristic zero is then a consequence.

Theorem 1.10. *Assume that every singularity over $K$ has a resolution. Then every irreducible variety $X$ over $K$ that can be embedded in a nonsingular ambient space has a desingularization, i.e. a proper birational map from a nonsingular variety to $X$.*

Proof. Let $X \subset W$ be a variety embedded in a nonsingular ambient space $W$. If $X$ is a hypersurface, then we simply resolve the singularity $(\mathrm{Ideal}(X), 2)$. The proper transform of $X$ is then a subscheme of the transform of $(\mathrm{Ideal}(X), 2)$. Since the transform of $(\mathrm{Ideal}(X), 2)$ has no points of order 1, also the proper transform has no points of order 1, and therefore it is a nonsingular hypersurface.

In higher codimension, there exist singular varieties with an ideal of order 1, namely varieties that are embedded in some smooth hypersurface; so it is not enough to resolve the $(\mathrm{Ideal}(X), 2)$. Instead, we resolve the singularity $(\mathrm{Ideal}(X), 1)$ and take only the part of the resolution where the proper transform of $X$ is not yet blown up. In the next step, when the proper transform is blown up, it must be a nonsingular subvariety. So the thread defines a sequence of blowing ups such that the proper transform is nonsingular, and this is a desingularization. □

Remark 1.11. The resolutions obtained in this way are *embedded resolutions*: the singular variety $X$ is embedded in a nonsingular ambient space $W$, and one constructs a proper birational morphism $\pi : \tilde{W} \to W$ such that the proper transform of $X$ is nonsingular. Moreover, the morphism $\pi$ is an isomorphism at the points outside $X$ and at the smooth points of $X$. In the hypersurface case, the singularity is already resolved in a neighborhood of these points. In the general case, the singularity is resolved at the points outside $X$, and it can be resolved by a single blowing up step locally in a neighborhood of a smooth point of $X$. But this is precisely the step where the resolution of the singularity is truncated.

Definition 1.12. Let $A_1$ and $A_2$ be two singularities on the same habitat. Then their sum $A_1 + A_2$ is defined as the singularity defined by the Rees algebra generated by $A_1$ and $A_2$.

REMARK 1.13. If $A_1 = (I_1, b)$ and $A_2 = (I_2, b)$ are ideal-type singularities with the same generating degree, then $A_1 + A_2 = (I_1 + I_2, b)$. For ideal-type singularities with different generating degrees, there is no such easy construction.

LEMMA 1.14. *The singular locus of* $A_1 + A_2$ *is equal to* $\mathrm{Sing}(A_1) \cap \mathrm{Sing}(A_2)$. *If* $Z$ *is a straight subvariety contained in this intersection, then* $\mathrm{Transform}_Z(A_1 + A_2) = \mathrm{Transform}_Z(A_1) + \mathrm{Transform}_Z(A_2)$.

PROOF. Straightforward. □

As a consequence, resolution of $A+B$ separates the singular loci. More precisely, the resolution of $A + B$ defines threads starting with $A$ and $B$, and the singular sets of the final singularities of these threads have disjoint singular loci.

DEFINITION 1.15. A singularity $A = \oplus_{i=0}^{\infty} A_i$ is *differentially closed* iff $\Delta(A_{i+1}) \subseteq A_i$ for all $i \geq 0$.

The *differential closure* of a singularity $A$ is the smallest differentially closed singularity containing $A$.

A priori it is not clear if the differential closure exists, or in other words if the intersection of all differentially closed finitely generated Rees algebras containg $A$ is again finitely generated. Assume that $A$ has generators $f_i$ in degree $d_i$, $i = 1, \ldots, N$. Then one can use the Leibniz rule to show that the differential closure is generated by all partial derivatives of order $j < d_i$ in degree $d_i - j$.

REMARK 1.16. The notion of differential closure is closely related to differential Rees algebras used in [**17**] and with differential saturation of an idealistic filtration defined in [**11**]. These two cases are different but both use higher order differential operators. Here we only use first order differential operators; this would not work for positive characteristic.

DEFINITION 1.17. Two singularities $A, B$ on the same habitat are *equivalent* iff there exists $N > 0$ such that $\mathrm{Closure}(A)_{kN} = \mathrm{Closure}(B)_{kN}$ for all $k \in \mathbb{Z}_+$.

LEMMA 1.18. *If two singularities* $A$ *and* $B$ *are equivalent, then their singular loci coincide.*

*Assume that* $A$ *and* $B$ *are equivalent, and let* $Z$ *be a straight subvariety in the singular locus. Then the transforms of* $A$ *and* $B$ *on the blowup at* $Z$ *are again equivalent.*

PROOF. The first statement is straightforward. For the second statement, we use [**7**, Lemma 9]: If $C$ is the differential closure of $A$, and $A'$ and $C'$ are the transforms of $A$ and $C$ along a center inside the singular locus, then the differential closures of $A'$ and $C'$ are equal. (In general, the transform of a differentially closed singularity may not be differentially closed.) □

DEFINITION 1.19. A number $b > 0$ is a *generating degree* of a singularity $A$ iff $A$ is equivalent to the ideal-type singularity $(A_b, b)$.

If $A$ is generated by elements of $A_b$ as an algebra over $\mathcal{O}_W$, then it is an easy exercise that $b$ is a generating degree for $A$.

DEFINITION 1.20. A *subhabitat* of a habitat $(W, (E_1, \ldots, E_n))$ is a straight subvariety $V \subset W$ which does not have components that are contained in one of the $E_i$, together with the sequence of the intersections $(V \cap E_1, \ldots, V \cap E_n)$.

If $Z$ is a straight subvariety of a subhabitat $(V,*)$ of $(W,*)$, then it is also a straight subvariety of $(W,*)$. The blowup of $(V,*)$ at $Z$ is a subhabitat of the blowup of $(W,*)$ at $Z$; its underlying variety $V'$ is the proper transform of $V$.

Example 1.21. Let $0 \le l \le m \le n$. On the habitat $(\mathbb{A}^n,())$, we have the subhabitat $(V,())$ where $V$ is the hypersurface defined by $x_{m+1} = \cdots = x_n = 0$ (say that $x_1,\dots,x_n$ are the coordinate variables). Note that $V$ is isomorphic to $\mathbb{A}^m$. Let $Z$ be the subvariety defined by $x_{l+1} = \cdots = x_n = 0$. Then the blowup is covered by $n-l$ charts with coordinate functions $x_1,\dots,x_l,x_k,\frac{x_{l+1}}{x_k},\dots,\frac{x_n}{x_k}$, where $k = l+1,\dots,n$. The proper transform of $V$ has a non-empty intersection with the $m-l$ charts corresponding to $k = l+1,\dots,m$. It is isomorphic to to the blowup of $\mathbb{A}^m$ at the subvariety defined by the last $m-l$ coordinates.

Definition 1.22. Let $i : V \to W$ be the inclusion map of a subhabitat $(V,*)$ of $(W,*)$. The *restriction* of a singularity $B = \oplus_{i=0}^{\infty} B_i$ on $(W,*)$ to $(V,*)$ is defined as the singularity $A = \oplus_{i=0}^{\infty} A_i$ where $A_i := i^*(B_i)\mathcal{O}_V$ and $i^*(B_i)$ denotes the pullback of $B_i$ along the inclusion map.

If $(\mathrm{Ideal}(V),1)$ is a subalgebra of $B$, then we say that $B$ *restricts properly* to $V$.

Example 1.23. Let $A$ be the ideal-type singularity $(\langle x, y^2 - z^3\rangle, 1)$ on the habitat $(\mathbb{A}^3,())$. Then the hyperplane $x = 0$ is a subhabitat to which $A$ restricts properly.

Remark 1.24. If $V$ has codimension 1, then the statement "*$B$ restricts properly to $V$*" is equivalent to the statement "*$V$ is a hypersurface of maximal contact*" in [**16**].

If $B$ restricts properly to $V$, then the singular locus of $B$ is contained in $V$ and is equal to the singular locus of the restriction of $B$ to $V$. The proof is straightforward.

Assume that $B$ restricts properly to $V$, and let $A$ be the restriction. Let $Z \subset \mathrm{Sing}(B) \subset V$ be a straight subvariety. Then the transform of $A$ on the blowup $(V',*)$ is equal to the restriction of the transform of $B$ to the restriction to the subhabitat $(V',*)$.

Definition 1.25. Let $i : V \to W$ be the inclusion map of a subhabitat $(V,*)$ of $(W,*)$. Let $A$ be a differentially closed singularity on $(V,*)$. Then the *extension* of $A$ to $(W,*)$ is defined as the largest differentially closed algebra which is contained in $\oplus_{i=0}^{\infty}(i^*)^{-1}(A_i)$.

Example 1.26. On the habitat $(\mathbb{A}^2,())$, we consider the singularity $(\langle x,y^2\rangle,1)+(\langle y^3\rangle,2)$ (this is the differential closure of $(\langle x^2+y^3\rangle,2)$). Its restriction to the subhabitat defined by $x$ is equal to $(\langle y^2\rangle,1)+(\langle y^3\rangle,2)$. The inverse of the pullback is $(\langle x,y^2\rangle,1)+(\langle x,y^3\rangle,2)$. It is not differentially closed, because $\partial_x(x) = 1$ is not contained in the degree 1 component. When we remove $x$ from the list of generators in degree 2, we get $(\langle x,y^2\rangle,1)+(\langle y^3\rangle,2)$, and this is the extension.

Lemma 1.27. *Let $i : V \to W$ be the inclusion map of a subhabitat $(V,*)$ of $(W,*)$. Let $B$ be a differentially closed singularity on $(W,*)$ which restricts properly to $V$. Then the extension of the restriction of $B$ to $V$ is equal to $B$.*

*Let $A$ be a differentially closed singularity on $(V,*)$. Then the extension of $A$ restricts properly to $V$, and its restriction is equal to $A$.*

PROOF. This is [**7**, Theorem 11]. It compares to the "commutativity" statement in [**4**]. □

DEFINITION 1.28. Let $(W,*)$ be a habitat, and let $m \le \dim(W)$ be a non-negative integer. A *gallimaufry* of dimension $m$ on $(W,*)$ is a differentially closed singularity $A$, such that for every point $p$ in the singular locus, there is an open subset $U \subset W$ and a subhabitat $(V,*)$ of the open restriction $(U,*)$ of dimension $m$, such that $A|_U$ restricts properly to $V$. Such an open subhabitat is called *zoom* for $A$ at $p$.

Let $A$ be a gallimaufry of dimension $m > 0$. Assume that there exists an open cover of the habitat of $A$ such that for every open subset $U$, there is a subhabitat of dimension $m-1$ to which $A|_U$ properly restricts; in other words, $A$ can also be considered as a gallimaufry of dimension $m-1$. Then we say that the gallimaufry $A$ descends to dimension $m-1$.

Any singularity can be considered as a gallimaufry of dimension $\dim(W)$. Any gallimaufry of dimension $m < \dim(W)$ can also be considered as a gallimaufry of dimension $m+1$. The dimension of the singular locus of a gallimaufry is less than or equal to the dimension of the gallimaufry. The unit singularity on $(W,*)$ can be considered as a gallimaufry of any dimension $m \le \dim(W)$.

EXAMPLE 1.29. Here is an example that shows that the passage to local covers is really necessary. Let $C \subset \mathbb{A}^3$ be an affine smooth space curve which is a complete intersection with ideal generated by $F, G \in K[x,y,z]$. Let $f : C \to \mathbb{P}^1$ be a regular map that cannot be extended to $\mathbb{A}^3$. For instance, we could set $K = \mathbb{R}$ and $C$ to be the circle $x^2+y^2-1 = z = 0$ and $f$ as the map $(x,y,z) \to (y : 1-x) = (1+x : y)$. Let $I$ be the ideal of all functions $g$ vanishing along $C$ such that for all $p \in C$, the gradient of $p$ is a multiple of the gradient of $f_1(p)F + f_2(p)G$, where $f(p) = (f_1(p) : f_2(p))$. In the concrete case of the unit circle and $f$ as above, the ideal is generated by $(x^2+y^2)(1+x)+yz, (x^2+y^2-1)y+z(1-x), (x^2+y^2-1)z, z^2$.

The ideal-type singularity $(I,1)$ is locally analytically isomorphic to $(\langle x,y\rangle, 1)$, bacause locally analytically we can assume $C$ is the line $x = y = 0$ and the gradient of elements in the ideal are multiples of the gradient of $x$. Still locally analytically, the hyperplane defined by $x$ is a subhabitat with proper restriction. In the concrete example of the unit circle, one can cover $\mathbb{A}^3$ by the three open subsets: $U_1$ is defined by $x+1 \ne 0$ and removing the point $(x,y,z) = \left(\frac{1}{3},0,0\right)$, $U_2$ is defined by $x-1 \ne 0$, and $U_3$ is the complement of the unit circle. In $U_1$, the restriction to the subhabitat defined by $(x^2+y^2-1)(1+x)+yz$ is proper (the only singular point of the habitat has been removed from $U_1$); in $U_2$, the restriction to the subhabitat $(x^2+y^2-1)y+z(1-x)$ is proper; in $U_3$, the singularity is resolved, so the restriction to any subhabitat is proper. Therefore we can consider $(I,1)$ as a gallimaufry in dimension 2.

On the other hand, we can show that $(I,1)$ does not globally restrict properly to a subhabitat of dimension 2. Assume, indirectly, that $H$ is the equation of such a surface. Then $H$ lies in the ideal of $C$, hence we can write $H = AF+BG$ for some $A, B \in K[x,y,z]$. Then $(x,y,z) \mapsto (A(x,y,z) : B(x,y,z))$ would be an extension of $f : C \to \mathbb{P}^1$, contradicting our assumption that such an extension does not exist.

LEMMA 1.30. *Let $A$ be a gallimaufry of dimension $m > 0$. If $A$ descends to $m-1$, then any transform of $A$ also descends to $m-1$.*

PROOF. If $V$ is a subhabitat of dimension $m-1$ to which $A$ properly restricts, then the transform restricts properly to the proper transform of $V$. □

DEFINITION 1.31. A gallimaufry $A$ of dimension $m$ is *bold* if $\dim(\mathrm{Sing}(A)) = m$.

EXAMPLE 1.32. A gallimaufry $A$ of maximal dimension $n = \dim(W)$ is bold if and only if there is an irreducible component of $W$ on which $A$ is the zero singularity.

LEMMA 1.33. *Let $A$ be a bold singularity. Then the $m$-dimensional locus $Z$ of* $\mathrm{Sing}(A)$ *is straight, and the transform of $A$ on the blowup along $Z$ is not bold.*

PROOF. Let $V$ be a subhabitat of dimension $m$ to which $A$ properly restricts. Then $\mathrm{Sing}(A)$ is the equal to the union of all irreducible components $V_0$ of $V$ such that the restriction to $V_0$ is the zero singularity. Since $V$ is straight, it follows that $Z$ is straight.

Let $\pi : W' \to W$ be the blowup along $Z$. Locally at some neighbourhood of a point $p$ in an irreducible component $V_0$ of $Z$, the blowup manifold of the subhabitat $V$ is empty. Hence the blowup of $V$ along $Z$ just removes all components in $Z$. It follows that the transform of the restricted singularity is not bold. Hence the transform of the $A$ as a gallimaufry of dimension $m$ is not bold. □

The next definition introduces a numerical invariant of the order of the singularity. It is based on Hironaka's order function. Other authors used iterated order functions to construct an invariant governing the resolution process. Here, the resolution process should not be determined by an invariant, but we still need some order concept.

DEFINITION 1.34. Let $(I, b)$ be an ideal-type singularity on $(W, (E_1, \ldots, E_r))$ which is not bold as a gallimaufry in dimension $\dim(W)$, i.e. $I$ is not the zero ideal on any component of $W$. Let $\{i_1, \ldots, i_k\}$ be the set of all hypersurface indices $i$ such that $E_i \cap \mathrm{Sing}(A) \neq \emptyset$. The *monomial factor* of $(I, b)$ is defined as the sequence $\left(\frac{e_1}{b}, \ldots, \frac{e_k}{b}\right)$ such that $I \subseteq \mathrm{Ideal}(E_{i_1})^{e_1} \cdots \mathrm{Ideal}(E_{i_k})^{e_k}$, with integers $e_1, \ldots, e_k$ chosen as large as possible.

The *maxorder* of $(I, b)$ is defined as $\frac{\min\{a \mid \Delta^a(\tilde{I}) = \langle 1 \rangle\}}{b}$, where $\tilde{I}$ is the ideal sheaf $\mathrm{Ideal}(E_1)^{-e_1} \cdots \mathrm{Ideal}(E_r)^{-e_r} I$. Note that this the maximum of the function $p \mapsto \mathrm{ord}_p(\tilde{I})/b$.

For an arbitrary singularity that is not bold, monomial factor and maxorder are defined by passing to an equivalent ideal-type singularity.

For a gallimaufry that is not bold, monomial factor and maxorder are defined by restricting to a subhabitat of correct dimension.

In order to show that the definitions are valid for arbitrary singularities, we use that any singularity is equivalent to an ideal-type singularity, by the comment after Definition 1.19. Moreover, one needs to show that two equivalent ideal-type singularities have the same monomial factors and order; in our setup, this is a straightforward consequence of the statement that if the ideal-type singularities $(I_1, b_1)$ and $(I_2, b_2)$ are equivalent, then there exist positive integers $n_1, n_2$ such that $b_1 n_1 = b_2 n_2$ and $I_1^{n_1} = I_2^{n_2}$. One may compare with [**9**], where te independence of the choice of ideal-type representative is shown for a similar equivalence relation. The validity of the definitions for gallimaufries (independence of the choice of the local subhabitat) is a consequence of [**7**, Proposition 13]. The proof, which uses local isomorphisms of restrictions to different subhabitats, is far from being trivial.

The idea to use local isomorphisms to compare orders on coefficient ideals defined on different hypersurfaces of maximal contact has been introduced in [**18**].

EXAMPLE 1.35. We consider the singularity $(\langle x^2-y^3\rangle,1)$ on the habitat $(\mathbb{A}^2,())$. We blowup the origin $(0,0)$, which is contained in the singular locus. The blowup variety can be covered by two charts, which already occured in Example 1.8. In the first chart, the transform is $(\langle(1-x\tilde{y}^3)x\rangle,1)$; in the second chart, it is the ideal-type singularity $(\langle(\tilde{x}^2-y)y\rangle,1)$. For the transform, the monomial factor is $(1)$; and the ideal sheaf $\tilde{I}$ is $\langle 1-x\tilde{y}^3\rangle$ in the first chart and $\langle\tilde{x}^2-y\rangle$ in the second chart, hence the maxorder is 1.

LEMMA 1.36. *Let $A$ be a non-bold gallimaufry of dimension $m$ on a habitat $(W,(E_1,\dots,E_r))$. Assume, for simplicity, that all hypersurfaces $E_i$ have a non-empty intersection with the singular locus. Let $(a_1,\dots,a_r)$ be the monomial factor of $A$, and let $o$ be the maxorder.*

*Let $\{i_1,\dots,i_k\}$ be a subset of $\{1,\dots,r\}$. If $a_{i_1}+\cdots+a_{i_k}+o<1$, then $E_{i_1}\cap\cdots\cap E_{i_k}\cap\operatorname{Sing}(A)=\emptyset$. If $a_{i_1}+\cdots+a_{i_k}\geq 1$, then $E_{i_1}\cap\cdots\cap E_{i_k}\subset\operatorname{Sing}(A)$.*

*Let $Z\subset\operatorname{Sing}(A)$ be a straight subvariety of $(W,(E_1,\dots,E_r))$, and let $\{i_1,\dots,i_k\}$ be the subset of $\{1,\dots,r\}$ of all $i$ such that $Z\subset E_i$. Then the monomial factor of the transform on the blowup at $Z$ is of the form $(a_1,\dots,a_r,a_{r+1})$, where*

$$a_{i_1}+\cdots+a_{i_k}-1\leq a_{r+1}\leq a_{i_1}+\cdots+a_{i_k}+o-1.$$

PROOF. Straightforward (using local analytic coordinates where the center and all hypersurfaces are defined by coordinate functions). □

EXAMPLE 1.37. On the habitat $(\mathbb{A}^3,(x,y,z))$, we consider the singularity $(\langle x^3y^5z^7(x^2+y^5)\rangle,1)$. Its monomial factor is $(3,5,7)$, and its maxorder is 2. When we blowup the line $x=y=0$, we get two charts. In the chart with coordinate functions $(\tilde{x}=\frac{x}{y},y,z)$, the transform is $(\langle\tilde{x}^3y^9z^7(\tilde{x}^2+y^3)\rangle,1)$, and in the chart with coordinate functions $(x,\tilde{y}=\frac{y}{x},z)$, the transform is $(\langle x^9\tilde{y}^5z^7(1+x^3\tilde{y}^5)\rangle,1)$. The monomial factor of the transform is $(3,5,7,9)$.

In this example, the maxorder of the transform is again equal to 2, and the transform of the ideal-type singularity $(\langle x^2+y^5\rangle,2)$ is $(\langle\tilde{x}^2+y^3\rangle,2)$ in the first chart and $(\langle 1+x^3\tilde{y}^5\rangle,2)$ in the second chart (see also Lemma 1.43 below).

DEFINITION 1.38. A gallimaufry is called *monomial* if its maxorder is 0.

LEMMA 1.39. *The transform of a monomial gallimaufry is monomial. If $(a_1,\dots,a_r)$ is the monomial factor, and if $\{i_1,\dots,i_k\}$ is the subset of hypersurface indices such that $Z\subset E_i$, then $(a_1,\dots,a_r,a_{i_1}+\cdots+a_{i_k}-1)$ is the monomial factor of the transform on the blowup at $Z$.*

PROOF. This is a consequence of Lemma 1.36. □

DEFINITION 1.40. A non-bold and non-resolved singularity/gallimaufry is *tight* if it has trivial monomial factor $(0,\dots,0)$ and maxorder 1.

Let $(I,b)$ be an ideal-type singularity such that $I$ is not zero on any component of the habitat. Let $\left(\frac{e_1}{b},\dots,\frac{e_r}{b}\right)$ be its monomial factor, and let $o$ be the maxorder. Assume $o>0$. The *tightification* of a $(I,b)$ is defined as the differential closure of $(\tilde{I},ob)+(I,b)$, where $\tilde{I}:=\operatorname{Ideal}(E_1)^{-e_1}\cdots\operatorname{Ideal}(E_r)^{-e_r}I$ (note that $ob$ is an integer).

The tightification of a general singularity is defined by passing to an equivalent ideal type singularity followed by tightification as defined above.

The tightification of a gallimaufry is defined by restriction to a zoom, singularity tightification, and extension. The tightification of a gallimaufry $A$ is denoted by $\mathrm{Tightify}(A)$. If $A$ is bold or resolved, then $\mathrm{Tightify}(A)$ is not defined.

EXAMPLE 1.41. Let $m, n$ be positive integers. On the habitat $(\mathbb{A}^1, ())$, the ideal-type singularity $(\langle x^m \rangle, n)$ has maxorder $\frac{m}{n}$, so it is tight if and only if $m = n$. If $m < n$, then the singularity is resolved. If $m \geq n$, then the tightification is equal to $(\langle x^m \rangle, m)$.

It is not apparent that the tightification is well-defined for gallimaufries, one has to show independence of the local choice of the subhabitat of dimension $m$. We refer to [**7**, Proof of Theorem 19] for the proof. This proof uses local analytic isomorphisms between restrictions to different habitats (see [**18**]).

REMARK 1.42. For given gallimaufry $A$, it would also be possible to compute monomial factor and maxorder using Jacobian ideals. Hence the computation of the tightification is the only construction of the resolution algorithm which uses subhabitats and therefore local choices (the result is independent of the local choices by the statement before). If we had a different construction without local choices we would have a global algorithm for resolution of singularities that never passes to local coverings. The author does not have an idea for such a construction.

LEMMA 1.43. *The transform of a tight gallimaufry is either tight or resolved.*

*Let $A$ be a non-bold gallimaufry. Let $a = (a_1, \dots, a_r)$ be its monomial factor. Assume that $A$ has maxorder $o > 0$. Let $Z$ be a straight subvariety in the singular locus of* $\mathrm{Tightify}(A)$*. Let $\{i_1, \dots, i_k\}$ be the subset of hypersurface indices such that $Z \subset E_i$. Then the monomial factor of* $\mathrm{Transform}_Z(A)$ *is $a' := (a_1, \dots, a_r, a_{i_1} + \dots + a_{i_k} + o - 1)$, and if $o'$ is the maxorder, then $o' \leq o$. Equality holds if and only if* $\mathrm{Transform}_Z(\mathrm{Tightify}(A))$ *is not resolved; and in this case,* $\mathrm{Transform}_Z(\mathrm{Tightify}(A))$ *is equivalent to* $\mathrm{Tightify}(\mathrm{Transform}_Z(A))$.

PROOF. Locally at any open subset in which the maximal order is assumed, we can restrict to a subhabitat of dimension $m$, and it suffices to show the statement for singularities. In this situation, the proof is straightforward (compare with Example 1.37).

In any other open subset, we get a monomial factor $a''$ with last exponent $, a_{i_1} + \dots + a_{i_k} + o_1 - 1)$ with some $o_1 < o$. In the transform of this open subset, the order is bounded by $o_1$. □

EXAMPLE 1.44. Let $A$ be the singularity $(\langle (x^2 - y^n) y^m \rangle, 1)$ on the habitat $(\mathbb{A}^2, x)$, where $n \geq 2, m \geq 0$. Its monomial factor is $(m)$, its maxorder is 2, and its tightification is $(\langle x^2 - y^n \rangle, 2)$. Let $Z$ be the point $(0,0)$. The blowup can be covered by two charts, which already occured in Example 1.8. In the first, the transform of $A$ is $(\langle (\tilde{x}^2 - y^{n-2}) y^{m+1} \rangle, 1)$. In the second, the transform is $(\langle (1 - x^{n-2}\tilde{y}^2) x^{m+1} \tilde{y}^m \rangle, 1)$. The monomial factor $(m, m+1)$. The transform of $\mathrm{Tightify}(A)$ is $(\langle \tilde{x}^2 - y^{n-2} \rangle, 2)$ in the first chart and $(\langle 1 - x^{n-2}\tilde{y}^2 \rangle, 2)$ in the second chart. If $n \geq 4$, then the transform of the tightification is the tightification of the transform, and the maxorder of $\mathrm{Transform}_Z(A)$ is 2. If $n = 2, 3$, then the transform of the tightification is resolved, and the maxorder is 1. The new tightification is $(\langle \tilde{x}^2 - y^{n-2} \rangle, 1)$ in the first chart.

LEMMA 1.45. *Let $A$ be a tight gallimaufry of dimension $m > 0$ on a habitat $(W, (E_1, \dots, E_r))$. Assume that $\mathrm{Sing}(A) \cap E_i = \emptyset$ for $i = 1, \dots, r$. Then $A$ descends to dimension $m - 1$.*

PROOF. It suffices to show the statement for singularities. Moreover, we may assume that $A$ is differentially closed. Then $\Delta(A_1) = \mathcal{O}_W$. For any $p \in \mathrm{Sing}(A)$, there exists $f \in (A_1)_p$ of order 1 at $p$. This local section is defined and still in some open neighbourhood $U$ of $p$. Moreover, the zero set $X$ of $f$ is a hypersurface in $p$ is a nonsingular point of $X$. We define $U'$ as $U$ minus the singular locus of $X$. Then $A|_{U'}$ restricts properly to the hypersurface defined by $f$. □

## 2. The Game and How to Win It

In this section, we explain the combinatoric part of the resolution.

In any step of the game, the player is given some combinatorial information on a main gallimaufry which is to be resolved, as well as additional gallimaufries which are related in various ways, for instance if a gallimaufry is not bold and has positive maxorder then there might be a tightification. During the game, threads are created; a blowup step adds one gallimaufry to each existing thread. In the beginning of the game, there is only one thread of length zero with a single gallimaufry $A$. In the end, this thread should be extended to a resolution of $A$.

This is the combinatorial information on a gallimaufry $A$ on a habitat $(W, (E_1, \dots, E_r))$ which is given to the player:

- a simplicial complex $\Xi$ with vertices in the set $\{1, \dots, r\}$, consisting of all faces $\{i_1, \dots, i_k\}$ such that $E_{i_1} \cap \dots E_{i_k} \cap \mathrm{Sing}(A) \neq \emptyset$;
- the gallimaufry dimension $m$;
- a generating degree of $A$;
- the information whether $A$ is bold or not;
- if $A$ is not bold, then the monomial factor $a : \mathrm{Vertices}(\Xi) \to \mathbb{Q}_{\geq 0}$. This is just a labelling og the vertices by rational numbers;
- the maxorder $o \in \frac{1}{b}\mathbb{Z}$, where $b$ is the generating degree provided as specified above.

Note that we have to distinguish the *empty set complex* that has no vertices and consists of one -1-complex whose set of vertices is the empty set, and the *empty complex* that has no faces at all. A gallimaufry is resolved if and only if its complex is the empty complex.

Now the player has to choose a move. There are six possible moves, two blowup moves and four moves that create additional gallimaufries. We may distinguish two types of blowups.

**Type I:** For a gallimaufry $A$ on a habitat $(W, (E_1, \dots, E_r))$ and monomial factor $(a_1, \dots, a_r)$ and indices $i_1, \dots, i_k$ such that $a_{i_1} + \cdots + a_{i_k} \geq 1$, the intersection of $E_1, \dots, E_r$ and some locally defined zoom is a straight subvariety contained in the singular locus. It is independent of the choice of the zoom because it can also be defined as the intersection of $E_1, \dots, E_r$ and the singular locus. A type I blowup is a blowup of such a subvariety. In the winning strategy we describe here, type I blowups are only needed when $A$ is monomial. However, one has to keep in mind that the blowup not only transforms $A$ but also other gallimaufries that are given at the same time in the game.

**Type II:** For a bold gallimaufry $A$ of gallimaufry dimension $m$, the union of all $m$-dimensional components of the singular locus is a straight subvariety, by Lemma 1.33. A type II blowup is a blowup at such a subvariety. By Lemma 1.33, the transform of $A$ is not bold, but again, the blowup also transforms other gallimaufries that are given at this step.

Here is an overview on the possible moves of the player at each turn.

(1) If a gallimaufry complex has a face with sum of labels greater than or equal to 1, then she may issue a blowup of type I on that face.
(2) If a gallimaufry is bold, then she may issue a blowup of type II on that gallimaufry.
(3) If a gallimaufry is tight and its complex $\Xi$ is the empty set complex, then she may issue a descent.
(4) If a gallimaufry is not bold and has maxorder $o > 0$, then she may issue a tightification.
(5) For some gallimaufry with complex $\Xi$ and vertex $j \in \text{Vertices}(\Xi)$, she may issue a relaxation. This will create a gallimaufry with a smaller sequence of hypersurfaces, as defined below.
(6) For some gallimaufry with complex $\Xi$ and vertex $j \in \text{Vertices}(\Xi)$, she may issue an intersection.

A blowup move includes a unique specification of the blowup center: for type I, the center is the intersection of the singular locus with all hypersurfaces in the habitat sequence corresponding to the vertices of the face occuring in the description of the move; for type II, it is the $m$-dimensional locus of the singular locus, where $m$ is the gallimaufry dimension. The consequences of a blowup move are that the habitat is blown up at the indicated center $Z$ and all gallimaufries with $Z \subset \text{Sing}(A)$ are transformed, so that their threads are prolongued. The remaining threads are removed, their threads are differentially closed. The combinatorial data of the transformed gallimaufries are partially determined by the properties of gallimaufries in the previous sections. The dimension and generating degree are never changed. Also, the relation between two gallimaufries in two threads (descent, quotient, intersection) are kept. The remaining data (for instance maxorder of non-tight gallimaufries) are again given to the player.

In the remaining moves, a new gallimaufry is created and a thread is opened starting with it. In a descent move, the new gallimaufry is given by the same Rees algebra on the same habitat, but it is considered as a gallimaufry in dimension one less; this has an effect on the monomial factor, on the maxorder, and on the boldness property. In a relaxation move with gallimaufry $A$ and vertex $j$, the new gallimaufry is defined by the same Rees algebra $A$, but the habitat is changed: $E_j$ is replaced by $\emptyset$. In an intersection move with gallimaufry $A$ and vertex $j$, the new gallimaufry is $A + (\text{Ideal}(E_j), 1)$.

Remark 2.1. It is possible that an intersection move follows a relaxation move for the same vertex. In this situation, we do not want to form the sum with $(\text{Ideal}(\emptyset), 1)$ because this is the unit singularity. So by convention, the added summand in the intersection move is always computed from the habitat in the main thread.

Lemma 2.2. *There is a winning strategy for monomial gallimaufries.*

PROOF. The winning strategy is to blowup a minimal face among all faces with sum of labels greater than or equal to 1. Then the complex of the transformed gallimaufry is a subdivision, where the label sum of any of the new faces is strictly smaller than the label sum of the old face which contains that new face topologically and which has disappeared in the subdivision. This is only possible a finite number of times because the labels are in $\frac{1}{b}\mathbb{Z}_{\geq 0}$, where $b$ is the generating degree. □

LEMMA 2.3. *Let $m \geq 0$ be an integer. If there is a winning strategy for tight gallimaufries of dimension $m$, then there is a winning strategy for all gallimaufries of dimension $m$.*

PROOF. By a type II blowup, we may reduce to a non-bold gallimaufry $A$. Then we have a maxorder $o \in \frac{1}{b}\mathbb{Z}_{\geq 0}$, where $b$ is the generating degree. If $o > 0$, then we create the tightification Tightify($A$). By assumption, there is a resolution of Tightify($A$). The sequence of blowups defines a thread starting with $A$. By Lemma 1.43, the last singularity $A'$ of this thread has maxorder $o' < o$. If $o' > 0$, then we start a new thread starting with Tightify($A'$) and apply the winning strategy to the tight gallimaufry. The resolution of the Tightify($A'$) induces a prolongation of the thread of $A$ passing $A'$ and ending with a singularity $A''$ of maxorder $o'' < o'$. Since the maxorder can only drop finitely many times, we eventually achieve the monomial case $o = 0$, which can be won by Lemma 2.2. □

LEMMA 2.4. *Let $m > 0$ be an integer. If there is a winning strategy for gallimaufries of dimension $m-1$, then there is a winning strategy for tight gallimaufries of dimension $m$.*

PROOF. Let $A$ be a gallimaufry of dimension $m$, and let $\Xi$ be its simplicial complex. If $\Xi$ is the empty set complex, then $A$ is tight, so it descends to dimension $m-1$. Then we can construct a resolution by assumption (the resolution for the descent is also a resolution for $A$ itself).

In general, we create a relaxation gallimaufry $B$ on $(W, ())$. Since $B$ is tight and its complex is the empty set complex, $B$ descends to dimension $m-1$. In the following, we construct a "careful" resolution of $B$. The extra care is necessary to avoid blowing up centers that are not straight for the habitat of $A$.

Let $f$ be a maximal face of $\Xi$. We form the intersection gallimaufry $B_f$ with respect to all vertices of $f$ and construct a resolution for $B_f$. The blowup centers are contained in the intersection of the hypersurfaces corresponding to $f$, therefore they are also straight for the habitat of the singularity in the thread of $A$. Then we set $A'$ and $B'$ to be the last singularities of the threads of $A$ and $B$, and $\Xi'$ to be the complex obtained by removing the face $f$ from $\Xi$. Again, the vertices of $\Xi'$ are the indices of hypersurfaces that have been relaxed in the thread of $B$, and the faces correspond to sets of hypersurfaces which have a non-empty intersection within the singular locus in the singularity in the thread of $A$.

If $\Xi'$ is not the empty complex, then we choose a maximal face $f'$ of $\Xi'$ and repeat. In each step, the number of faces of $\Xi$ drops. After finitely many steps, we get the empty complex, and $A$ is resolved. □

REMARK 2.5. There are no tight gallimaufries of dimension $m = 0$. Actually, a gallimaufry of dimension 0 is either bold or resolved, hence it can be resolved in at most one step.

THEOREM 2.6. *Every gallimaufry has a resolution.*

PROOF. This is now an obvious consequence of the three lemmas and the remark above. □

As a consequence of Theorem 2.6 and Theorem 1.10, every irreducible variety over a field of characteristic zero has a resolution.

EXAMPLE 2.7. Let $A$ be the differential closure of $(\langle xy(x-y)\rangle, 2)$ on the habitat $(\mathbb{A}^2, ())$. In order to resolve the singularity it, one would start the game by giving the singularity to Mephisto. He would then give the following information to Dido: *In thread* $T_0$ *(the main thread), we have dimension* 2 *and generating degree* 2. *Currently, its complex is the empty set complex, the gallimaufry is not bold, the monomial factor is obvious, and the maxorder is 1.*

Since $T_0$ is tight, Dido will now descend $T_0$, creating a thread $T_1$ of dimension 1 (using Lemma 2.4 for winning the game). Mephisto would then tell Dido that the maxorder of $T_1$ is again 1.

Since $T_1$ is tight, Dido will descend $T_1$, creating a thread $T_2$ of dimension 0. Mephisto will then tell Dido that $T_2$ is bold.

Now, Dido will issue a type II blowup on $T_2$. The blowup at the 0-dimensional part of the singular locus (which in this case coincides with the whole singular locus) will resolve all threads. The game is won and the singularity in thread $T_0$ is resolved.

The game described in this section was played in the Clay Summer School in Obergurgl. The participants formed two teams, called Mephisto and Dido. Each team was working on a blackboard that was not readable by the other team. The actual player was Dido, and it was Mephisto's task to provide the combinatorial information by computation. The singularity was not revealed to Dido, but the team did guess it.

(1) Mephisto was given the habitat $(\mathbb{A}^2, ())$ and the singularity $(\langle x^2 - y^3\rangle, 1)$. After a short computation, Mephisto gave Dido a piece of paper with the following information: *In thread* $T_0$ *(the main thread), you have dimension* 2 *and generating degree* 1. *Currently, its complex is the empty set complex, the gallimaufry is not bold, the monomial factor is obvious, and the maxorder is 2.*
(2) Dido decided to tightify $T_0$, creating the thread $T_1$. The dimension of $T_1$ is 2, the monomial factor is trivial and the maxorder is 1; this is already clear.
(3) Mephisto computed the tightification of $T_0$: it is $(\langle x^2 - y^3\rangle, 2)$. The information given to Dido was: $T_1$ *has generating degree* 2. *Its complex is the empty set complex* (this could have been deduced by Dido, because the complex of the tightification is always a non-empty subcomplex).
(4) Dido decided to descend $T_1$, creating the thread $T_2$. Its dimension is 1. The generating degree and the complex is inherited from $T_1$.
(5) To compute the maxorder, Mephisto restricted $T_2$ to the subvariety defined by $x$. The restriction is $(\langle y^3\rangle, 2)$. Mephisto told Dido: $T_2$ *is not bold, has trivial monomial factor, and maxorder* $\frac{3}{2}$.
(6) Dido decided to tightify $T_2$, creating the thread $T_3$.
(7) Mephisto computed the tightification $(\langle y\rangle, 1)$ and told Dido: *the generating degree of* $T_3$ *is 1, and the complex is the empty set complex.*
(8) Dido decided to descend $T_3$, creating the thread $T_4$ of dimension 0.

(9) Mephisto told Dido: $T_4$ *is bold.*
(10) Dido demanded a type II blowup on $T_4$.
(11) Mephisto computed the blowup and transforms. In the first chart (which is the interesting one), the habitat is $(\mathbb{A}^2, (y))$, and the main singularity in the thread $T_0$ is $(\langle(\tilde{x}^2 - y)y\rangle, 1)$. The other singularities are resolved. Mephisto told Dido: *The complex in* $T_0$ *consist of the 0-face* $\{\mathtt{1}\}$ *and the empty face. The threads* $T_1, T_2, T_3, T_4$ *are resolved. The monomial factor of* $T_0$ *is* $(1)$*, and the maxorder is 1.*
(12) Dido decided to tightify $T_0$, creating a thread $T_5$.
(13) Mephisto computed the tightification $(\langle\tilde{x}^2 - y\rangle, 1)$ and told Dido: *The complex of* $T_5$ *is currently the full complex of* $T_0$*. The generating degree is 2000. This is correct, it is not required to give the minimal generating degree to the player.*
(14) Dido decided to intersect $T_5$ with vertex `1`, creating thread $T_6$. For an intersection move, the combinatorial data can be inferred, so Mephisto does not need to provide information.
(15) Dido decided to relax vertex `1` from $T_6$, creating $T_7$. Also here, no additional information from Mephisto is needed.
(16) Dido decided to descend $T_7$, creating the thread $T_8$ of dimension 1.
(17) The restricted singularity is $(\langle\tilde{x}^2\rangle, 1)$. Mephisto told Dido: *Currently, the monomial factor of* $T_8$ *is trivial and the maxorder is 2.*
(18) Dido decided to tightify $T_8$, creating a thread $T_9$.
(19) For $T_8$, Mephisto gave the generating degree 2000. The complex of $T_8$ is currently the empty set complex (because `1` was relaxed).
(20) Dido decided to tightify $T_9$, creating the thread $T_{10}$ of dimension 0.
(21) Mephisto announced that $T_{10}$ is bold.
(22) Dido demanded a type II blowup at $T_{10}$.

The game went on for some time, but after 90 minutes the game was interrupted without a victory of Dido. No doubt Dido would have won when given more time, because the members of the team already had a clear strategy.

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Clay Mathematics Proceedings
Volume **20**, 2014

# Resolution of singularities in Characteristic p and monomialization

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*Dedicated to the memory of Professor Shreeram S. Abhyankar*

Abstract. In this article we survey the problems of resolution of singularities in positive characteristic and of local and global monomialization of algebraic mappings. We discuss the differences in resolution of singularities from characteristic zero and some of the difficulties. We outline Hironaka's proof of resolution for positive characteristic surfaces, and mention some recent results and open problems.

Monomialization is the process of transforming an algebraic mapping into a mapping that is essentially given by a monomial mapping by performing sequences of blow ups of nonsingular subvarieties above the target and domain. We discuss what is known about this problem and give some open problems.

## 1. Resolution of singularities in characteristic $p$

We will restrict to the case of positive characteristic $p > 0$, referring to other lectures for a discussion of the extensive results on resolution in equicharacteristic zero. We will restrict to discussion of classical results in the subject, and again, we refer to other lectures for discussion of the exciting programs and methods currently being pursued towards the general proof in all dimensions and positive characteristic.

We restrict our consideration to irreducible, reduced varieties over an algebraically closed field (this case is already hard enough).

Resolution of singularities is true quite generally in dimension 1. There are several proofs which are not difficult.

Resolution of singularities is true in dimension 2. This is a very difficult problem. The first proof was given by Abhyankar [**2**]. The proof appeared in 1956. Some other particularly important proofs in print are by Hironaka [**19**], Lipman [**20**] and by Cossart, Jannsen and Saito [**4**].

Resolution of singularities is true in dimension 3. This was first proven by Abhyankar over fields of characteristic $\geq 7$. The proof is given in the book [**3**] which appeared in 1966, and in a series of earlier papers by Abhyankar, which are refered to in the book. A greatly simplified proof is given in my article [**15**]. A proof valid in all characteristics was recently given by Cossart and Piltant [**5**] and [**6**].

2010 *Mathematics Subject Classification.* 14B05, 14E22.
Partially supported by NSF.

Suppose that $X$ is a subvariety of a nonsingular variety $W$. An *embedded resolution* of $X$ is a sequence of blowups of nonsingular subvarieties $W' \to W$ such that the strict transform $X'$ of $X$ in (the nonsingular variety) $W'$ is nonsingular.

In dimension 2 embedded resolution of singularities is known. The first proofs are the proof of Abhyankar in [**3**] and earlier papers, and the proof of Hironaka in [**19**]. A very general theorem on embedded resolution of surfaces has recently been given by Cossart, Jannsen and Saito in [**4**]. Embedded resolution is not known in dimension 3.

Suppose that $K/k$ is an algebraic function field. *local uniformization* holds on $K$ if whenever $\nu$ is a valuation of $K$, there exists a regular, algebraic local ring $R$ of $K$ such that the valuation ring $V_\nu$ dominates $R$. The words used in this definition will be explained in detail in the beginning of the third lecture.

If $X$ is a projective variety with function field $K$ (a model of $K$) and $\nu$ is a valuation of $K$, then there is a unique point $p \in X$ (which is not necessarily closed) such that the valuation ring $V_\nu$ dominates the local ring $\mathcal{O}_{X,p}$. The point $p$ is called the center of $\nu$ on $X$. Further, if $\phi : X' \to X$ is a birational morphism, and $p'$ is the center of $\nu$ on $X'$, then $\phi(p') = p$. Thus resolution along a valuation involves only looking at local issues in resolution, and is thus a little easier. It is also possible to make special use of properties of a particular valuation.

The introduction of valuation theory into algebraic geometry, and the above observations are due to Zariski. Zariski made ingenious use of these techniques in his beautiful papers [**21**] and [**22**], where he proves resolution of singularities of surfaces and 3-folds in characteristic zero.

in [**3**], Abhyankar makes use of a classical method using projections, by Albanese, to prove the following statement:

*Embedded resolution in dimension $n$ implies local uniformization in dimension $n+1$ and characteristic $> (n+1)!$.*

Abhyankar uses an extension of a classical method using projections of Albanese, to construct a model of an $n+1$ dimensional algebraic function field which has only singularities of multiplicity $\leq (n+1)!$. Then the assumption that the characteristic is smaller than the multiplicity is used to prove local uniformization. Abhyankar uses a generalization of Jung's method to prove this result. Embedded resolution is used to make the branch divisor of a generic projection onto a nonsingular $n+1$ dimensional germ a simple normal crossings divisor. The assumption on the multiplicity implies that the degree of the projection is less than the characteristic, so the ramification is tame.

This is enough to prove resolution of singularities in dimension three, by using extensions of methods that Zariski used in his characteristic zero proof [**22**]. The proof involves a patching argument which has not been extended to higher dimensions.

More discussion of this proof can be found in my paper [**15**].

Using standard bases of ideals, Hironaka gives a general method of reducing embedded resolution of singularites to embedded resolution for a hypersurface (one defining equation locally). This method is developed in [**18**]. This reduction does not depend on the characteristic.

A significant method in most proofs of resolution in characteristic zero is the existence of hypersurfaces of maximal contact. This allows the reduction (locally) of resolution to a resolution problem in one dimension less. This method was discussed in the lectures on resolution of singularities in characteristic zero. Hypersurfaces of maximal contact do not exist for varieties of dimension $\geq 2$ in positive characteristic. Hauser gave some examples of this failure in his lectures.

If a hypersurface singularity (of any dimension) has multiplicity less than the characteristic $p$ of the ground field, then a hypersurface of maximal contact does exist. This can be found, for instance, by applying a Tschirnhausen transformation. The first cases which really involve essential problems to resolve are hypersurfaces of degree $p$. The two essential cases are the inseparable degree $p$ case,

$$f = x_n^p + g(x_1, \ldots, x_{n-1}) = 0 \tag{1.1}$$

with $g^{\frac{1}{p}} \notin k[[x_1, \ldots, x_{n-1}]]$, and the Artin-Schreier case,

$$f = x_n^p + g^{p-1}x_n + h \tag{1.2}$$

with $0 \neq g, h \in k[[x_1, \ldots, x_n]]$ and $h^{\frac{1}{p}} \notin k[[x_1, \ldots, x_{n-1}]]$. Using ramification theory, Abhyankar reduces the proof of local uniformization in [**2**] for algebraic surfaces to the case of an Artin-Schreier extension, and then gives a direct proof for both the inseparable degree $p$ and Artin-Schreier cases. Cossart and Piltant extend this method to dimension three in [**5**], to reduce local uniformization in dimension three to the inseparable degree $p$ and Artin-Schreier cases. In [**6**], they give an extremely long proof of embedded local uniformization of inseparable degree $p$ and Artin-Schreier equations of dimension three.

## 2. Resolution of surface singularities in characteristic $p$

We give an exposition of a proof of Hironaka. Hironaka gives a sketch of the proof in "Desingularization of Excellent Surfaces" [**19**]. Our exposition is given in more detail in our book "Resolution of Singularities" [**14**] and in our paper on resolution of embedded surfaces and of 3-folds [**15**]. Another exposition from a different perspective is given by Hauser in [**17**]. A general proof of embedded resolution of excellent surfaces, starting from Hironaka's algorithm, has been given recently by Cossart, Jannsen and Saito [**4**].

Let $S$ be an irreducible surface over an algebraically closed field $K$ of characteristic $p > 0$. Suppose that $S$ is embedded in an nonsingular 3-fold $V$. Let

$$r = \max\{\nu_p(S) \mid p \in S\},$$

$$\mathrm{Sing}_r(S) = \{q \in S \mid \nu_q(S) = r\}.$$

Suppose that $f = 0$ is a local equation of $S$ at $q$. Let $x, y, z$ be regular parameters in $\mathcal{O}_{V,q}$.

$$\hat{\mathcal{O}}_{V,q} = K[[x, y, z]].$$

$$f = \sum_{i+j+k=r} a_{ijk}x^iy^jz^k$$

with $a_{ijk} \in K$. The leading form of $f$ is

$$L = \sum_{i+j+k=r} a_{ijk}x^iy^jz^k.$$

**2.1. The $\tau$ invariant.** For $q \in S$, $\tau(q)$ is the dimension of the smallest $K$-linear subspace $M$ of the $K$-span of $x, y, z$ in $K[[x, y, z]]$ such that $L \in K[M]$.

$N = V(M) \subset \mathrm{Spec}(\mathcal{O}_{V,q})$ is called an approximate manifold to $S$ at $q$.

**Recall that a hypersurface of maximal contact does not always exist in characteristic $p$.**

We have that

$$1 \leq \tau(q) \leq 3.$$

Example 2.1. Let

$$f = y^2 + 2xy + x^2 + z^2 + z^5$$

with $\mathrm{char}(K) > 2$. The leading form of $f$ is

$$L = (y + x)^2 + z^2,$$

and

$$\tau = 2.$$

$$x + y = z = 0$$

are local equations of an approximate manifold.

If $q \in C \subset \mathrm{Sing}_r(S)$ is a nonsingular curve, then there exist an approximate manifold $M$ at $q$ such that $C \subset M$. In this case we have $\tau(q) \leq 2$.

Lemma 2.2. *Suppose that $Y \subset \mathrm{Sing}_r(S)$ is a nonsingular subvariety (a point or a curve). Let $\pi_1 : V_1 \to V$ be the blow up of $Y$ and $S_1$ be the strict transform of $S$ on $V_1$. Suppose that $\alpha \in Y$ and let $M_\alpha$ be an approximate manifold to $S$ at $\alpha$ containing the germ of $Y$ at $\alpha$ and $\beta \in \pi_1^{-1}(\alpha)$. Then*

(1) $\nu_\beta(S_1) \leq r$.
(2) $\nu_\beta(S_1) = r$ *implies $\beta$ is on the strict transform $M'_\alpha$ of $M_\alpha$ and $\tau(\beta) \geq \tau(\alpha)$.*
(3) *Suppose that $\nu_\beta(S_1) = r$ and $\tau(\beta) = \tau(\alpha)$. Then there exists an approximate manifold $M_\beta$ to $S$ at $\beta$ such that $M_\beta \cap \pi^{-1}(\alpha) = M'_\alpha \cap \pi_1^{-1}(\alpha)$.*

Example 2.3.

$$f = z^p + x^{2p-1}y$$

is a local equation of a surface $S$. $z = 0$ is an approximate manifold $M_\alpha$ at the origin $\alpha$.

Let $\pi_1 : V_1 \to V$ be the blow ups of $\alpha$, and suppose that $\beta \in \pi_1^{-1}(\alpha) \cap \mathrm{Sing}_p(S_1)$. Since $z = 0$ is an approximate manifold, $\beta$ must have regular parameters $x_1, y_1, z_1$ of one of the following forms:

$$x = x_1, y = x_1(y_1 + c), z = x_1 z_1$$

with $c \in K$ or

$$x = x_1 y_1, y = y_1, z = y_1 z_1.$$

There is a point $\beta \in \pi_1^{-1}(\alpha) \cap \mathrm{Sing}_p(S_1)$, with regular parameters defined by

$$x = x_1, y = x_1(y_1 + 1), z = x_1 z_1.$$

$f_1 = 0$ is a local equation of the strict transform of $S$ at $\beta$, where

$$f_1 = z_1^p + x_1^p + x_1^p y_1.$$

$z_1 = 0$ is a local equation of the strict transform of $M_\alpha$ at $\beta$, which is not an approximate manifold. However, $z_1 + x_1 = 0$ is an approximate manifold at $\beta$.

The substitution $\overline{z}_1 = z_1 + x_1$ into

$$f_1 = z_1^p + x_1^p + x_1^p y_1$$

to obtain

$$f_1 = \overline{z}_1^p + x_1^p y_1$$

is called preparation (cleaning).

This suggests an algorithm to resolve an inseparable equation

$$z^p + g(x, y) = 0.$$

First clean making a substitution $z = z' + h(x, y)$ to remove $p$-th powers from $g$ ($z' = 0$ is then an approximate manifold). Then blow up to obtain a new inseparable equation

$$z_1^p + g_1(x_1, y_1).$$

Clean again (giving a new approximate manifold) and then blow up. Repeat this process.

An obvious candidate for an invariant is $\mathrm{ord}(g)$. A fundamental problem in resolution is to make $\mathrm{ord}(g)$ go down.

In the example we just looked at, $\mathrm{ord}(g_1) < \mathrm{ord}(g)$. However, $\mathrm{ord}(g)$ can go up, as is illustrated in the following example.

EXAMPLE 2.4. Let

$$f = z^2 + x^3 y + x y^3.$$

Here $\mathrm{char}(K) = 2$, $g = x^3 y + x y^3$. $\mathrm{ord}(g) = 4$. Blow up the origin, by making the substitution

$$x = x_1, y = x_1(y_1 + 1), z = x_1 z_1.$$

The strict transform of $f = 0$ has the local equation $f_1 = 0$ where

$$\begin{array}{rcl} f_1 & = & z_1^2 + x_1^2((y_1 + 1) + (y_1 + 1)^3) \\ & = & z_1^2 + x_1^2(y_1^2 + y_1^3). \end{array}$$

Clean, substituting $\overline{z}_1 = z_1 + x_1 y_1$, to obtain

$$f_1 = \overline{z}_1^2 + x_1^2 y_1^3.$$

Here $g_1 = x_1^2 + y_1^3$, and

$$\mathrm{ord}(g_1) = 5 > 4 = \mathrm{ord}(g).$$

Hauser discusses this kind of phenomenon in his lecture (Kangaroo points).

**2.2. The algorithm for resolution.** Using the theory of approximate manifolds, it is not so difficult to show that there exists a sequence of blow ups of points and nonsingular curves $V_1 \to V$ so that on the strict transform $S_1$ of $S$, $\mathrm{Sing}_r(S_1)$ is a finite set of points, so we can assume that $S$ satisfies this condition.

THEOREM 2.5 (Hironaka). *Suppose that $\mathrm{Sing}_r(S)$ is a finite set. Consider a sequence of blow ups*

$$\cdots \to V_n \to \cdots \to V_2 \to V_1 \to V \tag{2.1}$$

*where each $V_{i+1} \to V_i$ is the blow up of a curve in $\mathrm{Sing}_r(S_i)$ if such a curve exists, and the blow up of a point in $\mathrm{Sing}_r(S_i)$ otherwise. Then this sequence is finite (There exists $V_n$ such that $\mathrm{Sing}_r(S_n) = \emptyset$).*

This is the algorithm of Beppo Levi.

The proof of the Theorem easily reduces to the following statement ($\tau$-reduction theorem)

THEOREM 2.6. *Suppose that $q \in \mathrm{Sing}_r(S)$ is a point. Then there exists an $n$ such that all points $q_n \in \mathrm{Sing}_r(S_n)$ such that $q_n$ maps to $q$ satisfy $\tau(q_n) > \tau(q)$.*

When $\tau(q) = 3$, $q$ is isolated in $\mathrm{Sing}_r(S)$, and the blow up of $q$ leads to a reduction of order.

The case $\tau(q) = 2$ is not so difficult (it is similar to reduction for plane curve singularities). The essential case is when $\tau(q) = 1$. This is the hard case.

Assume that we have an infinite sequence (2.1)

$$\cdots \to V_n \to \cdots \to V_2 \to V_1 \to V$$

. and there exists an infinite sequence of points $q_n \in \mathrm{Sing}_r(S_n)$ such that $q_n$ maps to $q_{n-1}$ for all $n$ and $\tau(q_n) = 1$ for all $n$. We will show that this gives a contradiction.

We have a sequence of regular local rings

$$R_0 = \mathcal{O}_{V,q} \to R_1 = \mathcal{O}_{V_1,q_1} \to \cdots \to R_n = \mathcal{O}_{V_n,q_n} \to \cdots$$

Taking completions, we get a sequence of $K$-algebra homorphisms of power series rings over $K$:

$$\hat{R}_0 \to \hat{R}_1 \to \cdots$$

We will study how formal equations of the strict transform of $S$ transform under this sequence.

Suppose that $f = 0$ is a formal equation of $S$ at $q$. Regular parameters $x, y, z$ in $\hat{R}_0 = K[[x, y, z]]$ are *good parameters* if

$$\mathrm{ord}\ f(0, 0, z) = \mathrm{ord}\ f(= \nu_p(S) = r).$$

By the Weierstrass Preparation Theorem, after multiplying $f$ by a unit, we have

$$\begin{aligned} f &= z^r + b_1(x, y)z^{r-1} + \cdots + b_r(x, y) \\ &= z^r + \textstyle\sum_{k<r} a_{ijk}x^i y^j z^k. \end{aligned}$$

Let

$$\Delta = \Delta(x, y, z) = \left\{ \left( \frac{i}{r-k}, \frac{j}{r-k} \right) \mid a_{ijk} \neq 0 \right\},$$

and let $|\Delta|$ be the smallest convex set in $\mathbb{R}^2$ containing $\cup_{v \in \Delta(x,y,z)} v + \mathbb{R}^2_+$, where $\mathbb{R}^2_+$ is the positive quadrant in $\mathbb{R}^2$.

This is a projection of the usual Newton Polygon. Define

$$\Omega(q,x,y,z) = (\beta, \frac{1}{\epsilon}, \alpha) \in (\frac{1}{r!}\mathbb{N}) \times (\mathbb{Q} \cup \{\infty\}) \times (\frac{1}{r!}\mathbb{N})$$

with lex order. $\alpha$ is the distance of $|\Delta|$ from the $x$-axis, and $\beta$ is the smallest value such that $(\alpha,\beta) \in |\Delta|$. $\epsilon$ is the negative of the slope of the line on the boundary of $|\Delta|$ which contains $(\alpha,\beta)$ and has noninfinite slope.

$|\Delta|$ is *well prepared* if no vertices can be removed by substituting $z = z' + \eta x^a y^b$ with $\eta \in K$. Such a substitution only effects the vertex $(a,b)$ in $\Delta$.

$|\Delta|$ can be well prepared by a formal substitution $z = z' + \sum \eta_{ab} x^a y^b$.

$|\Delta|$ well prepared implies that $z = 0$ is a (formal) approximate manifold.

A translation is a substitution $\overline{y} = y + \alpha x^n$ with $\alpha \in K$.

$$\begin{aligned} x^i y^j z^k &= x^i(\overline{y} - \alpha x^n)^j z^k \\ &= \textstyle\sum_{l=0}^{j} (-\alpha)^l \binom{j}{l} x^{nl+i} y^{j-l} z^k. \end{aligned}$$

Translation plus well preparation does not change $(\alpha,\beta)$.

By making a possibly infinite sequence of translation, leading to a substitution

$$\overline{y} = y + \text{ series in } x,$$

and then performing a well preparation

$$\overline{z} = z + \text{ series in } x \text{ and } y,$$

we can make $|\Delta|$ very well prepared.

We introduce two new numbers associated to $|\Delta|$. $\gamma$ is the largest rational number such that the line $i + j = \gamma$ intersects $|\Delta|$. $\delta$ is the smallest number such that $(\gamma - \delta, \delta) \in |\Delta|$.

$|\Delta|$ is very well prepared if one of the following holds:

(1) $(\gamma - \delta, \delta) \neq (\alpha, \beta)$ and if we make a substitution $y_1 = y = \eta x$ with subsequent well preparation $z_1 = z - \Psi(x,y)$, and if $\alpha', \beta', \gamma', \delta'$ are the new invariants, then $\alpha' = \alpha$, $\beta' = \beta$, $\gamma' = \gamma$ and $\delta' \leq \delta$.

(2) $(\gamma - \delta, \delta) = (\alpha, \beta)$ and one of the following holds:
   (a) $\epsilon = 0$
   (b) $\epsilon \neq 0$ and $\frac{1}{\epsilon} \notin \mathbb{Z}$.
   (c) $\epsilon \neq 0$, $\eta = \frac{1}{\epsilon}$ is a positive integer and for any $\eta \in K$, if $y_1 = y - \eta x^n$ is a translation, with subsequent well preparation $z_1 = z - \Psi(x,y)$, then $\epsilon' = \epsilon$. Further, if $(c,d)$ is the lowest point on the line through $(\alpha,\beta)$ with slope $-\epsilon$ in $\Delta(x,y,z)|$ and $(c',d')$ is the lowest point on this line in $|\Delta(x,y_1,z_1)|$, then $d' \leq d$.

Each $R_i \to R_{i+1}$ in (2.1) has one of the forms:

(1) First well prepare (possibly very well prepare)

(2) Perform one of the following substitutions:
   T1 $x = x_1, y = x_1(y_1 + \eta)$, $\mathrm{Sing}_r(R_i) = V(x,y,z)$.
   T2 $x = x_1 y_1, y = y_1, z = y_1 z_1$, $\mathrm{Sing}_r(R_i) = V(x,y,z)$.
   T3 $x = x_1, z = x_1 z_1$, $\mathrm{Sing}_r(R_i) = V(x,z)$.
   T4 $x = x_1, y = y_1, z = y_1 z_1$, $\mathrm{Sing}_r(R_i) = V(y,z)$.

Theorem 2.7.

$$\Omega(q_i) = (\beta_i, \frac{1}{\epsilon_i}, \alpha_i)$$

*satisfies*

$$\Omega(q_i) < \Omega(q_{i+1})$$

*for all* $i$. *If* $\beta_{i+1} = \beta_i$ *and* $\frac{1}{\epsilon_{i+1}} \neq \frac{1}{\epsilon_i}$, *then*

$$\frac{1}{\epsilon_{i+1}} \leq \frac{1}{\epsilon_i} - 1.$$

Thus the sequence (2.1) is actually finite.

The theorem follows from following how $|\Delta|$ changes under the substitutions T1, T2, T3 and T4.

Subtle case: T1 with $\eta \neq 0$.

## 3. Monomialization and Ramification of Valuations

Suppose that $K$ is an algebraic function field over a base field $k$. A valuation $\nu$ of $K/k$ is a surjective group homomorphism

$$\nu : K^\times \to \Gamma_\nu$$

where $\Gamma_\nu$ is a totally ordered abelian group such that

$$\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$$

for $a, b \in K^\times$, and $\nu(c) = 0$ for $c \in k^\times$.

Set $\nu(0) = \infty$. The valuation ring of $\nu$ is

$$V_\nu = \{f \in K \mid \nu(f) \geq 0\}.$$

The ring $V_\nu$ is a (generally non noetherian) local ring with maximal ideal

$$\mathfrak{m}_\nu = \{f \in K \mid \nu(f) > 0\}.$$

Suppose that $(R, \mathfrak{m}_R)$ is a local ring contained in $K$. $R$ is an algebraic local ring of $K$ if $R$ is essentially of finite type over $k$ and the quotient field of $R$ is $K$. The valuation ring $V_\nu$ dominates $R$ if $R \subset V_\nu$ and $\mathfrak{m}_\nu \cap R = \mathfrak{m}_R$.

Suppose that $\mathfrak{p} \subset R$ is a regular prime ($R/\mathfrak{p}$ is a regular local ring). If $f \in \mathfrak{p}$ is an element of minimal value then $R[\frac{\mathfrak{p}}{f}]$ is contained in $V_\nu$, and if $\mathfrak{q} = R[\frac{\mathfrak{p}}{f}] \cap \mathfrak{m}_\nu$, then

$$R_1 = R\left[\frac{\mathfrak{p}}{f}\right]_{\mathfrak{q}}$$

is an algebraic local ring of $K$ which is dominated by $V_\nu$. We say that $R \to R_1$ is a monoidal transform along $\nu$.

If $R$ is a regular local ring, then $R_1$ is a regular local ring. In this case there exists a regular system of parameters $(x_1, \ldots, x_n)$ in $R$ such that if height$(\mathfrak{p}) = r$, then

$$R_1 = R\left[\frac{x_2}{x_1}, \ldots, \frac{x_r}{x_1}\right]_{\mathfrak{q}}.$$

Suppose that $K \to K^*$ is a finitely generated extension of algebraic function fields over $k$ and $V^*$ is a valuation ring of $K^*/k$ of a valuation $\nu^*$ with value group

$\Gamma^*$. Then the restriction $\nu = \nu^*|K$ of $\nu^*$ to $K$ is a valuation of $K/k$ with valuation ring $V = K \cap V^*$. Let $\Gamma$ be the value group of $\nu$. There is a commutative diagram

$$\begin{array}{lcl} K & \rightarrow & K^* \\ \uparrow & & \uparrow \\ V = K \cap V^* & \rightarrow & V^* \end{array}$$

The fact that the valuation ring $V = V^* \cap K$ has the quotient field $K$ is a highly desirable property of valuations rings. There exist algebraic regular local rings $R^*$ in a finite field extension of $K^*/K$ such that $K \cap R^* = k$; geometrically this means that there does not exist a germ of a (localization) of a finite map from spec$(R^*)$ to a variety with function field $K$.

THEOREM 3.1 (local monomialization, [**7**], [**11**]). *Let notation be as above, and assume that $k$ has characteristic zero. Suppose that $S^*$ is an algebraic local ring of $K^*$ which is dominated by $\nu^*$ and $R^*$ is an algebraic local ring of $K$ which is dominated by $S^*$, so that there is a commutative diagram:*

$$\begin{array}{lcl} K & \rightarrow & K^* \\ \uparrow & & \uparrow \\ V = K \cap V^* & \rightarrow & V^* \\ \uparrow & & \uparrow \\ R^* & \rightarrow & S^* \end{array}$$

*Then there exist sequences of monoidal transforms $R^* \rightarrow R_0$ and $S^* \rightarrow S$ such that $\nu^*$ dominates $S$ and $S$ dominates $R_0$ so that there is a commutative diagram*

$$\begin{array}{lcl} V & \rightarrow & V^* \\ \uparrow & & \uparrow \\ R_0 & \rightarrow & S \\ \uparrow & & \uparrow \\ R^* & \rightarrow & S^*, \end{array}$$

*and there are regular parameters $(x_1, \ldots, x_m)$ in $R_0$, $(y_1, \ldots, y_n)$ in $S$, units $\delta_1, \ldots \delta_m$ in $S$ and a matrix $A = (a_{ij})$ of nonnegative integers such that rank$(A) = m$ such that*

$$x_i = \delta_i \prod_{j=1}^{n} y_j^{a_{ij}} \text{ for } 1 \leq i \leq m.$$

The significance of the rank$(A) = m$ condition is that formally, even in an appropriate étale extension $\tilde{S}$ of $S$, $R_0 \rightarrow \tilde{S}$ is truly a monomial mapping, as there exist regular parameters $(\overline{y}_1, \ldots, \overline{y}_n)$ in $\tilde{S}$ such that

$$x_i = \prod_{j=1}^{n} \overline{y}_j^{a_j} \text{ for } 1 \leq i \leq m.$$

In the special case that $K = k$, The local monomialization theorem recovers the local uniformization theorem of Zariski [**23**].

The starting point of our theory of ramification of general valuations is the following theorem, which proves "weak simultaneous local resolution", which was conjectured by Abhyankar [**1**], for fields of characteristic zero (Abhyankar gave a counterexample to "strong simultaneous local resolution" [**1**]).

THEOREM 3.2 (weak simultaneous local resolution, [**8**]). *Let $k$ be a field of characteristic zero, $K$ an algebraic function field over $k$, $K^*$ a finite algebraic extension of $K$, $\nu^*$ a valuation of $K^*/k$,with valuation ring $V^*$. Suppose that $S^*$ is an algebraic local ring of $K^*$ which is dominated by $\nu^*$ and $R^*$ is an algebraic local ring of $K$ which is dominated by $S^*$.*

*Then there exists a commutative diagram*

$$\begin{array}{ccccc} R_0 & \rightarrow & R & \rightarrow & S \subset V^* \\ \uparrow & & & & \uparrow \\ R^* & & \rightarrow & & S^* \end{array}$$

*where $S^* \rightarrow S$ and $R^* \rightarrow R_0$ are sequences of monoidal transforms along $\nu^*$ such that $R_0 \rightarrow S$ have regular parameters of the form of the conclusions of the local monomialization theorem, $R$ is a normal algebraic local ring of $K$ with toric singularities which is the localization of the blowup of an ideal in $R_0$, and the regular local ring $S$ is the localization at a maximal ideal of the integral closure of $R$ in $K^*$.*

PROOF. By resolution of singularities, we first reduce to the case where $R^*$ and $S^*$ are regular, and then construct, by the local monomialization theorem, a sequence of monoidal transforms along $\nu^*$,

$$\begin{array}{ccc} R_0 & \rightarrow & S \subset V^* \\ \uparrow & & \uparrow \\ R^* & \rightarrow & S^* \end{array}$$

so that $R_0$ is a regular local ring with regular parameters $(x_1, \ldots, x_n)$, $S$ is a regular local ring with regular parameters $(y_1, \ldots, y_n)$, there are units $\delta_1, \ldots, \delta_n$ in $S$, and a matrix $A = (a_{ij})$ of natural numbers with nonzero determinant $d$ such that

$$x_i = \delta_i y_1^{a_{i1}} \cdots y_n^{a_{in}}$$

for $1 \leq i \leq n$. After possibly reindexing the $y_i$, we many assume that $d > 0$.

Let $B = (b_{ij})$ be the adjoint matrix of $A$. Set

$$f_i = \prod_{j=1}^{n} x_j^{b_{ij}} = \left( \prod_{j=1}^{n} \delta_j^{b_{ij}} \right) y_i^d$$

for $1 \leq i \leq n$. Let $R$ be the integral closure of $R_0[f_1, \ldots, f_n]$ in $K$, localized at the center of $\nu^*$. Since $\sqrt{\mathfrak{m}_R S} = \mathfrak{m}_S$, Zariski's Main Theorem shows that $R$ is a normal local ring of $K$ such that $S$ lies over $R$. $\square$

The global version of weak simultaneous resolution is:

Suppose that $f : X \rightarrow Y$ is a proper, generically finite morphism of $k$-varieties. Does there exist a commutative diagram

$$\begin{array}{ccc} X_1 & \stackrel{f_1}{\rightarrow} & Y_1 \\ \downarrow & & \downarrow \\ X & \stackrel{f}{\rightarrow} & Y \end{array}$$

such that $f_1$ is finite, $X_1$ and $Y_1$ are proper $k$-varieties such that $X_1$ is nonsingular, $Y_1$ is normal and the vertical arrows are birational?

We give an example showing that the answer is "no" [**10**].

Our main theorem on ramification of general valuations is joint work with Olivier Piltant.

THEOREM 3.3 ([**16**]). *Suppose that k has characteristic zero, and we are given a diagram where $K^*$ is finite over $K$ and $S^*$ is an algebraic local ring of $K^*$:*

$$\begin{array}{lcl} K & \to & K^* \\ \uparrow & & \uparrow \\ V = V^* \cap K & \to & V^* \\ & & \uparrow \\ & & S^*. \end{array}$$

*Let $k'$ be an algebraic closure of the residue field $k(V^*)$ of $V^*$. Let $\Gamma$ be the value group of $\nu$ and $\Gamma^*$ be the value group of $V^*$. Let $k(V)$, $k(V^*)$ be the respective residue fields. Then there exists a regular algebraic local ring $R_1$ of $K$ such that if $R_0$ is a regular algebraic local ring of $K$ which contains $R_1$ such that*

$$\begin{array}{ll} R_0 \to R \to & S \subset V^* \\ & \uparrow \\ & S^* \end{array}$$

*is a diagram satisfying the conclusions of the weak simultaneous local resolution theorem, then*

(1) *There is a natural isomorphism $\mathbb{Z}^n/A^t\mathbb{Z}^n \cong \Gamma^*/\Gamma$.*
(2) *$\Gamma^*/\Gamma$ acts on $\hat{S}\hat{\otimes}_{k(S)}k'$ and the invariant ring is $\hat{R}\hat{\otimes}_{k(R)}k' \cong (\hat{S}\hat{\otimes}_{k(S)}k')^{\Gamma^*/\Gamma}$.*
(3) *The reduced ramification index is $e = |\Gamma^*/\Gamma| = |Det(A)|$.*
(4) *The relative degree is $f = [k(V^*) : k(V)] = [k(S) : k(R)]$.*

By assumption, we have regular parameters $(x_1, \ldots, x_n)$ in $R_0$ and $(y_1, \ldots, y_n)$ in $S$ which satisfy the equations

$$x_i = \delta_i \prod_{j=1}^{n} y_j^{a_{ij}} \text{ for } 1 \le i \le n,$$

where $\mathrm{Det}(A) \neq 0$. We have relations

$$\nu(x_j) = \sum_{j=1}^{n} a_{ij}\nu^*(y_j) \in \Gamma$$

for $1 \le i \le n$. Thus there is a group homomorphism $\mathbb{Z}^n/A^t\mathbb{Z}^n \to \Gamma^*/\Gamma$ defined by

$$(b_1, \ldots, b_n) \mapsto b_1\nu^*(y_1) + \cdots + b_n\nu^*(y_n).$$

We have that $\hat{R}\hat{\otimes}_{k(R)}k'$ is a quotient singularity, by a group whose invariant factors are determined by $\Gamma^*/\Gamma$.

**3.1. Characteristic p.** Are the conclusions of the local monomialization theorem true in characteristic $p > 0$? This is not known, even in dimension two.[1]

The only case where local monomialization is not known to hold in dimension 2 is for rational rank 1 valuations (the usual trouble case). Here everything is OK unless we have a defect extension.

[1]I have recently found a counterexample to local monomialization in all positive characteristics and dimensions greater than or equal to 2 in “Ramification of valuations and counterexamples to local monomialization in positive characteristic”, arXiv:1404.7459

In [**16**] we give stable forms of mappings which can be attained after enough blowing up in the image and domain, but we do not know if these will eventually be monomial. We give an example showing failure of "strong monomialization". In characteristic 0, in the case of rational rank 1, after enough blowing up we get stable forms

$$x_1 = \delta y_1^n, \;\; x_2 = y_2$$

where $\delta$ is a unit. We give a characteristic $p$ example where such a good form cannot be attained.

Suppose that $K \to K^*$ is a finite and separable Galois extension, and $V^*$ is a valuation ring of $K^*$, $V = V^* \cap K$ is the induced valuation ring of $K$. The defect $p^{\delta(V^*/V)}$ is defined by the equality

$$|G^s(V^*/V)| = p^{\delta(V^*/V)}[k(V^*) : k(V)]|\Gamma^*/\Gamma|$$

where $G^s(V^*/V)$ is the splitting (decomposition) group.

If $K^*$ is finite and separable, but not Galois, then the defect is defined by taking a Galois closure $K'$ of $K^*$ over $K$, and an extension of $V^*$ to a valuation ring $V'$ of $K'$. Define

$$\delta(V^*/V) = \delta(V'/V) - \delta(V'/V^*).$$

In the case where $K \to K^*$ is a separable extension of two dimensional algebraic function fields and $R \to S$ are regular algebraic local rings of these fields, we construct a diagram of regular local rings where the horizontal arrows are products of quadratic transforms along $V^*$,

$$\begin{array}{ccccccccc} S & \to & S_1 & \to & \cdots & \to & S_n & \to & \cdots \\ \uparrow & & \uparrow & & & & \uparrow & & \\ R & \to & R_1 & \to & \cdots & \to & R_n & \to & \cdots \end{array}$$

such that $R_n$, $S_n$ have regular parameters $u_n, v_n$ and $x_n, y_n$ such that for $n \gg 0$,

$$u_n = \gamma_n x_n^{\overline{a}_n p^{\alpha_n}}, \;\; v_n = x_n^{b_n}(\tau_n y_n^{\overline{d}_n p^{\beta_n}} + x_n\Omega_n)$$

where $\overline{a}_n, \overline{d}_n$ are relatively prime to $p$, $\gamma_n$ and $\tau_n$ are units, $\alpha_n + \beta_n$ is a constant and the defect $p^{\delta(V^*/V)}$ satisfies

$$\beta_n \le \delta(V^*/V) \le \alpha_n + \beta_n.$$

**3.2. Global Monomialization.** Suppose that $\Phi : X \to Y$ is a dominant morphism of nonsingular varieties, over a field $k$ of characteristic zero. $\Phi$ is *monomial* at $p \in X$ if there exist regular parameters $(y_1, \ldots, y_m)$ in $\mathcal{O}_{Y,\Phi(p)}$ and an étale cover $U$ of an affine neighborhood of $p$, uniformizing parameters $(x_1, \ldots, x_n)$ on $U$ and a matrix $(a_{ij})$ of nonnegative integers such that

$$\begin{array}{rcl} y_1 & = & x_1^{a_{11}} \cdots x_n^{a_{1n}} \\ & \vdots & \\ y_m & = & x_1^{a_{m1}} \cdots x_n^{a_{mn}} \end{array}$$

As a consequence of the local monomialization theorem, we obtain:

THEOREM 3.4. *Suppose that $\Phi : X \to Y$ is a dominant morphism of proper $k$-varieties where $k$ is a field of characteristic zero. Suppose that $\nu$ is a valuation of the function field of $X$. Then there exists a commutative diagram*

$$\begin{array}{ccc} X_1 & \stackrel{\Psi}{\to} & Y_1 \\ \alpha\downarrow & & \downarrow\beta \\ X & \stackrel{\Phi}{\to} & Y \end{array}$$

*such that $\alpha$ and $\beta$ factor as products of blowups of nonsingular subvarieties, and $\Psi$ is monomial at the center of $\nu$ on $X_1$.*

We have the following global result for monomialization.

THEOREM 3.5 ([**9**], [**12**], [**13**]). *Suppose that $k$ is an algebraically closed field of characteristic zero, $X$ is a 3-fold over $k$ , $Y$ is a $k$-variety, and $\Phi : X \to Y$ is a dominant morphism. Then there exists a commutative diagram*

$$\begin{array}{ccc} X_1 & \stackrel{\Psi}{\to} & Y_1 \\ \alpha\downarrow & & \downarrow\beta \\ X & \stackrel{\Phi}{\to} & Y \end{array}$$

*such that $\alpha$ and $\beta$ factor as products of blowups of nonsingular subvarieties, and $\Psi$ is monomial at all points of $X_1$. In fact, we can make $\Psi$ to be a "toroidal morphism".*

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Clay Mathematics Proceedings
Volume **20**, 2014

# Resolution of toric varieties

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Abstract. The topic of this survey is an algorithm of embedded desingularization for toric varieties. The procedure uses the binomial equations defining the toric variety as an embedded subvariety in an affine space. We define a resolution function in terms of these binomials. The values of the resolution function determine, at each step, the center to be modified and all centers are combinatorial in the toric sense. To make this survey as self-contained and readable as possible, we include a short introduction to toric varieties. Also several examples are worked out and presented together with the involved computations.

## Introduction

In this paper all algebraic varieties are irreducible. Fix a field $k$, consider the affine space of dimension $n$ over $k$, say $\mathbb{A}^n_k$. Any algebraic variety $X \subset \mathbb{A}^n_k$ is the zero set of some polynomials in $k[x_1, \ldots, x_n]$. Toric varieties in $\mathbb{A}^n_k$ are algebraic varieties which can be defined by binomials in the polynomial ring. The name toric comes from the fact that an algebraic torus $\mathbb{T}$ is embedded in $X$ as an open dense set. The complement of the torus $X \setminus \mathbb{T}$ turns out to be a divisor in $X$.

The advantage of considering toric varieties is that we have fixed coordinates, given by the inclusion of the torus, and that the geometry and properties of the toric variety may be expressed in terms of combinatorial objects. However, this last observation does not imply that the geometry of toric varieties is *easy*, since the involved combinatorics may be very tricky.

We will focus on embedded desingularization of toric varieties. If $\text{char}(k) = 0$ we know that there are algorithms to obtain a desingularization by blowing up smooth centers, see [**Vil13, Hau13**] for references.

If $\text{char}(k) > 0$ then embedded desingularization is an open problem in general, there are results for dimension $\leq 3$ and for special classes of varieties, like toric varieties. The classical definition of toric varieties in terms of cones and fans, see section 4, implies that toric varieties are normal. In this setting, one may use the so-called subdivision of fans in order to obtain a non-embedded desingularization of the variety, see section 2.6 in [**Ful93**]. Recently, other approaches have extended the notion of toric varieties to be non-normal in general, see [**CLS11**, p. 150], [**MS05**] and also [**GPT12**]. We restrict ourselves to toric varieties $X$ which are embedded in a smooth toric variety $W$, in such a way that the inclusion preserves

2010 *Mathematics Subject Classification.* Primary 14E15, 14M25.
*Key words and phrases.* Resolution of singularities, toric varieties.

the tori in $X$ and $W$, this is what we call a toric embedding in Definition 5.1, see [**BM06**] for details.

A very convenient feature of smooth toric varieties is the fact that they provide a natural choice of coordinates. Consider the embedded torus $\mathbb{T} \subset W$ in a smooth $n$-dimensional toric variety, we have that $W \setminus \mathbb{T}$ is a simple normal crossings divisor. If we cover $W$ by affine open sets $U \subset W$ with $\mathbb{T} \subset U$, then every $U$ is an open set of the affine space $\mathbb{A}^n$ and $\mathbb{T} = \mathbb{T}^n \subset \mathbb{A}^n$ is the usual algebraic $n$-dimensional torus of $\mathbb{A}^n$. The divisor $U \setminus \mathbb{T}$ is defined by some coordinate variables in $\mathbb{A}^n$. Therefore every toric embedding $X \subset W$ has *natural* coordinates and we want to express our computations in terms of these coordinates.

In what follows, we will fix a perfect field $k$. We recall that for perfect fields the notions of regularity and smoothness over $k$ are equivalent, which allows us to use partial derivatives in order to detect regular points. In general, desingularization of varieties is only stated over perfect fields. For toric varieties we also restrict to perfect fields, see [**BM06**, Sec. 3] for more details.

The embedded desingularization of a toric embedding $X \subset W$ may be obtained by toric morphisms. Here we have two approaches, namely: directly define a toric morphism between two toric varieties as in [**GPT02**] or proceed step by step and construct the toric morphism $W' \to W$ as a sequence of blowing ups having combinatorial centers as in [**BM06**] or [**BE11**]. A combinatorial center is defined, locally in every affine chart, by some of the coordinates in $W$. The blowing up in a combinatorial center is quite easy to compute and a sequence of combinatorial blowing ups produces toric varieties.

In this paper we will follow [**BE11**], the difference with [**BM06**] is the type of invariant or function we define in order to determine the combinatorial center. In [**BM06**] the main invariant is the Hilbert-Samuel function, which is an upper-semi-continuous function and the stratum with the maximum value may be described using monomials. In [**BE11**] the main invariants are the functions Hcodim and E-ord. The function Hcodim acts as a toric embedding codimension, and the goal is to achieve the case Hcodim $= 0$ which means that in the toric embedding $X \subset W$, the variety $X$ is smooth and transversal to the normal crossing divisor $W \setminus \mathbb{T}$. The function E-ord is a simplification of the usual function order for ideals, adapted to the toric situation. This function was first introduced in [**Bla12a**] in order to define log-resolution of ideals generated by binomials and monomials.

In the first part of this paper we introduce the notion of toric varieties, beginning with the definition of a torus and defining affine and non-affine toric varieties. Affine toric varieties are affine varieties defined by binomials. Here binomial means difference of two monomials, $x_1^{\alpha_1} \cdots x_n^{\alpha_n} - x_1^{\beta_1} \cdots x_n^{\beta_n}$. An affine variety in $\mathbb{A}_k^n$ is toric if and only if its associated ideal is prime and may be generated by binomials (Theorem 2.3). Affine toric embeddings $X \subset W$ are equivalent to surjective homomorphisms $\mathbb{Z}^n \to \mathbb{Z}^d$, where $d = \dim(X)$ and $n = \dim(W)$ (Proposition 2.7). In section 3 we compute examples of some toric varieties in terms of the parametrization of the torus and in terms of the binomial ideal. A very short introduction to the language of cones and fans is given in section 4 in order to be able to consider normal non-affine toric varieties, in particular smooth toric varieties. Section 5 is devoted to non-affine toric embeddings.

The second part is devoted to embedded desingularization of toric varieties after [**BE11**]. In section 6 we recall the notion of normal crossing of a divisor, and we define the function $\mathrm{Hcodim}_X$. This function is upper-semi-continuous (Definition 6.7) and it will be the first coordinate of our desingularization function. The combinatorial center will be determined by the maximum value of the desingularization function. After any combinatorial blowing up it follows that the values of the function Hcodim do not increase (Proposition 6.10). So that the goal is to achieve the condition $\mathrm{Hcodim} = 0$, since then we will have a transversal and smooth embedded toric variety.

To achieve the decrease of the function Hcodim, we need to rephrase the problem in terms of binomial ideals. Upon each drop of the value of Hcodim we take the binomial ideal associated to our toric variety $X$ embedded in a space of codimension Hcodim. At this point the function E-ord comes into play, this function is defined in section 7. The function E-ord is also upper-semi-continuous and does not increase after combinatorial blowing up, provided the combinatorial center is permissible. Permissibility means that the function E-ord is constant along the combinatorial center. Hence our goal is a drop of the function $(\mathrm{Hcodim}_X, \text{E-ord}_J)$ w.r.t the lexicographical ordering. The procedure is analogous to the general procedure of log-resolution for ideals [**EV00**], see also [**BEV05**], but the procedure for toric varieties is much simpler since we do not have to take care of transversality of centers because all centers are combinatorial. We proceed by induction on the dimension of the ambient space by choosing a hypersurface of $E$-maximal contact (Definition 7.4), which turns out to be a variable, and we produce a coefficient ideal having one variable less (Definition 7.5).

Finally we may define a function E-inv, the first coordinate is Hcodim and the rest of the coordinates are functions E-ord of several coefficient ideals. The maximum value of the function E-inv determines a combinatorial center and after blowing up the value of E-inv strictly drops (Theorem 7.12). After finitely many steps one obtains a desingularization. Section 8 includes some explicit examples with the precise values and computations for the function E-inv.

The author wants to thank Orlando Villamayor, Rocío Blanco, Ana Bravo, Angélica Benito and Mariluz García for interesting mathematical discussions. The author also wants to thank David Ellwood and Herwig Hauser for the organization of the Clay Summer School at Obergurgl (Austria) and for providing the appropriate ambience for mathematical discussions and also non-mathematical fun. Finally the author is very grateful to the referees for their careful reading and many valuable corrections and comments.

## Part 1. Crash course on toric varieties

In what follows, $k$ will denote a perfect field. In this first part we will give a very short introduction to the basic notions of toric varieties.

### 1. Torus

We recall some known facts on tori, see [**Hum75**] and [**Bor91**] for proofs and more references. See also [**CLS11**, 1.1] for a more extended introduction.

Definition 1.1. The *torus* of dimension $d$ over the field $k$ is $\mathbb{T}^d = (k^*)^d$.

Every element of $\mathbb{T}^d$ is a $d$-uple $(a_1, \ldots, a_d)$ where every $a_i \in k^*$, $i = 1, \ldots, d$.

We consider $k^*$ as a multiplicative group. The torus $\mathbb{T}^d$ is also an abelian group with multiplication at each coordinate:

$$\begin{array}{ccccc} \mathbb{T}^d & \times & \mathbb{T}^d & \longrightarrow & \mathbb{T}^d \\ (a_1,\dots,a_d) & , & (b_1,\dots,b_d) & \longrightarrow & (a_1b_1,\dots,a_db_d) \end{array}. \tag{1.1}$$

The above definition just defines the $d$-dimensional torus as a group and the elements correspond to *closed* points. We may also consider the torus as an algebraic variety and the group operation will be a morphism of varieties.

Therefore set $\mathbb{T}^d$ to be the spectrum of the ring of Laurent polynomials

$$\mathbb{T}^d = \operatorname{Spec}\left(k\left[x_1, x_1^{-1}, \dots, x_d, x_d^{-1}\right]\right) = \operatorname{Spec}\left(k\left[x_1^{\pm}, \dots, x_d^{\pm}\right]\right).$$

The group operation corresponds to the ring homomorphism

$$\begin{array}{ccc} k\left[x_1^{\pm},\dots,x_d^{\pm}\right] & \longrightarrow & k\left[x_1^{\pm},\dots,x_d^{\pm}\right] \otimes_k k\left[x_1^{\pm},\dots,x_d^{\pm}\right] \\ x_i & \longrightarrow & x_i \otimes x_i \end{array}.$$

Equivalently this ring homomorphism may be expressed renaming the variables

$$\begin{array}{ccc} k\left[x_1^{\pm},\dots,x_d^{\pm}\right] & \longrightarrow & k\left[y_1^{\pm},\dots,y_d^{\pm},z_1^{\pm},\dots,z_d^{\pm}\right] \\ x_i & \longrightarrow & y_iz_i \end{array}. \tag{1.2}$$

Remark 1.2. If the field $k$ is algebraically closed, then the homomorphism in (1.2), at every closed point, corresponds to the map in (1.1).

Note that the operation in (1.1) may be considered as an action of the group $\mathbb{T}^d$ on the variety $\mathbb{T}^d$. For advanced readers, a torus is a connected diagonalizable algebraic group [**Hum75, Bor91**].

A morphism of tori as algebraic groups is a morphism as algebraic varieties which is also a group homomorphism.

Proposition 1.3. [**Bor91**, cf. 8.5] *Let $\varphi : \mathbb{T}^m \to \mathbb{T}^n$ be a morphism of tori. Then $\operatorname{Im}\varphi \subset \mathbb{T}^n$ is a torus, in particular $\operatorname{Im}\varphi$ is closed.*

Tori are related to free $\mathbb{Z}$-modules using the group of characters [**CLS11**, p. 11]. Moreover this relationship is functorial, so that morphisms of tori correspond to homomorphisms of $\mathbb{Z}$-modules.

Definition 1.4. Let $L \subset \mathbb{Z}^n$ be a $\mathbb{Z}$-submodule. We say that $L$ is *saturated* if for any $\alpha \in \mathbb{Z}^n$ and $\lambda \in \mathbb{Z} \setminus \{0\}$, $\lambda\alpha \in L$ implies that $\alpha \in L$.

The *saturation* of $L$ is

$$\operatorname{Sat}(L) = \{\alpha \in \mathbb{Z}^n \mid \lambda\alpha \in L \text{ for some } \lambda \in \mathbb{Z} \setminus \{0\}\}.$$

It follows that $L$ is saturated if and only if $\operatorname{Sat}(L) = L$. Equivalently $L \subset \mathbb{Z}^n$ is saturated if and only if the quotient $\mathbb{Z}^n/L$ is (torsion) free.

Proposition 1.5. *Fix $n, d \in \mathbb{N}$, $d \leq n$. There is a 1-1 correspondence between the following sets:*

(1) *Closed reduced immersions $\mathbb{T}^d \to \mathbb{T}^n$,*
(2) *surjective homomorphisms of $\mathbb{Z}$-modules $\mathbb{Z}^n \to \mathbb{Z}^d$,*
(3) *saturated $\mathbb{Z}$-submodules $L \subset \mathbb{Z}^n$ of rank $n - d$.*

Proof. Equivalence of (2) and (3) follows easily by setting $L$ to be the kernel of the homomorphism $\mathbb{Z}^n \to \mathbb{Z}^d$.

For the equivalence of (1) and (2), given a surjective homomorphism $\varphi : \mathbb{Z}^n \to \mathbb{Z}^d$, set $m_i = \varphi(e_i)$, $i = 1, \dots, n$, where $e_i = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$. Then we may consider the morphism $\mathbb{T}^d \to \mathbb{T}^n$ defined by the surjective ring homomorphism

$$\begin{array}{ccc} k[x_1^{\pm}, \dots, x_n^{\pm}] & \longrightarrow & k[y_1^{\pm}, \dots, y_d^{\pm}] \\ x_i & \longrightarrow & y^{m_i}. \end{array} \tag{1.3}$$

Conversely, if $\mathbb{T}^d \to \mathbb{T}^n$ is a closed immersion, then it follows that the corresponding ring homomorphism has to be as in 1.3, see [**Bor91**]. □

DEFINITION 1.6. Let $L \subset \mathbb{Z}^n$ be a submodule. Consider the Laurent polynomials $k[x^{\pm}] = k[x_1^{\pm}, \dots, x_n^{\pm}]$ and define the ideal

$$I_L = \langle x^\alpha - 1 \mid \alpha \in L \rangle \subset k[x^{\pm}].$$

If $L$ is saturated $I_L$ is called a *Laurent toric ideal.*

If $\alpha_1, \dots, \alpha_r \in \mathbb{Z}^n$ are generators of $L$ as $\mathbb{Z}$-submodule then one may prove that $x^{\alpha_1} - 1, \dots, x^{\alpha_n} - 1$ are generators of the ideal $I_L$.

Note that $L$ is saturated if and only if $I_L$ is a prime ideal in $k[x^{\pm}]$, see [**ES96**].

## 2. Affine toric varieties

Let $X$ be an algebraic variety of finite type over $k$. Recall that $X$ is said to be affine iff $X$ may be embedded as a closed subset in the affine space $\mathbb{A}_k^n$ for some suitable $n$. We will denote by $I(X)$ the ideal defining $X$ as a closed set of $\mathbb{A}_k^n$. The ideal $I(X)$ is an ideal in the polynomial ring $k[x_1, \dots, x_n]$ with $n$ variables.

We will say that a polynomial in $k[x_1, \dots, x_n]$ is a binomial if it is a difference of two monomials

$$x^\alpha - x^\beta = x_1^{\alpha_1} \cdots x_n^{\alpha_n} - x_1^{\beta_1} \cdots x_n^{\beta_n},$$

where $\alpha, \beta \in \mathbb{Z}_{\geq 0}^n$.

DEFINITION 2.1. An *affine toric variety* of dimension $d$ over the field $k$ is an affine algebraic variety $X$ such that the $d$-dimensional torus $\mathbb{T}^d$ is an open dense subset of $X$ and the action of the torus on $\mathbb{T}^d$ extends to an action on $X$:

$$\mathbb{T}^d \times X \longrightarrow X.$$

DEFINITION 2.2. Let $L \in \mathbb{Z}^n$ be a saturated submodule. Set $I_L \subset k[x_1, \dots, x_n]$ to be the ideal generated by the binomials

$$\{x^\alpha - x^\beta \mid \alpha - \beta \in L\}.$$

The ideals of the form $I_L$, with $L$ saturated, are called *toric ideals.*

Note that we are using $I_L$ to denote both the Laurent toric ideal (Definition 1.6) and the toric ideal. It will be clear from the context where we consider the ideal $I_L$ in $k[x]$ or in $k[x^{\pm}]$. In fact the toric ideal is the intersection of the Laurent toric ideal with $k[x_1, \dots, x_n]$.

If $\alpha \in \mathbb{Z}^n$, there are unique $\alpha^+, \alpha^- \in \mathbb{Z}_{\geq 0}^n$ such that $\alpha = \alpha^+ - \alpha^-$ and $\alpha_i^+ \alpha_i^- = 0$ for $i = 1, ..., n$. It can be proven that the ideal $I_L$ may be generated by the binomials

$$\{x^{\alpha^+} - x^{\alpha^-},\ \alpha \in L\}.$$

THEOREM 2.3. *Let $X$ be an affine algebraic variety over $k$ of dimension $d$. The following are equivalent:*

(1) *$X$ is an affine toric variety as in Definition* 2.1.

(2) *$X \cong \operatorname{Spec}(k[t^{a_1}, \dots, t^{a_n}])$, for some $n$, where $a_i \in \mathbb{Z}^d$, $i = 1, \dots, n$ and these generate $\mathbb{Z}^d$ as $\mathbb{Z}$-module,*

$$\langle a_1, \dots, a_n \rangle_{\mathbb{Z}} = \mathbb{Z}^d.$$

(3) *$X$ is a closed subvariety of $\mathbb{A}^n_k$ and $I(X)$ is a toric ideal (Definition 2.2).*

(4) *$X$ is a closed subvariety of $\mathbb{A}^n_k$ (for some $n$), the ideal $I(X)$ is prime and it may be generated by binomials.*

See [**MS05**], [**CLS11**] or [**KKMSD73**] for a proof.

The $k$-algebra $k[t^{a_1}, \dots, t^{a_n}]$ is the $k$-algebra associated to the semigroup generated by $a_1, \dots, a_n \in \mathbb{Z}^d$.

Definition 2.4. Let $E \subset \{1, \dots, n\}$ be a finite set. We define $\mathbb{TA}^n_E$ to be the open set of the affine space $\mathbb{A}^n_k = \operatorname{Spec}(k[x_1, \dots, x_n])$:

$$\mathbb{TA}^n_E = \mathbb{A}^n_k \setminus \bigcup_{i \notin E} H_i,$$

where $H_i = V(x_i)$ is the hypersurface defined by $x_i = 0$. Set $k[x]_E$ to be the affine coordinate ring of $\mathbb{TA}^n_E$.

$$k[x]_E = k[x_1, \dots, x_n, x_i^{-1} \mid i \notin E].$$

$$\mathbb{TA}^n_E = \operatorname{Spec}(k[x]_E) = \operatorname{Spec}\left(k[x_1, \dots, x_n, x_i^{-1} \mid i \notin E]\right).$$

In particular, if $E = \emptyset$ then $\mathbb{TA}^n_E = \mathbb{T}^n$. If $E = \{1, \dots, n\}$ then $\mathbb{TA}^n_E = \mathbb{A}^n_k$.

The variety $\mathbb{TA}^n_E$ is toric since $\mathbb{T}^n \subset \mathbb{TA}^n_E$ is an open dense subset (Definition 2.1). Moreover the complement of the torus is the union of hypersurfaces defined by $E$.

Toric varieties of the form $\mathbb{TA}^n_E$ are in fact all possible regular affine toric varieties.

Theorem 2.5. *Let $X$ be an affine toric variety of dimension $d$. If $X$ is regular then $X \cong \mathbb{TA}^d_E$ for some subset $E \subset \{1, \dots, d\}$.*

See [**CLS11**, Th. 1.3.12] for a proof.

We recall that since we assume that $k$ is a perfect field, regularity and smoothness over $k$ are equivalent notions.

Definition 2.6. An *affine toric embedding* is a closed embedding $X \subset W$ where $W \cong \mathbb{TA}^n_E$ is a regular affine toric variety, $E \subset \{1, \dots, n\}$ and $X$ is an affine toric variety of dimension $d$. Moreover we want that the immersion has to be toric, so that the $d$-dimensional torus of $X$ is a closed subset of the $n$-dimensional torus of $W$.

Note that once we have the embedding of tori $\mathbb{T}^d \to \mathbb{T}^n \subset \mathbb{TA}^n_E = W$, the affine toric variety $X$ is uniquely determined, since $X$ is the Zariski closure of $\mathbb{T}^d$ in $W$.

Theorem 2.3 may be extended now to the following

Proposition 2.7. *Fix $d \le n$ and a subset $E \subset \{1, \dots, n\}$. There is a one to one correspondence between the following sets:*

(1) *The set of affine toric embeddings $X \subset \mathbb{TA}^n_E$, with $d = \dim(X)$.*

(2) *The set of closed reduced immersions $\mathbb{T}^d \to \mathbb{T}^n$.*

(3) *The set of surjective homomorphisms $\mathbb{Z}^n \to \mathbb{Z}^d$.*

(4) *The set of saturated $\mathbb{Z}$-submodules $L \subset \mathbb{Z}^n$ of rank $n - d$.*

Let us consider the correspondences more explicit:

- Given a saturated submodule $L \subset \mathbb{Z}^n$, we know that $\mathbb{Z}^n/L \cong \mathbb{Z}^d$ and this defines a surjective homomorphism $\mathbb{Z}^n \to \mathbb{Z}^d$.
- Given a surjective homomorphism $\varphi : \mathbb{Z}^n \to \mathbb{Z}^d$, we may define the morphism of tori $\mathbb{T}^d \to \mathbb{T}^n$ parametrized by the columns of the matrix of $\varphi$ as in Proposition 1.5 and vice versa.
- If $\mathbb{T}^d \to \mathbb{T}^n$ is a closed immersion, consider the inclusion on $\mathbb{T}^n \subset \mathbb{T}\mathbb{A}^n_E$ and then the toric variety $X$ is the closure of the image of $\mathbb{T}^d$ in $\mathbb{T}\mathbb{A}^n_E$.
- If $X \subset \mathbb{T}\mathbb{A}^n_E$ is an affine toric embedding then the torus included in $X$ as open dense gives an immersion $\mathbb{T}^d \to \mathbb{T}^n$.

Remark 2.8. Proposition 2.7 gives equivalent ways for determining an affine toric embedding. Can we make these correspondences ven more explicit? Let us see how to compute the ideal $I(X)$, generators of the saturated submodule $L$ or the matrix of an homomorphism $\mathbb{Z}^n \to \mathbb{Z}^d$ if one of them is given.

- If $X \subset \mathbb{A}^n_k$ is an affine toric embedding and we have binomials generating the ideal $I(X) \in k[x_1, \ldots, x_n]$, say
$$x^{\alpha_i^+} - x^{\alpha_i^-}, \qquad i = 1, \ldots, r.$$
Then $\alpha_1, \ldots, \alpha_r$ are generators of a saturated submodule $L \subset \mathbb{Z}^n$.
- Assume that we have $\alpha_1, \ldots, \alpha_{n-d} \in \mathbb{Z}^n$ and that they generate a saturated submodule $L$ of rank $n - d$. By the saturation of $L$, we may also assume that for every $i = 1, \ldots, n-d$ the gcd of the coordinates of $\alpha_i$ is 1.

  Consider the matrix whose columns are the $\alpha_i$
$$M = (\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_{n-d}).$$
It can be proved that there is an invertible $n \times n$ matrix with integer entries $P$ such that $PM = D$ where $D$ is in the form
$$D = \left( \frac{I_{n-d}}{0} \right),$$
where $I_{n-d}$ is the $(n-d)\times(n-d)$ identity matrix. Now the matrix formed by the last $d$ rows of the matrix $P$ defines a surjective homomorphism $\mathbb{Z}^n \to \mathbb{Z}^d$.

  Note that $D$ is the Smith normal form of $M$, which is uniquely determined [**New72**, Theorem II.9]. But the matrix $P$ is not unique in general. This is coherent with the fact that the isomorphism $\mathbb{Z}^n/L \cong \mathbb{Z}^d$ is not canonical. In fact it is enough to have a matrix $P$ such that $PM$ is an upper-triangular matrix with ones in the principal diagonal, see Example 3.5.
- Let $A = (a_1 \mid a_2 \mid \cdots \mid a_n)$ be the matrix, where each $a_i$ is the $i$-th column of $A$. Assume that the matrix $A$ determines a surjective homomorphism $\mathbb{Z}^n \to \mathbb{Z}^d$. The morphism of tori is
$$\begin{array}{ccc} \mathbb{T}^d & \longrightarrow & \mathbb{T}^n \\ t = (t_1, \ldots, t_d) & \longrightarrow & (t^{a_1}, \ldots, t^{a_n}) \end{array}.$$
To compute the ideal $I(X) \subset k[X_1, \ldots, X_n]$ defining the toric variety, it is enough to compute the intersection, or elimination ideal,
$$\langle x_1 - t^{a_1}, \ldots, x_n - t^{a_n} \rangle \cap k[x_1, \ldots, x_n] \subset k[x_1, \ldots, x_n, t_1, \ldots, t_d].$$

This computation is an elimination of the variables $t_1, \dots, t_d$ which can be achieved by a standard basis computation with respect to a suitable monomial ordering, see [**GP08**, p. 69] for details.

## 3. Examples

We will illustrate with some examples the equivalence in Proposition 2.7. If $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ are polynomials, let denote by $V(f_1, \dots, f_r) \subset \mathbb{A}^n_k$ the zero locus defined by $f_1, \dots, f_r$.

EXAMPLE 3.1. Set $X = V(x^2 - y^3) \subset \mathbb{A}^2_k$. $X$ is a toric variety of dimension one. The torus embedding

$$\begin{array}{ccc} \mathbb{T}^1 & \longrightarrow & \mathbb{T}^2 \\ t & \longrightarrow & (t^3, t^2) \end{array}$$

corresponds to the saturated submodule $L \subset \mathbb{Z}^2$ generated by $(2, -3)$. It is easy to check that $L$ is the kernel of the surjective homomorphism

$$\begin{array}{ccc} \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} \\ (\alpha_1, \alpha_2) & \longrightarrow & 3\alpha_1 + 2\alpha_2 \end{array} .$$

EXAMPLE 3.2. Set $X = V(x - y^2) \subset \mathbb{A}^2_k$. The parabola $X$ is a toric variety, the torus embedding is

$$\begin{array}{ccc} \mathbb{T}^1 & \longrightarrow & \mathbb{T}^2 \\ t & \longrightarrow & (t^2, t) \end{array}$$

corresponding to the saturated submodule generated by $(1, -2)$.

EXAMPLE 3.3. Set $X = V(xy - zw) \subset \mathbb{A}^4_k$. We associate to $X$ the saturated submodule $L \subset \mathbb{Z}^4$ generated by $(1, 1, -1, -1)$. Using the method described in Remark 2.8, we find a $4 \times 4$ matrix such that

$$\left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c} 1 \\ 1 \\ -1 \\ -1 \end{array}\right) = \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right).$$

The last three rows of the matrix define a $\mathbb{Z}$-linear homomorphism $\mathbb{Z}^4 \to \mathbb{Z}^3$ corresponding to the morphism of tori

$$\begin{array}{ccc} \mathbb{T}^3 & \longrightarrow & \mathbb{T}^4 \\ (t_1, t_2, t_3) & \longrightarrow & (t_1^{-1} t_2 t_3, t_1, t_2, t_3) \end{array} .$$

Alternatively, we can use the $\mathbb{Z}$-homomorphism $\mathbb{Z}^4 \to \mathbb{Z}^3$ with matrix

$$\left(\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right),$$

obtaining the equivalent parametrization

$$\begin{array}{ccc} \mathbb{T}^3 & \longrightarrow & \mathbb{T}^4 \\ (t_1, t_2, t_3) & \longrightarrow & (t_2 t_3, t_1, t_1 t_2, t_3) \end{array} .$$

The image of $\mathbb{T}^3$ by both parametrizations is the same torus embedded in $\mathbb{T}^4$.

EXAMPLE 3.4. The Whitney umbrella $X = V(x^2 - y^2 z) \subset \mathbb{A}^3$ is also a toric variety. The associated saturated submodule is generated by $v = (2, -2, -1)$. Following Remark 2.8 one may find a matrix $P$ such that $P \cdot v$ is the column vector $(1, 0, 0)$. The matrix $P$ is not unique, for instance

$$\left(\begin{array}{ccc} 0 & 0 & -1 \\ \hline 0 & 1 & -2 \\ 1 & 0 & 2 \end{array}\right)\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \left(\begin{array}{ccc} 0 & 0 & -1 \\ \hline 1 & 1 & 0 \\ 1 & 0 & 2 \end{array}\right)\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

both show the desired behaviour. The last two rows of the matrices define an homomorphism $\mathbb{Z}^3 \to \mathbb{Z}^2$

$$\begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

and the corresponding morphisms of tori are

$$\begin{array}{ccc} \mathbb{T}^2 & \longrightarrow & \mathbb{T}^3 \\ (t, s) & \longrightarrow & (s, t, t^{-2} s^2) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{T}^2 & \longrightarrow & \mathbb{T}^3 \\ (t, s) & \longrightarrow & (ts, t, s^2) \end{array}.$$

Both parametrizations determine the same subvariety in $\mathbb{A}^3_k$.

EXAMPLE 3.5. Let $X \subset \mathbb{A}^3$ be the monomial curve parametrized by $t \to (t^3, t^4, t^5)$. One may compute the elimination ideal [**GP08**, p. 69]

$$\langle x - t^3, y - t^4, z - t^5 \rangle \cap k[x, y, z] = \langle y^2 - xz, x^2 y - z^2, x^3 - yz \rangle = I(X).$$

From the binomials generating $I(X)$ we obtain vectors

$$(1, -2, 1), (2, 1, -2), (3, -1, -1)$$

generating a saturated submodule $L \subset \mathbb{Z}^3$. In fact

$$L = \langle (1, -2, 1), (2, 1, -2) \rangle_{\mathbb{Z}},$$

since $(3, -1, -1) = (1, -2, 1) + (2, 1, -2)$. Set $M$ to be the matrix with columns $(1, -2, 1)$ and $(2, 1, -2)$. We may find a $3 \times 3$ matrix $P$ such that

$$PM = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 1 \\ \hline 3 & 4 & 5 \end{array}\right)\begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 1 & -2 \end{pmatrix} = \left(\begin{array}{cc} 1 & 2 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right).$$

And we recover the original parametrization of $X$ in the last row of $P$.

EXAMPLE 3.6. Consider the morphism of tori

$$\begin{array}{ccc} \mathbb{T}^2 & \longrightarrow & \mathbb{T}^4 \\ (t, s) & \longrightarrow & (t^3, t^2, s^4, s^3) \end{array}.$$

This parametrization defines a surface $X \subset \mathbb{A}^4_k$ which is a toric variety. The ideal $I(X) \subset k[x, y, z, w]$ is

$$\langle x - t^3, y - t^2, z - s^4, w - s^3 \rangle \cap k[x, y, z, w] = \langle x^2 - y^3, z^3 - w^4 \rangle.$$

EXAMPLE 3.7. The ideal

$$J = \langle xy - zw, z^4 - xw, yz^3 - w^2 \rangle$$

defines a toric variety in $X \subset \mathbb{A}^4$ of dimension 2. The associated saturated submodule $L \subset \mathbb{Z}^4$ is generated by $(1, 1, -1, -1)$ and $(-1, 0, 4, -1)$. Note that the third binomial gives the vector $(0, 1, 3, -2) = (1, 1, -1, -1) + (-1, 0, 4, -1)$.

Set $M$ the matrix with columns $(1,1,-1,-1)$ and $(-1,0,4,-1)$. We may find a $4\times 4$ matrix $P$ such that

$$PM=\left(\begin{array}{cccc} 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 2 & 1 & 1 & 2 \\ 1 & 3 & 1 & 3 \end{array}\right)\left(\begin{array}{cc} 1 & -1 \\ 1 & 0 \\ -1 & 4 \\ -1 & -1 \end{array}\right)=\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right).$$

The torus homomorphism $\mathbb{T}^2\to\mathbb{T}^4$ associated to the toric embedding $X\subset\mathbb{A}^4$ is

$$\begin{array}{ccc} \mathbb{T}^2 & \longrightarrow & \mathbb{T}^4 \\ (t,s) & \longrightarrow & (t^2s, ts^3, ts, t^2s^3) \end{array}.$$

## 4. Normal toric varieties

In the classical theory of toric varieties [**Ful93**] on usually assumes that a toric variety is normal. However one may see in the examples in section 3 that, according to our definition, this is not always the case for affine toric varieties defined as in Definition 2.1.

For non-normal varieties some non trivial difficulties appear, see [**CLS11**, p.150]. The description of non-affine toric varieties is simpler assuming normality. We will describe briefly how normal toric varieties are defined using the terminology of cones and fans and we refer to [**Ful93**] or [**CLS11**] for details.

DEFINITION 4.1. Let $u_1,\dots,u_r\in\mathbb{Z}^n$. The *rational convex polyhedral cone* generated by $u_1,\dots,u_r$ is the subset $\sigma\subset\mathbb{R}^n$

$$\sigma=\left\{\sum_{i=1}^{r}\lambda_i u_i \mid \lambda_i\in\mathbb{R}_{\geq 0}\right\}.$$

For short we will say that $\sigma$ is a *cone*.

We identify the dual of $\mathbb{Z}^n$ (resp. $\mathbb{R}^n$) with $\mathbb{Z}^n$ (resp. $\mathbb{R}^n$). If $u\in\mathbb{R}^n$ and $m$ is in the dual lattice, then usual pairing is just

$$\langle m,u\rangle=\sum_{i=1}^{n}m_iu_i.$$

DEFINITION 4.2. Let $\sigma$ be a (rational convex polyhedral) cone in $\mathbb{R}^n$. The *dual* of $\sigma$ is

$$\sigma^\vee=\{m\in\mathbb{R}^n \mid \langle m,u\rangle\geq 0, \text{ for all } u\in\sigma\}.$$

A *face* of a (rational convex polyhedral) cone $\sigma$ is a subset $\tau\subset\sigma$ such that there exists $m\in\sigma^\vee$ with

$$\tau=\{u\in\sigma\mid\langle m,u\rangle=0\}=m^\perp\cap\sigma.$$

Any face $\tau$ of $\sigma$ is again a (rational convex polyhedral) cone. Using $m=0$ we can regard $\sigma$ as a face of $\sigma$. The intersection of two faces $\tau_1$, $\tau_2$ of $\sigma$ is again a face of $\sigma$. More precisely, if $\tau_1=m_1^\perp\cap\sigma$ and $\tau_2=m_2^\perp\cap\sigma$ then $\tau_1\cap\tau_2=(\lambda_1m_1+\lambda_2m_2)^\perp\cap\sigma$ for any $\lambda_1,\lambda_2>0$.

If $\{0\}$ is a face of $\sigma$ we say that the cone $\sigma$ is strongly convex.

PROPOSITION 4.3. [**CLS11**, Proposition 1.2.4] *If $\sigma$ is a cone in $\mathbb{R}^n$ then $\sigma^\vee$ is also a (rational convex polyhedral) cone in $\mathbb{R}^n$.*

In particular, the cone $\sigma^\vee$ is finitely generated (Definition 4.1) by some elements $m_1, \dots, m_s \in \mathbb{Z}^n$. Moreover, as we are dealing with proper cones here, the dual of the dual cone not just contains the original cone, but coincides with it $(\sigma^\vee)^\vee = \sigma$.

The intersection of a cone with the lattice $\mathbb{Z}^n$ is a semigroup. In the classical setting this intersection is constructed using the dual cone. Thus, given a rational convex polyhedral cone $\sigma$, consider the dual $\sigma^\vee$ and then the intersection $\sigma^\vee \cap \mathbb{Z}^n$ is a semigroup. It makes sense to consider the semigroup algebra $k[\sigma^\vee \cap \mathbb{Z}^n]$ associated to this semigroup. This algebra is generated by elements $\mathcal{X}^m$ called *characters*, where $m \in \sigma^\vee \cap \mathbb{Z}^n$. By Gordan's lemma [**CLS11**, Prop. 1.2.17] the semigroup $\sigma^\vee \cap \mathbb{Z}^n$ is finitely generated. If $m_1, \dots, m_s$ are generators of this semigroup then the $k$-algebra $k[\sigma^\vee \cap \mathbb{Z}^n]$ is generated, as $k$-algebra by the characters $\mathcal{X}^{m_1}, \dots, \mathcal{X}^{m_s}$.

Now we arrive at the classical definition of affine toric variety using polyhedral cones.

Definition 4.4. Given a rational convex polyhedral cone $\sigma \subset \mathbb{R}^n$, set

$$U_\sigma = \operatorname{Spec}\left(k[\sigma^\vee \cap \mathbb{Z}^n]\right).$$

This can be indeed be used to describe all possible normal affine toric varieties.

Theorem 4.5. [**CLS11**, Th. 1.3.5] *$X$ is a normal affine toric variety (Definition* 2.1*) if and only if $X = U_\sigma$ for some strongly rational convex polyhedral $\sigma$.*

Normality of the $k$-algebra $k[\sigma^\vee \cap \mathbb{Z}^n]$ comes from the fact that the semigroup $\sigma^\vee \cap \mathbb{Z}^n$ is always saturated.

The dimension of $U_\sigma$ is $n$, the dimension of the lattice $\mathbb{Z}^n$. The torus embedding $\mathbb{T}^n \subset U_\sigma$ is given by the $k$-algebra inclusion

$$k[\sigma^\vee \cap \mathbb{Z}^n] \subset k[x_1^{\pm}, \dots, x_n^{\pm}]$$

coming from the semigroup inclusion $\sigma^\vee \cap \mathbb{Z}^n \subset \mathbb{Z}^n$.

Smoothness of the toric variety $U_\sigma$ can also be detected in the cone $\sigma$. On one hand, we say that a cone $\sigma$ is smooth if it can be generated, as cone (Definition 4.1), by part of a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^n$. On the other hand, by [**CLS11**, Th. 1.3.12] we have that $U_\sigma$ is a smooth variety if and only if $\sigma$ is a smooth cone. In this case $U_\sigma \cong \mathbb{T}\mathbb{A}^n_E$, for some $E \subset \{1, \dots, n\}$, see Theorem 2.5.

If $\tau$ is a face of $\sigma$, $\tau \subset \sigma$ implies $\sigma^\vee \subset \tau^\vee$. The $k$-algebra inclusion

$$k[\sigma^\vee \cap \mathbb{Z}^n] \subset k[\tau^\vee \cap \mathbb{Z}^n]$$

is a localization and $U_\tau \subset U_\sigma$ is an open immersion. These open immersions will allow gluing several affine toric varieties. Thus we obtain normal toric varieties, which, in general, are non-affine.

Definition 4.6. A *fan* $\Sigma$ in $\mathbb{R}^n$ is a finite collection of cones $\sigma \subset \mathbb{R}^n$ such that:

- If $\sigma \in \Sigma$ then $\sigma$ is a strongly convex rational polyhedral cone.
- If $\sigma \in \Sigma$ and $\tau$ is a face of $\sigma$, then $\tau \in \Sigma$.
- If $\sigma_1, \sigma_2 \in \Sigma$ then $\sigma_1 \cap \sigma_2$ is a face of $\sigma_i$, for $i = 1, 2$ (hence $\sigma_1 \cap \sigma_2 \in \Sigma$).

Given a fan $\Sigma$ we may construct a variety $X_\Sigma$ by gluing all toric varieties $U_\sigma$ where $\sigma \in \Sigma$. The gluing maps are constructed by the open immersions defined by the faces of the cones. If $\sigma_1, \sigma_2 \in \Sigma$ we glue the varieties $U_{\sigma_1}$ and $U_{\sigma_2}$ identifying the open set defined by $\tau = \sigma_1 \cap \sigma_2$:

$$U_{\sigma_1} \xleftarrow{\varphi_1} U_\tau \xrightarrow{\varphi_2} U_{\sigma_2},$$

where $\varphi_i$ is the open immersion defined by $\tau$ as face of $\sigma_i$, $i = 1, 2$.

The fact that $X_\Sigma$ is toric comes from the fact that $\{0\}$ is a common face of all $\sigma \in \Sigma$ and $U_{\{0\}} = \mathbb{T}^n$. Every $U_\sigma$ is a separated variety, but a variety obtained by gluing several affine varieties along open subsets does not need to be separated. Fortunately we are in a positive case:

Theorem 4.7. [**CLS11**, Th. 3.1.5] *If $\Sigma$ is a fan, then $X_\Sigma$ is a normal separated toric variety.*

In fact every normal affine toric variety, according to Definition 2.1, is of the form $X_\Sigma$, for some fan $\Sigma$ [**CLS11**, Cor. 3.1.8].

## 5. Embedded toric varieties

By the above results, a smooth toric variety $W$ of dimension $n$ will be the toric variety associated to a fan $\Sigma$, $W = X_\Sigma$, such that for every $\sigma \in \Sigma$, $U_\sigma$ is smooth. Thus, by Theorem 2.5 we have $U_\sigma \cong \mathbb{T}\mathbb{A}^n_E$, for some $E \subset \{1, \ldots, n\}$. We will denote $W_\sigma = U_\sigma$ for every $\sigma \in \Sigma$, and we will say $W = W_\Sigma$ in order to express that $W$ is defined by the fan $\Sigma$.

The complement of the torus embedding $\mathbb{T}^n \subset W$ is a set of smooth hypersurfaces having only normal crossings (Definition 6.1)

$$E = W \setminus \mathbb{T}^n.$$

By abuse of notation, we are using $E$ as a union of divisor having normal crossings in $W$ and also as a subset of $\{1, \ldots, n\}$ representing a divisor in the affine space $\mathbb{A}^n$ (Definition 2.4).

Definition 5.1. A *toric embedding* is a closed subvariety $X \subset W$ such that:

- $W = W_\Sigma$ is a smooth toric variety given by a fan $\Sigma$ as above.
- If for every cone $\sigma \in \Sigma$ we set $X_\sigma = X \cap W_\sigma$, then $X_\sigma \subset W_\sigma$ is an affine toric embedding (Definition 2.6).

Definition 5.2. Let $\sigma$ be a smooth cone. The variety $U_\sigma$ is smooth and by Theorem 2.5 $U_\sigma \cong \mathbb{T}\mathbb{A}^n_E$, for some subset $E \subset \{1, \ldots, n\}$.

The *distinguished point* is the point

$$(\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{T}\mathbb{A}^n_E \subset \mathbb{A}^n_k, \qquad \varepsilon_i = \begin{cases} 0 & \text{if} \quad i \in E \\ 1 & \text{if} \quad i \notin E \end{cases}$$

For example, the distinguished point of $\mathbb{A}^n_k$ is the origin.

Remark 5.3. For every cone $\sigma \subset \mathbb{R}^n$, recall that the torus $\mathbb{T}^n$ acts on the affine toric variety $U_\sigma$. Thus, every point $\xi \in U_\sigma$ belongs to a unique orbit of this action.

Definition 5.2 depends on the isomorphism $U_\sigma \cong \mathbb{T}\mathbb{A}^n_E$. Each isomorphism will set different distinguished points, but all of them belong to the same orbit by the torus action. Thus, only the orbit of the distinguished point is well defined.

The distinguished point of $U_\sigma$ is defined in general using the correspondence of closed points of $U_\sigma$ and semigroups homomorphisms, see [**CLS11**, page 116]. The orbit, by the torus action, of the distinguished point is the unique closed orbit in $U_\sigma$.

Proposition 5.4. [**BE11**, Rem. 28] *Let $X \subset W$ be a toric embedding (Definition 5.1), where $W = W_\Sigma$, for some fan $\Sigma$ in $\mathbb{R}^n$.*

*For any point $\xi \in W$ (non-necessarily closed) there exists a unique cone $\sigma \in \Sigma$ such that $\xi \in W_\sigma$ and $\xi$ belongs to the orbit of the distinguished point of $W_\sigma$.*

Consider the open covering $W = \cup_{\sigma\in\Sigma} W_\sigma$. It is enough to set $W_\sigma$ to be the smallest open subset of the covering with $\xi \in W_\sigma$.

## Part 2. Embedded toric resolution

Following [**BE11**], we will describe a procedure to obtain an embedded desingularization of a toric embedding $X \subset W$. The goal is the construction of a sequence

$$\begin{array}{ccccccccc} W = & W^{(0)} & \longleftarrow & W^{(1)} & \longleftarrow & \cdots & \longleftarrow & W^{(N)} \\ & \uparrow & & \uparrow & & & & \uparrow \\ X = & X^{(0)} & \longleftarrow & X^{(1)} & \longleftarrow & \cdots & \longleftarrow & X^{(N)} \end{array} \tag{5.1}$$

of combinatorial blowing ups $W^{(i+1)} \to W^{(i)}$, $i = 0, 1, \ldots, N-1$, with center $Z_i \subset W^{(i)}$. Each $X^{(i+1)}$ is the strict transform of $X^{(i)}$, $i = 0, 1, \ldots, N-1$. The final toric embedding $X^{(N)} \subset W^{(N)}$ is such that $X^{(N)}$ is smooth and transversal to $E^{(N)} = W^{(N)} \setminus \mathbb{T}^n$, where $n = \dim(W)$.

### 6. Transversality

We recall the definition of normal crossings and transversality for general varieties.

Definition 6.1. Let $W$ be a smooth variety over $k$ of dimension $n$, let $X \subset W$ be a closed subvariety and let $E$ be a finite set of smooth hypersurfaces in $W$.

- We say $E$ has only *normal crossings* if for every closed point $\xi \in W$ there is a regular system of parameters $x_1, \ldots, x_n$ in the local ring $\mathcal{O}_{W,\xi}$ such that
  - for every $H \in E$ with $\xi \in H$ one has $I(H) = (x_i)$ for some $i = 1, \ldots, n$.
- We say that $X$ has only *normal crossings with* $E$ if for every closed point $\xi \in X$ there is a regular system of parameters $x_1, \ldots, x_n$ in the local ring $\mathcal{O}_{W,\xi}$ such that
  - $I(X)_\xi = (x_1, \ldots, x_m)$ where $m = \operatorname{codim}(X)$ and
  - for every $H \in E$ with $\xi \in H$ one has $I(H) = (x_i)$ for some $i = 1, \ldots, n$.
- We say that $X$ is *transversal to* $E$ if for every closed point $\xi \in X$ there is a regular system of parameters $x_1, \ldots, x_n$ in the local ring $\mathcal{O}_{W,\xi}$ such that
  - $I(X)_\xi = (x_1, \ldots, x_m)$ where $m = \operatorname{codim}(X)$ and
  - for every $H \in E$ with $\xi \in H$ one has $I(H) = (x_i)$ for some $i = 1, \ldots, n$, with $m < i \leq n$.

Note that if $X$ has normal crossings with $E$ then automatically $X$ is smooth. Note also that if $X$ has normal crossings with $E$, but not transversal to $E$, then $X$ can be contained in some hypersurface $H \in E$.

Remark 6.2. Transversality may be detected using logarithmic differentials. Let $W$ be a smooth variety over $k$ and $E$ a set of hypersurfaces having only normal crossings. Fix a closed point $\xi \in X$ and a regular system of parameters $x_1, \ldots, x_n$ in $\mathcal{O}_{W,\xi}$ such that for every $H \in E$ with $\xi \in H$ then $I(H) = (x_i)$ for some $i = 1, \ldots, n$ (as in Definition 6.1). The dual of the logarithmic differentials are the logarithmic derivatives along $E$, which form a free $\mathcal{O}_W$-module of rank $n$. At the point $\xi$ a basis of this module is

$$\left\{x_i^{\varepsilon_i}\frac{\partial}{\partial x_i}, \mid i=1,\ldots,n\right\} \qquad \text{where} \qquad \varepsilon_i = \begin{cases} 0 & \text{if} \quad V(x_i)\notin E \\ 1 & \text{if} \quad V(x_i)\in E \end{cases}$$

The jacobian criterion for smoothness may be adapted in order to detect transversality.

THEOREM 6.3. *Let $X \subset W$ be closed subvariety of $W$ and $E$ a set of hypersurfaces in $W$ having only normal crossings.*

*Fix a closed point $\xi \in X$ and a regular system of parameters $x_1,\ldots,x_n$ in $\mathcal{O}_{W,\xi}$ as in Remark* 6.2*.*

*Set $d = \dim(X)$ and consider generators $f_1,\ldots,f_s \in \mathcal{O}_{W,\xi}$ of the ideal of $X$ in $W$ at the point $\xi$, $I(X)_\xi = (f_1,\ldots,f_s)$. The variety $X$ is transversal to $E$ at the point $\xi$ if and only if the matrix*

$$\left(x_i^{\varepsilon_i}\frac{\partial f_j}{\partial x_i}\right)_{i,j} \qquad \textit{has rank } n-d \textit{ at the point } \xi.$$

In the case that $X \subset W$ is an affine toric embedding (Definition 2.6) we set $E = W \setminus \mathbb{T}^n$. Transversality of $X$ with $E$ is equivalent to finding good generators for the ideal $I(X)$ (Theorem 6.4).

If $E \subset \{1,\ldots,n\}$ we set

$$\mathbb{Z}^n_E = \{\alpha = (\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}^n \mid \alpha_i \geq 0, \text{ for all } i\in E\}.$$

Monomials of the ring $k[x]_E$ are of the form $X^\alpha$ with $\alpha \in \mathbb{Z}^n_E$. Recall from Definition 2.4 that we will identify the set $E \subset \{1,\ldots,n\}$ with the set of hypersurfaces defined by the variables $x_i$, for $i \in E$, in the regular affine toric variety $\mathbb{TA}^n_E = \operatorname{Spec}(k[x]_E)$.

THEOREM 6.4. [**BE11**, Th. 23] *Let $V \subset W = \mathbb{TA}^n_E$ be an affine toric embedding (Definition* 2.6*). The toric variety $V$ is transversal to $E$ if and only if the ideal $I(V)$ is generated by hyperbolic equations*

$$I(V) = (x^{\alpha_1}-1,\ldots,x^{\alpha_\ell}-1),$$

*where $\ell = n - \dim V$, $\alpha_1,\ldots,\alpha_\ell \in \mathbb{Z}^n_E$ and they generate a saturated lattice of rank $\ell$.*

THEOREM 6.5. [**BE11**, Th. 26] *Let $X \subset W = \mathbb{TA}^n_E$ be an affine toric embedding, $E \subset \{1,\ldots,n\}$.*

*There is a unique affine toric variety $V$ such that the embeddings $X \subset V$ and $V \subset W$ are affine toric embeddings (Definition* 2.6*) and $V$ is the smallest smooth toric variety containing $X$ and transversal to $E$.*

The proof of this theorem reduces to considering the associated saturated lattice $L$ of the embedding $X \subset \mathbb{TA}^n_E$. The embedding $V \subset \mathbb{TA}^n_E$ will correspond to a saturated lattice $L_0 \subset L$, where $L_0$ may be generated by elements in $\mathbb{Z}^n_E$ (Theorem 6.4). The key fact is to prove that there is a unique maximal lattice $L_0$ with the above conditions.

The difference $\dim(V) - \dim(X)$ will be an invariant in our procedure for desingularization of toric varieties.

Definition 6.6. Let $X \subset W = W_\Sigma$ be a toric embedding (Definition 5.1).

Let $\xi \in X$ be any point, and let $\sigma \in \Sigma$ be the unique cone such that $\xi$ is in the orbit of the distinguished point of $W_\sigma$ (Proposition 5.4). Consider the affine toric embedding $X_\sigma \subset W_\sigma$ and let $V_\sigma$ be the smooth toric variety given by Theorem 6.5. We set the *hyperbolic codimension* of $X$ at $\xi$ to be

$$\mathrm{Hcodim}_X(\xi) = \dim(V_\sigma) - \dim(X_\sigma).$$

Note that, by construction, the function $\mathrm{Hcodim}_X : X \to \mathbb{N}$ is equivariant under the torus action. All the points in the same orbit have the same value.

The hyperbolic codimension may be understood as a toric embedding dimension. The value $\mathrm{Hcodim}_X(\xi)$ gives the minimum dimension of a regular toric variety $V$ (defined locally), $V$ transversal to $E$ and with $X \subset V$.

Definition 6.7. Let $X$ be an algebraic variety and $(\Lambda, \leq)$ be a totally ordered set. A function $h : X \to \Lambda$ is said to be *upper-semi-continuous* if $h$ takes only finitely many values in $\Lambda$ and for every $\lambda \in \Lambda$ the set

$$\{h \geq \lambda\} = \{\xi \in X \mid h(\xi) \geq \lambda\} \qquad \text{is a closed set.}$$

Proposition 6.8. *Let $X \subset W = W_\Sigma$ be a toric embedding (Definition 5.1). The function $\mathrm{Hcodim}_X : X \to \mathbb{N}$ is upper-semi-continuous.*

Proof. Since $\mathrm{Hcodim}_X$ takes only finitely many values, it is enough to prove that for any $m$, the set

$$\{\mathrm{Hcodim}_X \leq m\} = \{\xi \in X \mid \mathrm{Hcodim}_X(\xi) \leq m\} \qquad \text{is an open set.}$$

Let $\xi \in \{\mathrm{Hcodim}_X \leq m\}$ be a point. There exist a cone $\sigma \in \Sigma$ and an affine toric variety $V_\sigma$ tranversal to $E$, such that $\xi \in X_\sigma \subset V_\sigma \subset W_\sigma$ and $\dim(V_\sigma) - \dim(X) \leq m$. Note that $X_\sigma \subset X$ is an open set. For any point $\eta \in X_\sigma$ we have that $\mathrm{Hcodim}_X(\eta) \leq \dim(V_\sigma) - \dim(X) \leq m$. $\square$

Definition 6.9. [**BM06**] Consider the smooth toric variety $\mathbb{TA}^n_E$ where $E \subset \{1, \ldots, n\}$. A *combinatorial center* in $\mathbb{TA}^n_E$ is a closed subvariety $Z \subset \mathbb{TA}^n_E$ defined by some coordinates $x_i$ with $i \in E$, thus the ideal defining $Z$ is

$$I(Z) = (x_{j_1}, \ldots, x_{j_s}) \subset k[x_1, \ldots x_n, x_{i_1}^{-1}, \ldots, x_{i_r}^{-1}]$$

for some $j_1, \ldots, j_s \in E$, where $\{i_1, \ldots, i_r\} = \{1, \ldots, n\} \setminus E$ (Definition 2.4).

In general, if $W$ is a smooth toric variety defined by some fan $\Sigma$ in $\mathbb{R}^n$, then every cone $\Delta \in \Sigma$ defines a *combinatorial center* $Z_\Delta \subset W$.

For every open set $W_\sigma$, $\sigma \in \Sigma$, the intersection $Z_\Delta \cap W_\sigma$ is set as follows:

If $\Delta$ is not a face of $\sigma$ then $Z_\Delta \cap W_\sigma = \emptyset$.

If $\Delta$ is a face of $\sigma$, note that $W_\Delta \subset W_\sigma$, and then $Z_\Delta \cap W_\sigma$ is the closure (in $W_\sigma$) of the orbit of the distinguished point of $W_\Delta$.

Combinatorial centers will be the centers chosen by the algorithm of desingularization of toric varieties. The desingularization function will be upper-semi-continuous and its maximum value will determine a combinatorial center.

In general, the blowing up may be defined for any smooth center $Z \subset W$. See [**Har77**, page 160] for the general definition. When we restrict to smooth centers $Z$ in smooth varieties $W$ then the blowing up $W' \to W$ with center $Z$ is such that $W'$ is again smooth.

In our setting, if $W$ is a smooth toric variety and $Z_\Delta$ is a combinatorial center then the blowing up $W' \to W$ with center $Z_\Delta$ is such that $W'$ is again toric and smooth.

In fact, locally at every open set $W_\sigma$, $\sigma \in \Sigma$, with $W_\sigma \cong \mathbb{TA}^n_E$, a combinatorial blowing up $(\mathbb{TA}^n_E)' \to \mathbb{TA}^n_E$ is such that $(\mathbb{TA}^n_E)'$ is covered by $r = \operatorname{codim}(Z)$ affine charts, each one isomorphic to $\mathbb{TA}^n_E$, see [**Vil13**].

Moreover, if $X \subset W$ is a toric embedding and $Z$ is a combinatorial center, we may consider the combinatorial blowing up $W' \to W$ and let $X' \subset W'$ be the strict transform of $X$ in $W'$. The variety $X'$ is again a toric variety and $X' \subset W'$ is a toric embedding. We have normal crossing divisors $E$ in $W$ and $E'$ in $W'$ defined by the torus embedding

$$E = W \setminus \mathbb{T}^n, \qquad E' = W' \setminus \mathbb{T}^n.$$

Note that combinatorial centers $Z$ always have empty intersection with the torus $\mathbb{T}^n$. The irreducible components of the normal crossing divisor $E'$ are the strict transforms of the components of $E$ together with the exceptional divisor of the blowing up $W' \to W$.

A key fact in algorithms of desingularization is to prove that functions used for resolution do not increase after any (permissible) blowing up. In our case the first coordinate of the desingularization function is the function Hcodim.

PROPOSITION 6.10. [**BE11**, Prop. 33] *Let $X \subset W$ be a toric embedding, and $Z = Z_\Delta$ be a combinatorial center in $W$. Denote by $\varphi : W' \to W$ the combinatorial blowing up with center $Z$ and let $X' \subset W'$ be the toric embedding where $X'$ is the strict transform of $X$.*

*Let $\xi' \in X'$ be any point and set $\xi = \varphi(\xi') \in X$, then*

$$\mathrm{Hcodim}_{X'}(\xi') \leq \mathrm{Hcodim}_X(\xi).$$

PROOF. We may assume that $X \subset W$ is an affine toric embedding and $\xi$ is in the orbit of the distinguished point. Let $V$ be the minimal toric variety transversal to $E$ and with $X \subset V \subset W$ (Theorem 6.5). Let $V' \subset W'$ be the strict transform of $V$ in $W'$. Since $V$ is transversal to $E$ then $V'$ is again transversal to $E'$. This may be proved by direct computation using Theorem 6.4.

Then at the point $\xi'$ we have a transversal variety $V'$ including $X'$, so that

$$\mathrm{Hcodim}_{X'}(\xi') \leq \dim(V') - \dim(X') = \mathrm{Hcodim}_X(\xi).$$

□

## 7. E-order

The desingularization function E-inv for toric varieties will have several coordinates. The first coordinate is the function Hcodim and the other coordinates are functions E-ord of suitable ideals.

The function E-ord appeared first in [**Bla12a**] and [**Bla12b**] as part of the log-resolution algorithm for general binomial ideals.

DEFINITION 7.1. Let $f \in k[x]_E$ be any polynomial, where $E \subset \{1, \ldots, n\}$ (Definition 2.4). We set the E-ord of $f$ at the distinguished point of $\mathbb{TA}^n_E$ to be

$$\text{E-ord}(f) = \max\{b \mid f \in \mathfrak{m}_E^b\},$$

where $\mathfrak{m}_E = (x_i \mid i \in E)$.

Let $J \subset k[x]_E$ be an ideal, we set

$$\text{E-ord}(J) = \max\{b \mid J \subset \mathfrak{m}_E^b\}.$$

For any point $\xi$ in the orbit of the distinguished point we set the E-ord of $J$ at $\xi$, E-ord$(J)(\xi)$, to the E-ord of $J$ at the distinguished point.

In general, if $W = W_\Sigma$ is a smooth toric variety defined by a fan $\Sigma \subset \mathbb{R}^n$, we set $E = W \setminus \mathbb{T}^n$. Consider an ideal $J \subset \mathcal{O}_W$, for any $\xi \in W$ we set the E-ord of $J$ at $\xi$ to be the E-ord$(J)$ at the distinguished point of $W_\sigma$ where $\sigma \in \Sigma$ is as in Proposition 5.4.

By our definition, the function E-ord is constant along the orbits of $W$. There is a more intrinsic way for defining E-ord, see [**BE11**, Def. 17]. The value E-ord$(J)(\xi)$ is the usual order of the ideal $J$ along the strata defined by the normal crossing divisor $E$.

EXAMPLE 7.2. Consider $k[x_1, x_2, x_3]_E$ and $f = x_1^2 x_2^3 - x_3^7$. The value of the E-ord$(f)$ at the distinguished point is given in the following table for all possibilities of $E$:

| $E$ | E-ord$(f)$ | $E$ | E-ord$(f)$ |
|---|---|---|---|
| $1,2,3$ | 5 | 1 | 0 |
| $1,2$ | 0 | 2 | 0 |
| $1,3$ | 2 | 3 | 0 |
| $2,3$ | 3 | $\emptyset$ | 0 |

Note that a binomial $f$ has E-ord$(f) = 0$ if and only if $f = x^\gamma(1 - X^\alpha)$ as element in $k[x]_E$, where $\alpha, \gamma \in \mathbb{Z}_E^n$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ with $\gamma_i = 0$ for all $i \in E$. The monomial $x^\gamma$ is invertible in the ring $k[x]_E$.

If $f = x^\alpha - x^\beta$ is a binomial, then the coefficient ideal $C(f, c)$ (Definition 7.5) is generated either by a binomial or by two monomials. This fact forces to extend our definition of binomial ideal to ideals generated by monomials and binomials.

DEFINITION 7.3. An ideal $J \subset k[x]_E$ is called *general binomial* if $J$ may be generated (as ideal) by monomials $x^\gamma$ and powers of binomials $(x^\alpha - x^\beta)^\ell$, where $\alpha, \beta, \gamma \in \mathbb{Z}_E^n$ and $\ell \in \mathbb{N}$.

In the general procedure of log-resolution or desingularization for arbitrary ideals in characteristic zero there are two key concepts: Maximal contact and coefficient ideals, see [**EV00**] and [**EH02**]. In our binomial setting, we consider a combinatorial concept of $E$-maximal contact and the same definition for coefficient ideals. An $E$-maximal contact hypersurface will be one of the hypersurfaces in the normal crossing divisor $E$

DEFINITION 7.4. Let $J \subset k[x]_E$ be a general binomial ideal and set $c =$ E-ord$(J)$. Let $f \in J$ be a monomial or binomial such that E-ord$(f) = c$.

- If $f = x^\gamma$ is a monomial, $\gamma \in \mathbb{Z}_E^n$, then every variable $x_i$ with $\gamma_i > 0$ and $i \in E$ defines a *hypersurface of $E$-maximal contact* for $J$.
- If $f = x^\alpha - x^\beta$ is a binomial, we may assume that E-ord$(X^\alpha) = c$. Then any hypersurface of $E$-maximal contact for the monomial $x^\alpha$, as above, is a *hypersurface of $E$-maximal contact* for the ideal $J$.

Below we define the coefficient ideal. In fact the coefficient ideal of an ideal $J$ is a pair $(C(J), c)$. Here $C(J)$ is a general binomial ideal and $c \in \mathbb{Q}$ is a positive rational number. There are two important observations: The first one is that $C(J)$ is an ideal in one variable less, since the variable of $E$-maximal contact does not appear in $C(J)$. And the second observation is that the points where the ideal $C(J)$ has $E$-order $\geq c$ is the set of points where $J$ has maximum $E$-order. Moreover, this last relation is stable after permissible transformation [**Bla12a**].

Definition 7.5. Let $f \subset K[x]_E$ be any polynomial and $x_i$ a variable with $i \in E$. We may express $f$ as

$$f = \sum_{m \geq 0} a_m(x) x_i^m,$$

where $a_m(x)$ is a polynomial in $k[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]_{E\setminus\{i\}}$.

Fix a positive integer rational number $c \in \mathbb{Q}$, the coefficient ideal of $(f, c)$ with respect to $x_i$ is $(C(f), c!)$ where

$$C(f,c) = \left\langle a_m(x)^{\frac{c!}{c-m}} \mid m = 0, 1, \ldots, c-1 \right\rangle.$$

If $J \subset k[x]_E$ is an ideal then the *coefficient ideal* of $(J, c)$ is

$$C(J,c) = \langle C(f,c) \mid f \in J \rangle.$$

The coefficient ideal of $(f, c)$ comes from considering the pairs $(a_m, c-m)$ for $m = 0, 1, \ldots, c$ and normalizing all the numbers $c - m$ to a common multiple, for instance the factorial of $c$. In fact for computations it is much simpler to accept fractional powers and set

$$C(f,c) = \left\langle a_m(x)^{\frac{c}{c-m}} \mid m = 0, 1, \ldots, c-1 \right\rangle.$$

If $J$ is the ideal generated by $f_1, \ldots, f_s$ then it can be proved that $C(J, c)$ is generated by $C(f_1, c), \ldots, C(f_s, c)$.

Note that if $J$ is a general binomial Definition 7.3 ideal then $C(J, c)$ is also a general binomial. This may be checked by direct computation.

Definition 7.6. The function E-ord may be defined for a pair $(J, c)$

$$\text{E-ord}(J,c) = \frac{\text{E-ord}(J)}{c}.$$

Remark 7.7. Given a toric embedding $X \subset W$ we will define an upper-semi-continuous function $\text{E-inv}_X : X \to \Lambda$ such that:

- E-inv is constant along the orbits of the torus action.
- The set of points where the maximum value $\max \text{E-inv}_X$ is achieved,

  $$\underline{\text{Max}}\,\text{E-inv}_X = \{\xi \in X \mid \text{E-inv}_X(\xi) = \max \text{E-inv}_X\},$$

  defines a combinatorial center $Z \subset W$.
- If $W' \to W$ is the blowing up with center $Z$ and $X' \subset W'$ is the strict transform of $X \subset W$ then

  $$\max \text{E-inv}_X > \max \text{E-inv}_{X'}.$$

- By repeating the above procedure the maximum value of E-inv may not decrease infinitely many times and we eventually obtain an embedded desingularization of $X \subset W$.

The set $\Lambda$ will be *essentially* $\mathbb{Q}^{n+1}_{\geq 0}$ with the lexicographical ordering. In fact the values of E-inv in $\mathbb{Q}^{n+1}_{\geq 0}$ have bounded denominators and termination of the algorithm will be clear. We say that $\Lambda$ is not exactly $\mathbb{Q}^{n+1}_{\geq 0}$ since once we arrive to a simpler case (Remark 7.8) we use a different function $\Gamma$, see [**BE11**, Definition 41] for details.

If characteristic of $k$ is zero then it is possible to obtain an embedded desingularization by blowing up centers contained in the strict transform of our variety which need not to be toric, see [**EV00**]. In this paper we consider toric varieties and our goal is to obtain a sequence (5.1) of blowing ups with combinatorial centers. The blowing up with a combinatorial center is easier to compute, but if we restrict to choose only combinatorial centers then some of them can be not included in the strict transform of the original variety. For the toric variety in Example 3.6 it is not possible to achieve an embedded desingularization with combinatorial centers. Observe that in Example 8.6 the second center is not included in the strict transform of the variety.

Remark 7.8. **The case of one monomial.**
If $J$ is an ideal generated by only one monomial, say $x^\alpha$, with $\alpha \in \mathbb{Z}^n_E$, then resolution of the ideal $J$ is *easier* to obtain and will be given by a special function $\Gamma$ in terms of $\alpha$. See [**EV00**, Page 165] for details and examples.

Remark 7.9. Fix a toric embedding $X \subset W = W_\Sigma$, where $\Sigma \subset \mathbb{R}^n$ a fan.

To define function E-inv$_X$ it is enough to set the value E-inv$_X(\xi)$ for $\xi \in X_\sigma \subset W_\sigma$, $\sigma \in \Sigma$, where we assume that $\xi$ is the distinguished point of $W_\sigma$. Let $V_\sigma \subset W_\sigma$ be the smooth toric variety transversal to $E$ and minimal with $X_\sigma \subset V_\sigma$ (Theorem 6.5).

At the first stage we set

$$\text{E-inv}_X = (\text{Hcodim}_X, \text{E-ord}(J_n), \text{E-ord}(J_{n-1}, c_n), \ldots, \text{E-ord}(J_r, c_{r-1}), \infty, \ldots, \infty)\,,$$

where $J_n = I_{V_\sigma}(X_\sigma)$ is the ideal of $X_\sigma$ in $V_\sigma$. Recall that $V_\sigma \cong \mathbb{TA}^m_{E_{V_\sigma}}$ where $m = \dim(V_\sigma) = n - \text{Hcodim}_{X_\sigma}$ and $E_{V_\sigma}$ is obtained by intersection with $V_\sigma$.

Assume that we have defined inductively $J_n, \ldots, J_i$ and the numbers $c_n, \ldots, c_{i+1}$. Then $c_i = \text{E-ord}(J_i)$ and $J_{i-1} = C(J_i, c_i)$ with respect to some $E$-maximal contact hypersurface of $J_i$.

At some index $r \leq n$ it may happen that $C(J_r, c_{r-1}) = 0$; in that case the remaining entries are filled with $\infty$.

Note that we have constructed a finite sequence of ideals

$$(J_n, J_{n-1}, \ldots, J_{r+1}, J_r, J_{r-1}, \ldots, J_1),$$

where $J_{r-1} = \cdots = J_1 = 0$

The function E-inv$_X : X \to \Lambda$ is well defined. Note that there may exist several hypersurfaces having $E$-maximal contact with the ideal $I_i$. However the value of E-ord$(J_{i-1})$ does not depend on the choice of the hypersurface. Moreover the function E-inv is upper-semi-continuous, see [**BE11**] and [**Bla12a**].

The intersection of all hypersurfaces of maximal contact that we needed to define the coefficient ideals will be the combinatorial center $Z$. The center $Z$ is transversal to $V$ and it is not included in $V$, so that if $\text{Hcodim}_X < n - \dim(X)$ then $Z \not\subset X$.

Remark 7.10. Assume that we have constructed a sequence of blowing ups

$$\begin{array}{ccccccc} W=W^{(0)} & \longleftarrow & W^{(1)} & \longleftarrow & \cdots & \longleftarrow & W^{(\ell)} \\ \uparrow & & \uparrow & & & & \uparrow \\ X=X^{(0)} & \longleftarrow & X^{(1)} & \longleftarrow & \cdots & \longleftarrow & X^{(\ell)} \end{array}.$$

Recall that for every $i=0,\dots,\ell$ we have the divisor with normal crossings $E^{(i)}$ with support $W^{(i)}\setminus\mathbb{T}^n$.

Fix a point $\xi_\ell\in X^{(\ell)}$. In order to define the value $\text{E-inv}_{X^{(\ell)}}(\xi)$, we may assume that the point $\xi$ is the distinguished point of some affine $(X^{(\ell)})_\sigma\subset X^{(\ell)}$. We will denote by $\xi_i\in X^{(i)}$ the image of $\xi_\ell$. For simplicity we remove the subscript $\sigma$ for the cones, so that we have a diagram of affine toric varieties and distinguished points

$$\begin{array}{ccccccc} W=W^{(0)} & \longleftarrow & W^{(1)} & \longleftarrow & \cdots & \longleftarrow & W^{(\ell)} \\ \uparrow & & \uparrow & & & & \uparrow \\ V=V^{(0)} & \longleftarrow & V^{(1)} & \longleftarrow & \cdots & \longleftarrow & V^{(\ell)} \\ \uparrow & & \uparrow & & & & \uparrow \\ X=X^{(0)} & \longleftarrow & X^{(1)} & \longleftarrow & \cdots & \longleftarrow & X^{(\ell)} \\ \xi_0 & \longleftarrow & \xi_1 & \longleftarrow & \cdots & \longleftarrow & \xi_\ell \end{array}.$$

For $i=0,1,\dots,\ell-1,\ell$ we have ideals

$$(J_n^{(i)},J_{n-1}^{(i)},\dots,J_1^{(i)}).$$

Every ideal will be factorized as $J_j^{(i)}=M_j^{(i)}I_j^{(i)}$. The value $\text{E-inv}_{X^{(i)}}(\xi_i)$ will be defined as

$$\text{E-inv}_{X^{(i)}}=\left(\text{Hcodim}_{X^{(i)}},\text{E-ord}(I_n^{(i)}),\text{E-ord}(I_{n-1}^{(i)},c_n^{(i)}),\dots,\text{E-ord}(I_1^{(i)},c_2^{(i)})\right).$$

To be more precise, we need to treat two situations separately:

- if for some $r$ we have $I_{r-1}^{(i)}=0$ then

$$\text{E-inv}_{X^{(i)}}=\left(\text{Hcodim}_{X^{(i)}},\text{E-ord}(I_n^{(i)}),\text{E-ord}(I_{n-1}^{(i)},c_n^{(i)}),\dots,\text{E-ord}(I_r^{(i)},c_{r+1}^{(i)}),\infty,\dots,\infty\right)$$

- and if $I_{r-1}^{(i)}=1$ we set

$$\text{E-inv}_{X^{(i)}}=\left(\text{Hcodim}_{X^{(i)}},\text{E-ord}(I_n^{(i)}),\text{E-ord}(I_{n-1}^{(i)},c_n^{(i)}),\dots,\text{E-ord}(I_r^{(i)},c_{r+1}^{(i)}),\Gamma,\infty,\dots,\infty\right),$$

  where $\Gamma$ is the function of Remark 7.8 for the monomial $M_{r-1}^{(i)}$.

Consider the truncation of the function $\text{E-inv}_{X^{(i)}}$ to the first $j$ coordinates
(7.1)

$$\text{E-inv}_{X^{(i)},\geq j}=\left(\text{Hcodim}_{X^{(i)}},\text{E-ord}(I_n^{(i)}),\text{E-ord}(I_{n-1}^{(i)},c_n^{(i)}),\dots,\text{E-ord}(I_j^{(i)},c_{j+1}^{(i)})\right)$$

and set the birth of this value as the minimum index $b(i,j)\leq i$ such that

$$\text{E-inv}_{X^{(i)},\geq(j+1)}=\text{E-inv}_{X^{(i-1)},\geq(j+1)}=\cdots=\text{E-inv}_{X^{(b(i,j))},\geq(j+1)}.$$

The birth is used to make partitions of $E^{(i)}=E_j^{(i),+}\sqcup E_j^{(i),-}$. Such that

- $E_j^{(b(i,j)),+}=\emptyset$, $E_j^{(b(i,j)),-}=E\setminus E_j^{(b(i,j)),+}$ and
- $E_j^{(i+1),-}$ are the strict transforms of $E_j^{(i),-}$, $E_j^{(i+1),+}=E^{(i)}\setminus E_j^{(i+1),-}$.

Remark 7.11. The ideals $J_j^{(i)}=M_j^{(i)}I_j^{(i)}$ and the numbers $c_j^{(i)}$ are defined inductively as follows

- The ideal $M_j^{(i)}$ is generated by one monomial with support at the components of $E_j^{(i),+}$ such that the ideal $I_j^{(i)}$ may not be divided by any of the variables corresponding to $E_j^{(i),+}$.
- The number $c_j^{(i)} = \text{E-ord}(I_j^{(i)})$.
- The ideal $J_j^{(i)}$ is the coefficient ideal $C(I_{j+1}^{(i)}, c_{j+1}^{(i)})$ with respect to a $E$-maximal contact hypersurface of $I_{j+1}^{(i)}$.

To be precise, the ideal $J_j^{(i)}$ is the coefficient ideal of $I_{j+1}^{(i)}$ if $c_{j+1}^{(i)} \geq c_{j+2}^{(i)}$. If $c_{j+1}^{(i)} < c_{j+2}^{(i)}$ then one has to add the pair $(M_{j+1}^{(i)}, c_{j+2}^{(i)} - c_{j+1}^{(i)})$ to the ideal $(I_{j+1}^{(i)}, c_{j+1}^{(i)})$, see [**Bla12a**, Def. 4.2] for details. In fact this last construction is the same as in the general non-toric case, see [**EH02**].

Using the previous setting, by ascending induction on $i$ and descending induction on $j$, it is possible to construct the ideals $J_j^{(i)}$ and the values $\text{E-inv}_{X^{(i)}}$.

For a blowing up $W^{(i+1)} \to W^{(i)}$ the transformation law for every $j = n, n-1, \ldots, 1$ is as follows: The ideal $J_j^{(i+1)}$ is the controlled transform of $(J_j^{(i)}, c_{j+1}^{(i)})$. Consider the total transform of $J_j^{(i)}$ and factor out $c_{j+1}^{(i)}$ times the exceptional divisor:

$$\left(J_j^{(i)}\right)^* = I(H)^{c_{j+1}^{(i)}} J_j^{(i+1)},$$

where the asterisk means total transform of the ideal. For $j = n$ we may set $c_{n+1}^{(i)} = 0$, so that the controlled transform is the total transform.

By [**Bla12a**] we have a commutativity diagram. If

$$\text{E-inv}_{X^{(i+1)}, \geq j+1} = \text{E-inv}_{X^{(i)}, \geq j+1}$$

then the coefficient ideal $C(I_{j+1}^{(i+1)}, c_{j+1}^{(i+1)})$ is equal to the controlled transform of $(J_j^{(i)}, c_{j+1}^{(i)})$. The commutativity diagram is

$$\begin{array}{ccc} J_{j+1}^{(i)} & \longleftrightarrow & J_{j+1}^{(i+1)} \\ \downarrow & & \downarrow \\ J_j^{(i)} & \longrightarrow & J_j^{(i+1)} \end{array}$$

where horizontal arrows are controlled transforms and vertical arrows correspond to coefficient ideals.

The commutativity diagram for the ideal $J_n$ depends on the first coordinate Hcodim. If the first coordinate of E-inv remains constant, that is $\text{Hcodim}_{X^{(i+1)}} = \text{Hcodim}_{X^{(i)}}$ then $J_n^{(i+1)}$ is the total transform of $J_n^{(i)}$. If the first coordinate of E-inv drops, $\text{Hcodim}_{X^{(i+1)}} < \text{Hcodim}_{X^{(i)}}$, then the we reset the ideal $J_n^{(i+1)}$ to be the ideal of $X^{(i+1)}$ in $V^{(i+1)}$, where $\dim(V^{(i+1)}) = \dim(X^{(i)}) + \text{Hcodim}_{X^{(i+1)}}$.

THEOREM 7.12. [**BE11**] *The functions* $\text{E-inv}_{X^{(i)}}$ *are well defined for every point in* $X^{(i)}$ *and are upper-semi-continuous.*

*At every step $i$, the value* $\max \text{E-inv}_{X_i}$ *defines a combinatorial center* $Z_i \subset W_i$ *such that*

$$\max \text{E-inv}_{X_i} > \max \text{E-inv}_{X_{i+1}},$$

*and after finitely many steps we obtain an embedded desingularization of* $X \subset W$.

In general, the center $Z^{(i)}$ is not included in $X^{(i)}$, see [**BM06**] for a discussion on why combinatorial centers not included in the toric variety are needed in order to achieve the embedded desingularization. An embedded desingularization of a toric embedding $X \subset W$ may not be obtained, in general, using combinatorial centers included in $X^{(i)}$.

The center $Z^{(i)}$ is the intersection of all hypersurfaces of maximal contact used for the construction of the value $\max \text{E-inv}_{X^{(i)}}$. Thus, at every affine open set the center $Z^{(i)}$ is transversal to the smooth subvariety $V^{(i)} \supset X^{(i)}$. If the function $\Gamma$ appears in the value E-inv then some more variables have to be added for defining the center $Z^{(i)}$, depending on the monomial $M_{r-1}^{(i)}$.

## 8. Examples

We will describe how the algorithm works for the examples in section 3. For every example we only need the ideal defining the toric variety $X \subset \mathbb{A}_k^n$.

EXAMPLE 8.1. Let $X \subset \mathbb{A}_k^2$ be defined by the ideal $J = \langle x^2 - y^3 \rangle$.

We compute the value of $\text{E-inv}_X$ at the origin, the distinguished point. First note that $\text{Hcodim}_X = 1$, here $V = \mathbb{A}_k^2$.

| $j$ | Contact | $J_j^{(0)}$ | $E_j^{(0),+}$ | $M_j^{(0)}$ | $I_j^{(0)}$ | $c_j^{(0)}$ |
|---|---|---|---|---|---|---|
| 2 | | $x^2 - y^3$ | $\emptyset$ | 1 | $x^2 - y^3$ | 2 |
| 1 | $x$ | $y^3$ | $\emptyset$ | 1 | $y^3$ | 3 |

So that the value is

$$\text{E-inv}_{X^{(0)}} = \left( \text{Hcodim}_{X^{(0)}}, \text{E-ord}(I_2^{(0)}), \frac{\text{E-ord}(I_1^{(0)})}{2} \right) = \left(1, 2, \frac{3}{2}\right).$$

The center is the origin, defined by the ideal $\langle x, y \rangle$. The hypersurface defined by the variable $x$ is a hypersurface having $E$-maximal contact with $I_2^{(0)}$ and $y$ is has $E$-maximal contact with $I_1^{(0)}$.

Let $W^{(1)} \to \mathbb{A}_k^2$ be the blowing up at the origin. The variety $W^{(1)}$ may be covered by two affine charts, the ring homomorphisms are the following

$$\begin{array}{ccc} & x-\text{chart} & \\ \hline k[x,y] & \longrightarrow & k[x,y] \\ x & \longrightarrow & x \\ y & \longrightarrow & xy \end{array} \qquad \begin{array}{ccc} & y-\text{chart} & \\ \hline k[x,y] & \longrightarrow & k[x,y] \\ x & \longrightarrow & xy \\ y & \longrightarrow & y \end{array}.$$

In the $x$-chart the strict transform $X^{(1)}$ is defined by the equation $1 - xy^3$, here $\text{Hcodim}_{X^{(0)}} = 0$ and we are done. The interesting points are in the $y$-chart, where $X^{(1)}$ is defined by the equation $x^2 - y$. Note that $\text{Hcodim}_{X^{(1)}} = 1$ but the $\text{E-ord}(I_2^{(1)})$ has dropped so that we have to reset the set $E_1^{(1),+}$.

| $j$ | Contact | $J_j^{(1)}$ | $E_j^{(1),+}$ | $M_j^{(1)}$ | $I_j^{(1)}$ | $c_j^{(1)}$ |
|---|---|---|---|---|---|---|
| 2 | | $y^2(x^2 - y)$ | $y$ | $y^2$ | $x^2 - y$ | 1 |
| 1 | $y$ | $x^2$ | $\emptyset$ | 1 | $x^2$ | 2 |

The value of $\text{E-inv}_{X^{(1)}}$ is

$$\text{E-inv}_{X^{(1)}} = \left( \text{Hcodim}_{X^{(1)}}, \text{E-ord}(I_2^{(1)}), \frac{\text{E-ord}(I_1^{(1)})}{1} \right) = (1, 1, 2).$$

The center is the origin of the $y$-chart, it is given by equation $y$, the hypersurface of $E$-maximal contact with $I_2^{(1)}$ and equation $x$ having $E$-maximal contact with $E_1^{(1)}$.

We consider the blowing up $W^{(2)} \to W^{(1)}$. In the $y$-chart the equation of $X^{(2)}$ is $1 - x^2 y$ and we have $\mathrm{Hcodim}_{X^{(2)}} = 0$. In the $x$-chart the equation of $X^{(2)}$ is $x - y$, note that $\mathrm{Hcodim}_{X^{(2)}} = 1$.

| $j$ | Contact | $J_j^{(2)}$ | $E_j^{(2),+}$ | $M_j^{(2)}$ | $I_j^{(2)}$ | $c_j^{(2)}$ |
|---|---|---|---|---|---|---|
| 2 | | $x^3y^2(x-y)$ | $x, y$ | $x^3y^2$ | $x-y$ | 1 |
| 1 | $y$ | $x$ | $\emptyset$ | 1 | $x$ | 1 |

.

The value of $\text{E-inv}_{X^{(2)}}$ is

$$\text{E-inv}_{X^{(2)}} = \left( \mathrm{Hcodim}_{X^{(2)}}, \text{E-ord}(I_2^{(2)}), \frac{\text{E-ord}(I_1^{(2)})}{1} \right) = (1,1,1)\,.$$

The center is the origin of this chart and after the blowing up $W^{(3)} \to W^{(2)}$ we obtain that $\mathrm{Hcodim}_{X^{(3)}} = 0$ and we have reached an embedded desingularization.

Example 8.2. Consider the parabola $x^2 - y$ defining $X^{(0)} \subset W^{(0)} = \mathbb{A}_k^2$. This case it is almost the same as in the $y$-chart in $W^{(1)}$ for the cusp, only some hypersurfaces $E_2^{(i)}$ and the ideal $J_2^{(i)}$ are different. The parabola is already smooth but it is not transversal with $E^{(0)} = \{x, y\}$. The computation values are:

| $j$ | Contact | $J_j^{(0)}$ | $E_j^{(0),+}$ | $M_j^{(0)}$ | $I_j^{(0)}$ | $c_j^{(0)}$ |
|---|---|---|---|---|---|---|
| 2 | | $(x^2-y)$ | $\emptyset$ | 1 | $x^2-y$ | 1 |
| 1 | $y$ | $x^2$ | $\emptyset$ | 1 | $x^2$ | 2 |

,

$$\text{E-inv}_{X^{(0)}} = \left( \mathrm{Hcodim}_{X^{(0)}}, \text{E-ord}(I_2^{(0)}), \frac{\text{E-ord}(I_1^{(0)})}{1} \right) = (1,1,2)\,.$$

After blowing up the origin $W^{(1)} \to W^{(0)}$, in the $x$-chart one has $\mathrm{Hcodim}_{X^{(1)}} = 1$.

| $j$ | Contact | $J_j^{(1)}$ | $E_j^{(1),+}$ | $M_j^{(1)}$ | $I_j^{(1)}$ | $c_j^{(1)}$ |
|---|---|---|---|---|---|---|
| 2 | | $x(x-y)$ | $x$ | $x$ | $x-y$ | 1 |
| 1 | $y$ | $x$ | $\emptyset$ | 1 | $x$ | 1 |

,

$$\text{E-inv}_{X^{(1)}} = \left( \mathrm{Hcodim}_{X^{(1)}}, \text{E-ord}(I_2^{(1)}), \frac{\text{E-ord}(I_1^{(1)})}{1} \right) = (1,1,1)\,.$$

The center is the origin of the $x$-chart and after the blowing up $W^{(2)} \to W^{(1)}$ we obtain that $\mathrm{Hcodim}_{X^{(2)}} = 0$ and we have reached an embedded desingularization.

Example 8.3. Let $xy - zw \in k[x, y, z, w]$ be the equation defining a toric variety $X^{(0)} \subset W^{(0)} = \mathbb{A}_k^4$.

Here $\mathrm{Hcodim}_{X^{(0)}} = 1$, we compute the ideals $J_j^{(0)}$, $j = 4, 3, 2, 1$:

| $j$ | Contact | $J_j^{(0)}$ | $E_j^{(0),+}$ | $M_j^{(0)}$ | $I_j^{(0)}$ | $c_j^{(0)}$ |
|---|---|---|---|---|---|---|
| 4 | | $xy-zw$ | $\emptyset$ | 1 | $xy-zw$ | 2 |
| 3 | $x$ | $y^2, zw$ | $\emptyset$ | 1 | $y^2, zw$ | 2 |
| 2 | $y$ | $zw$ | $\emptyset$ | 1 | $zw$ | 2 |
| 1 | $z$ | $w^2$ | $\emptyset$ | 1 | $w^2$ | 2 |

.

Here for the equation $xy-zw$ all variables $x$, $y$, $z$, $w$ define hypersurfaces of maximal contact. In the above table we have chosen first $x$ but the invariant E-inv does not depend on this choice.

$$\text{E-inv}_{X^{(0)}} = \left(\text{Hcodim}_{X^{(0)}}, \text{E-ord}(I_4^{(0)}), \frac{\text{E-ord}(I_3^{(0)})}{2}, \frac{\text{E-ord}(I_2^{(0)})}{2}, \frac{\text{E-ord}(I_1^{(0)})}{2}\right) = \\ = (1,2,1,1,1)\,.$$

The center is the origin, we will consider the $w$-chart, the other charts are analogous. The equation of $X^{(1)}$ is $xy-z$ and $\text{Hcodim}_{X^{(1)}} = 1$.

| $j$ | Contact | $J_j^{(1)}$ | $E_j^{(1),+}$ | $M_j^{(1)}$ | $I_j^{(1)}$ | $c_j^{(1)}$ |
|---|---|---|---|---|---|---|
| 4 | | $w^2(xy-z)$ | $w$ | $w^2$ | $xy-z$ | 1 |
| 3 | $z$ | $xy$ | $\emptyset$ | 1 | $xy$ | 2 |
| 2 | $x$ | $y^2$ | $\emptyset$ | 1 | $y^2$ | 2 |
| 1 | $y$ | 0 | $\emptyset$ | 1 | 0 | $\infty$ |

,

$$\text{E-inv}_{X^{(1)}} = \left(\text{Hcodim}_{X^{(1)}}, \text{E-ord}(I_4^{(1)}), \frac{\text{E-ord}(I_3^{(1)})}{1}, \frac{\text{E-ord}(I_2^{(1)})}{2}, \infty)\right) = \\ = (1,1,2,1,\infty)\,.$$

Now the center is the line $V(x,y,z)$ corresponding to the intersection of the hypersurfaces of maximal contact. In the $z$-chart we obtain transversality. Consider the $y$-chart, here $\text{Hcodim}_{X^{(2)}} = 1$ and the equation defining $X^{(2)}$ is $x-z$.

| $j$ | Contact | $J_j^{(2)}$ | $E_j^{(2),+}$ | $M_j^{(2)}$ | $I_j^{(2)}$ | $c_j^{(2)}$ |
|---|---|---|---|---|---|---|
| 4 | | $yw^2(x-z)$ | $y, w$ | $yw^2$ | $x-z$ | 1 |
| 3 | $z$ | $x$ | $y$ | 1 | $x$ | 1 |
| 2 | $x$ | 0 | $\emptyset$ | 1 | 0 | $\infty$ |
| 1 | | | | | | $\infty$ |

,

$$\text{E-inv}_{X^{(2)}} = \left(\text{Hcodim}_{X^{(2)}}, \text{E-ord}(I_4^{(2)}), \frac{\text{E-ord}(I_3^{(2)})}{1}, \infty, \infty)\right) = (1,1,1,\infty,\infty)\,.$$

The center is the surface $V(x,z)$ and after this blowing up we obtain embedded desingularization.

EXAMPLE 8.4. Now consider the Whitney umbrella $x^2-y^2z$, $X^{(0)} \subset W^{(0)} = \mathbb{A}_k^3$.

We have $\text{Hcodim}_{X^{(0)}} = 1$.

| $j$ | Contact | $J_j^{(0)}$ | $E_j^{(0),+}$ | $M_j^{(0)}$ | $I_j^{(0)}$ | $c_j^{(0)}$ |
|---|---|---|---|---|---|---|
| 3 | | $x^2-y^2z$ | $\emptyset$ | 1 | $x^2-y^2z$ | 2 |
| 2 | $x$ | $y^2z$ | $\emptyset$ | 1 | $y^2z$ | 3 |
| 1 | $y$ | $z^3$ | $\emptyset$ | 1 | $z^3$ | 3 |

,

$$\text{E-inv}_{X^{(0)}} = \left(\text{Hcodim}_{X^{(0)}}, \text{E-ord}(I_3^{(0)}), \frac{\text{E-ord}(I_2^{(0)})}{2}, \frac{\text{E-ord}(I_1^{(0)})}{3}\right) = \left(1,2,\frac{3}{2},1\right).$$

The first center is the origin. The $x$-chart is resolved, we will study both the $y$-chart and the $z$-chart of $W^{(1)}$.

Consider first the $z$-chart. The equation of $X^{(1)}$ here looks the same as before, $x^2 - y^2 z$, but there are differences in the factorization of the ideal $J_2^{(1)} = M_2^{(1)} I_2^{(1)}$.

| $j$ | Contact | $J_j^{(1)}$ | $E_j^{(1),+}$ | $M_j^{(1)}$ | $I_j^{(1)}$ | $c_j^{(1)}$ |
|---|---|---|---|---|---|---|
| 3 | | $z^2(x^2 - y^2 z)$ | $z$ | $z^2$ | $x^2 - y^2 z$ | 2 |
| 2 | $x$ | $y^2 z$ | $z$ | $z$ | $y^2$ | 2 |
| 1 | $y$ | 0 | $\emptyset$ | 1 | 0 | $\infty$ |

,

$$\text{E-inv}_{X^{(1)}} = \left(\text{Hcodim}_{X^{(1)}}, \text{E-ord}(I_3^{(1)}), \frac{\text{E-ord}(I_2^{(1)})}{2}, \infty\right) = (1, 2, 1, \infty).$$

The center is the line $V(x, y)$. The $x$-chart is already resolved. In the $y$-chart the equation of $X^{(2)}$ is $x^2 - z$.

| $j$ | Contact | $J_j^{(2)}$ | $E_j^{(2),+}$ | $M_j^{(2)}$ | $I_j^{(2)}$ | $c_j^{(2)}$ |
|---|---|---|---|---|---|---|
| 3 | | $y^2 z^2(x^2 - z)$ | $y, z$ | $y^2 z^2$ | $x^2 - z$ | 1 |
| 2 | $z$ | $x^2$ | $\emptyset$ | 1 | $x^2$ | 2 |
| 1 | $x$ | 0 | $\emptyset$ | 1 | 0 | $\infty$ |

.

Note that $E_2^{(2),+} = \emptyset$ since $\text{E-ord}(I_3^{(2)}) < \text{E-ord}(I_3^{(1)})$.

$$\text{E-inv}_{X^{(2)}} = \left(\text{Hcodim}_{X^{(2)}}, \text{E-ord}(I_3^{(2)}), \frac{\text{E-ord}(I_2^{(2)})}{1}, \infty\right) = (1, 1, 2, \infty).$$

The next center is $V(x, z)$, the hypersurface $X^{(3)}$ is defined by the equation $x - z$ in the $x$-chart.

| $j$ | Contact | $J_j^{(3)}$ | $E_j^{(3),+}$ | $M_j^{(3)}$ | $I_j^{(3)}$ | $c_j^{(3)}$ |
|---|---|---|---|---|---|---|
| 3 | | $x^3 y^2 z^2(x - z)$ | $x, y, z$ | $x^3 y^2 z^2$ | $x - z$ | 1 |
| 2 | $z$ | $x$ | $x$ | $x$ | 1 | 0 |

.

Note that $J_2^{(3)} = M_2^{(3)}$ and then function $\Gamma$ for the monomial case comes into play in the function E-inv.

$$\text{E-inv}_{X^{(3)}} = \left(\text{Hcodim}_{X^{(3)}}, \text{E-ord}(I_3^{(3)}), \Gamma, \infty\right) = (1, 1, \Gamma, \infty).$$

The function $\Gamma$ will pick a component of the closed set of all points where the monomial $M_2^{(3)}$ has order greater or equal than $c_3^{(1)} = 1$. In this particular case there is only one component defined by $x$. Thus the center is given by the ideal $\langle x, z\rangle$ and after this blowing up we are done.

Now, come back to $X^{(1)}$ and consider the $y$-chart. The equation of $X^{(1)}$ in this chart is $x^2 - yz$

| $j$ | Contact | $J_j^{(1)}$ | $E_j^{(1),+}$ | $M_j^{(1)}$ | $I_j^{(1)}$ | $c_j^{(1)}$ |
|---|---|---|---|---|---|---|
| 3 | | $y^2(x^2 - yz)$ | $y$ | $y^2$ | $x^2 - yz$ | 2 |
| 2 | $x$ | $yz$ | $y$ | $y$ | $z$ | 1 |
| 1 | $z$ | $y$ | $\emptyset$ | 1 | 1 | $\infty$ |

.

Here it appears a new auxiliary ideal $P_2^{(1)} = \langle y, z\rangle = I_2^{(1)} + M_2^{(1)}$. Note that $\max \text{E-ord}(I_3^{(1)}) = 2$ and the set $\underline{\text{Max}}\, \text{E-ord}(I_3^{(1)}) = V(x, y, z)$. The ideal $I_2^{(1)}$ has maximum E-ord equal to 1 but the set $\underline{\text{Max}}\, \text{E-ord}(I_2^1) = V(x, z)$, recall that $x$ comes from the hypersurface of maximal contact. To make consistent the algorithm we want that $\underline{\mathrm{Max}}\,\text{E-ord}(I_2^{(1)})$ has to be contained in $\underline{\mathrm{Max}}\,\text{E-ord}(I_3^{(1)})$. The ideal $P_2^{(1)}$ will make this possible: $\underline{\mathrm{Max}}\,\text{E-ord}(P_2^{(1)}) \subset \underline{\mathrm{Max}}\,\text{E-ord}(I_3^{(1)})$.

$$\text{E-inv}_{X^{(1)}} = \left(\text{Hcodim}_{X^{(1)}}, \text{E-ord}(I_3^{(1)}), \frac{\text{E-ord}(I_2^{(1)})}{2}, \frac{\text{E-ord}(I_2^{(1)})}{1}\right) = \left(1, 2, \frac{1}{2}, 1\right).$$

The center is the origin. The $x$-chart is resolved, for the $y$-chart and the $z$-chart the centers will be lines and after two more steps one achieves resolution.

EXAMPLE 8.5. Consider the monomial curve $(t^3, t^4, t^5)$ in $\mathbb{A}_k^4$. The ideal defining the curve is $\langle y^2 - xz, x^2y - z^2, x^3 - yz\rangle$. We note that these three polynomials form a Gröbner basis and one may compute the strict transform of the ideal just by taking the strict transform of each generator. We have $\text{Hcodim}_{X^{(0)}} = 2$.

| $j$ | Contact | $J_j^{(0)}$ | $E_j^{(0),+}$ | $M_j^{(0)}$ | $I_j^{(0)}$ | $c_j^{(0)}$ |
|---|---|---|---|---|---|---|
| 3 | | $y^2 - xz$, $x^2y - z^2$, $x^3 - yz$ | $\emptyset$ | 1 | $y^2 - xz$, $x^2y - z^2$, $x^3 - yz$ | 2 |
| 2 | $x$ | $y^2, z^2$, $z^2$, $yz$ | $\emptyset$ | 1 | $y^2, z^2$, $z^2$, $yz$ | 2 |
| 1 | $y$ | $z^2$ | $\emptyset$ | 1 | $z^2$ | 2 |

,

$$\text{E-inv}_{X^{(0)}} = \left(\text{Hcodim}_{X^{(0)}}, \text{E-ord}(I_3^{(0)}), \frac{\text{E-ord}(I_2^{(0)})}{2}, \frac{\text{E-ord}(I_1^{(0)})}{2}\right) = (2, 2, 1, 1).$$

The center is the origin. In the $y$-chart the curve $X^{(1)}$ is transversal to $E^{(1)}$, the ideal of $X^{(1)}$ is

$$\langle 1 - xz, z^3 - xy, z^2 - x^2y\rangle = \langle 1 - xz, 1 - x^4y\rangle.$$

The same occurs for the $z$-chart. We will thus focus on the more interesting $x$-chart.

| $j$ | Contact | $J_j^{(1)}$ | $E_j^{(1),+}$ | $M_j^{(1)}$ | $I_j^{(1)}$ | $c_j^{(1)}$ |
|---|---|---|---|---|---|---|
| 3 | | $x^2(y^2 - z)$, $x^2(xy - z^2)$, $x^2(x - yz)$ | $x$ | $x^2$ | $y^2 - z$, $xy - z^2$, $x - yz$ | 1 |
| 2 | $x$ | $y^2 - z$, $z^2$, $yz$ | $\emptyset$ | 1 | $y^2 - z$, $z^2$, $yz$ | 1 |
| 1 | $z$ | $y^2$ | $\emptyset$ | 1 | $y^2$ | 2 |

,

$$\text{E-inv}_{X^{(1)}} = \left(\text{Hcodim}_{X^{(1)}}, \text{E-ord}(I_3^{(1)}), \frac{\text{E-ord}(I_2^{(1)})}{1}, \frac{\text{E-ord}(I_1^{(1)})}{1}\right) = (2, 1, 1, 2).$$

The center is again the origin. In the $x$-chart and the $z$-chart we are done. In the $y$-chart we have $\text{Hcodim}_{X^{(2)}} = 2$.

$$I(X^{(2)}) = \langle y - z, x - z^2, x - yz\rangle,$$

| $j$ | Contact | $J_j^{(2)}$ | $E_j^{(2),+}$ | $M_j^{(2)}$ | $I_j^{(2)}$ | $c_j^{(2)}$ |
|---|---|---|---|---|---|---|
| 3 | | $x^2y^3(y-z)$, $x^2y^4(x-z^2)$, $x^2y^3(x-yz)$ | $x, y$ | $x^2y^3$ | $y-z$, $y(x-z^2)$, $x-yz$ | 1 |
| 2 | $x$ | $y-z$, $yz^2$, $yz$ | $y$ | 1 | $y-z$, $z^2$, $yz$ | 1 |
| 1 | $z$ | $y$ | $y$ | $y$ | 1 | $\Gamma$ |

Note that $I_3^{(2)} \neq I(X^{(2)})$ and we see the difference between the strict transform $X^{(2)}$, and our transformed ideal $I_3^{(2)}$. We do not have to reset the ideal $J_3^{(2)}$ since $\mathrm{Hcodim}_{X^{(1)}} = \mathrm{Hcodim}_{X^{(2)}}$.

$$\text{E-inv}_{X^{(2)}} = \left(\mathrm{Hcodim}_{X^{(2)}}, \text{E-ord}(I_3^{(2)}), \frac{\text{E-ord}(I_2^{(2)})}{1}, \Gamma\right) = (2,1,1,\Gamma)\,.$$

The center is the origin and after the blowing up $W^{(3)} \to W^{(2)}$ we obtain an embedded resolution.

EXAMPLE 8.6. Let $J = \langle x^2 - y^3, z^3 - w^4\rangle$ be the ideal defining the toric surface $X^{(0)} \subset W^{(0)} = \mathbb{A}_k^4$.

Note that $\mathrm{Hcodim}_{X^{(0)}} = 2$.

| $j$ | Contact | $J_j^{(0)}$ | $E_j^{(0),+}$ | $M_j^{(0)}$ | $I_j^{(0)}$ | $c_j^{(0)}$ |
|---|---|---|---|---|---|---|
| 4 | | $x^2-y^3$, $z^3-w^4$ | $\emptyset$ | 1 | $x^2-y^3$, $z^3-w^4$ | 2 |
| 3 | $x$ | $y^3$, $z^3-w^4$ | $\emptyset$ | 1 | $y^3$, $z^3-w^4$ | 3 |
| 2 | $y$ | $z^3-w^4$ | $\emptyset$ | 1 | $z^3-w^4$ | 3 |
| 1 | $z$ | $w^4$ | $\emptyset$ | 1 | $w^4$ | 4 |

$$\text{E-inv}_{X^{(0)}} = \left(\mathrm{Hcodim}_{X^{(0)}}, \text{E-ord}(I_4^{(0)}), \frac{\text{E-ord}(I_3^{(0)})}{2}, \frac{\text{E-ord}(I_2^{(0)})}{3}, \frac{\text{E-ord}(I_1^{(0)})}{3}\right) = \left(2,2,\frac{3}{2},1,\frac{4}{3}\right).$$

We blow up the origin and look in the $x$-chart. The variety $X^{(1)}$ is defined, in the $x$-chart by the ideal $\langle 1 - xy^3, z^3 - xw^4\rangle$. Note that here $\mathrm{Hcodim}_{X^{(1)}} = 1$, since $X^{(1)} \subset V$ where $I(V) = \langle 1 - xy^3\rangle$. We have that $V \cong \mathrm{Spec}(k[y^{\pm}, z, w])$ and the ideal of $X^{(1)}$ in $V$ is $J^{(1)} = \langle z^3 - y^3w^4\rangle$.

| $j$ | Contact | $J_j^{(1)}$ | $E_j^{(1),+}$ | $M_j^{(1)}$ | $I_j^{(1)}$ | $c_j^{(1)}$ |
|---|---|---|---|---|---|---|
| 4 | | $z^3-y^3w^4$ | $\emptyset$ | 1 | $z^3-y^3w^4$ | 3 |
| 3 | $z$ | $y^3w^4$, | $\emptyset$ | 1 | $y^3w^4$ | 4 |
| 2 | $w$ | 0 | $\emptyset$ | 1 | 0 | $\infty$ |

$$\text{E-inv}_{X^{(1)}} = \left(\mathrm{Hcodim}_{X^{(1)}}, \text{E-ord}(I_4^{(1)}), \frac{\text{E-ord}(I_3^{(1)})}{3}, \infty, \infty\right) = \left(1,3,\frac{4}{3},\infty,\infty\right).$$

The next center is $V(z,w)$.

The ideal of $X^{(1)}$ in the $y$-chart is $\langle x^2-y, z^3-yw^4\rangle$. Here $\mathrm{Hcodim}_{X^{(1)}}=2$ and we do not reset the ideal $J_2^{(1)}$

| $j$ | Contact | $J_j^{(1)}$ | $E_j^{(1),+}$ | $M_j^{(1)}$ | $I_j^{(1)}$ | $c_j^{(1)}$ |
|---|---|---|---|---|---|---|
| 4 | | $y^2(x^2-y)$, $y^3(z^3-yw^4)$ | $y$ | $y^2$ | $x^2-y$, $y(z^3-yw^4)$ | 1 |
| 3 | $y$ | $x^2$ | $\emptyset$ | 1 | $x^2$ | 2 |
| 2 | $x$ | 0 | $\emptyset$ | 1 | 0 | $\infty$ |

,

$$\text{E-inv}_{X^{(1)}}=\left(\mathrm{Hcodim}_{X^{(1)}},\text{E-ord}(I_4^{(1)}),\frac{\text{E-ord}(I_3^{(1)})}{1},\infty,\infty\right)=(2,1,2,\infty,\infty)\,.$$

The center in this chart is $V(x,y)$.

EXAMPLE 8.7. Consider the toric variety $X\subset W=\mathbb{A}_k^4$ defined by the ideal $J=\langle xy-zw, z^4-xw, yz^3-w^2\rangle$. We have $\dim(X^{(0)})=2$ and $\mathrm{Hcodim}_{X^{(0)}}=2$.

| $j$ | Contact | $J_j^{(0)}$ | $E_j^{(0),+}$ | $M_j^{(0)}$ | $I_j^{(0)}$ | $c_j^{(0)}$ |
|---|---|---|---|---|---|---|
| 4 | | $xy-zw$, $z^4-xw$, $yz^3-w^2$ | $\emptyset$ | 1 | $xy-zw$, $z^4-xw$, $yz^3-w^2$ | 2 |
| 3 | $x$ | $y^2, zw$, $z^4, w^2$ $yz^3-w^2$ | $\emptyset$ | 1 | $y^2, zw$, $z^4, w^2$ $yz^3-w^2$ | 2 |
| 2 | $y$ | $zw$, $z^4, w^2$ $z^6, w^2$ | $\emptyset$ | 1 | $zw$, $z^4, w^2$ $z^6, w^2$ | 2 |
| 1 | $z$ | $w^2$ | $\emptyset$ | 1 | $w^2$ | 2 |

,

$$\text{E-inv}_{X^{(0)}}=\left(\mathrm{Hcodim}_{X^{(0)}},\text{E-ord}(I_4^{(0)}),\frac{\text{E-ord}(I_3^{(0)})}{2},\frac{\text{E-ord}(I_2^{(0)})}{2},\frac{\text{E-ord}(I_1^{(0)})}{2}\right)=$$
$$=(2,2,1,1,1)\,.$$

The center is the origin, we describe the situation in the $x$-chart The ideal of $X^{(1)}$ is $\langle y-zw, x^2z^4-w, x^2yz^3-w^2\rangle$ and $\mathrm{Hcodim}_{X^{(1)}}=2$.

| $j$ | Contact | $J_j^{(1)}$ | $E_j^{(1),+}$ | $M_j^{(1)}$ | $I_j^{(1)}$ | $c_j^{(1)}$ |
|---|---|---|---|---|---|---|
| 4 | | $x^2(y-zw)$, $x^2(x^2z^4-w)$, $x^2(x^2yz^3-w^2)$ | $x$ | $x^2$ | $y-zw$, $x^2z^4-w$, $x^2yz^3-w^2$ | 1 |
| 3 | $y$ | $zw$, $x^2z^4-w$ $w^2$ | $\emptyset$ | 1 | $zw$, $x^2z^4-w$ $w^2$ | 1 |
| 2 | $w$ | $x^2z^4$ | $\emptyset$ | 1 | $x^2z^4$ | 6 |
| 1 | $z$ | $x^6$ | $\emptyset$ | 1 | $x^6$ | 6 |

,

$$\text{E-inv}_{X^{(1)}}=\left(\mathrm{Hcodim}_{X^{(1)}},\text{E-ord}(I_4^{(1)}),\frac{\text{E-ord}(I_3^{(1)})}{1},\frac{\text{E-ord}(I_2^{(0)})}{1},\frac{\text{E-ord}(I_1^{(0)})}{6}\right)=$$
$$=(2,1,1,6,1)\,.$$

The next center will be the origin.

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Clay Mathematics Proceedings
Volume **20**, 2014

# Desingularization in computational applications and experiments

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Abstract. After briefly recalling some computational aspects of blowing up and of representation of resolution data common to a wide range of desingularization algorithms (in the general case as well as in special cases like surfaces or binomial varieties), we shall proceed to computational applications of resolution of singularities in singularity theory and algebraic geometry, also touching on relations to algebraic statistics and machine learning. Namely, we explain how to compute the intersection form and dual graph of resolution for surfaces, how to determine discrepancies, the log-canoncial threshold and the topological Zeta-function on the basis of desingularization data. We shall also briefly see how resolution data comes into play for Bernstein-Sato polynomials, and we mention some settings in which desingularization algorithms can be used for computational experiments. The latter is simply an invitation to the readers to think themselves about experiments using existing software, whenever it seems suitable for their own work.

## 1. Introduction

This article originated from the notes of an invited talk at the Clay Mathematics Institute summer school on "The Resolution of Singular Algebraic Varieties" in Obergurgl, Austria, 2012. As the whole school was devoted to desingularization, the focus in this particular contribution is on applications and on practical aspects. A general knowledge of resolution of singularities and different approaches to this task is assumed and can be acquired from other parts of this proceedings volume. The overall goal of this article is to give readers a first impression of a small choice of applications and point them to good sources for further reading on each of the subjects. A detailed treatment of each of the topics would fill an article by itself and is thus beyond the scope here.

One focus here is on the practical side. To this end, we first revisit desingularization algorithms in section 2 and have a closer look at the representation of resolution data: as a consequence of the heavy use of blowing up, the data is distributed over a rather large number of charts which need to be glued appropriately. Glueing, however, only describes the theoretical side of the process; from the practical point of view, it is closer to an identification of common points in charts.

*Key words and phrases.* algorithmic resolution of singularities, applications of desingularization, topological zeta function, b-function, log-canonical threshold, computational experiments.

partially supported by DFG priority program SPP 1489 'Algorithmic and Experimental Methods in Algebra, Geometry and Number Theory'.

In section 3 we focus on applications needing different amounts of resolution data. Using an abstract resolution of singularities only, the computation of the intersection form and dual graph of the resolution for surface singularities requires the smallest amount of data. For determining discrepancies and the log-canonical threshold, we already need an embedded resolution which is also required for the third application, the computation of the topological zeta function.
In the rather short last section, we only sketch two settings in which one might want to use algorithmic desingularization as an experimental tool: the roots of the Bernstein-Sato polynomial and resolution experiments in positive characteristic. This last part is not intended to provide actual research projects. It is only intended to help develop a feeling for settings in which experiments can be helpful.
All parts of the article are illustrated by the same simple example which is desingularized using a variant of Villamayor's algorithm available in SINGULAR. Based on this resolution data, all further applications are also accompanied by the corresponding SINGULAR-code. The SINGULAR code is not explained in detail, but hints and explanations on the appearing commands and their output are provided as comments in the examples. To further familiarize with the use of SINGULAR in this context, we recommend that the readers try out the given session themselves and use the built-in manual of SINGULAR to obtain further information on the commands (e.g.: `help resolve;` returns the help page of the command `resolve`).
I would like to thank the organizers of the summer school for the invitation. Insights from conversations with many colleagues have contributed to the content of these notes. In particular, I would like to thank Herwig Hauser, Gerhard Pfister, Ignacio Luengo, Alejandro Melle, Frank-Olaf Schreyer, Wolfram Decker, Hans Schönemann, Duco van Straten, Nobuki Takayama, Shaowei Lin, Frank Kiraly, Bernd Sturmfels, Zach Teitler, Nero Budur, Rocio Blanco and Santiago Encinas for comments, of which some led to applications explained here and some others helped seeing the applications in broader context. For reading earlier versions of this article and many helpful questions on the subject of this article, I am indepted to Frithjof Schulze and Bas Heijne.

## 2. Desingularization from the computational side

Before turning toward applications of resolution of singularities, we need to review certain aspects of algorithmic desingularization to understand the way in which the computed resolution data is represented. Although there are various settings in which different resolution algorithms have been created, we may discern three main approaches suitable for the purposes of this article: the algorithms based on Hironaka's proof in characteristic zero in any dimension (see e.g. [**9**], [**6**], [**15**]), the algorithms for binomial and toric ideals (see e.g. [**18**], [**8**], [**7**]) and the algorithms for 2-dimensional varieties and schemes (such as [**2**] – based on [**23**] – or [**28**]).[1] The algorithms of the first kind of approach involve embedded desingularization, i.e. they blow up a smooth ambient space and consider the strict transform of the variety and the exceptional divisors inside the new ambient space. The algorithms for 2-dimensional varieties on the other hand, do not consider the

[1]Of course this list of approaches is far from exhaustive, but it is intended to narrow down our scope to those which lead to similar forms of resolution data allowing similar applications later on. In particular, the third kind of approaches is restricted to non-embedded resolution here; approaches to embedded resolution of singularities may e.g. be found in [**22**], [**3**], [**10**].

embedded situation, but blow up the variety itself and consider exceptional divisors inside the blown up variety.

It would be beyond the scope of this article to cover all these algorithms in depth, but they all have certain ingredients in common. In the first two situations, a desingularization is achieved by finite sequences of blow-ups at suitable non-singular centers. The differences between these algorithms then lie in the choice of center, which is the key step of each of these, but does not affect the structure of the practical representation of resolution data. For the third class of algorithms, blow-ups are not the only tool, but are combined with other tools, in particular normalization steps. However, the exceptional divisors to be studied arise from blow-ups and additionally only require proper tracing through the normalization steps if necessary. Therefore the technique to focus on in this context is blowing up; more precisely blowing up at non-singular centers.

**2.1. Blowing up – the computational side.** Let us briefly recall the definition of blowing up, as it can be found in any textbook on algebraic geometry (e.g. [**19**]), before explaining its computational side:

DEFINITION 2.1. Let $X$ be a scheme and $Z \subset X$ a subscheme corresponding to a coherent ideal sheaf $\mathcal{I}$. The blowing up of $X$ with center $Z$ is

$$\pi : \overline{X} := Proj(\bigoplus_{d\geq 0} \mathcal{I}^d) \longrightarrow X.$$

Let $Y \overset{i}{\hookrightarrow} X$ be a closed subscheme and $\pi_1 : \overline{Y} \longrightarrow Y$ the blow up of $Y$ along $i^{-1}\mathcal{I}\mathcal{O}_Y$. Then the following diagram commutes

$$\begin{array}{ccc} \overline{Y} & \hookrightarrow & \overline{X} \\ \pi_1 \downarrow & & \downarrow \pi \\ Y & \hookrightarrow & X \end{array}$$

$\overline{Y}$ is called the strict transform of $Y$, $\pi^*(Y)$ the total transform of $Y$.

To make $\overline{X}$ accessible to explicit computations of examples in computer algebra systems, it should best be described as the zero set of an ideal in a suitable ring. We shall assume now for simplicity of presentation that $X$ is affine because schemes are usually represented in computer algebra systems by means of affine covers. So we are dealing with the following situation: $J = \langle f_1, \ldots, f_m \rangle \subset A$ is the vanishing ideal of the center $Z \subset X = Spec(A)$ and the task is to compute

$$Proj(\bigoplus_{d\geq 0} J^d).$$

To this end, we consider the canonical graded $A$-algebra homomorphism

$$\Phi : A[y_1, \ldots, y_m] \longrightarrow \bigoplus_{n\geq 0} J^n t^n \subset A[t]$$

defined by $\Phi(y_i) = tf_i$. The desired object $\bigoplus_{d\geq 0} J^d$ is then isomorphic to

$$A[y_1, \ldots, y_m]/ker(\Phi)$$

or from the more geometric point of view $\overline{X}$ is isomorphic to $V(ker(\Phi)) \subset Spec(A) \times \mathbb{P}^{m-1}$.

The computation of the kernel in the above considerations is a standard basis computation and further such computations arise during the calculation of suitable centers in the different algorithms. Moreover, each blowing up introduces a further set of new variables as we have just seen and desingularization is hardly ever achieved with just one or two blow-ups – usually we are seeing long sequences thereof. The performance of standard bases based algorithms, on the other hand, is very sensitive to the number of variables as its complexity is doubly exponential in this number. Therefore it is vital from the practical point of view to pass from the $\mathbb{P}^{m-1}$ to $m$ affine charts and pursue the resolution process further in each of the charts[2]; consistency of the choice of centers does not pose a problem at this stage as this follows from the underlying desingularization algorithm. Although this creates an often very large tree of charts with the final charts being the leaves and although it postpones a certain part of the work, this approach has a further important advantage: it allows parallel computation by treating several charts on different processors or cores at the same time.

As a sideremark, we also want to mention the computation of the transforms, because they appeared in the definition cited above; for simplicity of notation we only state the affine case. Without any computational effort, we obtain the exceptional divisor as $I(H) = J\mathcal{O}_{\overline{X}}$ and the total transform $\pi^*(I) = I\mathcal{O}_{\overline{X}}$ for a subvariety $V(I)$ in our affine chart. The strict transform is then obtained by a saturation, i.e. an iteration of ideal quotients until it stabilizes: $I_{\overline{V(I)}} = (\pi^*(I) : I(H)^\infty)$; for the weak transform, the iteration may stop before it stabilizes, namely at the point where multiplying with the ideal of the exceptional divisor gives back the result of the previous iteration step for the last time. These ideal quotient computations are again based on standard bases.

**2.2. Identification of Points in Different Charts.** As we have just seen, it is more useful for the overall performance to pass to affine charts after each blow-up, even though this leads to an often rather large tree of charts. As a consequence, it is not possible to directly work with the result without any preparation steps, namely identifying points which are present in more than one chart – a practical step equivalent to the glueing of the charts. This is of particular interest for the identification of exceptional divisors which are present in more than one chart.

For the identification of points in different charts, we need to pass through the tree of charts – from one final chart all the way back to the last common ancestor of the two charts and then forward to the other final chart. As blowing up is an isomorphism away from the center, this step does not pose any problems as long as we do not need to identify points lying on an exceptional divisor which was not yet created in the last common ancestor chart. In the latter case, however, we do not have a direct means of keeping track of points originating from the same point of the center. The way out of this dilemma is a representation of points on the exceptional divisor as the intersection of the exceptional divisor with an auxilliary

[2] In practice, it is very useful to discard all charts not appearing in any other chart and not containing any information which is relevant for the desired application.

variety (or a suitable constructible set) not contained in the exceptional divisor (see [**17**], section 4.2.5 for details on finding such an auxilliary variety).

Given this means of identification of points, we can now also identify the exceptional divisors or, more precisely, the centers leading to the respective exceptional divisors. To avoid unnecessary comparisons between centers in different charts, we can a priori rule out all comparisons involving centers lying in different exceptional divisors. If the desingularization is controlled by an invariant as in [**9**], [**6**] or [**8**], we can also avoid comparisons with different values of the controlling invariant, because these cannot give birth to the same exceptional divisor either.

Example 2.2. To illustrate the explanations given so far and to provide a practical example to be used for all further applications, we now consider an isolated surface singularity of type $A_4$ at the origin. We shall illustrate this example using the computer algebra system Singular ([**13**]).

```
> // load the appropriate libraries for resolution of singularities
> // and applications thereof
> LIB"resolve.lib";
> LIB"reszeta.lib";
> LIB"resgraph.lib";

> // define the singularity
> ring R=0,(x,y,z),dp;
> ideal I=x5+y2+z2;                // an A4 surface singularity

> // compute a resolution of the singularity (Villamayor-approach)
> list L=resolve(I);
> size(L[1]);                      // final charts
6
> size(L[2]);                      // all charts
11
> def r9=L[2][9];                  // go to chart 9
> setring r9;
> showBO(BO);                      // show data in chart 9

==== Ambient Space:
_[1]=0

==== Ideal of Variety:
_[1]=x(2)^2+y(0)+1

==== Exceptional Divisors:
[1]:
   _[1]=1
[2]:
   _[1]=y(0)
[3]:
   _[1]=1
[4]:
```

```
  _[1]=x(1)

==== Images of variables of original ring:
_[1]=x(1)^2*y(0)
_[2]=x(1)^5*x(2)*y(0)^2
_[3]=x(1)^5*y(0)^2
> setring R;                          // go back to old ring
```

This yields a total number of 11 charts of which 6 are final charts. All blow-ups have zero-dimensional centers except the two line blow-ups leading from chart 6 to charts 8 and 9 and from chart 7 to charts 10 and 11. As it would not be very useful to reproduce all data of this resolution here, we show the total tree of charts as figure 1 and give the content of chart 9 as an example:

strict transform[3]: $V(x_2^2 + y_0 + 1)$

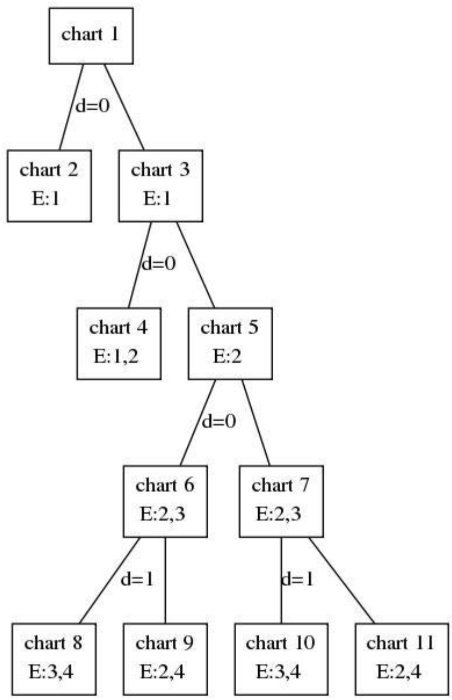


FIGURE 1. Tree of charts of an embedded desingularization of an $A_4$ surface singularity. The numbers given in the second line in each chart are the labels of the exceptional divisors visible in the respective chart. The numbers stated as $d = 0$ or $d = 1$ state the dimension of the center of the corresponding blow-up. Charts providing only data which is also present in other charts are not shown.

exceptional divisors: $V(y_0)$ from 2nd blow-up
$V(x_1)$ from last blow-up

[3]Weak and strict transforms coincide for hypersurfaces.

images of variables of original ring:

$x \longmapsto x_1^2 y_0$
$y \longmapsto x_1^5 x_2 y_0^2$
$z \longmapsto x_1^5 y_0^2$

For example, the exceptional divisor originating from the last blow-up leading to chart 9 needs to be compared to the exceptional divisor originating from the last blow-up leading to chart 10. To this end, one needs to consider the centers computed in charts 6 and 7 which turn out to be the intersection of the exceptional divisors labeled 2 and 3, if one considers the output of the resolution process in detail[4]. Therefore the last exceptional divisors in charts 9 and 10 coincide. This identification is implemented in SINGULAR and can be used in the following way:

```
> // identify the exceptional divisors
> list coll=collectDiv(L);
> coll[1];
0,0,0,0,  // no exc. div. in chart 1
1,0,0,0,  // first exc. div. is first in chart 2
1,0,0,0,  // .....
1,2,0,0,  // data too hard to read, better
0,2,0,0,  // use command below to create figure 1
0,2,3,0,
0,2,3,0,
0,0,3,4,
0,2,0,4,
0,0,3,4,
0,2,0,4

> //present the tree of charts as shown in figure 1
> ResTree(L,coll[1]);
```

## 3. Applications of Resolution of Singularities

The applications we present in this section originate from different subfields of mathematics ranging from algebraic geometry to singularity theory and $D$-modules. For each application we shall revisit our example from the previous section and also show how to perform the corresponding computation using SINGULAR.

**3.1. Intersection Form and Dual Graph of Resolution.** Given a resolution of an isolated surface singularity, we want to compute the intersection matrix of the excpetional divisors. This task does not require an embedded resolution of singularities, only an abstract one. Given such a desingularization, it can then be split up into 3 different subtasks:

(1) computation of the intersections $E_i.E_j$ for exceptional curves $E_i \neq E_j$
(2) computation of the self-intersection numbers $E_i^2$ for the exceptional curves $E_i$
(3) representation of the result as the dual graph of the resolution

[4]We encourage the reader to verify this by typing the above sequence of commands into SINGULAR and then exploring the data in the different charts.

If the given resolution, is not an abstract one, but an embedded one - like the result of Villamayor's algorithm - we need to add a preliminary step

(0) determine an abstract resolution from an embedded one. [5]

Although the definition of intersection numbers of divisors on surfaces can be found in many textbooks on algebraic geometry (e.g. in [**19**], V.1), we give a brief summary of the used properties for readers' convenience:

DEFINITION 3.1. Let $D_1, D_2$ be divisors in general position[6] on a non-singular surface $X$. Then the intersection number of $D_1$ and $D_2$ is defined as

$$D_1.D_2 := \sum_{x \in D_1 \cap D_2} (D_1.D_2)_x$$

where $(D_1.D_2)_x$ denotes the intersection multiplicity of $D_1$ and $D_2$ at $x$.

LEMMA 3.2. *For any divisors $D_1$ and $D_2$ on a non-singular surface $X$, there exist divisors $D_1'$ and $D_2'$, linearly equivalent to $D_1$ and $D_2$ respectively, such that $D_1'$ and $D_2'$ are in general position.*

LEMMA 3.3. *Intersection numbers have the following basic properties:*

(a) *For any divisors $C$ and $D$: $C.D = D.C$.*
(b) *For any divisors $C$, $D_1$ and $D_2$: $C.(D_1 + D_2) = C.D_1 + C.D_2$*
(c) *For any divisors $C$, $D_1$ and $D_2$, such that $D_1$ and $D_2$ are linearly equivalent: $C.D_1 = C.D_2$.*

At this point, we know what we want to compute, but we have to take care of another practical problem before proceeding to the actual computation which is then a straight forward calculation of the intersection numbers of exceptional curves $E_i \neq E_j$. The practical problem is that computations in a computer algebra system usually take place in polynomial rings over the rationals or algebraic extensions thereof. So we easily achieve a decomposition of the exceptional divisor into $\mathbb{Q}$-irreducible components, but we need to consider $\mathbb{C}$-irreducible components to obtain the intersection matrix we expect from the theoretical point of view. To this end, passing to suitable extensions of the ground field may be necessary – explicitly by introducing a new variable and a minimal polynomial (slowing down subsequent computations) or implicitly by taking into account the number of components over $\mathbb{C}$ for each $\mathbb{Q}$-component.

EXAMPLE 3.4. Revisiting our example of a desingularization of an $A_4$ surface singularity, we first need to pass to an abstract resolution.

```
\\ compute part of the tree of charts relevant for abstract resolution
> abstractR(L)[1];
   0,1,0,1,1,0,0,0,0,0,0   //final charts are 2,4,5
> abstractR(L)[2];
   0,0,0,0,0,1,1,1,1,1,1   //charts 6 and higher are irrelevant
                           //for non-embedded case
```

[5] This is achieved by canceling all trailing blow-ups in our tree of charts which are unnecessary for the non-embedded case. Then the intersection of the remaining excpetional divisors with the strict transform yields the exceptional locus of the non-embedded resolution.

[6] $D_1$ and $D_2$ are in general position, if their intersection is either empty or a finite set of points.

So we only see 2 exceptional divisors in the final charts, the ones labeled 1 and 2 (cf. figure 1). But looking at the charts in more detail, we would see e.g. in chart 4 that the first divisor is given by $V(y_2, y_0^2+1)$ and the second one by $V(x_2, y_0^2+1)$. So each of these has two $\mathbb{C}$-irreducible components. Considering these $\mathbb{C}$-components, we can obtain the following intersection data (seeing two of the intersections directly in chart 4 and the remaining one in chart 5):

$$\begin{pmatrix} * & 0 & 1 & 0 \\ 0 & * & 0 & 1 \\ 1 & 0 & * & 1 \\ 0 & 1 & 1 & * \end{pmatrix}$$

Hence only the self-intersection numbers – marked as * in the matrix – are still missing.

For the self-intersection numbers of the exceptional curves, we need to make use of another property of divisors in the context of desingularization:

LEMMA 3.5. *Let $\pi : \tilde{X} \longrightarrow X$ be a resolution of singularities of a surface $X$. Let $D_1$ be a divisor on $\tilde{X}$ all of whose components are exceptional curves of $\pi$ and let $D_2$ be a divisor on $X$, then*

$$\pi^*(D_2).D_1 = 0.$$

Denoting by $E_1, \ldots, E_s$ the $\mathbb{C}$-irreducible exceptional curves, we can hence consider a linear form $h : X \longrightarrow \mathbb{C}$ passing through the only singular point of $X$ and the divisor $D$ defined by it. We then know

$$\pi^*(D) = \sum_{i=1}^{s} c_i E_i + H$$

where $H$ denotes the strict transform of $D$ and the $c_i$ are suitable integers. From the lemma we additionally know that

$$0 = \pi^*(D).E_j = \sum_{i=1}^{s} c_i E_i.E_j + H.E_j \quad \forall 1 \leq j \leq s$$

where all intersection numbers are known or directly computable in each of the equations except the self-intersection numbers $E_j.E_j$ which we can compute in this way. For the dual graph of the resolution, each divisor is represented by a vertex (those with self-intersection -2 are unlabeled, the other ones labeled by their self-intersection number), each intersection is represented by an edge linking the two vertices corresponding to the intersecting exceptional curves.

EXAMPLE 3.6. The computation of the intersection form is implemented in SINGULAR and can be used as follows:

```
> // intersection matrix of exceptional curves
> // (no previous abstractR is needed, this is done automatically
> // in the procedure intersectionDiv)
> list iD=intersectionDiv(L);
> iD[1];
-2,0,1,0,
0,-2,0,1,
1,0,-2,1,
```

```
0,1,1,-2

> // draw the dual graph of the resolution
InterDiv(iD[1]);
```

This yields the expected intersection matrix (as entry iD[1] of the result)

$$\begin{pmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix}$$

and the dual graph of the resolution which is just the Dynkin diagram of the $A_4$ singularity.

**3.2. Discrepancies and Log-canonical Threshold.** In contrast to the last task, which only required an abstract resolution of the given surface singularity, the task of computing (log-)discrepancies and the log-canonical threshold requires embedded desingularization (or principalization of ideals). This is provided by Villamayor's algorithm. As before we first recall the definitions and some basic properties (see e.g. [**29**] for a direct and accessible introduction to the topic). To keep the exposition as short as possible and the considerations directly accesible to explicit computation, we restrict our treatment here to the case of a singular affine variety.

Definition 3.7. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-zero polynomial defining a hypersurface $V$ and let $\pi : X \longrightarrow \mathbb{C}^n$ be an embedded resolution of $V$. Denote by $E_j$, $j \in J$, the irreducible components of the divisor $\pi^{-1}(f^{-1}(0))$. Let $N(E_j)$ denote the multiplicity of $E_j$, $j \in J$, in the divisor of $f \circ \pi$ and let $\nu(E_j) - 1$ be the multiplicity of $E_j$ in the divisor $K_{X/\mathbb{C}^n} = \pi^*(dx_1 \wedge \ldots \wedge dx_n)$. Then the log-discrepancies of the pair $(\mathbb{C}^n, V)$ w.r.t. $E_j$, $j \in J$, are

$$a(E_j; \mathbb{C}^n, V) := \nu(E_j) - N(E_j).$$

The minimal log discrepancy of the pair $(\mathbb{C}^n, V)$ along a closed subset $W \subset \mathbb{C}^n$ is the minimum over the log-discrepancies for all $E_j$ with $\pi(E_j) \subset W$, i.e. originating from (sequences of) blow-ups with centers in $W$. The log-canonical threshold of the pair $(\mathbb{C}^n, V)$ is defined as

$$lct(\mathbb{C}^n, V) = inf_{j \in J} \frac{\nu(E_j)}{N(E_j)}.$$

Remark 3.8. The above definition of log-discrepancies and log-canonical threshold holds in a far broader context. Allowing more general pairs $(Y, V)$ it is also the basis for calling a resolution of singularities log-canonical, if the minimal log discrepancy of the pair along all of $Y$ is non-negative, and log-terminal, if it is positive.

As we already achieved an identification of exceptional divisors in a previous section, the only computational task here is the computation of the multiplicities $N(E_i)$ and $\nu(E_i)$. The fact that we might be dealing with $\mathbb{Q}$-irreducible, but $\mathbb{C}$-reducible $E_i$ does not pose any problem here, because we can easily check that the respective multiplicities coincide for all $\mathbb{C}$-components of the same $\mathbb{Q}$-component. To compute these multiplicities from the resolution data in the final charts (i.e. without moving through the tree of charts), we can determine $N(E_i)$ by finding the highest exponent $j$ such that $I(E_j)^j : J$ is still the whole ring where $J$ denotes the ideal of the total transform of the original variety. To compute the $\nu(E_i)$ we can use the same approach, but taking into account the appropriate Jacobian determinant.

EXAMPLE 3.9. We now simply continue our SINGULAR session on the basis of the data already computed in the previous examples

```
>// identify exceptional divisors (embedded case)
> list iden=prepEmbDiv(L);

>// multiplicities N(Ei)
> intvec cN=computeN(L,iden);
> cN;           // last integer is strict transform
2,4,5,10,1

>// multiplicities v(Ei)
> intvec cV=computeV(L,iden);
> cV;           // last integer is strict transform
3,5,7,12,1

>// log-discrepancies
> discrepancy(L);
0,0,1,1

>// compute log-canonical threshold
>// as an example of a loop in Singular
> number lct=number(cV[1])/number(cN[1]);
> number lcttemp;
> for (int i=1; i < size(cV); i++)
> {
>   lcttemp=number(cV[i])/number(cN[i]);
>   if(lcttemp < lct)
>   {
>      lct=lcttemp;
>   }
> }
> lct;
6/5
```

The log-canonical threshold, in particular, is a very important invariant which appears in many different contexts, ranging from a rather direct study of properties of pairs to the study of multiplier ideals (cf. [**26**]), to motivic integration or to Tian's $\alpha$-invariant which provides a criterion for the existence of Kähler-Einstein metrics (cf. [**32**]).

A real analogue to the log-canonical threshold, the real log-canonical threshold appears when applying resolution of singularities in the real setting [**30**]. In algebraic statistics, more precisely in model selection in Bayesian statistics, desingularization plays an important role in understanding singular models by monomializing the so-called Kullback-Leibler function at the true distribution. In this context the real log-canonical threshold is then used to study the asymptotics of the likelihood integral.[**27**]. It also appears in singular learning theory as the learning coefficient.

**3.3. Topological Zeta Function.** Building upon the multiplicities of $N(E_i)$ and $\nu(E_i) - 1$ of exceptional curves appearing in $f \circ \pi$ and $K_{X/\mathbb{C}^n} = \pi^*(dx_1 \wedge \ldots \wedge dx_n)$ which already appeared in the previous section, we can now define and compute the topological Zeta-function. As before, we first recall the definitions and properties (see e.g. [**14**]) and then continue with the computational aspects – listing the corresponding SINGULAR commands in the continued example of an $A_4$ surface singularity.

DEFINITION 3.10. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-zero polynomial defining a hypersurface $V$ and let $\pi : X \longrightarrow \mathbb{C}^n$ be an embedded resolution of $V$. Denote by $E_i$, $i \in I$, the irreducible components of the divisor $\pi^{-1}(f^{-1}(0))$ . To fix notation, we define for each subset $J \subset I$

$$E_J := \cap_{j \in J} E_j \text{ and } E_J^* := E_J \setminus \cup_{j \notin J} E_{J \cup \{j\}}$$

and denote for each $j \in I$ the multiplicity of $E_j$ in the divisor of $f \circ \pi$ by $N(E_j)$. We further set $\nu(E_j) - 1$ to be the multiplicity of $E_j$ in the divisor $K_{X/\mathbb{C}^n} = \pi^*(dx_1 \wedge \ldots \wedge dx_n)$. In this notation the topological Zeta-function of $f$ is

$$Z_{top}^{(d)}(f, s) := \sum_{\substack{J \subset I \text{s.th.} \\ d | N(E_j) \forall j \in J}} \chi(E_J^*) \prod_{j \in J} (\nu(E_j) + N(E_j)s)^{-1} \in \mathbb{Q}(s).$$

Intersecting the $E_J^*$ with the preimage of zero in the above formula leads to the local topological Zeta-function

$$Z_{top,0}^{(d)}(f, s) := \sum_{\substack{J \subset I \text{s.th.} \\ d | N(E_j) \forall j \in J}} \chi(E_J^* \cap \pi^{-1}(0)) \prod_{j \in J} (\nu(E_j) + N(E_j)s)^{-1} \in \mathbb{Q}(s).$$

(The local and global topological Zeta-function are independent of the choice of embedded resolution of singularities of $V$.)

Here, it is again important to observe that in the above context the irreducible components are taken over $\mathbb{C}$, while practical calculations usually take place over $\mathbb{Q}$ and further passing to components taken over $\mathbb{C}$ is rather expensive. The following lemma shows that considering $\mathbb{Q}$-irreducible components already allows the computation of the Zeta-function:

LEMMA 3.11. *Let $D_l$ , $l \in L$, be the $\mathbb{Q}$-irreducible components of the divisor $\pi^{-1}(f^{-1}(0))$. For each subset $J \subset L$ define $D_J$ and $D_J^*$ as above. Then the topological (global and local) Zeta-function can be computed by the above formulae using the $D_J$ and $D_J^*$ instead of the $E_J$ and $E_J^*$.*

As we already identified the exceptional divisors and computed the multiplicities $N(E_i)$ and $\nu(E_i)$, the only computational data missing is the Euler characteristic of the exceptional components in the final charts. If the intersection of exceptional divisors is zero-dimensional, this is just a matter of counting points using the identification of points in different charts. For 1-dimensional intersections the Euler characteristic can be computed using the geometric genus of the curve (using $\chi(C) = 2 - 2g(C)$). Starting from dimension two on, this becomes more subtle.

EXAMPLE 3.12. In our example which we have been treating throughout this article, we are dealing with a surface in a three-dimensional ambient space. So the only further Euler characteristics which need to be determined are those of the exceptional divisors themselves. At the moment of birth of an exceptional divisor, it will either be a $\mathbb{P}^2$ with Euler characteristic 3 or a $\mathbb{P}^1 \times C$ (for a one-dimensional center $C$) leading to Euler characteristic $4 - 4g(C)$. Under subsequent blow-ups the tracking of the changes to the Euler characteristic is then no difficult task.

```
// compute the topological zeta-function for our isolated
// surface singularity (global and local zeta-function coincide)
> zetaDL(L,1);           // global zeta-function
[1]:
   (s+6)/(5s2+11s+6)

> zetaDL(L,1,"local");   // local zeta-function
Local Case: Assuming that no (!) charts were dropped
during calculation of the resolution (option "A")
[1]:
   (s+6)/(5s2+11s+6)

// zetaDL also computes the characteristic polynomial
// of the monodromy, if additional parameter "A" is given
> zetaDL(L,1,"A");
Computing global zeta function
[1]:
   (s+6)/(5s2+11s+6)
[2]:
   (s4+s3+s2+s+1)
```

## 4. Desingularization in Experiments

The previous section showed some examples in which desingularization was a crucial step in the calculation of certain invariants and was hence used as a theoretical and practical tool. We now turn our interest to a different kind of settings: experiments on open questions which involve desingularization. Here we only sketch two such topics and the way one could experiment in the respective setting.[7]

**4.1. Bernstein-Sato polynomials.** In the early 1970s J.Bernstein [**5**] and M.Sato [**31**] independently defined an object in the theory of D-modules, which is nowadays called the Bernstein-Sato polynomial. The roots of such polynomials have a close, but still somewhat mysterious relation to the multiplicities of exceptional divisors in a related desingularization. For briefly recalling the definition of Bernstein-Sato polynomials, we shall follow the article of Kashiwara [**25**], which also introduces this relation. After that we sketch what computer algebra tools are available in SINGULAR for experiments on this topic.

[7] Neither of the two topics should be seen as a suggestion for a short term research project! Both questions, however, might gain new insights from someone playing around with such experiments just for a short while and stumbling into examples which open up new perspectives, insight or conjectures.

Definition 4.1. Let $f$ be an analytic function defined on some complex manifold $x$ of dimension $n$ and let $\mathcal{D}$ be the sheaf of differential operators of finite order on $X$. The polynomials in an additional variable $s$ satisfying

$$b(s)f^s \in \mathcal{D}[s]f^{s+1}$$

form an ideal. A generator of this ideal is called the Bernstein-Sato polynomial and denoted by $b_f(s)$.

Kashiwara then proves the rationality of the roots of the Bernstein-Sato polynomial by using Hironaka's desingularization theorem in the following way:

Theorem 4.2 ([**25**]). *Let $f$ be as above and consider a blow-up $F : X' \longrightarrow X$ and a new function $f' = f \circ F$. Then $b_f(s)$ is a divisor of $\prod_{k=0}^{N} b_{f'}(s+k)$ for a sufficiently large $N$.*

More precisely, a principalization of the ideal generated by $f$ leads to a polynomial $f' = \prod_{i=1}^{m} t_i^{r_i}$ with a local system of coordinates $t_1, \ldots, t_m$. For $f'$ the Bernstein-Sato polynomial is known to be

$$b_{f'}(s) = \prod_{i=1}^{m} \prod_{k=1}^{r_i} (r_i s + k).$$

Therefore the roots of the Bernstein-Sato polynomial are negative rational numbers of the form $-\frac{k}{r_i}$ with exceptional mulitplicities $r_i$ and $1 \leq k \leq r_i$. Yet it is not clear at all whether there are any rules or patterns which exceptional multiplicities $r_i$ actually appear as denominators of roots of the Bernstein-Sato polynomial. (As a sideremark: If this were known, then such knowledge could be used to speed up computations of Bernstein-Sato polynomials whenever a desingularization is known.)

Today there are implemented algorithms for computing the Bernstein-Sato polynomial of a given $f$ available: by Ucha and Castro-Jiminez [**33**], Andres, Levandovskyy and Martin-Morales [**1**] and by Berkesh and Leykin [**4**]. All of these algorithms do not use desingularization techniques, but rather rely on Gröbner-Bases and annihilator computations. The computed objects, however, are also used (theoretically and in examples) in the context of multiplier ideals and thus have relation to invariants like the log-canonical threshold.

Given algorithmic ways to independently compute the Bernstein-Sato polynomial and the exceptional multiplicities, one could now revisit the exploration of the interplay between these and try to spot patterns to get a better understanding.

**4.2. Positive Characteristic.** Desingularization in positive characteristic is one of the long standing, central open problems in algebraic geometry. In dimensions up to three there is a positive answer (see e.g. [**11**], [**12**]), but in the general case there are several different approaches (e.g. [**24**], [**34**], [**21**]) each of which has run into obstacles which are currently not resolved.

About a decade ago, Hauser started studying the reasons why Hironaka's approach of characteristic zero fails in positive characteristic [**20**]. Among other findings, he singled out two central points which break down:

(1) failure of maximal contact:
Hypersurfaces of maximal contact are central to the descent in ambient dimension which in turn is the key to finding the correct centers for blowing up. In positive characteristic, it is well known that hypersurfaces of maximal contact need not exist; allowing hypersurfaces satisfying only slightly weaker conditions is one of the central steps in the approach of Hauser and Wagner for dimension 2 [**21**]. (For higher dimensions this definition requires a little bit more care [**16**].)

(2) increase of order of coefficient ideal:
As the improvement of the singularities is measured by the decrease of the order and of orders for further auxilliary ideals constructed by means of descent in ambient dimension, it is crucial for the proof of termination of resolution that these orders cannot increase under blowing up. Unfortunately this does no longer hold for the orders of the auxilliary ideals in positive characteristic as has again been known since the 1970s. Hauser characterized the structure of polynomials which can exhibit such behaviour in [**20**].

Although problems of desingularization in positive characteristic are known, this knowledge seems to be not yet broad enough to provide sufficient feedback for suitable modification of one of the approaches to overcome the respective obstacles. Experiments could prove to be helpful to open up a new point of view. In particular, the approach of Hauser and Wagner for surfaces is sufficiently close to the characteristic zero approach of Hironaka (and hence to algorithmic approaches like the one of Villamayor) to allow modification of an existing implementation to provide a tool for a structured search for examples with special properties also in higher dimensions. This has e.g. been pursued in [**16**].

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Clay Mathematics Proceedings
Volume **20**, 2014

# Introduction to the Idealistic Filtration Program with emphasis on the radical saturation

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Abstract. This article is an expository account of what we call the Idealistic Filtration Program (IFP), an approach toward resolution of singularities of a variety defined over an algebraically closed field of positive characteristic, with emphasis on the role of the radical saturation. Some new result concerning the monomial case of the IFP using the radical saturation is also presented.

The purpose of this article is to give an informal and expository account of the Idealistic Filtration Program (abbreviated as IFP). Our goal is to explain the motivations behind the new concepts and main ideas, rather than to give the precise formulations and rigid arguments. One of the main ideas of the IFP is, given the original algebraic data (called an idealistic filtration), to take various *saturations* of the data, namely, the $\mathfrak{D}$-saturation (differential saturation), the $\mathfrak{R}$-saturation (radical saturation), the integral closure, and their combinations. Accordingly, the IFP is expected to have several variations, depending upon which saturations we use in the algorithm. In this article, we mainly discuss the IFP using the $\mathfrak{D}$-saturation, while we give some analysis of the IFP using the $\mathfrak{R}$-saturation at the same time. We remark, however, that it is not a trivial matter to incorporate various saturations into one coherent algorithm, and that some problems and obstacles, which we also discuss in this article, may appear in the process. So far, we have succeeded in completing a coherent algorithm only up to dimension three, using the notion of being relatively $\mathfrak{D}$-saturated. For the detail of this algorithm in dimension three, we refer the reader to our research articles [**16**], [**17**] and [**18**].

**Convention in this article**. Throughout this article, we assume that $k$ is an algebraically closed field and all varieties are defined over $k$.

## 0. Introduction

**0.1. Introduction to IFP.** The problem of resolution of singularities requires us, given a variety $X$, to construct a proper birational morphism $\pi\colon \widetilde{X} \to X$ from a nonsingular variety $\widetilde{X}$. It is one of the most important problems in the subject of algebraic geometry. In characteristic zero, Hironaka established its existence in arbitrary dimension, a theorem which is considered to be a mathematical milestone

2010 *Mathematics Subject Classification.* Primary 14E15.
*Key words and phrases.* Algebraic geometry, resolution of singularities.
The author is supported in part by JSPS Grant-in-Aid for Young Scientists (B) 23740016.

in the last century [**12**]. However, in positive characteristic, it is solved only up to dimension three ([**1**],[**7**]). Recently, various approaches toward its solution in arbitrary dimension have been proposed by several authors, including Hironaka, Villamayor, and Włodarczyk, among others. The main subject of this article, the IFP, is one of such approaches. It is first proposed by the author, and then developed further in collaboration with Matsuki. We list the novelty of the IFP in the following:

- Introduction of the new algebraic object called an *idealistic filtration*: It is a natural generalization of the notion of an idealistic exponent initiated by Hironaka. However, the critical difference is that we use *saturations* in order to analyze an idealistic filtration, while one uses the Hironaka equivalence in order to analyze an idealistic exponent.
- Introduction of the new notion of a *leading generator system* (called an *LGS* for short): While it is known that there is *no* hypersurface of maximal contact in positive characteristic, an LGS provides a collective substitute in arbitrary characteristic for the notion of a hypersurface of maximal contact in characteristic zero along the line first considered by Giraud [**9**].[1]
- Introduction of a new *Nonsingularity Principle* of the center: In characteristic zero, it is the existence of a hypersurface of maximal contact that guarantees the nonsingularity of the center. Confronted by the non-existence in positive characteristic, we establish a new principle which guarantees that the center of blow up in our algorithm is always nonsingular.

We discuss the details of the novelties in §1 and §2.

**0.2. Hypersurface of maximal contact.** After Hironaka's work [**12**], the proof of resolution of singularities in characteristic zero has been extensively refined and simplified by Bierstone-Milman[**4**], Villamayor[**24**], Włodarczyk [**25**] among others, ultimately leading to some beautiful and constructive algorithm. The structure of the proof is based upon the inductive scheme on dimension, for which the notion of a hypersurface of maximal contact plays the central role. We call this algorithm in characteristic zero as the *classical* algorithm in this article.

We give a short and rough description of a hypersurface of maximal contact as follows: Let $X \subset M$ be a closed subscheme of a nonsingular variety $M$, with its defining ideal $\mathcal{I}_X \subset \mathcal{O}_M$. Fix a closed point $P \in X$ and set $\mu = \mu_P(\mathcal{I}_{X,P})$ the order of $\mathcal{I}_X$ at $P$. Then a hypersurface of maximal contact of $X$ at $P$ is a *nonsingular* hypersurface, locally defined in a neighborhood of $P$, such that it contains the locus where the order of $\mathcal{I}_X$ is at least $\mu$. Moreover, we require this property persists to hold after any sequence of permissible blow ups.

A hypersurface of maximal contact always exists in characteristic zero. We can see its existence utilizing the derivations. Actually, it is easy to see that the set $\{\partial_1\partial_2\cdots\partial_{\mu-1}f \mid \partial_i \in \mathrm{Der}_k(\mathcal{O}_{M,P}),\ f \in \mathcal{I}_{X,P}\}$ contains an element $h \in \mathcal{O}_{M,P}$ with $\mu_P(h) = 1$, and that such $h$ defines a hypersurface of maximal contact of $X$ at $P$. In positive characteristic, on the other hand, it does not exist in general. We present the following examples in characteristic 2 and 3, which are variations of the famous example due to R. Narasimhan [**21**].

[1]The origin of the notion of an LGS can also be traced back to the works of Hironaka [**13**] and Oda [**23**] concerning the additive group schemes.

EXAMPLE 0.2.1. A Narasimhan-type example in $\mathrm{char}(k) = 2$. For $\ell \in \mathbb{Z}_{\geq 0}$, set

$$F_\ell = x^2 + yz^3 + zw^3 + wy^{2\ell+3} \in k[x, y, z, w].$$

Then $F_\ell$ defines a hypersurface $V_\ell = \mathrm{V}(F_\ell) \subset \mathbb{A}^4_k$. The order of $F_\ell$ on $\mathbb{A}^4_k$ is at most 2, and the locus where the order is 2 is nothing but the singular locus of $V_\ell$, i.e.,

$$\mathrm{Sing}(V_\ell) = \mathrm{V}(f, f_x, f_y, f_z, f_w) = \mathrm{V}(x^2 + zw^3, z^3 + wy^{2\ell+2}, yz^2 + w^3, zw^2 + y^{2\ell+3}).$$

It is easy to see that $\mathrm{Sing}(V_\ell)$ contains the curve $C_\ell$ defined as follows:

$$C_\ell = \left\{(t^{9\ell+14}, t^7, t^{6\ell+7}, t^{4\ell+7}) \mid t \in k\right\} \subset \mathbb{A}^4_k.$$

Now assume $\ell \notin 7\mathbb{Z}$. Then, none of $\{9\ell + 14, 7, 6\ell + 7, 4\ell + 7\}$ is expressed as a $\mathbb{Z}_{\geq 0}$-linear combination of the others, a fact which implies that no local nonsingular hypersurface at the origin can contain $C_\ell$. Therefore, for $\ell \in \mathbb{Z}_{\geq 0} \setminus 7\mathbb{Z}$, $V_\ell$ has no hypersurface of maximal contact at the origin. Note that the original Narasimhan's example corresponds to the case $\ell = 2$.

EXAMPLE 0.2.2. A Narasimhan-type example in $\mathrm{char}(k) = 3$. Set

$$\begin{aligned} F = &-y_0^3 + y_1y_2y_3^4 + y_2y_3y_4^4 + y_3y_4y_5^4 + y_4y_5y_1^4 + y_5y_1y_2^7 \\ &+ y_1y_3y_5y_6^3 + y_2y_4y_1y_7^3 + y_3y_5y_2y_8^3 + y_4y_1y_3y_9^3 + y_5y_2y_4y_{10}^3 \in k[y_0, \ldots, y_{10}] \end{aligned}$$

Then, $\mathrm{V}(F) \subset \mathbb{A}^{11}$ has no hypersurface of maximal contact at the origin. Actually, the locus where the order of $F$ is at least 3 contains the curve

$$C = \left\{(t^{625}, t^{310}, t^{181}, t^{346}, t^{337}, t^{298}, t^{307}, t^{349}, t^{350}, t^{294}, t^{353}) \mid t \in k\right\}.$$

The rest of the argument is identical to the previous example.

REMARK 0.2.3. For each characteristic, Narasimhan [**22**] gives an example where there is no hypersurface of maximal contact. The construction in [**22**] is based upon the method different from the one in [**21**], and hence different from ours used to construct the examples given above.

**0.3. Basic strategy of IFP.** As we have seen in §0.2 a hypersurface of maximal contact does not exist in positive characteristic in general. Since a hypersurface of maximal contact plays the central role in the inductive scheme on dimension in characteristic zero, how should we find an inductive scheme to establish resolution of singularities in positive characteristic?

Our answer to this question is the following. We introduce an algebraic object, called an *idealistic filtration*, and define its various *saturations*. In characteristic zero, we observe that the generators of "the order 1-part" of a $\mathfrak{D}$-saturated idealistic filtration can be taken all from level 1, and that they correspond to hypersurfaces of maximal contact. In positive characteristic, these generators may not be concentrated at level 1 even when the idealistic filtration is $\mathfrak{D}$-saturated. However, we regard a set of these generators satisfying certain conditions, called a *leading generators system* (abbreviated as LGS), as a collective substitute in arbitrary characteristic for the notion of a hypersurface of maximal contact.

We remark that an element in an LGS may be located at level higher than 1, and then it defines a *singular* hypersurface. In other words, we need to deal with "a *singular* hypersurface of maximal contact" in positive characteristic. This may be considered as a major drawback. In fact, the nonsingularity of the center, which follows from the nonsingularity of a hypersurface of maximal contact in characteristic zero, is not automatic in positive characteristic in our setting. However, by analyzing a $\mathfrak{D}$-saturated idealistic filtration, we can establish a new *Nonsingularity Principle*, which guarantees that the center of blow up in our algorithm is always nonsingular. We also need further tune-ups to establish the basic properties, e.g., the upper semi-continuity of the invariant and so on, in order to carry out our algorithm in positive characteristic in a way parallel to the classical algorithm.

We give a brief comparison of the IFP with the approaches by Villamayor and Hironaka below. For Włodarczyk's approach, we refer the reader to his slides on the web page[2] of the workshop, held at RIMS in 2008.

In the approach by Villamayor and his collaborators, the main algebraic object is a Rees algebra. Their basic strategy is to carry out the inductive scheme on dimension by taking a generic projection. Starting from a Rees algebra on a nonsingular ambient space, by taking a generic projection, they construct another Rees algebra, called an elimination algebra, on a nonsingular ambient space of dimension one less. Even though an idealistic filtration and a Rees algebra are similar in nature (see, for example, Blanco and Encinas [**2**]), the inductive schemes are quite different in our approach and theirs. It is a quite interesting fact that, in spite of the difference, there seems to be a dictionary between the two approaches. For example, some of their analysis in the monomial case in dimension three can be translated into the language of the IFP (see §2.5). Bravo's article in this volume would be a good introduction to their approach.

Hironaka introduced the notion of edge generators, whose configuration gives rise to the first invariant in his approach. This corresponds to the part of the IFP where we take an LGS, whose configuration gives rise to our first invariant $\sigma$. However, the similarity between the two approaches seems to stop here. In Hironaka's approach, the second invariant is computed from the residual orders of the edge generators, while in our approach the second invariant is the order modulo the LGS. The behavior of the residual order is subtle; it may increase after permissible blow ups. The analysis of the structure of this increase seems to be the key step in Hironaka's approach (cf. the notion of a *metastable singularity* in Hironaka's manuscript. See also Hauser [**11**]), while our second invariant does not increase after permissible blow ups as long as we compute the invariants for the transformation of an idealistic filtration without taking further saturations (see §2.4(2) for more detail). However, the author has to confess that he does not have a full understanding of Hironaka's approach, and the reader is encouraged to look into Hironaka's manuscript on the web page[3] of the CMI Summer School 2012.

**0.4. Structure of this article.** The contents of §1 and §2 of this article correspond to the first and second lectures given by the author at the CMI Summer School 2012, respectively. In §1, we focus our attention on the subject of an *idealistic filtration*. We present its precise definition, define various saturations, and then discuss their properties. Even though the notion of an idealistic filtration was conceived in order to solve the problem of resolution of singularities, the author feels it is an interesting object of the study in its own right. In §2, we discuss the details of the IFP. We show how an LGS plays the role of a collective substitute in positive characteristic for the notion of a hypersurface of maximal contact in characteristic zero. We also compare our algorithm constructed according to the IFP with the classical one in characteristic zero.

[2] `http://www.kurims.kyoto-u.ac.jp/~kenkyubu/proj08-mori/`

[3] `http://www.claymath.org/programs/summer_school/2012/`

**0.5. Acknowledgments.** This article is based on the author's two lectures at the Clay Mathematics Institute Summer School 2012 "The Resolution of Singular Algebraic Varieties" in Obergurgl, Austria. The author is grateful to the Clay Mathematics Institute and all the participants. Special thanks go to the organizers: Professors Ellwood, Hauser, Mori and Schicho, for this wonderful summer school. He thanks greatly Professor Matsuki, his longtime collaborator, for making numerous suggestions to improve the first draft of this article. This article is written during the author's stay at the Erwin Schrödinger International Institute for Mathematical Physics as a member of Research in Teams "Resolution of Surface Singularities in Positive Characteristic", and also at Vienna University visiting Professor Hauser. His best appreciation goes to their kind hospitality and support. It is also a pleasure to acknowledge the support of the Global COE Program "Fostering top leaders in mathematics", Ministry of Education, Culture, Sports, Science and Technology, in Japan. Finally, the author would like to thank the referee and Doctor Stefan Perlega for many helpful comments.

## 1. Idealistic Filtration

First we give the precise definition of an idealistic filtration, the main algebraic object of our study in the IFP (§1.1). Then we introduce its various saturations, namely, the $\mathfrak{D}$-saturation, the $\mathfrak{R}$-saturation and the integral closure (§1.2, §1.3). Finally we investigate their properties and relations among them (§1.3, §1.4). The main source of reference is [**16**], where one can find the details of all the arguments.

**Convention in §1**. In this section, the symbol $R$ represents a regular ring essentially of finite type over $k$.

**1.1. Definition of idealistic filtration.** First we give a brief review on the classical algorithm, in order to explain the motivation behind the definition of an idealistic filtration. The problem of resolution of singularities is reduced to the following problem of decreasing the order of an ideal by blow ups: We are given an ideal $\mathcal{I}$ on a nonsingular variety $W$. Fix a positive integer $a \in \mathbb{Z}_{>0}$. When we choose a *permissible* nonsingular center $C \subset W$ with respect to a pair $(\mathcal{I}, a)$, namely, when $C$ is contained in the locus where the order of $\mathcal{I}$ is at least $a$, we define the transformation of the ideal by the formula $\widetilde{\mathcal{I}} = I(\pi^{-1}(C))^{-a} \cdot \pi^{-1}(\mathcal{I})\mathcal{O}_{\widetilde{W}}$, where $\pi \colon \widetilde{W} \to W$ is the blow up along the center $C$. We are required to construct a sequence of transformations, with all the centers having only normal crossings with the exceptional divisor, so that the order of the final transformation is everywhere smaller than $a$.

The classical algorithm to solve this problem in characteristic zero adopts the inductive scheme on dimension, by taking a hypersurface of maximal contact which contains the locus where the order of the ideal is the maximum (cf. §0.2). Moreover, instead of aiming at decreasing the order directly, which may increase after blow up, the algorithm is designed to decrease the so called *weak order* first, namely, the order of the reduced part of the ideal obtained by subtracting the exceptional factors as much as possible. When the algorithm reduces the weak order to be 0, therefore, the ideal is actually the monomial of the defining variables of the components of the exceptional divisor. We say we are in the *monomial case.* Since the exceptional divisor has only simple normal crossings, we can now construct a sequence of blow ups choosing the centers by a simple and combinatorial method, to finally reduce the order of the ideal below the required level $a$. This is how the classical algorithm works. We refer the readers to the expository articles in this volume for the details of the classical algorithm.

Recall that the pair $(\mathcal{I}, a)$ of an ideal sheaf over $W$ and a fixed positive integer $a \in \mathbb{Z}_{>0}$, called the *level*, is at the core of the classical analysis. If we restrict ourselves to the local settings, this is equivalent to considering a collection of pairs $(f, a)$ with $f \in I \subset R$, where $R$ is the coordinate ring of an affine open subset or the local ring at the closed point of $W$, and where $I$ is the ideal of $R$ corresponding to the ideal sheaf $\mathcal{I}$. In our setting, we let the level $a$ vary among all the real numbers $\mathbb{R}$. Actually, the rational levels have already appeared implicitly in the classical algorithm, while we also allow the real levels to consider the limit of the levels, an operation which appears when we introduce the $\mathfrak{R}$-*saturation*.

After all, the consideration of a subset $S \subset R \times \mathbb{R}$ is at the core of our analysis. Since the goal is to reduce the order, we may interpret an element $(f, a) \in S$ as a statement "the order of $f$ is at least $a$". In view of this interpretation, we would like to consider the collection $\mathrm{G}(S) \subset R \times \mathbb{R}$ of all the statements which deduced from the statements in $S$. Needless to say, $\mathrm{G}(S)$ should contain $S$. As the order of any function is non-negative, we require (1) $R \times \mathbb{R}_{\leq 0} \subset \mathrm{G}(S)$. Since the order of the sum (resp. the product) of functions is at least the minimum (resp. the sum) of their orders, we also require that (2) $(f, a), (g, b) \in \mathrm{G}(S)$ implies $(f+g, \min\{a, b\}) \in \mathrm{G}(S)$ and that (3) $(f, a), (g, b) \in \mathrm{G}(S)$ implies $(fg, a+b) \in \mathrm{G}(S)$. Thus we define $\mathrm{G}(S)$ to be the set of all the elements in $R \times \mathbb{R}$, which we obtain, starting from the initial data $S \cup (R \times \mathbb{R}_{\leq 0})$, by applying the rules of sums and products corresponding to the above conditions (2) and (3) finitely many times.

The above line of thoughts leads to the following abstract definition of an idealistic filtration.

Definition 1.1.1. We say a subset $\mathbb{I} \subset R \times \mathbb{R}$ is an idealistic filtration over $R$ if it satisfies the following conditions:

(1) $f \in R$, $a \in \mathbb{R}_{\leq 0} \Rightarrow (f, a) \in \mathbb{I}$,
(2) $(f, a), (g, a) \in \mathbb{I} \Rightarrow (f + g, a) \in \mathbb{I}$,
(3) $(f, a), (g, b) \in \mathbb{I} \Rightarrow (fg, a + b) \in \mathbb{I}$.

We denote by $\mathbb{I}_a$ the set of all the elements in $R$ lying at the level $a \in \mathbb{R}$, namely, $\mathbb{I}_a = \{f \in R \mid (f, a) \in \mathbb{I}\}$. It follows from the definition that $\mathbb{I}_a$ is an ideal of $R$ for any $a$ and that $\mathbb{I}_a \supset \mathbb{I}_b$ if $a \leq b$. Given a subset $S \subset R \times \mathbb{R}$, we denote by $\mathrm{G}(S)$ the smallest idealistic filtration containing $S$, and call it the idealistic filtration generated by $S$.

In the rest of this article, the symbol $\mathbb{I}$ always represents an idealistic filtration over $R$.

Example 1.1.2. Let $I \subset R$ be an ideal of $R$ and $a \in \mathbb{R}_{>0}$ a positive real number. Then the idealistic filtration $\mathbb{I} = \mathrm{G}(I \times \{a\})$ is characterized by the description $\mathbb{I}_b = I^{\lceil b/a \rceil}$ for any $b \in \mathbb{R}_{>0}$ and $\mathbb{I}_b = R$ for any $b \in \mathbb{R}_{\leq 0}$.

Remark 1.1.3. Many authors in the subject of resolution of singularities consider an algebraic object similar to the notion of an idealistic filtration, e.g., the notion of a *presentation* by Bierstone-Milman, the notion of a *basic object* by Villamayor, and so on, all of which find their origin in the notion of an *idealistic exponent* initiated by Hironaka. These authors also introduce an equivalence relation among

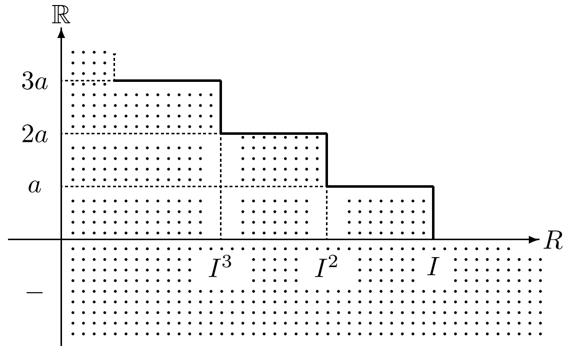


FIGURE 1. $\mathbb{I} = \mathrm{G}(I \times \{a\})$ in Example 1.1.2.

these objects, usually called the Hironaka equivalence, by considering the behavior of all the possible sequences of permissible blow ups (and smooth morphisms). However, we would like to emphasize that, in our setting, we investigate various saturations of an idealistic filtration extensively instead of considering the Hironaka equivalence. We discuss the relation between the Hironaka equivalence and the saturations in our setting briefly in Remark 1.6.3.

We define the order and the support of a subset $S \subset R \times \mathbb{R}$, and in particular of an idealistic filtration $\mathbb{I} \subset R \times \mathbb{R}$.

DEFINITION 1.1.4. Let $S \subset R \times \mathbb{R}$ be a subset. Then we define the order $\mu_P(S)$ of $S$ at $P \in \operatorname{Spec} R$ and the support $\operatorname{Supp}(S) \subset \operatorname{Spec} R$ as follows:

$$\begin{aligned} \mu_P(S) &= \inf\{\mu_P(f)/a \mid (f,a) \in S,\ a \in \mathbb{R}_{>0}\}, \\ \operatorname{Supp}(S) &= \{P \in \operatorname{Spec} R \mid \mu_P(S) \geq 1\}, \end{aligned}$$

We remark that $\mu_P(S)$ is upper semi-continuous as a function of $P \in \operatorname{Spec} R$, a fact which easily follows from the assumption that $R$ is a regular ring essentially of finite type over $k$ (cf. Remark 1.2.7 (1)). In particular, it follows that $\operatorname{Supp}(S)$ is a closed subset of $\operatorname{Spec} R$.

EXAMPLE 1.1.5. Assume that an idealistic filtration $\mathbb{I}$ is generated by an ideal $I \subset R$ at level $a \in \mathbb{R}$, i.e., $\mathbb{I} = \mathrm{G}(I \times \{a\})$. Then $\operatorname{Supp}(\mathbb{I})$ is the locus where $I$ has order at least $a$.

We start from a given idealistic filtration, which incorporates the initial data of the problem of resolution of singularities. In order to extract the "good" information, such as the invariants we use in our algorithm, which should be independent of the choice of the original idealistic filtration, we make the idealistic filtration as large as possible, by taking various saturations, *provided that this enlargement process leaves the problem of resolution of singularities "intact"*. The larger an idealistic filtration is, the better it is. This is the philosophy. However, we note that this philosophy has to be taken with a grain of salt. Taking the $\mathfrak{D}$-saturation, the $\mathfrak{R}$-saturation and the integral closure at the beginning of the resolution process, we leave the Hironaka equivalence class of a given idealistic filtration intact. Therefore, we would like to say these saturations are good candidates to realize our philosophy. However, taking these saturations may affect the invariants we use in our algorithm and hence the construction of a resolution sequence. To make the situation worse, taking these saturations after blow ups may end up increasing these invariants. This is why we say in the prologue that it is not a trivial matter to incorporate various saturations into one coherent algorithm. For more details, see §2.

**1.2. The differential operators and the differential saturation.** In order to introduce the notion of the $\mathfrak{D}$-saturation of an idealistic filtration, we first give a brief review of the basics of the theory of the differential operators. Recall that a derivation $\partial \in \mathrm{Der} = \mathrm{Der}_k(R)$ is characterized as a $k$-linear map $\partial \in \mathrm{End}_k(R)$ which satisfies the Leibniz rule $\partial(\alpha\beta) = \partial(\alpha)\beta + \alpha\partial(\beta)$. A differential operator is the generalization of a derivation, as defined below. In characteristic zero, a differential operator is obtained as an $R$-linear combination of the compositions of the derivations. However, this is no longer the case in positive characteristic (cf. Remark 1.2.7 (2)). This is the major difference between the theory of the differential operators in characteristic zero and that in positive characteristic.

Definition 1.2.1. Let $n \in \mathbb{Z}_{\geq 0}$ be a non-negative integer. A $k$-linear map $\partial\colon R \to R$ is called a differential operator of degree $\leq n$ (on $R$ over $k$) if $\partial$ satisfies the generalized Leibniz rule of degree $n$, i.e.,

$$\sum_{\Gamma\subset\Lambda}(-1)^{\#\Gamma}\left(\prod_{\lambda\in\Lambda\setminus\Gamma} r_\lambda\right)\partial\left(\prod_{\lambda\in\Gamma} r_\lambda\right) = 0$$

for any $r = (r_0, r_1, \ldots, r_n) \in R^{n+1}$, where $\Lambda = \{0, 1, 2, \ldots, n\}$. We denote by $\mathrm{Diff}^{\leq n} = \mathrm{Diff}_R^{\leq n}$ the set of all the differential operators of degree $\leq n$. It is easy to see that $\mathrm{Diff}^{\leq n} \subset \mathrm{Diff}^{\leq n+1}$ (exercise). Given a subset $T \subset R$, we denote by $\mathrm{Diff}^{\leq n}(T)$ the ideal of $R$ generated by all the elements of $T$ after applying the differential operators of degree $\leq n$, i.e., $\mathrm{Diff}^{\leq n}(T) = (\partial t \in R \mid \partial \in \mathrm{Diff}^{\leq n},\ t \in T)$.

Example 1.2.2. We write down the generalized Leibniz rule for $n \leq 1$ explicitly.

$\underline{n=0}$: $\quad d \in \mathrm{Diff}^{\leq 0} \Leftrightarrow r_0 d(1) - 1\cdot d(r_0) = 0 \Leftrightarrow d(r_0) = d(1)r_0 \quad \forall r_0 \in R.$

Therefore, a differential operator $d$ of degree $\leq 0$ is a multiplication of a constant $d = d(1)\cdot \mathrm{id}_R$, i.e., an $R$-linear homomorphism of $R$.

$\underline{n=1}$: $\quad d \in \mathrm{Diff}^{\leq 1} \Leftrightarrow r_0r_1d(1) - r_0d(r_1) - r_1d(r_0) + d(r_0r_1) = 0 \quad \forall r_0, r_1 \in R.$

Therefore, a differential operator $d$ of degree $\leq 1$ is expressed as $d = d(1)\cdot\mathrm{id}_R + d'$ for some derivation $d' \in \mathrm{Der}$. Note that $\mathrm{Der} = \{\partial \in \mathrm{Diff}^{\leq 1} \mid \partial(1) = 0\}$.

In order to carry out some explicit calculations involving the differential operators, the following notion of the *partial differential operators* is quite useful.

Definition 1.2.3. A subset $\{x_\lambda \mid \lambda \in \Lambda\} \subset R$ is called a set of differential coordinates for $R$ (over $k$) if the following equality holds:

$$\Omega^1_{R/k} = \bigoplus_{\lambda\in\Lambda} R dx_\lambda.$$

Lemma 1.2.4. *Let $W$ be a nonsingular variety and $P \in W$ a closed point. Then, a regular system of parameters (abbreviated as r.s.p.) for the local ring $\mathcal{O}_{W,P}$ at $P$ is a set of differential coordinates. Moreover, it expands to a set of differential coordinates for the coordinate ring of some affine open neighborhood of $P$ in $W$.*

PROPOSITION 1.2.5. *Assume $R$ has a set of differential coordinates $X = \{x_\lambda \mid \lambda \in \Lambda\} \subset R$. Then, there exists a set of differential operators $\{\partial_{X^I} \in \mathrm{Diff}^{\leq |I|} \mid I \in \mathbb{Z}^\Lambda_{\geq 0}\}$ on $R$ satisfying the following conditions:*

(1) $\partial_{X^I}(X^J) = \binom{J}{I} X^{J-I}$ *for any* $I, J \in \mathbb{Z}^\Lambda_{\geq 0}$,

(2) $\mathrm{Diff}^{\leq n} = \bigoplus_{|I| \leq n} R\partial_{X^I}$ *for any* $n \in \mathbb{Z}_{\geq 0}$,

(3) $\partial_{X^I}(fg) = \sum_{J+K=I} (\partial_J f)(\partial_K g)$ *for any* $I \in \mathbb{Z}^\Lambda_{\geq 0}$ *and* $f, g \in R$.

*Note that we use the multi-index notation above. Namely, setting $I = (i_\lambda \mid \lambda \in \Lambda)$, we have $|I| = \sum_{\lambda \in \Lambda} i_\lambda$, $X^I = \prod_{\lambda \in \Lambda} x_\lambda^{i_\lambda}$, $\binom{J}{I} = \prod_{\lambda \in \Lambda} \binom{j_\lambda}{i_\lambda}$, and so on.*

DEFINITION 1.2.6. We call $\{\partial_{X^I} \mid I \in \mathbb{Z}^\Lambda_{\geq 0}\}$ the partial differential operators with respect to a set of differential coordinates $X$. The property (2) says they form a basis of the differential operators, while the property (3) is called the generalized product rule.

REMARK 1.2.7.
(1) Let $I \subset R$ be an ideal. For given point $P \in \operatorname{Spec} R$, Proposition 1.2.5 implies

$$\mu_P(I) \geq n \Leftrightarrow P \in \mathrm{V}(\mathrm{Diff}^{\leq n-1}(I)).$$

From this it follows that the order function $\operatorname{Spec} R \ni P \mapsto \mu_P(I) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ of an ideal $I$ is upper semi-continuous.
(2) Note that $\{\partial_{X^I} \mid |I| = 1\} = \{\frac{\partial}{\partial x_\lambda} \mid \lambda \in \Lambda\}$ forms a basis of Der as an $R$-module. If $\mathrm{char}(k) = 0$, all the partial differential operators are also written as the compositions of the derivations, i.e.,

$$\partial_{X^I} = \prod_{\lambda \in \Lambda} \frac{1}{i_\lambda !} \left( \frac{\partial}{\partial x_\lambda} \right)^{i_\lambda} \qquad \text{for any } I = (i_\lambda \mid \lambda \in \Lambda) \in \mathbb{Z}^\Lambda_{\geq 0} \quad \text{if } \mathrm{char}(k) = 0.$$

However, in $\mathrm{char}(k) = p > 0$, there are differential operators which cannot be expressed in terms of the compositions of the derivations. For example, $\partial_{x_\lambda^{p^e}}$ with $e \in \mathbb{Z}_{>0}$ never has such an expression. In fact, $\partial_{x_\lambda^{p^e}}(1) = 0$ and $\partial_{x_\lambda^{p^e}}(x_\lambda^{p^e}) = 1$, but $\partial(x_\lambda^{p^e}) = 0$ for any derivation $\partial \in \mathrm{Der}$.
(3) The logarithmic differential operators are defined as follows: Let $E \subset \operatorname{Spec} R$ be a simple normal crossing divisor and $Y \subset R$ the defining variables of the components of $E$. A differential operator $\partial \in \mathrm{Diff}^{\leq n}$ of degree $\leq n$ is called logarithmic with respect to $E$ if $\partial(y^m) \subset (y^m)$ for any $y \in Y$ and $m \in \mathbb{Z}_{\geq 0}$. The set of all the logarithmic differential operators of degree $\leq n$ with respect to $E$ is denoted by $\mathrm{Diff}^{\leq n}_E = \mathrm{Diff}^{\leq n}_{R,E}$.

We conclude this brief review by giving a characterization of an ideal whose generators can be taken from the set of the $p^e$-th powers in terms of the differential operators.

PROPOSITION 1.2.8. *Assume* $\mathrm{char}(k) = p > 0$. *Let $I \subset R$ be an ideal of $R$, $e \in \mathbb{Z}_{\geq 0}$ and $R^{[p^e]} = \{f^{p^e} \mid f \in R\}$. Then, the following conditions are equivalent:*
*(1)* $\mathrm{Diff}^{\leq p^e - 1}(I) = I$.
*(2) $I$ is generated by some finite subset of $R^{[p^e]}$.*

Sketch of the proof. Assume $R = k[[X]] = k[[x_1, \dots, x_n]]$ is a formal power series ring over $k$. Then $R$ has an r.s.p. $X$. Observing that the operators $\{\partial_{X^I} \mid |I| < p^e\}$ commute with the multiplication by any element in $R^{[p^e]} = \{f^{p^e} \mid f \in R\}$, we see that (2) implies (1). Next, take $g \in I$ and express it as $g = \sum_{J \in \Delta^n} g_J^{p^e} X^J$, where $\Delta = \{0, \dots, p^e - 1\}$. Then, (1) implies $\{g_J^{p^e} \mid J\} \subset I$. Thus $I \cap R^{[p^e]}$ generates $I$. For a general $R$, see [**16**]. □

Before defining the $\mathfrak{D}$-saturation of an idealistic filtration, we would like to discuss the motivation behind it. Take $f \in R$ and $\partial \in \mathrm{Diff}^{\leq t}$. Suppose $\mu_P(f) \geq a$ for $P \in \operatorname{Spec} R$. Then from Proposition 1.2.5 it follows easily that $\mu_P(\partial f) \geq a - t$. Therefore, $(\partial f, a - t)$, which we call the "differentiation" of $(f, a)$ by $\partial$, is deduced from $(f, a)$ as the statement on the order. Therefore the enlargement of the idealistic filtration by adding all the differentiations of its elements is expected to leave the problem of resolution of singularities "intact" (cf. §1.1).

Definition 1.2.9. We say an idealistic filtration $\mathbb{I}$ is differentially saturated (or $\mathfrak{D}$-saturated) if the following condition holds:

$$\mathrm{Diff}^{\leq t}(\mathbb{I}_a) \subset \mathbb{I}_{a-t} \quad \forall a \in \mathbb{R},\ \forall t \in \mathbb{Z}_{\geq 0}.$$

The smallest $\mathfrak{D}$-saturated idealistic filtration containing $\mathbb{I}$ is called the differential saturation (or $\mathfrak{D}$-saturation) of $\mathbb{I}$, denoted by $\mathfrak{D}(\mathbb{I})$.

Remark 1.2.10.
(1) We see by the generalized product rule that the generators of $\mathfrak{D}(\mathbb{I})$ are given by the differentiations of the generators of $\mathbb{I}$. For example, if $\mathbb{I} = \mathrm{G}(\{(x^2 + y^5, 2)\})$ over $R = k[x, y]$, then $\mathfrak{D}(\mathbb{I})$ is generated by $\{(x^2 + y^5, 2), (2x, 1), (5y^4, 1)\}$. Note that the second (resp. the third) generator of $\mathfrak{D}(\mathbb{I})$ vanishes when $\mathrm{char}(k) = 2$ (resp. 5).
(2) Assume $\mathrm{char}(k) = 0$. Let $I \subset R$ be an ideal of $R$ and $n \in \mathbb{Z}_{>0}$ a positive integer. Then, $\mathfrak{D}(\mathrm{G}(I \times \{n\}))_a = \mathrm{Diff}^{\leq n - \lceil a \rceil}(I)$ for $0 \leq a \leq n$. Consequently, a hypersurface of maximal contact of $I$ at a closed point $P \in \operatorname{Spec} R$ is defined by an element $h \in \mathfrak{D}(\mathrm{G}(I \times \{\mu_P(I)\}))_1$ such that $\mu_P(h) = 1$ as explained in §0.2. This demonstrates the virtue of the notion of the $\mathfrak{D}$-saturation in the context of the problem of resolution of singularities, as it gives a natural characterization of a hypersurface of maximal contact in characteristic zero. This characterization leads to the key idea of the IFP, the notion of a *leading generator system*, in §2.1.
(3) We can also define the logarithmic $\mathfrak{D}$-saturation $\mathfrak{D}_E(\mathbb{I})$ of $\mathbb{I}$ with respect to $E$ in a similar manner (cf. Remark 1.2.7 (3)).

**1.3. Radical saturation and integral closure.** We introduce the notion of the $\mathfrak{R}$-saturation of an idealistic filtration. We discuss the motivation behind it. For $f \in R$, $a \in \mathbb{R}_{\geq 0}$ , $n \in \mathbb{Z}_{>0}$ and $P \in \operatorname{Spec} R$, observe that $\mu_P(f^n) \geq a$ and $\mu_P(f) \geq a/n$ have the equivalent information as the statements on the order. Accordingly, we consider the enlargement of an idealistic filtration $\mathbb{I}$ by the radical operation, that is, by adding $(f, a/n)$ to $\mathbb{I}$ if $(f^n, a) \in \mathbb{I}$. This is the naïve definition of the $\mathfrak{R}$-saturation. Unfortunately, as in (1) of Example 1.3.3 below, this radical operation does not preserve the property that an idealistic filtration is finitely generated. In order to overcome this problem, we define the $\mathfrak{R}$-saturation by combining the radical operation above with the limit operation, another natural operation described as follows. For $f \in R$, a sequence $\{a_i \mid i \in \mathbb{Z}_{>0}\} \subset \mathbb{R}$ with its limit converging to $\lim_{i \to \infty} a_i = a$, and $P \in \operatorname{Spec} R$, observe that $\mu_P(f) \geq a_i$ for any $i \in \mathbb{Z}_{>0}$ implies

$\mu_P(f) \geq a$. Therefore $(f,a)$ is deduced from $\{(f,a_i) \mid i \in \mathbb{Z}_{>0}\}$ as the statement on the order. Thus we consider the enlargement of $\mathbb{I}$ by the limit operation, that is, by adding $(f,a)$ to $\mathbb{I}$ if $\{(f,a_i) \mid i \in \mathbb{Z}_{>0}\} \subset \mathbb{I}$. The radical operation, the limit operation, and hence the $\mathfrak{R}$-saturation, are expected to leave the problem of resolution of singularities "intact" (cf. §1.1).

Definition 1.3.1. We say an idealistic filtration $\mathbb{I}$ is radically saturated (or $\mathfrak{R}$-saturated) if the following conditions hold:

(1) $(f^n,a) \in \mathbb{I} \Rightarrow (f,a/n) \in \mathbb{I} \quad (\forall n \in \mathbb{Z}_{>0})$

(2) $\{(f,a_i) \mid i \in \mathbb{Z}_{>0}\} \subset \mathbb{I}$ and $\lim_{i\to\infty} a_i = a \Rightarrow (f,a) \in \mathbb{I}$.

The smallest $\mathfrak{R}$-saturated idealistic filtration containing $\mathbb{I}$ is called the radical saturation (or $\mathfrak{R}$-saturation) of $\mathbb{I}$, denoted by $\mathfrak{R}(\mathbb{I})$.

Remark 1.3.2. We remark that, instead of using the condition (1) above where we let $n$ vary among all the positive integers, we may use the condition $(1)_m$ below where we use a fixed $m \in \mathbb{Z}_{>1}$.

$$(1)_m \quad (f^m,a) \in \mathbb{I} \Rightarrow (f,a/m) \in \mathbb{I}.$$

It is straight forward to show that the $\mathfrak{R}$-saturation defined by using the conditions (1) and (2) is the same as the one defined by using the conditions $(1)_m$ and (2) (exercise, cf. [**16**] 2.1.3.2). Note that the latter is quite useful in positive characteristic if we set $m = p = \mathrm{char}(k)$.

Example 1.3.3.

(1) We note that, without the condition (2) in Definition 1.3.1, the enlargement may not be finitely generated even if we start from a finitely generated idealistic filtration. Let $\mathbb{I} = \mathrm{G}(\{(x^2+xy,2),(y,1)\})$. Then, $x^2+xy, xy \in \mathbb{I}_1$, and $x^2 \in \mathbb{I}_1$. Thus $(x,1/2) \in \mathfrak{R}(\mathbb{I})$ and $(xy,3/2) \in \mathfrak{R}(\mathbb{I})$. Next, $x^2+xy, xy \in \mathfrak{R}(\mathbb{I})_{3/2}$ and $x^2 \in \mathfrak{R}(\mathbb{I})_{3/2}$. Thus $(x,3/4) \in \mathfrak{R}(\mathbb{I})$ and $(xy,7/4) \in \mathfrak{R}(\mathbb{I})$. Repeating this argument, we see $(x,1-2^{-n}) \in \mathfrak{R}(\mathbb{I})$ for any $n \in \mathbb{Z}$. By the condition (2), we conclude $\mathfrak{R}(\mathbb{I}) = \mathrm{G}(\{(x,1),(y,1)\})$. If we drop the condition (2) but keep only the condition (1), then our enlargement will be $\mathrm{G}(\{(x^2+xy,2),(y,1)\} \cup \{(x,a) \mid a \in \mathbb{R}_{<1}\})$, which is never finitely generated, even though the original $\mathbb{I}$ is finitely generated.

(2) Let $\mathbb{I} = \mathrm{G}(\{(x^2+y^5,2)\})$ as in Remark 1.2.10 in $\mathrm{char}(k) = 2$. Then

$$\mathfrak{R}(\mathfrak{D}(\mathbb{I})) = \mathfrak{R}(\mathrm{G}(\{(x^2+y^5,2),(y^4,1)\})) = \mathrm{G}\left(\{(x^2+y^5,2),(x,5/8),(y,1/4)\}\right).$$

Verify that the right-hand side is $\mathfrak{R}$-saturated by using Remark 1.3.2 (exercise).

As the last of our various saturations, we introduce the notion of the integral closure of an idealistic filtration. It is analogous to the notion of the integral closure of an ideal. We would like to discuss the motivation behind it in terms of the order. Assume we have a monic equation $f^n + a_1 f^{n-1} + \cdots + a_n = 0$. Then there must be some $a_i f^{n-i}$ $(1 \leq i \leq n)$ whose order does not exceed the order of $f^n$. Otherwise the orders of the two sides of the equation above would not coincide. Thus the order of $f$ is at least the order of $a_i$ divided by $i$. Rephrasing this observation, we see that, with the monic equation as above and for a fixed $c \in \mathbb{R}$, if each $a_i$ has order at least $ic$, then $f$ has order at least $c$. Therefore, if $\{(a_i, ic) \mid 1 \leq i \leq n\} \subset \mathbb{I}$ and if the monic equation holds, then adding the element $(f,c)$ to $\mathbb{I}$ is expected to leave the problem of resolution of singularities "intact" (cf. §1.1).

Definition 1.3.4. We say an idealistic filtration $\mathbb{I}$ is integrally closed if the following condition holds:

$$f^n + \sum_{i=1}^{n} a_i f^{n-i} = 0 \text{ and } \{(a_i, ic) \mid 1 \le i \le n\} \subset \mathbb{I} \Rightarrow (f, c) \in \mathbb{I}$$

The smallest integrally closed idealistic filtration containing $\mathbb{I}$ is called the integral closure of $\mathbb{I}$, denoted by $\mathrm{IC}(\mathbb{I})$.

Example 1.3.5. Let $\mathbb{I} = \mathrm{G}(\{(x^2 + xy, 2), (y, 1)\})$ as in Example 1.3.3 (1). Then, $\mathrm{IC}(\mathbb{I}) = \mathrm{G}(\{(x, 1), (y, 1)\}) = \mathfrak{R}(\mathbb{I})$. Actually $T = x$ satisfies a monic equation $T^2 + (y) \cdot T + (-x^2 - xy) = 0$, thus $(x, 1) \in \mathrm{IC}(\mathbb{I})$.

Remark 1.3.6. We make some supplemental remarks on the saturations.
(1) The saturations $\mathfrak{D}(\mathbb{I})$, $\mathfrak{R}(\mathbb{I})$ and $\mathrm{IC}(\mathbb{I})$ do exist for a given idealistic filtration $\mathbb{I}$.
(2) The saturations $\mathfrak{D}(\mathbb{I})$, $\mathfrak{R}(\mathbb{I})$ and $\mathrm{IC}(\mathbb{I})$ preserve the support, i.e.,

$$\mathrm{Supp}(\mathbb{I}) = \mathrm{Supp}(\mathfrak{D}(\mathbb{I})) = \mathrm{Supp}(\mathfrak{R}(\mathbb{I})) = \mathrm{Supp}(\mathrm{IC}(\mathbb{I})).$$

In fact, for the $\mathfrak{D}$-saturation, we observe that $\mu_P(\mathfrak{D}(\mathbb{I})) = \mu_P(\mathbb{I})$ if $\mu_P(\mathbb{I}) \ge 1$ and $\mu_P(\mathfrak{D}(\mathbb{I})) = 0$ if $\mu_P(\mathbb{I}) < 1$. For the other saturations, the order $\mu$ does not change.
(3) The Hironaka equivalence is a powerful tool to study the problem of resolution of singularities when one uses Hironaka's notion of an idealistic exponent or its variations. Given an idealistic exponent, the operations of adding radicals, differentials or integral elements in a suitable manner do not change its Hironaka equivalence classes. These operations correspond to taking various saturations in our setting. However, the definition of the Hironaka equivalence requires one to consider *all* the sequence of permissible blow ups, a task theoretically easy but practically difficult. Instead, we take an alternative path of analyzing these saturations, which does not require us the formidable task mentioned above but which is expected to capture all the same information as the one obtained by considering the Hironaka equivalence. See Remark 1.6.3 for the related results.
(4) We see in Proposition 1.6.1 that the integral closure coincide with the $\mathfrak{R}$-saturation under some mild condition (cf. Definition 1.4.1).

**1.4. Finiteness property and saturations.** Here we introduce the notion of *rationally and finitely generated type* (abbreviated as *r.f.g. type*), representing some finiteness property of an idealistic filtration, which is always satisfied by the idealistic filtrations we consider in the context of the problem of resolution of singularities. We show that this property is preserved under $\mathfrak{D}$-saturation and $\mathfrak{R}$-saturation.

Definition 1.4.1. We say an idealistic filtration $\mathbb{I}$ is of rationally and finitely generated type (abbreviated as r.f.g. type) if there exists a finite subset $S \subset R \times \mathbb{Q}$ which generates $\mathbb{I}$, i.e., $\mathbb{I} = \mathrm{G}(S)$.

It is easy to see that the property of being r.f.g. type is preserved under $\mathfrak{D}$-saturation.

Proposition 1.4.2. *If $\mathbb{I}$ is of r.f.g. type, then so is $\mathfrak{D}(\mathbb{I})$.*

Sketch of the proof. Covering $\operatorname{Spec} R$ by small affine open subsets, we may assume $R$ has a set of differential coordinates. As $\mathbb{I}$ is of r.f.g. type, there exists $S \subset R \times \mathbb{Q}$ such that $\#S < \infty$ and $\mathbb{I} = \mathrm{G}(S)$. By the generalized product rule, we see $\mathfrak{D}(\mathbb{I}) = \mathrm{G}(S')$ where $S' = \{(\partial_{X^I} f, a - |I|) \mid (f, a) \in S,\ |I| < a\}$. Thus $\mathfrak{D}(\mathbb{I})$ is of r.f.g. type. □

It is not so easy to see that the property of being r.f.g. type is preserved under $\mathfrak{R}$-saturation, since we cannot construct the generators of the $\mathfrak{R}$-saturation directly from those of the original idealistic filtration. The key point of our proof is the following theorem of Nagata, which is later reproved by Lejeune-Teissier:

THEOREM 1.4.3 ([**19**],[**20**]). *Let $S$ be a noetherian domain and $J$ an ideal of $S$. Define a map $\phi_J \colon S \to \mathbb{R} \cup \{\infty\}$ by*

$$\phi_J(f) = \sup\{m/n \mid m \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}_{>0},\ f^n \in J^m\} \quad (f \in R).$$

*Then there exists a positive integer $\rho \in \mathbb{Z}_{>0}$ such that* $\mathrm{Image}(\phi_J) \subset \rho^{-1}\mathbb{Z} \cup \{\infty\}$.

SKETCH OF THE PROOF. Take the normalized blow up of $\operatorname{Spec} S$ along $J$ and show that the denominator of an element in $\mathrm{Image}(\phi_J)$ is expressed in terms of the valuations with respect to the divisors appearing in the exceptional locus. □

The limit operation described in the condition (2) in Definition 1.3.1 makes it unclear that, even if we start from an idealistic filtration which has generators at the rational levels, the generators of the $\mathfrak{R}$-saturation can be taken all at the rational levels. However, the boundedness assertion of Theorem 1.4.3 reduces the problem to the usual finiteness statement of an integral closure of an integral domain.

THEOREM 1.4.4. *If $\mathbb{I}$ is of r.f.g. type, then so is $\mathfrak{R}(\mathbb{I})$.*

SKETCH OF THE PROOF. We assume $\mathbb{I} = \mathrm{G}(I \times \{1\})$ for an ideal $I \subset R$, which is the essential case. Note that $(f, a) \in \mathfrak{R}(\mathbb{I}) \Leftrightarrow a \leq \sup\{m/n \mid f^n \in I^m\}$ by the definition of the $\mathfrak{R}$-saturation. Thus, by Theorem 1.4.3, there exists $\rho \in \mathbb{Z}_{>0}$ such that $\mathfrak{R}(\mathbb{I})$ is generated by the elements whose levels are in $\rho^{-1}\mathbb{Z}$, namely, $\mathfrak{R}(\mathbb{I}) = \mathrm{G}(\bigsqcup_{n \in \mathbb{Z}_{>0}} \mathfrak{R}(\mathbb{I})_{n/\rho} \times \{n/\rho\})$. Set the graded algebras $A$ and $B$ as below:

$$A = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} I^n X^n \subset R[X], \quad B = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathfrak{R}(\mathbb{I})_{n/\rho} X^{n/\rho} \subset R[X^{1/\rho}].$$

Then $B$ coincides with the integral closure of $A$ in $R[X^{1/\rho}]$, i.e., $B = \overline{A}^{(R[X^{1/\rho}])}$. Thus $B$ is finitely generated as an $A$-module, and hence as an $R$-algebra. Since the generators of $B$ as an $R$-algebra give rise to the generators of $\mathfrak{R}(\mathbb{I})$ as an idealistic filtration, we conclude that $\mathfrak{R}(\mathbb{I})$ is of r.f.g. type. □

**1.5. Localization and saturations.** In order to develop the theory of a sheaf of idealistic filtrations, it is indispensable to consider their localization. While the compatibility with localization of the $\mathfrak{D}$-saturation follows immediately as we expect, that of the $\mathfrak{R}$-saturation only follows assuming that the idealistic filtration is of r.f.g. type (cf. Example 1.5.3).

DEFINITION 1.5.1. Let $S \subset R$ be a multiplicative subset. Then, localization $\mathbb{I}_S \subset R_S \times \mathbb{R}$ of $\mathbb{I}$ by $S$ is defined by the formula $(\mathbb{I}_S)_a = (\mathbb{I}_a)_S = \mathbb{I}_a \otimes_R R_S$ $(a \in \mathbb{R})$.

PROPOSITION 1.5.2. *Let $S \subset R$ be a multiplicative subset.*
*(1) Localization and $\mathfrak{D}$-saturation are compatible, i.e., $(\mathfrak{D}(\mathbb{I}))_S = \mathfrak{D}(\mathbb{I}_S)$.*
*(2) If $\mathbb{I}$ is of r.f.g. type, then localization and $\mathfrak{R}$-saturation are compatible, i.e., $(\mathfrak{R}(\mathbb{I}))_S = \mathfrak{R}(\mathbb{I}_S)$.*

SKETCH OF THE PROOF.
(1) By the generalized Leibniz rule, we have $\mathrm{Diff}^{\leq n}(I)R_S = \mathrm{Diff}^{\leq n}(IR_S)$ for any ideal $I \subset R$. Thus $(\mathfrak{D}(\mathbb{I}))_S$ is already $\mathfrak{D}$-saturated, and so $(\mathfrak{D}(\mathbb{I}))_S = \mathfrak{D}(\mathbb{I}_S)$.

(2) We assume $\mathbb{I} = \mathrm{G}(I \times \{1\})$ for an ideal $I \subset R$ as in the proof of Theorem 1.4.4. Then $\mathfrak{R}(\mathbb{I})$ is described by $B$, which is the integral closure of $A$. $\mathfrak{R}(\mathbb{I}_S)$ is also described by $B'$, which is the integral closure of $A_S = A \otimes_R R_S$. Now, by the commutativity of integral closure and localization, we see $B' = B \otimes_R R_S$. □

Example 1.5.3. Let $R = k[x, y]$ and $\mathfrak{m} = (x, y) \subset R$. Set $\mathbb{I} = \mathrm{G}(S)$ over $R$, where $S = \left\{ \left( y \prod_{j=1}^{i}(x - j), 1 - i^{-1} \right) \mid i \in \mathbb{Z}_{>0} \right\}$. Then it is easy to see that $\mathbb{I}_\mathfrak{m} = \mathrm{G}(\{(y, 1 - i^{-1}) \mid i\})$ and $\mathfrak{R}(\mathbb{I}_\mathfrak{m}) = \mathrm{G}(\{(y, 1)\})$. However, one can show that $(y, 1) \notin (\mathfrak{R}(\mathbb{I}))_\mathfrak{m}$ (exercise). Thus $\mathfrak{R}(\mathbb{I}_\mathfrak{m}) \supsetneqq (\mathfrak{R}(\mathbb{I}))_\mathfrak{m}$. (See [**16**] 2.4.2.2 for more details.)

**1.6. Relation among saturations.** We introduced the three kinds of operations to enlarge an idealistic filtration, the $\mathfrak{D}$-saturation, the $\mathfrak{R}$-saturation, and the integral closure. We may ask the following natural question: What is the biggest enlargement we can obtain by composing these three operations? We discuss the basic relations among these operations, and also give an answer to this question under some mild conditions.

Proposition 1.6.1.
*(1) The integral closure is contained in the $\mathfrak{R}$-saturation, i.e.,* $\mathrm{IC}(\mathbb{I}) \subset \mathfrak{R}(\mathbb{I})$.
*(2) If $\mathbb{I}$ is of r.f.g. type, then they are same, i.e.,* $\mathrm{IC}(\mathbb{I}) = \mathfrak{R}(\mathbb{I})$.

Sketch of the proof.
(1) Let $\mathbb{I}' \subset R \times \mathbb{R}$ be the set of all integral elements over $\mathbb{I}$.

First we show that $\mathbb{I}' \subset \mathfrak{R}(\mathbb{I})$. Take $(f, b) \in \mathbb{I}'$. By the definition of $\mathbb{I}'$, there exists a monic equation $f^n + \sum_{i=1}^{n} a_i f^{n-i} = 0$ with $(a_i, ib) \in \mathbb{I}$. Set $\alpha = (n-1)/n$. Then we see $(f^n, b) \in \mathbb{I}$ and thus $(f, (1-\alpha)b) = (f, \frac{b}{n}) \in \mathfrak{R}(\mathbb{I})$. Plugging in this information to the above equation, we see $(f^n, (1+\alpha)b) = (f^n, b + (n-1)(1-\alpha)b) \in \mathfrak{R}(\mathbb{I})$ and thus $(f, (1-\alpha^2)b) \in \mathfrak{R}(\mathbb{I})$. Repeating this procedure, we have $(f^n, (\sum_{i=0}^{j} \alpha^i)b) \in \mathfrak{R}(\mathbb{I})$ and $(f, (1-\alpha^j)b) \in \mathfrak{R}(\mathbb{I})$ for any $j \in \mathbb{Z}_{\geq 0}$. Taking $j \to \infty$, we see $(f, b) \in \mathfrak{R}(\mathbb{I})$. Thus the condition $\mathbb{I}' \subset \mathfrak{R}(\mathbb{I})$ is verified.

Next, we show that $\mathbb{I}' = \mathrm{IC}(\mathbb{I})$, that is, $\mathbb{I}'$ is integrally closed. Take $(f, c) \in R \times \mathbb{R}$ which is integral over $\mathbb{I}'$. Then, there exists a monic equation $f^n + \sum_{i=1}^{n} a_i f^{n-i} = 0$ with $(a_i, ic) \in \mathbb{I}'$. Put $S = \bigoplus_{i=0}^{n} \mathbb{I}_{ic} T^i \subset R[T]$. Since $(a_i, ic) \in \mathbb{I}'$, $a_i T^i$ is integral over $S$. Denote the integral closure of $S$ in $R[T]$ as $\overline{S}$. Then, the monic equation above implies $fT$ is integral over $\overline{S}$. Therefore, we have $fT \in \overline{S}$, i.e., $(f, c) \in \mathbb{I}'$.

(2) We assume $\mathbb{I} = \mathrm{G}(I \times \{1\})$ for an ideal $I \subset R$. We use the notation of the proof of Theorem 1.4.4. Take $(f, a) \in \mathfrak{R}(\mathbb{I})$. Then, by the definition of $\rho$, we have $(f, \lceil \rho a \rceil / \rho) \in \mathfrak{R}(\mathbb{I})$. Therefore $f X^{\lceil \rho a \rceil / \rho} \in B$ is integral over $A$, and from this fact we can conclude that $(f, \lceil \rho a \rceil / \rho) \in \mathrm{IC}(\mathbb{I})$. Thus $(f, a) \in \mathrm{IC}(\mathbb{I})$. □

Now we discuss the question of "the biggest enlargement". Proposition 1.6.1 allows us to ignore the operation of taking the integral closure. The following theorem implies that, if we start from an idealistic filtration of r.f.g. type, then $\mathfrak{R}\mathfrak{D}(\mathbb{I})$ is the biggest such enlargement.

Theorem 1.6.2. *Assume that $R$ has a set of differential coordinates or that $\mathbb{I}$ is of r.f.g. type. Then, we have*

$$\mathfrak{D}\mathfrak{R}(\mathbb{I}) \subset \mathfrak{R}\mathfrak{D}(\mathbb{I}).$$

*In particular, under the assumption above, the biggest enlargement that can be obtained by composing the three operations is* $\mathfrak{R}\mathfrak{D}(\mathbb{I})$.

Sketch of the proof. If $\mathbb{I}$ is of r.f.g. type, then the operations of taking the two saturations are compatible with localization by Proposition 1.5.2. Therefore, in order to show the inclusion above, we have only to prove it assuming $R$ has a set of differential coordinates. Here, instead of giving the full proof, we demonstrate one example of how an element of the left hand side is included in the right hand side, which, nevertheless, captures the essence of the idea of the proof. Assume $(f^3, 6) \in \mathbb{I}$ and fix a differential coordinate $x$. Then $(f, 2) \in \mathfrak{R}(\mathbb{I})$ and $(f_1, 1) \in \mathfrak{D}\mathfrak{R}(\mathbb{I})$, where $f_i = \partial_{x^i} f$. Set $\mathbb{I}' = \mathfrak{R}\mathfrak{D}(\mathbb{I})$. We will show that $(f_1, 1) \in \mathbb{I}'$. By applying $\partial_{x^0}$, $\partial_{x^3}$, $\partial_{x^4}$ to $(f^3, 6)$, we have the elements $\alpha_0, \alpha_3, \alpha_4 \in \mathfrak{D}(\mathbb{I})$ as follows:

$$\begin{aligned} \alpha_0 &= (f_0^3, 6), \quad \alpha_3 = (f_1^3 + 6f_0f_1f_2 + 3f_0^2f_3, 3), \\ \alpha_4 &= (3f_1^2f_2 + 3f_0f_2^2 + 6f_0f_1f_3 + 3f_0^2f_4, 2). \end{aligned}$$

As $\alpha_0 \in \mathbb{I}'$, we have $(f_0, 2) \in \mathbb{I}'$. As $(f_0, 2), \alpha_4 \in \mathbb{I}'$, we have $(3f_1^2f_2, 2) \in \mathbb{I}'$. Therefore $(3^2f_1^2f_2^2, 2) \in \mathbb{I}'$ and $(3f_1f_2, 1) \in \mathbb{I}'$. Finally, as $(f_0 \cdot 3f_1f_2, 2+1), (f_0^2, 4), \alpha_3 \in \mathbb{I}'$, we have $(f_1^3, 3) \in \mathbb{I}'$, which implies $(f_1, 1) \in \mathbb{I}'$. □

Remark 1.6.3. Theorem 1.6.2 is closely related to Finite Presentation Theorem by Hironaka ([**15**]) or Canonicity Principle by Bravo, García-Escamilla and Villamayor ([**3**]). Roughly speaking, their theorem and principle can be interpreted as saying that $\mathfrak{R}\mathfrak{D}(\mathbb{I})$ gives the largest representative among all the idealistic filtrations which are Hironaka equivalent to a given idealistic filtration $\mathbb{I}$, i.e., $\mathbb{I} \sim \mathbb{I}' \Leftrightarrow \mathfrak{R}\mathfrak{D}(\mathbb{I}) = \mathfrak{R}\mathfrak{D}(\mathbb{I}')$. According to our philosophy, two idealistic filtrations should be "equivalent" if their largest saturations coincide. Thus Theorem 1.6.2 and their results mean that, in principle, our "equivalence" coincides with the Hironaka equivalence. However, one must remember the subtle yet definite difference between their settings and our setting in order to understand this interpretation in a precise manner:

(1) The Hironaka equivalence for an idealistic filtration $\mathbb{I}$ is well-defined provided $\mathbb{I}$ is of r.f.g. type, where the transformation $\mathbb{I}'$ of $\mathbb{I}$ by the blow up $\pi\colon \widetilde{W} \to W$ along the center $C$ is defined by the following formula.

$$\mathbb{I}'_a = \sum_{b \in \mathbb{R}_{\geq a}} \left( I(\pi^{-1}(C))^{-\lfloor b \rfloor} \cdot \pi^{-1}(\mathbb{I}_b)\mathcal{O}_{\widetilde{W}} \right) \quad (a \in \mathbb{R}_{\geq 0}).$$

(2) Our operation of taking the $\mathfrak{R}$-saturation, or the integral closure, of an idealistic filtration allows adding some extra elements at any rational levels, while their operation of taking the integral closure of a Rees algebra only add some extra elements at the integral levels, as the Rees algebra is graded by non-negative integers by definition. Therefore, Theorem 1.4.3 (Nagata's theorem) is important for us to control the denominators of the elements in the $\mathfrak{R}$-saturation, while there is no need for it in their setting.

We observe that a Rees algebra $A$ can be identified with the idealistic filtration $\mathbb{I} = \mathrm{G}(A)$ it generates, and that the integral closure of $A$ can be recovered by first taking the $\mathfrak{R}$-saturation of $\mathbb{I}$ and then looking at the elements at the integral levels. The precise interpretation is obtained through this observation.

## 2. Idealistic Filtration Program

In this section, we present our strategy to construct an algorithm for resolution of singularities in the framework of the IFP. How should we construct such an algorithm? Ideally the construction of such an algorithm should go as follows (§2.2). Given an idealistic filtration $\mathbb{I}$, we associate an invariant, whose maximum locus determines the center of blow up. After blow up, the value of the invariant drops. By showing the value of the invariant can not drop infinitely many times, we observe that the process must terminate after finitely many times, achieving resolution of singularities. How should we construct such an invariant? In characteristic zero, the invariant consists of the smaller units. Looking at $\mathbb{I}$, we compute the first unit. We look at the maximum locus of the first unit. Then we construct a hypersurface of maximal contact, and another idealistic filtration on the hypersurface, so that its support coincides with the maximum locus. By looking at this idealistic filtration on the ambient space of dimension one less, we compute the second unit and so forth, weaving the entire invariant. This is how the inductive scheme on dimension is manifested as the weaving of the invariant. In positive characteristic, we try to follow a similar path. The first issue is how to compute the unit. Here the invariant $\sigma$ and $\mu^{\sim}$ are the basic constituents of the unit. The second issue is how to proceed from the first unit to the next. Since there is no hypersurface of maximal contact (cf. §0.2), instead of taking the restriction, we take the modification of the original idealistic filtration (§2.3), sticking to the same nonsingular ambient space. The invariant $\sigma$ strictly decreases for this modification, and we compute the second unit and so forth, weaving the entire invariant "inv". This is how the inductive scheme on the invariant $\sigma$ is manifested as the weaving of the invariant "inv". Sitting at the core of the IFP is the notion of a leading generator system (§2.1), a collective substitute for the notion of a hypersurface of maximal contact. We discuss some fundamental results of the IFP in §2.4. We also present a new result, concerning the nonsingularity of the support of an idealistic filtration in the monomial case, assuming it is $\mathfrak{R}$-saturated (§2.5).

We refer the reader to [**16**](§2.1, §2.4), [**17**](§2.2, §2.4) and [**18**](§2.3, §2.5) for the details of the materials discussed in this section.

**Convention in §2**. In this section, since our analysis is focused on the invariants defined locally at a closed point $P \in W$ of a nonsingular variety $W$, we assume that $R = (R, \mathfrak{m})$ is a regular local ring essentially of finite type over $k$ with the residue field $\kappa(R) = R/\mathfrak{m} = k$. We also assume $P \in \mathrm{Supp}(\mathbb{I})$, i.e., $\mu(\mathbb{I}) \geq 1$, where we denote by $\mu$ the order $\mu_P$ at the unique closed point $P \in \mathrm{Spec}\, R$.

**2.1. Leading generator system.** In order to explain the motivation behind introducing the notion of a leading generator system, we look at the following simple example in characteristic zero. Consider the problem of resolution of singularities for the curve $\mathrm{V}(x^2 - y^3) \subset \mathrm{Spec}\, k[x,y]_{(x,y)}$ at the origin. It is the support of the idealistic filtration $\mathbb{I} = \mathrm{G}(\{(x^2 - y^3, 1)\})$. The maximum order is 2, which is nothing but the maximum of the order $\mu$ of $\mathbb{I}$. The hypersurface of maximal contact is given by $H = \mathrm{V}(x)$, which contains the maximum locus of the order (cf. §0.2). In terms of the IFP, the process of finding $H$ can be interpreted as follows: Take an enlargement $\mathbb{I}' = \mathrm{G}(\{(x^2 - y^3, 2 \cdot 1)\})$ of $\mathbb{I}$, and its $\mathfrak{D}$-saturation $\mathfrak{D}(\mathbb{I}')$. Then we observe that the element $(x, 1) \in \mathfrak{D}(\mathbb{I}')_1$ gives rise to $H$ (cf. Remark 1.2.10(2)). Moreover, by taking the restriction to $H$, we start seeing the higher order information $(x^2 - y^3, 2)|_H = (-y^3, 2)$, hidden at the beginning. Now we continue our process by considering the idealistic filtration $\mathrm{G}(\{(-y^3, 2)\})$ on $H$, and so on. The role of a hypersurface of maximal contact can be summarized in the two points below:

(1) It reflects the "order 1" part of the information of an idealistic filtration $\mathfrak{D}(\mathbb{I}')$.

(2) It builds the bridge to extract the higher order information.

The process is analogous to "peeling an onion to see the core": We peel the first visual part (order 1 part) by taking a hypersurface of maximal contact, revealing the inside, hidden and higher order information, which naturally leads to peeling of the second. According to the IFP in positive characteristic, we realize this peeling process in the following two steps:

(1) Given an idealistic filtration $\mathbb{I}$, after taking its $\mathfrak{D}$-saturation, we consider the *leading algebra* $\mathrm{L}(\mathbb{I})$ in order to extract the "order 1" part of $\mathbb{I}$. The *leading generator system* is a representative in $\mathbb{I}$ of a set of generators for $\mathrm{L}(\mathbb{I})$, defined in any characteristic, which is considered to be a collective substitute for the notion of a hypersurface of maximal contact.
(2) By computing the order modulo a leading generator system, we reveal the "higher order" information hidden in the idealistic filtration $\mathbb{I}$. We take the associated *modification* $\mathbb{I}'$ of $\mathbb{I}$, and go back to the step (1) by setting $\mathbb{I}'$ as $\mathbb{I}$, and continue the peeling process.

Definition 2.1.1. The leading algebra $\mathrm{L}(\mathbb{I})$ of an idealistic filtration $\mathbb{I}$ is a graded subalgebra of the graded algebra $\mathrm{Gr}(R) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ associated to $R$, defined by the formula

$$\mathrm{L}(\mathbb{I}) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \left( \mathbb{I}_n \mod \mathfrak{m}^{n+1} \right) \subset \mathrm{Gr}(R).$$

Example 2.1.2. Let $\mathbb{I} = \mathfrak{D}(\mathrm{G}(\{(x^2+y^5,2)\}))$ over $R = k[x,y]_{(x,y)}$. Then, as we have seen in Remark 1.2.10 (1), $\mathbb{I} = \mathrm{G}(\{(x^2+y^5,2),(2x,1),(5y^4,1)\})$. The leading algebra of $\mathbb{I}$ is $\mathrm{L}(\mathbb{I}) = k[\overline{x}]$ if $\mathrm{char}(k) \neq 2$, and $\mathrm{L}(\mathbb{I}) = k[\overline{x^2}]$ if $\mathrm{char}(k) = 2$.

From now on till the end of §2.4, we assume that an idealistic filtration of our concern is $\mathfrak{D}$-saturated, by taking the $\mathfrak{D}$-saturation if necessary. We will see that we extract a "good" information if an idealistic filtration is $\mathfrak{D}$-saturated, a typical example of the philosophy of the IFP explained at the ending of §1.1.

Recall that $\mathrm{Gr}(R)$ is isomorphic to a polynomial ring $k[X]$ over $k$, since $R$ is regular. Through this isomorphism $\mathrm{Gr}(R) \cong k[X]$, the leading algebra $\mathrm{L}(\mathbb{I})$ is regarded as a graded subalgebra $L \subset k[X]$, where the grading $k[X] = \bigoplus_{r \in \mathbb{Z}_{\geq 0}} k[X]_r$ is given by the total degree in $X$. Observe that $L$ is stable under differentiation, since $\mathbb{I}$ is $\mathfrak{D}$-saturated. This observation leads to the following structure theorem.

Proposition 2.1.3 (Hironaka-Oda). *Let $k[X]$ be a polynomial ring over $k$ and $L \subset k[X]$ its graded subalgebra. Assume $L$ is stable under differentiation, that is, $\partial_{X^J} L \subset L$ for any multi-index $J$. Then the following holds:*
*(1) If $\mathrm{char}(k) = 0$, then there exists a $k$-linearly independent subset $\{z_i \mid i\} \subset k[X]_1$ of linear forms such that $L = k[z_i \mid i]$.*
*(2) If $\mathrm{char}(k) = p > 0$, then there exist a $k$-linearly independent subset $\{z_i \mid i\} \subset k[X]_1$ of linear forms and also a set of non-negative integers $\{e_i \mid i\} \subset \mathbb{Z}_{\geq 0}$ such that $L = k[z_i^{p^{e_i}} \mid i]$.*

Sketch of the proof. We only sketch the proof in $\mathrm{char}(k) = p > 0$, since the proof in $\mathrm{char}(k) = 0$ is obtained just by setting $p^e = 1$ and $e = 0$ below. Set $S = \{\alpha^{p^e} \mid \alpha \in k[X]_1,\ e \in \mathbb{Z}_{\geq 0}\} \cap L$ and take $f \in L$. It suffices to show $f \in k[S]$. As $L$ is a graded subalgebra, we may assume $f \in k[X]_r$ with $r \in \mathbb{Z}_{>0}$. Set

$e = \max\{j \in \mathbb{Z}_{\geq 0} \mid f^{p^{-j}} \in k[X]\}$. Then, there exist an index $J$ with $|J| + 1 = r/p^e$ and a linear form $z \in k[X]_1 \setminus \{0\}$ such that $\partial_{X^{p^e J}} f = z^{p^e}$. Note $z^{p^e} \in S$. Take a new coordinate $\{z\} \sqcup Y$ and set $L' = L \cap k[Y]$. Expand $f$ as $f = \sum_{j=0}^{r} a_j^{p^e} z^{p^e j}$ with $a_j \in k[Y]$. Applying $\partial_{z^{p^e j}}$ to $f$, we see $\{a_j^{p^e} \mid j\} \subset L'$. Now, by induction on the number $\#Y$ of variables, we have $L' \subset k[S \cap L'] \subset k[S]$. Thus $f \in k[S]$. □

Remark 2.1.4. Sometimes we hold a view point that the case of characteristic zero is a special case of positive characteristic where $\mathrm{char}(k) = p = \infty$. Proposition 2.1.3 tells us that, in case (2), the generators of $\mathrm{L}(\mathbb{I})$ are distributed at the levels $\{p^e \mid e \in \mathbb{Z}_{\geq 0}\}$. If $p \to \infty$, all the levels $p^e$ with positive $e > 0$ go to $\infty$ and hence become invisible. Therefore, at the limit, i.e., in characteristic zero, we should only see the remaining generators at level $p^e = \infty^0 = 1$, which is the assertion in case (1). This confirms our view point, and also explains why a hypersurface of maximal contact exists in characteristic zero, but not always in positive characteristic.

We have seen in Remark 1.2.10 (2) that an element $(h, 1) \in \mathbb{I}$, whose image in $\mathrm{L}(\mathbb{I})_1$ is nonzero, gives a hypersurface of maximal contact. In view of Proposition 2.1.3, we interpret that a hypersurface of maximal contact corresponds to a representative of an element of generators of $\mathrm{L}(\mathbb{I})$ in $\mathrm{char}(k) = 0$. This interpretation is plausible in the context of the "peeling an onion" procedure (cf. beginning of §2.1). Therefore, we adopt the representative of generators of $\mathrm{L}(\mathbb{I})$ as the substitute of a hypersurface of maximal contact in arbitrary characteristic:

Definition 2.1.5. A finite subset $\mathbb{H} \subset \mathbb{I}$ is called a leading generator system (abbreviated as LGS) of $\mathbb{I}$, if it gives rise to a set of generators as described in Proposition 2.1.3. That is to say, take a set of generators for the leading algebra $\mathrm{L}(\mathbb{I}) = k[\overline{z_i}^{p^{e_i}} \mid i]$, as described in Proposition 2.1.3, where $\{z_i \mid i\} \subset R$ forms a part of an r.s.p. (Note that we set $p^{e_i} = 1$ for all $i$ in the case of characteristic zero.) Then we say that a subset $\mathbb{H} = \{(h_i, p^{e_i}) \mid i\} \subset \mathbb{I}$ is an LGS of $\mathbb{I}$ if the condition $h_i - z_i^{p^{e_i}} \in \mathfrak{m}^{p^{e_i}+1}$ holds for all $i$.

Remark 2.1.6.
(1) Note that an LGS $\mathbb{H}$ is not unique, while $\tau = \#\mathbb{H} = \#\{z_i \mid i\} \leq \dim R$ is independent of the choice of an LGS.
(2) In $\mathrm{char}(k) = p > 0$, an element $h_i = z_i^{p^{e_i}} +$ (higher order terms) in an LGS $\mathbb{H}$ defines a *singular* hypersurface if $e_i > 0$.
(3) In characteristic zero, an LGS $\mathbb{H}$ is of the form $\mathbb{H} = \{(z_i, 1) \mid i\}$ where $\{z_i \mid i\} \subset R$ forms a part of an r.s.p., and $\mathbb{H}$ gives rise to a basis of $\mathrm{L}(\mathbb{I})_1$ as a $k$-vector space. Therefore, in view of Remark 1.2.10 (2), we may say $\{z_i \mid i\}$ gives a "basis" of the defining equations of all hypersurfaces of maximal contact.

Example 2.1.7. Let $\mathbb{I} = \mathfrak{D}(\mathrm{G}(\{(x^2 + y^5, 2)\}))$ on $R = k[x, y]_{(x,y)}$ as before.

- If $\mathrm{char}(k) \neq 2$, then $\mathrm{L}(\mathbb{I}) = k[\overline{x}]$ and $\mathbb{H} = \{(x, 1)\}$ is an LGS.
- If $\mathrm{char}(k) = 2$, then $\mathrm{L}(\mathbb{I}) = k[\overline{x^2}]$ and $\mathbb{H} = \{(x^2 + y^5, 2)\}$ is an LGS.

**2.2. Basic unit of resolution invariant.** The setting is same as that in §2.1. In the classical algorithm, the long strands of invariants, which we call the *resolution invariant* in this article, consists of the basic units of the form $(\dim H, \mu_H)$, where $H$ is a hypersurface of maximal contact and $\mu_H$ is the weak order of an ideal (with a level), computed after taking the restriction to $H$. To be precise, we do not take the restriction of the pair itself but of its so-called coefficient ideal (with some appropriate level). The weak order represents the usual order minus the contribution from the exceptional divisor. We construct the basic units one after another, as we go through the peeling process as explained at the beginning of §2.1. In the IFP, the resolution invariant "inv" consists of the basic units of the form $(\sigma, \mu^{\sim})$, where the invariant $\sigma$ indicates not only the number $\tau$ of the elements in an LGS but also the configuration of their levels, and where $\mu^{\sim}$ is the weak order of an idealistic filtration modulo the ideal generated by the elements in an LGS. Note that, in the real algorithm in both settings above, the basic unit has the third factor, the number of "bad" components of the exceptional divisor, which we ignore in this article for simplicity of the presentation. Also note that, in characteristic zero, since all the elements of an LGS are concentrated at level 1, counting $\tau$ gives the same information as the distribution described in Proposition 2.1.3, while it does not in positive characteristic.

Table 1. The classical algorithm vs. the IFP

| | ambient space | the 1-st invariant | the 2-nd invariant |
|---|---|---|---|
| classical | max. cont. $H$ | $\dim H$: dimension of $H$ | $\mu_H$: order on $H$ |
| IFP | original $W$ | $\sigma(\mathbb{I})$: config. of an LGS $\mathbb{H}$ | $\mu^{\sim}(\mathbb{I})$: order modulo $\mathbb{H}$ |

We present how we define the invariants $\sigma(\mathbb{I})$ and $\mu^{\sim}(\mathbb{I})$ more precisely. Let $\mathbb{I}$ be an $\mathfrak{D}$-saturated idealistic filtration and $\mathbb{H} = \{(h_i, p^{e_i}) \mid i\}$ an LGS of $\mathbb{I}$.

Definition 2.2.1. The invariant $\sigma(\mathbb{I})$ is defined as follows:

$$\sigma(\mathbb{I}) = (\sigma_0, \sigma_1, \ldots) \in \mathbb{Z}_{\geq 0}^{\infty} \quad \text{where} \quad \sigma_j = \dim R - \#\{i \mid e_i \leq j\} \quad (\forall j \in \mathbb{Z}_{\geq 0}).$$

Remark 2.2.2.
(1) The sequence $\sigma(\mathbb{I})$ is infinite, but it stabilizes to a constant after finitely many. Actually, $\sigma(\mathbb{I})$ is a non-increasing sequence, and all entries $\sigma_j$ of $\sigma$ are integers in the range $0 \leq \sigma_j \leq \dim R$.
(2) Note that $\#\{i \mid e_i \leq j\}$ coincides with the dimension of $\mathrm{L}(\mathbb{I})_{p^j} \cap \overline{R^{[p^j]}} \subset \mathrm{Gr}(R)_{p^j}$ as a $k$-vector space. Thus $\sigma(\mathbb{I})$ is independent of the choice of an LGS $\mathbb{H}$.
(3) We introduce the lexicographical order on the values of the invariant $\sigma$. The invariant $\sigma$ according to this order does not increase after a permissible blow up.

Next we discuss how we define the invariant $\mu^{\sim}(\mathbb{I})$. Let $\{y_j \mid j\} \subset R$ be the defining variables of the components of the exceptional divisor $E$. (Note that, in the real algorithm, we only take into consideration the contribution of some components of $E$ and not all. We ignore this subtlety here.) We assume that the LGS $\mathbb{H}$ is transversal to $E$, that is to say, $\{y_j \mid j\} \sqcup \{z_i \mid i\}$ forms a part of an r.s.p. for $R$, where the leading form of $h_i$ is given by $z_i^{p^{e_i}}$ as before.

Definition 2.2.3. We define the order $\mu_{\mathbb{H}}(\mathbb{I})$ of $\mathbb{I}$ modulo $\mathbb{H}$ by the formula

$$\mu_{\mathbb{H}}(\mathbb{I}) = \inf_{a \in \mathbb{R}_{>0}} \frac{\mu_{\mathbb{H}}(I_a)}{a} \quad \text{where} \quad \mu_{\mathbb{H}}(I_a) = \sup\{n \in \mathbb{Z}_{\geq 0} \mid I_a \subset \mathfrak{m}^n + (h_i \mid i)\}.$$

For each irreducible component $E_j = \mathrm{V}(y_j)$ of $E$, we also define the order $\mu_{\mathbb{H},E_j}(\mathbb{I})$ of $\mathbb{I}$ with respect to $E_j$ modulo $\mathbb{H}$ by the formula

$$\mu_{\mathbb{H},E_j}(\mathbb{I}) = \inf_{a \in \mathbb{R}_{>0}} \frac{\mu_{\mathbb{H},E_j}(I_a)}{a} \text{ where } \mu_{\mathbb{H},E_j}(I_a) = \sup\{n \in \mathbb{Z}_{\geq 0} \mid I_a \subset (y_j^n) + (h_i \mid i)\}.$$

Finally, the invariant $\mu^{\sim}(\mathbb{I})$ is defined as follows:

$$\mu^{\sim}(\mathbb{I}) = \mu_{\mathbb{H}}(\mathbb{I}) - \sum_j \mu_{\mathbb{H},E_j}(\mathbb{I}).$$

EXAMPLE 2.2.4. Let $\mathbb{I} = \mathfrak{D}(\mathrm{G}(\{(x^2+y^5, 2)\}))$ over $R = k[x,y]_{(x,y)}$. We assume that there is no exceptional divisor involved for simplicity. Note that $\dim R = 2$.

- In $\mathrm{char}(k) \neq 2$, $\mathbb{H} = \{(x,1)\}$ is an LGS. We have $\sigma(\mathbb{I}) = (2-1, 2-1, \cdots)$ and $\mu^{\sim}(\mathbb{I}) = 5/2$. The infimum is attained by $(x^2+y^5, 2)$.
- In $\mathrm{char}(k) = 2$, $\mathbb{H} = \{(x^2+y^5, 2)\}$ is an LGS. We have $\sigma(\mathbb{I}) = (2-0, 2-1, 2-1, \cdots)$ and $\mu^{\sim}(\mathbb{I}) = 4$. The infimum is attained by $(y^4, 1)$.

We will show that the invariant $\mu^{\sim}(\mathbb{I})$ is independent of the choice of an LGS $\mathbb{H}$. This corresponds to, in the classical algorithm, the part where they show that the weak order $\mu_H$ is independent of the choice of a hypersurface of maximal contact $H$. Before explaining the independence, we introduce the following "Coefficient Lemma", concerning the expansion of $\mathbb{I}$ with respect to $\mathbb{H}$:

LEMMA 2.2.5 (Coefficient Lemma). *Let $\mathbb{H} = \{(h_i, p^{e_i}) \mid i\}$ be an LGS of $\mathbb{I}$ and $\mu \in \mathbb{R}_{>0}$ with $\mu \leq \mu_{\mathbb{H}}(\mathbb{I})$. Then, $\mathbb{I}$ has the following expansion with respect to $\mathbb{H}$:*

$$\mathbb{I}_a = \sum_B \mathbb{J}_{a-|[B]|} H^B \quad \text{where} \quad \mathbb{J}_c = \mathbb{I}_c \cap \mathfrak{m}^{\max(\lceil \mu c \rceil, 0)} \quad (a \in \mathbb{R}).$$

*Here we denote $H^B = \prod_i h_i^{b_i}$ and $[B] = (b_i p^{e_i} \mid i)$ for $B = (b_i \mid i) \in \mathbb{Z}_{\geq 0}^{\#\mathbb{H}}$.*

Note that the idealistic filtration $\mathbb{J} \subset \mathbb{I}$ is a set of the elements of $\mathbb{I}$ whose orders are at least $\mu$. By the definition of $\mu_{\mathbb{H}}(\mathbb{I})$, it is clear that we have $\mathbb{I}_a \subset (h_i \mid i) + \mathbb{J}_a$, proving that the constant term of the above expansion is $\mathbb{J}_a$ as expected. The key point of the lemma is that the coefficients of the higher power terms $H^B$ are also given by $\mathbb{J}_{a-|[B]|}$, using the condition that $\mathbb{I}$ is $\mathfrak{D}$-saturated.

SKETCH OF THE PROOF. Take $f \in \mathbb{I}_a$. Recall $h_i \in z_i^{p^{e_i}} + \mathfrak{m}^{p^{e_i}+1}$ and hence $H^B \in Z^{[B]} + \mathfrak{m}^{|[B]|+1}$. Thus the coefficient of $H^B$ in $f$ is the same as the constant term of $\partial_{Z^{[B]}} f$ modulo higher terms. As $\mathbb{I}$ is $\mathfrak{D}$-saturated, we see $\partial_{Z^{[B]}} f \in \mathbb{I}_{a-|[B]|} \subset (h_i \mid i) + \mathbb{J}_{a-|[B]|}$, which implies that the leading form of the coefficient of $H^B$ in the expansion of $f$ is in $\mathbb{J}_{a-|[B]|}$ as expected. By repeating this argument systematically and combining with Krull's intersection theorem, we can prove the assertion. $\square$

Now we state the independence of the invariant $\mu^{\sim}(\mathbb{I})$.

PROPOSITION 2.2.6. *The invariants $\mu_{\mathbb{H}}(\mathbb{I})$ and $\mu_{\mathbb{H},E_j}(\mathbb{I})$ are independent of the choice of an LGS $\mathbb{H}$, and hence so is the invariant $\mu^{\sim}(\mathbb{I})$.*

SKETCH OF THE PROOF. We only give the proof for $\mu_{\mathbb{H}}(\mathbb{I})$, under the assumption that $\mathrm{L}(\mathbb{I}) = k[z^{p^e}]$. (In fact, the general case is almost reduced to this case.)

Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two LGS's of $\mathbb{I}$. Denote $\alpha_i = \mu_{\mathbb{H}_i}(\mathbb{I})$. We have only to show $\alpha_1 \leq \alpha_2$ by symmetry. Set $\mathbb{H}_i = \{(h_i, p^e)\}$ with $h_i \in z^{p^e} + \mathfrak{m}^{p^e+1}$. Since $\alpha_i \geq \mu(\mathbb{I}) \geq 1$, we may assume $\alpha_1 > 1$. By the definition of $\alpha_1$, we can take $u \in R$ such that $h_2 - uh_1 \in \mathfrak{m}^{\lceil p^e \alpha_1 \rceil}$. Thus $(h_2 - h_1), (h_2 - uh_1) \in \mathfrak{m}^{p^e+1}$ and so $u \in R^{\times}$, which implies $h_1 \in (h_2) + \mathfrak{m}^{\lceil p^e \alpha_1 \rceil}$. Now applying Coefficient Lemma for $\mathbb{H}_1$, we see

$$\mathbb{I}_a \subset \sum_b \mathbb{J}_{a-p^e b} {h_1}^b \subset \sum_b \mathfrak{m}^{\lceil (a-p^e b)\alpha_1 \rceil} \left( (h_2) + \mathfrak{m}^{\lceil p^e \alpha_1 \rceil} \right)^b \subset (h_2) + \mathfrak{m}^{\lceil a \alpha_1 \rceil}$$

for any $a > 0$, thus we have $\alpha_2 = \mu_{\mathbb{H}_2}(\mathbb{I}) = \inf_{a>0} \frac{\mu_{\mathbb{H}_2}(\mathbb{I}_a)}{a} \geq \inf_{a>0} \frac{\lceil a\alpha_1 \rceil}{a} \geq \alpha_1$. □

REMARK 2.2.7.
(1) In fact, Lemma 2.2.5 and Proposition 2.2.6 are valid only assuming $\mathbb{I}$ is relatively $\mathfrak{D}$-saturated (cf. Definition 2.5.4). This assumption is weaker than assuming $\mathbb{I}$ is $\mathfrak{D}$-saturated.
(2) The independence of $\mu_H$ in characteristic zero is originally proved using *Hironaka's trick*, a technique in terms of the Hironaka equivalence. Later Włodarczyk gave a simpler proof, altering the original ideal by a slightly bigger ideal, which he calls its *homogenization* ([**25**]). In our context, the homogenization can be interpreted as a saturation, smaller than our $\mathfrak{D}$-saturation, yet big enough to be analytically isomorphic when restricted to two different hypersurfaces of maximal contact. In our setting, the condition of being $\mathfrak{D}$-saturated (or relatively $\mathfrak{D}$-saturated) is also essential and provides a simple proof for the independence of the invariant $\mu^{\sim}$.

**2.3. Construction of resolution invariant.** We explain the recipe for computing the resolution invariant of the IFP. For simplicity, we assume that there is no exceptional divisor involved.

First, we review again the classical algorithm. Recall that the resolution invariant in characteristic zero is the sequence of the pairs of the dimension $\dim H_i$ of $H_i$ and the order $\mu_{H_i}$ on $H_i$, where each $H_i$ is a hypersurface of maximal contact of an ideal (with a level) on $H_{i-1}$, and this sequence stops when the order becomes 0 or $\infty$, i.e.,

$$(\dim H_0, \mu_{H_0}; \dim H_1, \mu_{H_1}; \ldots; \dim H_r, \mu_{H_r}), \quad \mu_{H_r} = 0 \text{ or } \infty.$$

For example, consider the problem of resolution of singularity of a hypersurface $\mathrm{V}(f) \subset \mathbb{A}^3_k$ defined by $f = x^2 + y^3 + z^4$. Then the invariant at the origin $\mathbf{0} \in \mathbb{A}^3_k$ is

$$(3, 2\colon 2, 3/2\colon 1, 4/3\colon 0, \infty).$$

We divide the procedure of computing the resolution invariant into the following steps and analyze each of them.

(A) The initial ambient space is $H_0 = \mathbb{A}^3_k$. Note $\dim H_0 = \underline{3}$. The initial data is the pair $\alpha_0 = (f, 1)$ with $f = x^2 + y^3 + z^4 \in R_0 = \mathcal{O}_{\mathbb{A}^3_k,\mathbf{0}} = k[x,y,z]_{(x,y,z)}$. The order of the pair $\alpha_0$ is $\mu(\alpha_0) = \mu(f)/1 = \underline{2}$.
(B) Now we raise the level of the pair $\alpha_0$ by multiplying $\mu(\alpha_0)$ to obtain a simple object $\beta_0$, i.e., a pair of order 1. In this example, $\beta_0 = (f, 1 \times \mu(\alpha_0)) = (f, 2)$.
(C) The hypersurface $H_1 = \mathrm{V}(x) \subset H_0$ is of maximal contact for $\mathrm{V}(f)$ at $\mathbf{0}$. Note that $\mathrm{Supp}(\{\beta_0\}) \subset H_1$. Now $H_1$ is our new ambient space.
(D) We construct the coefficient ideal $\alpha_1$ on $H_1$, whose support coincides with $\mathrm{Supp}(\{\beta_0\})$ on $H_1$ in a canonical manner. In this example, it is the restriction of $\beta_0$ on $H_1$, i.e., $\alpha_1 = (f_1, 2)$ with $f_1 = y^3 + z^4 \in R_1 = R_0/(x) \cong k[y,z]_{(y,z)}$.
(*) Now we repeat the same procedure as in (A)∼(D) for $\alpha_1$ on $H_1$. Then we have $\dim H_1 = \underline{2}$, $\mu(\alpha_1) = \mu(f_1)/2 = \underline{3/2}$, $\beta_1 = (f_1, 3)$, $H_2 = \mathrm{V}(y) \subset H_1$ and $\alpha_2 = (f_2, 3)$ with $f_2 = z^4 \in R_2 = R_1/(y) \cong k[z]_{(z)}$.
(**) We continue this procedure till $\mu(\alpha_i)$ becomes 0 or $\infty$. In this example, we have $\dim H_2 = \underline{1}$, $\mu(\alpha_2) = \mu(f_2)/3 = \underline{4/3}$, $\beta_1 = (f_2, 4)$, $H_3 = \mathrm{V}(z) \subset H_2$ and $\alpha_3 = (0, 4)$. In the next step, $\dim H_3 = \underline{0}$ and $\mu(\alpha_3) = \underline{\infty}$, and here we stop.

By positioning the pairs $(\dim H_i, \mu_{H_i} = \mu(\alpha_i))$, we have the resolution invariant. Note that the first entry $\dim H_i = \dim W - i$ can be omitted, since it is already encoded as the position of the corresponding basic unit in the resolution invariant. However, we include it explicitly here, in order to emphasize the inductive scheme on dimension, as well as to compare it to the inductive scheme on $\sigma$ of the IFP.

In the case of the IFP, the resolution invariant is also the long strands of the units of invariants $(\sigma, \mu^{\sim})$ as mentioned in §2.2. We explain its construction according to the steps above, also with the same example $f = x^2 + y^3 + z^4$ in $\mathrm{char}(k) = 0$.

First, as in step (A) above, we also evaluate the unit of invariants $(\sigma, \mu^{\sim})$ of $\mathbb{I}_0$.

(A) The initial object is $\mathbb{I}_0 = \mathfrak{D}(\mathrm{G}(\{(f,1)\}))$ over $R_0$. Then $\mathbb{I}_0 = \mathrm{G}(\{(f,1)\})$, the order $\mu(\mathbb{I}_0) = 2$ and $\mathbb{H}_0 = \emptyset$ is an LGS of $\mathbb{I}_0$. Thus $\sigma(\mathbb{I}_0) = \underline{(3-0, 3-0, \dots)}$ and $\mu^{\sim}(\mathbb{I}_0) = \mu(\mathbb{I}_0) = \underline{2}$.

Instead of constructing the pair $\beta_0$ on $H_0$ by increasing the level as in step (B) above, we construct $\mathbb{I}_1$ as the *modification* $\mathbb{I}'_0$ of $\mathbb{I}_0$. Roughly speaking, the modification $\mathbb{I}'$ of $\mathbb{I}$ is constructed as follows. Divide $\mathbb{I}$ into an LGS $\mathbb{H}$ of $\mathbb{I}$ and "the remainder part of $\mathbb{I}$ by $\mathbb{H}$". Then $\mathbb{I}'$ is constructed as the $\mathfrak{D}$-saturation of the idealistic filtration generated by the remainder part with the levels multiplied by $\mu^{\sim}(\mathbb{I})$, and $\mathbb{H}$. We will give the precise definition of the modification afterwards.

(B) Recall that $\mathbb{H}_0 = \emptyset$. We write $\mathbb{I}_0 = \mathrm{G}(\mathbb{H}_0 \cup \{(f,1)\})$. Then $\mathbb{I}_1$ is defined as $\mathbb{I}_1 = \mathbb{I}'_0 = \mathfrak{D}(\mathrm{G}(\mathbb{H}_0 \cup \{(f, 1 \times \mu^{\sim}(\mathbb{I}_0))\})) = \mathfrak{D}(\mathrm{G}(\{(f,2)\}))$.

Recall that a hypersurface of maximal contact for $\mathrm{V}(f)$ is given by an element of $\mathfrak{D}(\mathrm{G}(\beta_0))_1$. Thus the counterpart of the IFP to step (C) above is to take an LGS $\mathbb{H}_1$ of $\mathbb{I}_1$.

(C) Note that $\mathbb{I}_1 = \mathfrak{D}(\mathrm{G}(\{(f,2)\})) = \mathrm{G}(\{(y^3+z^4,2),(x,1),(y^2,1),(z^3,1)\})$. Thus $\mathbb{H}_1 = \{(x,1)\}$ is an LGS of $\mathbb{I}_1$.

As we stick to the same ambient space, we do not need step (D) above in the IFP.

We repeat the same procedures as in (A)$\sim$(C) above, with $\mathbb{I}_1$ replaced by $\mathbb{I}_0$.

(*) We see $\sigma(\mathbb{I}_1) = \underline{(3-1, 3-1, \dots)}$ and $\mu^{\sim}(\mathbb{I}_1) = \underline{3/2}$, attained by $(y^3+z^4, 2)$. Since $\mathbb{I}_1 = \mathrm{G}(\mathbb{H}_1 \cup \{(y^3+z^4,2),(y^2,1),(z^3,1)\})$, the modification $\mathbb{I}_2$ is defined as

$$\begin{aligned}\mathbb{I}_2 = \mathbb{I}'_1 &= \mathfrak{D}(\mathrm{G}(\mathbb{H}_1 \cup \{(y^3+z^4, 2\cdot\frac{3}{2}),(y^2, 1\cdot\frac{3}{2}),(z^3,1\cdot\frac{3}{2})\})) \\ &= \mathrm{G}(\{(x,1),(y,1),(z^4,3),(z^3,2),(z^2,1)\}).\end{aligned}$$

We see that $\mathbb{H}_2 = \{(x,1),(y,1)\}$ is an LGS of $\mathbb{I}_2$.

We continue this procedure till $\mu^{\sim}(\mathbb{I}_i)$ becomes 0 or $\infty$.

(**) We see $\sigma(\mathbb{I}_2) = \underline{(3-2, 3-2, \dots)}$ and $\mu^{\sim}(\mathbb{I}_2) = \underline{4/3}$, attained by $(z^4,3)$. Since $\mathbb{I}_2 = \mathrm{G}(\mathbb{H}_2 \cup \{(z^4,3),(z^3,2),(z^2,1)\})$, the modification $\mathbb{I}_3$ is defined as

$$\mathbb{I}_3 = \mathbb{I}'_2 = \mathfrak{D}(\mathrm{G}(\mathbb{H}_2 \cup \{(z^4,4),(z^3,8/3),(z^2,4/3)\})) = \mathrm{G}(\{(x,1),(y,1),(z,1)\}).$$

Thus $\mathbb{H}_3 = \{(x,1),(y,1),(z,1)\}$. Next we see that $\sigma(\mathbb{I}_3) = \underline{(3-3, 3-3, \dots)}$ and $\mu^{\sim}(\mathbb{I}_3) = \underline{\infty}$, thus we stop here and obtain the resolution invariant.

$$\mathrm{inv}_{\mathbf{0}} = ((3,3,\dots),2;(2,2,\dots),3/2;(1,1,\dots),4/3;(0,0,\dots),\infty)$$

Now we give the precise definition of the modification $\mathbb{I}'$ of $\mathbb{I}$. Set an LGS of $\mathbb{I}$ as $\mathbb{H} = \{(h_i, p^{e_i}) \mid i\} \subset \mathbb{I}$ with a part of an r.s.p. $\{z_i \mid i\} \subset R$ such that $h_i = z_i^{p^{e_i}} + (\text{higher})$. Then, for $f \in R$, we can define the remainder $\mathrm{r}(f) \in \widehat{R}$ of $f$

TABLE 2. The classical algorithm vs. the IFP

| | resolution invariant | adjusting step of object |
|---|---|---|
| classical | seq. of $(\dim H, \mu_H)$ | level change on $H$: $(f, a) \to (f, a \times \mu_H)$ |
| IFP | seq. of $(\sigma(\mathbb{I}), \mu^{\sim}(\mathbb{I}))$ | modification on $W$: $\mathbb{I} \subset \mathbb{I}'$ |

by $\{h_i \mid i\}$ with respect to the degrees in $z_i$'s, i.e., $\mathrm{r}(f)$ satisfies $f - \mathrm{r}(f) \in (h_i \mid i)$ and $\deg_{z_i} \mathrm{r}(f) < p^{e_i}$ for all $i$. Then the modification $\mathbb{I}'$ of $\mathbb{I}$ is defined by

$$\mathbb{I}' = \mathfrak{D}(\mathrm{G}(\mathbb{H} \cup \{(\mathrm{r}(f), a \times \mu^{\sim}(\mathbb{I})) \mid (f, a) \in \mathbb{I}\})).$$

Note that $\mathbb{I}'$ is a priori defined over $\widehat{R}$. However, it is actually defined over the henselization of $R$. From this, by étale descent argument, we actually see $\mathbb{I}'$ is defined over $R$. We also mention that, in fact, the modification is independent of the choice of LGS. For the details, see [**18**].

EXAMPLE 2.3.1. Assume $\mathrm{char}(k) = 2$. Let $\mathbb{I} = \mathfrak{D}(\mathrm{G}(\{(x^2 + y^5, 2)\}))$ over $R = k[x, y]_{(x,y)}$. Then, $\mathbb{H} = \{(x^2 + y^5, 2)\}$ is an LGS of $\mathbb{I}$, $\sigma(\mathbb{I}) = (2, 1, 1, \ldots)$ and $\mu^{\sim}(\mathbb{I}) = 4$. Since $\mathbb{I} = \mathrm{G}(\mathbb{H} \cup \{(y^4, 1)\})$, the modification $\mathbb{I}'$ of $\mathbb{I}$ is

$$\mathbb{I}' = \mathfrak{D}(\mathrm{G}(\mathbb{H} \cup \{(y^4, 1 \times 4)\})) = \mathrm{G}(\{(x^2, 2), (y^4, 4)\}).$$

Then $\mathbb{H}' = \{(x^2, 2), (y^4, 4)\}$ is an LGS of $\mathbb{I}'$, $\sigma(\mathbb{I}') = (2, 1, 0, 0, \ldots)$ and $\mu^{\sim}(\mathbb{I}') = \infty$. Thus the invariant of $\mathbb{I}$ at the origin $\mathbf{0}$ is $\mathrm{inv}_{\mathbf{0}} = ((2, 1, 1, \ldots), 4; (2, 1, 0, 0, \ldots), \infty)$.

REMARK 2.3.2.
(1) So far we discussed the construction of the resolution invariant in the case where no exceptional divisor is involved, using the $\mathfrak{D}$-saturation. When we take a sequence of blow ups and hence the exceptional divisor is involved, the construction needs further tunings so that the resolution invariant does not increase. We discuss this issue in §2.4(3).
(2) We state the definition of the modification $\mathbb{I}'$ of an idealistic filtration $\mathbb{I}$ when the exceptional divisor is involved. Let $\mathbb{H}$ be an LGS of $\mathbb{I}$ and $Y = \{y_j \mid j\}$ the defining variables of the components of the exceptional divisor $E = \bigcup_j E_j$. Then the modification $\mathbb{I}'$ is defined by the formula

$$\mathbb{I}' = \mathfrak{D}(\mathrm{G}(\mathbb{I} \cup \{(\mathrm{r}(f) \prod_j y_j^{-\lfloor a\mu_{\mathbb{H}, E_j}(\mathbb{I}) \rfloor}, a \times \mu^{\sim}(\mathbb{I})) \mid (f, a) \in \mathbb{I}\})).$$

(3) In the simplest terms, the problem of resolution of singularities requires us to make a variety nonsingular, a condition which is sufficient to be checked at the closed points. Therefore, it should be sufficient to construct the resolution invariants only at the closed points. Our resolution invariants are thus constructed at the closed points. However, it is easy to extend the resolution invariants to the non-closed points using the upper semi-continuity (cf. Theorem 2.4.1).
(4) For the points outside the support, we formally introduce the absolute minimum value $\mathrm{inv}_{\min}$ of the resolution invariant, i.e., for any closed point $P$, we define

$$\mathrm{inv}_P = \mathrm{inv}_{\min} \Leftrightarrow P \notin \mathrm{Supp}(\mathbb{I}).$$

**2.4. Properties of resolution invariant.** A resolution invariant is expected to have the following properties, in order for our algorithm to work:

(1) **USC**: it should be upper semi-continuous,
(2) **Center**: it should indicate how to choose the nonsingular center of blow up, which meets the exceptional divisor with only simple normal crossings,
(3) **Decrease**: it should strictly decrease after permissible blow ups,
(4) **Termination**: it can not decrease infinitely many times, and
(5) **Happy End**: when it hits the absolute minimum value at some point $P$ after a sequence of blow ups, the resolution of singularity is achieved at $P$.

In the following, we discuss these properties of the resolution invariant in the IFP, comparing them to those in the classical algorithm. Theorems 2.4.1 and 2.4.2 are the fundamental results in the IFP appeared in [**16**] and [**17**].

(1) **USC**. We would like to decrease the maximum value of the resolution invariant by blow up. In order to achieve this goal, the center of blow up should contain the maximum locus of the resolution invariant. One of the most reasonable choices for the center is the maximum locus itself, where the center should be a closed subset. In order to guarantee this last condition, the issue of the upper semi-continuity naturally arises. In characteristic zero, the upper semi-continuity of the resolution invariant boils down to that of the basic unit $(\dim H, \mu_H)$, and hence to that of $\mu_H$, which follows easily from that of the order on a nonsingular space $H$. In positive characteristic according to the IFP, the upper semi-continuity of the resolution invariant boils down to that of the basic unit $(\sigma, \mu^{\sim})$. Even though the upper semi-continuity of the invariant $\sigma$ is not so difficult, that of the invariant $\mu^{\sim}$ is subtle. Namely, the invariant $\mu^{\sim}$ depends upon the order on the singular space defined by the elements in an LGS, where the upper semi-continuity is not automatic. To make the situation worse, an LGS may vary from point to point. However, the following theorem establishes the upper semi-continuity over the closed points.

THEOREM 2.4.1. *Let $S$ be a regular ring essentially of finite type over $k$ and $\mathbb{I}$ a $\mathfrak{D}$-saturated idealistic filtration over $S$. Then the function on* $\mathrm{Supp}(\mathbb{I}) \cap \mathrm{maxSpec}\, S$*, defined by the pair $P \mapsto (\sigma(\mathbb{I}_P), \mu^{\sim}(\mathbb{I}_P))$, is upper semi-continuous.*

SKETCH OF THE PROOF. The upper semi-continuity of $\sigma(\mathbb{I}_P)$ is easy. Note that $\dim S - \sigma_0 = \dim \mathrm{L}(\mathbb{I}_P)_1$ is the rank of the Jacobian matrix of $\mathbb{I}_1$ evaluated at $P$, thus $\sigma_0$ is upper semi-continuous. In general, we show the upper semi-continuity of $\sigma_j$ on the constant locus of $(\sigma_0, \sigma_1, \ldots, \sigma_{j-1})$. As mentioned in Remark 2.2.2 (2), we can describe $\sigma_j$ in terms of $\dim \mathrm{L}(\mathbb{I}_P)_{p^j} \cap \overline{S_P^{[p^j]}}$, which is controlled by the rank of some Jacobian-like matrix but using $\partial_{x_i^{p^j}}$'s instead of $\partial_{x_i}$'s. Next we show the upper semi-continuity of $\mu^{\sim}(\mathbb{I}_P)$ on the constant locus of $\sigma(\mathbb{I}_P)$, which is the harder part of the proof. For each $P \in \mathrm{Supp}(\mathbb{I})$, we construct a *uniform* LGS $\mathcal{H}$ on a small affine open neighborhood $U \subset \mathrm{Spec}\, S$ of $P$, that is, the subset $\mathcal{H} \subset \mathbb{I}_U$ such that $\mathcal{H}_Q$ is an LGS of $\mathbb{I}_Q$ for any $Q \in U$ satisfying $\sigma(\mathbb{I}_Q) = \sigma(\mathbb{I}_P)$. After that, we construct the *uniform* coefficients of an element of $S$ in the expansion with respect to $\mathcal{H}$, which yields the upper semi-continuity of the invariant $\mu^{\sim}(\mathbb{I}_P)$. □

(2) **Center**. Now we know that the maximum locus of the resolution invariant is closed. As it is the candidate for the center of blow up, we also expect its nonsingularity. In the classical algorithm, we distinguish the two cases according to the value of the second entry $\mu_{H_r}$ of the last basic unit of the resolution invariant.

In the case where $\mu_{H_r} = \infty$, the maximal locus of the resolution invariant is nothing but the support of the last object $\alpha_r$, and it locally coincides with $H_r$, since $\mu_{H_r} = \infty$. Thus it is nonsingular, since so is $H_r$. The case where $\mu_{H_r} = 0$ is called the *monomial case*. The center is chosen to be the intersection of the components of some simple normal crossing divisor in $H_r$, and hence it is nonsingular. In the IFP, we have a similar distinction of the following two cases according to the value of the second entry $\mu^{\sim}(\mathbb{I}_r)$ of the last basic unit of the resolution invariant. In the case where $\mu^{\sim}(\mathbb{I}_r) = \infty$, we take the center to be the support of the last modification. Since the elements of an LGS may define a singular subvariety, it seems that one cannot expect the nonsingularity of the support of the last modification. However, the following *nonsingularity principle* guarantees it is actually nonsingular. The case where $\mu^{\sim}(\mathbb{I}_r) = 0$ is also called the *monomial case*. We will discuss the monomial case of the IFP in §2.5.

THEOREM 2.4.2 (Nonsingularity principle for $\mathfrak{D}$-saturation). *Assume* $\mathrm{char}(k) = p > 0$. *Let* $\mathbb{I}$ *be a* $\mathfrak{D}$*-saturated idealistic filtration with* $\mu(\mathbb{I}) \geq 1$ *and* $\mu^{\sim}(\mathbb{I}) = \infty$. *Then there exist a part of an r.s.p.* $\{x_j \mid j\} \subset R$ *for* $R$ *and a set* $\{e_j \mid j\} \subset \mathbb{Z}_{\geq 0}$ *of non-negative integers such that* $\mathbb{H} = \{(x_j^{p^{e_j}}, p^{e_j}) \mid j\}$ *is an LGS of* $\mathbb{I}$ *and it generates* $\mathbb{I}$, *i.e.,* $\mathbb{I} = \mathrm{G}(\mathbb{H})$. *In particular, the support of* $\mathbb{I}$ *is* $\mathrm{Supp}(\mathbb{I}) = \mathrm{V}(x_j \mid j)$, *which is nonsingular.*

In the case where $\mu^{\sim}(\mathbb{I}) = \infty$, Theorem 2.5.2, which we will present later, gives a similar but slightly weaker assertion than Theorem 2.4.2. For the proof of Theorem 2.4.2, where Proposition 1.2.8 plays the key role, we only refer the reader to the appendix of [**17**], since it has the same flavor as that of Theorem 2.5.2.

We do not discuss here the property that the center meets the exceptional divisor with only simple normal crossings, which is realized by taking into account the third factor of the basic unit of the resolution invariant (cf. beginning of §2.2).

(3) **Decrease**. Next we discuss how the resolution invariant decreases after blow up. In the classical algorithm, the mechanism of the decrease is summarized as follows. First of all, we show the non-increase of the resolution invariant, which is in principle guaranteed by that of the weak order on a hypersurface of maximal contact. After that, we show the strict decrease of the resolution invariant. Roughly speaking, the center of blow up is defined by the intersection of the hypersurfaces of maximal contact corresponding to the basic units of the resolution invariant. After blow up, the point on the exceptional divisor is not contained in at least one strict transform of hypersurfaces $H$ of maximal contact, and the corresponding $\mu_H$ decreases by the definition of $H$. In positive characteristic according to the IFP, the first mechanism is more subtle, while the second mechanism works in a way parallel to the classical one. There are two major issues as follows:

(i) Recall that one of the "natural" definitions for the transformation of an idealistic filtration of r.f.g. type is given in Remark 1.6.3 (1). However, according to this definition, the invariant may increase after blow up, without taking any saturation.
(ii) The invariant for an idealistic filtration of r.f.g. type may increase when we incorporate too large saturations. Namely, if we always take the $\mathfrak{D}$-saturation, the $\mathfrak{D}_E$-saturation, or those combined with the $\mathfrak{R}$-saturation after blow up, the invariant may increase. See Example 2.5.3.

The first issue (i) is rather innocent, since it is caused by the discrepancy of the exponent of the exceptional factors due to taking the round down. There are two ways to resolve this issue. The first way is that we only deal with the idealistic filtrations generated by the elements at the integral levels to avoid the discrepancy. This is the method which we use in [**18**]. Another way is to allow the fractional powers of the exceptional factors by considering a "bigger" category of rings. Details of this approach will appear elsewhere, while this idea is already used in the construction of the modification in [**18**]. The second issue (ii) concerning the effect of saturations is more serious. One way to resolve this issue is to use the intermediate object between the transformation of the original idealistic filtration (without taking any saturations) and its $\mathfrak{D}$-saturation. We adopt this strategy in [**18**], by using the notion of being *relatively* $\mathfrak{D}$*-saturated* (see Definition 2.5.4). Another way, though the research in this direction is still in progress, is to use the full $\mathfrak{RD}_E$-saturation, but changing the definition of the weak order by factoring out monomials of the exceptional variables elementwise, while keeping the information of the exceptional factors. Anyway, the author does not have a decisive answer for the best tune-up of the algorithm to resolve these issues at this moment. We will deal with the second issue (ii) again in §2.5.

(4) **Termination**. We would like to design the resolution invariant not to decrease infinitely many times, in order for the algorithm to terminates after finitely many steps. We assume that the maximum of the resolution invariant decreases after each blow up, and that the center of blow up coincides with the maximum locus. We would like to have the property that the maximum of the resolution invariant reaches the absolute minimum after finitely many permissible blow ups. We remark that this property is local, namely, *if the resolution invariant cannot decrease infinitely many times in any sequence of local permissible blow ups, then the same property holds for the global maximum of the resolution invariant in any sequence of permissible blow ups.* It is verified by the following argument.

Sketch of the proof. Let $W = W_0 \overset{\pi_1}{\leftarrow} W_1 \overset{\pi_2}{\leftarrow} W_2 \ldots$ be an infinite sequence of blow ups along the nonsingular centers $C_i \subset W_i$ such that the global maximum $\alpha_i = \max_{P \in W_i} \operatorname{inv}_P$ of the resolution invariant is strictly decreasing, i.e., $\alpha_i > \alpha_{i+1}$ for any $i \in \mathbb{Z}_{\geq 0}$. We denote $\pi_{i,j} = \pi_{i+1} \circ \cdots \circ \pi_j \colon W_j \to W_i$ for any $i \leq j$. First we show that there exists an infinite sequence $t_1 < t_2 < \ldots$ of nonnegative integers satisfying the condition that $\Phi_\ell = \left\{ j \in \mathbb{Z}_{>t_\ell} \mid \pi_{t_\ell+1,j}(C_j) \cap \bigcap_{1 \leq i \leq \ell} \pi^{-1}_{t_i, t_\ell+1}(C_i) \neq \emptyset \right\}$ is an infinite set for any $\ell \in \mathbb{Z}_{>0}$. Set $t_0 = -1$. If $t_0 < \ldots < t_\ell$ are already constructed, then $t_{\ell+1}$ is constructed as follows. Set $S_\ell = \bigcap_{1 \leq i \leq \ell} \pi^{-1}_{t_i, t_\ell+1}(C_{t_i}) \subset W_{t_\ell+1}$. Observe that $\pi_{j+1}$ is isomorphic over $S_\ell$ for any $j \in \mathbb{Z}_{>t_\ell} \setminus \Phi_\ell$, since $\pi_{t_\ell+1,j}(C_j) \cap S_\ell = \emptyset$. Set $F = \{\operatorname{inv}_P \mid P \in S_\ell\}$. Note that $\#F < \infty$, since $\operatorname{inv}_P$ is upper semi-continuous. Set $s = \max\{i \in \Phi_\ell \mid \alpha_i \in F\}$. Then we have

$$\pi_{s,j}(C_j) \cap \pi^{-1}_{t_\ell+1,s}(S_\ell) \subset \bigcup \left\{ \pi^{-1}_{i,s}(C_i) \mid i \in \Phi_\ell \cap \mathbb{Z}_{\leq s} \right\} \quad \text{for all } j \in \mathbb{Z}_{\geq s},$$

since $\pi^{-1}_{t_\ell+1,s}(S_\ell) \setminus \bigcup_{i \in \Phi_\ell \cap \mathbb{Z}_{\leq s}} \pi^{-1}_{i,s}(C_i)$ is isomorphic to $S_\ell \setminus \bigcup_{i \in \Phi_\ell \cap \mathbb{Z}_{\leq s}} \pi_{t_\ell+1,i}(C_i)$ via $\pi_{t_\ell+1,s}$, where the invariants are disjoint from $\{\alpha_i \mid i \in \Phi_\ell\}$, while the invariants on $C_j$ is $\alpha_j$. Moreover, we have $\pi_{s,j}(C_j) \cap \pi^{-1}_{t_\ell+1,s}(S_\ell) \neq \emptyset$ for any $j \in \Phi_\ell \cap \mathbb{Z}_{\geq s}$. Therefore there exists some $t_{\ell+1} \in \Phi_\ell \cap \mathbb{Z}_{\leq s}$ such that $\pi_{s,j}(C_j) \cap \pi^{-1}_{t_\ell+1,s}(S_\ell) \cap \pi^{-1}_{t_{\ell+1},s}(C_{t_{\ell+1}}) \neq \emptyset$ for infinitely many $j \in \Phi_\ell \cap \mathbb{Z}_{\geq s}$, namely, $\#\Phi_{\ell+1} = \infty$. Set $V_{i,j} =$

$\bigcap_{1\le a\le i}\pi^{-1}_{t_a,t_i}(C_k)\cap\bigcap_{i<b\le j}\pi_{t_i,t_b}(C_{t_b})\subset W_{t_i}$ for any $i<j$ and $V_i=\bigcap_{j\in\mathbb{Z}_{\ge i}}V_{i,j}\subset W_{t_i}$ for any $i\in\mathbb{Z}_{>0}$. Then $V_{i,j}\ne\emptyset$ by the above assertion and $V_i\ne\emptyset$, since $V_i$ is the intersection of the decreasing sequence of nonempty closed subsets. It is also easy to see that $\pi_{t_i,t_j}(V_j)=V_i$ for any $i\le j$. Now choose $Q_1\in V_1$ and $Q_{i+1}\in\pi^{-1}_{t_i,t_{i+1}}(Q_i)$ for $i\in\mathbb{Z}_{>0}$ inductively. Then the local sequence $\cdots Q_i\mapsto\cdots\mapsto Q_{i-1}\mapsto\cdots\mapsto Q_1$ admits the infinitely many decreases of the invariant $\alpha_{t_1}>\alpha_{t_2}>\cdots$. □

Thus we have only to discuss the local termination of the decrease of the resolution invariant, and it is reduced to that of the basic unit. It is easy to see that the first entry of the basic unit, $\dim H$ in the classical setting or $\sigma$ in the IFP, does not decrease infinitely many times. Thus the problem is boiled down to the termination of the second entry of the basic unit, $\mu_H$ or $\mu^\sim$, under the assumption that the first entry stays constant. If the invariant is constructed by using only the $\mathfrak{D}$-saturation or its variation, it is easy to see the termination of $\mu^\sim$ just by an argument similar to the one in the classical algorithm. The key point is that the generators of the $\mathfrak{D}$-saturation is described explicitly in terms of the generators of the original idealistic filtration. Therefore we can control their levels, and it gives us the bound of the denominator of $\mu^\sim$. However, unfortunately we do not have the proof of the termination if the construction of the resolution invariant involves the $\mathfrak{R}$-saturation, where we do not have the explicit description of the generators.

(5) **Happy End**. By the definition of $\mathrm{inv}_{\min}$, the terminal situation is nothing but the case of $\mathrm{Supp}(\mathbb{I})=\emptyset$, where $\mathbb{I}$ is the transformation of the original idealistic filtration $\mathrm{G}(\mathcal{I}_X\times\{1\})$. The total transform of $X$ is now consisted of the exceptional divisor only, and the strict transform of $X$ has disappeared. Therefore, at some stage in the resolution sequence, the strict transform of $X$ itself coincides with a component of the center of blow up, which is nonsingular and meets the exceptional divisor with only simple normal crossings. Thus the embedded resolution of singularities for $X$ is achieved at this stage.

**2.5. Monomial case of IFP.** The last topic of this section is the monomial case of the IFP. As mentioned in §2.4, we have two cases according to the value of $\mu^\sim$ of the last basic unit of the resolution invariant; $\mu^\sim=\infty$ or $\mu^\sim=0$. The case of $\mu^\sim=\infty$ is already discussed in §2.4(2). The case of $\mu^\sim=0$, the *monomial case*, is more difficult. We discuss two ways to analyze the monomial case. The first way is to utilize the $\mathfrak{R}$-saturation and the second way is to utilize the notion of being relatively $\mathfrak{D}$-saturated. The former is an ongoing project, which has not evolved into a precise algorithm, even though the author feels it has a good potential. Actually, we show in Theorem 2.5.2 that the $\mathfrak{R}$-saturation yields the nonsingularity principle also in the monomial case. This is a new result and appears here for the first time. We also discuss the shortcomings of this approach. The latter leads us to a new invariant, providing an alternative proof for embedded resolution of surfaces [**18**]. This is done by incorporating the method of Benito-Villamayor [**5**] into the framework of the IFP.

Recall the monomial case in the classical setting, where we analyze a pair $(M,a)$ of a monomial $M$ of the defining variables of the components of the exceptional divisor and a level $a\in\mathbb{Z}_{>0}$. Note that $M$ lives on a hypersurfaces $H$ of maximal contact, which is nonsingular. Thus the support $\mathrm{Supp}(\{(M,a)\})$ is combinatorially described in terms of $a$ and the exponents of $M$. Therefore, it is easy to prescribe the way to handle the monomial case in the classical setting. However, in the setting of the IFP, the last modification in the monomial case is generated by a monomial at a certain level, on a possibly *singular* space defined by the elements in an LGS. Therefore the irreducible components of $\mathrm{Supp}(\mathbb{I})$ also may be singular, a fact which prevents us to apply the combinatorial argument as in the classical setting.

We review the monomial case in the IFP. Let $Y = \{y_j \mid j\}$ be the defining variables of the components of the exceptional divisor $E = \bigcup_j E_j$ and $\mathbb{H} = \{(h_i, p^{e_i}) \mid i\}$ an LGS of $\mathbb{I}$. Then $\mathbb{I}$ is in the monomial case if $\mu^{\sim}(\mathbb{I}) = 0$, i.e., $\mu_{\mathbb{H}}(\mathbb{I}) = \sum_j \mu_{\mathbb{H},E_j}(\mathbb{I})$. Symbolically saying, "$\mathbb{I}$ is generated by $\mathbb{H}$ and $(\prod_j y_j^{\mu_{\mathbb{H},E_j}(\mathbb{I})}, 1)$". More precisely, the monomial case is expressed as follows:

> There exists $(Y^\alpha, b) \in \mathbb{I}$, where $\alpha = (\alpha_i \mid i) \in \mathbb{Z}_{\geq 0}^{\#Y}$, $Y^\alpha = \prod_j y_j^{\alpha_j}$ and $b \in \mathbb{R}_{>0}$, such that $\mathbb{I}_a \subset (h_i \mid i) + (Y^{\lceil (a/b)\alpha \rceil})$ for any $a \in \mathbb{R}_{>0}$.

Note that, if all the elements in $\mathbb{H}$ sit at level 1, i.e., $\{h_i \mid i\}$ is a part of an r.s.p. for $R$, then we can apply the combinatorial argument as in the classical algorithm.

The first way to analyze the monomial case is to utilize the $\mathfrak{RD}_E$-saturation. This is the way which the author originally expected as a good candidate for tuning the algorithm. Finally he obtained two facts concerning this candidate, one good and one bad. The good fact is that the nonsingularity principle also holds if one uses this candidate. The bad fact is that the invariant may increase after blow up if we utilize the $\mathfrak{RD}_E$-saturation.

Before stating this new nonsingularity principle, we prepare one proposition, which is a generalization of Proposition 1.2.8.

PROPOSITION 2.5.1. *Assume* $\mathrm{char}(k) = p > 0$. *Let* $Y \subset R$ *be the defining variables of the components of a simple normal crossing divisor* $E \subset \operatorname{Spec} R$, $I \subset R$ *an ideal and* $e \in \mathbb{Z}_{\geq 0}$. *Then, the following conditions are equivalent:*

(1) $\mathrm{Diff}_E^{\leq p^e-1}(I) = I$,
(2) *there exist* $\{f_i \mid i\} \subset R^{[p^e]}$ *and* $\{\alpha_i \mid i\} \subset \mathbb{Z}_{\geq 0}^{\#Y}$ *such that* $I = (f_i Y^{\alpha_i} \mid i)$.

PROOF. Set $\Delta = \{0, \ldots, p^e - 1\}$ and let $M = \bigsqcup_{J \in \Delta^{\#Y}} R^{[p^e]} Y^J \subset R$ be a disjoint sum of $R^{[p^e]}$-submodule of $R$. Then (2) is equivalent to $I = (I \cap M)R$.

**Step 1**. First we show that we may replace $R$ by its completion $\widehat{R}$ with respect to the unique maximal ideal $\mathfrak{m}$ of $R$. Set $M' = \bigsqcup_{J \in \Delta^{\#Y}} \widehat{R}^{[p^e]} Y^J \subset \widehat{R}$. Since $\widehat{R}$ is faithfully flat over $R$, it suffices to show the following equations

$$(I \cap M)\widehat{R} = (I\widehat{R} \cap M')\widehat{R} \quad \text{and} \quad \mathrm{Diff}_E^{\leq p^e-1}(I)\widehat{R} = \mathrm{Diff}_E^{\leq p^e-1}(I\widehat{R}).$$

The former follows from the observation that $(I \cap R^{[p^e]} Y^J)\widehat{R}^{[p^e]} \supset I\widehat{R} \cap \widehat{R}^{[p^e]} Y^J$ for any $J \in \Delta^{\#Y}$, since $\widehat{R}^{[p^e]}$ is flat over $R^{[p^e]}$ (see [**16**] 1.3.1.3), and the latter follows from $\mathrm{Diff}_{\widehat{R},E}^{\leq p^e-1} = \mathrm{Diff}_{R,E}^{\leq p^e-1} \otimes_R \widehat{R}$.

Now we prove the assertion replacing $R$ with $\widehat{R}$. As $R$ is regular with $\kappa(R) = k$, $\widehat{R}$ is isomorphic to the formal power series ring over $k$. By taking an r.s.p. $X \sqcup Y$ for $R$, we regard $R = k[[X \sqcup Y]]$.

**Step 2**. We show that (2) implies (1). Note $\mathrm{Diff}_E^{\leq p^e-1}(I) = \mathrm{Diff}_E^{\leq p^e-1}(I \cap M)$, since $I = (I \cap M)$. Recall that $\mathrm{Diff}_E^{\leq p^e-1} = \bigoplus_{|\beta|+|\gamma| \leq p^e-1} R \cdot Y^\beta \partial_{(Y^\beta X^\gamma)}$ (see [**16**]

1.2.2.2). For $g \in R^{[p^e]}$ and $J \in \Delta^{\#Y}$, we have

$$Y^\beta \partial_{(Y^\beta X^\gamma)}(gY^J) = gY^\beta \partial_{(Y^\beta X^\gamma)}(Y^J) = gY^\beta \delta_{\gamma,\mathbf{0}} \binom{J}{\beta} Y^{J-\beta} \in \mathbb{Z} \cdot (gY^J).$$

Thus $\mathrm{Diff}_E^{\leq p^e-1}(I \cap M) = (I \cap M)R = I$, which concludes (1).

**Step 3**. We show that (1) implies (2). Take $g \in I$ and express it as $g = \sum_{J \in \Delta^{\#Y},\ K \in \Delta^{\#X}} g_{J,K} Y^J X^K$, where $\{g_{J,K} \mid J, K\} \subset R^{[p^e]}$. It suffices to show $\{g_{J,K} Y^J \mid J, K\} \subset I$. Assume it does not hold. Then there exists a maximal $J_0$ such that $\{g_{J_0,K} Y^{J_0} \mid K\} \not\subset I$, and also a maximal $K_0$ such that $g_{J_0,K_0} Y^{J_0} \notin I$. Now set $D = Y^{J_0} \partial_{Y^{J_0}} \partial_{X^{K_0}} = \prod_{y_u \in Y} \left( y_u^{j_{0,u}} \partial_{y_u^{j_{0,u}}} \right) \prod_{x_v \in X} \left( \partial_{x_v^{k_{0,v}}} \right)$. As $D$ is a composition of the operators in $\mathrm{Diff}_E^{\leq p^e-1}$, we have $Dg \in I$. Therefore we have

$$\sum_{J \in \Delta^{\#Y},\ K \in \Delta^{\#X}} \binom{J}{J_0} \binom{K}{K_0} g_{J,K} Y^J X^{K-K_0} \in I.$$

If $J_0 \not\leq J$, then $\binom{J}{J_0} = 0$ by its definition. If $J_0 < J$, then $\{g_{J,K} Y^J \mid K\} \subset I$ by the maximality of $J_0$. Thus we have

$$\sum_{K \in \Delta^{\#X}} \binom{K}{K_0} g_{J_0,K} Y^{J_0} X^{K-K_0} \in I.$$

If $K_0 \not\leq K$, then $\binom{K}{K_0} = 0$ by its definition. If $K_0 < K$, then $g_{J_0,K} Y^{J_0} \in I$ by the maximality of $K_0$. Therefore we have $g_{J_0,K_0} Y^{J_0} \in I$, which is a contradiction. $\square$

By using Proposition 2.5.1, we have the following theorem.

Theorem 2.5.2 (Nonsingularity Principle for $\mathfrak{RD}_E$-saturation). *Assume $p = \mathrm{char}(k) > 0$. Let $E \subset \operatorname{Spec} R$ be a simple normal crossing divisor. Let $Y, Z \subset R$ be two disjoint subsets of $R$ such that $Y$ is the set of the defining variables of the components of $E$ and $Y \sqcup Z$ is a part of an r.s.p. for $R$. Let $\mathbb{I}$ be an $\mathfrak{RD}_E$-saturated idealistic filtration and $\mathbb{H} = \{(h_i, p^{e_i}) \mid i\} \subset \mathbb{I}$ a finite subset satisfying $h_i - z_i^{p^{e_i}} \in \mathfrak{m}^{p^{e_i}+1}$ for any $i$. Assume one of the following holds:*

(1) *$\mathbb{I}_a \subset I(\mathbb{H})$ for any $a \in \mathbb{R}_{>0}$, where $I(\mathbb{H}) = (h_i \mid i)$.*

(2) *There exist $m \in \mathbb{R}_{>0}$ and $\alpha = (\alpha_j \mid j) \in \mathbb{Z}_{\geq 0}^{\#Y}$ such that $|\alpha| \geq m$, $Y^\alpha \in \mathbb{I}_m + I(\mathbb{H})$ and $\mathbb{I}_a \subset I(\mathbb{H}) + \left(Y^{\lceil (a/m)\ \alpha \rceil}\right)$ for any $a \in \mathbb{R}_{>0}$.*

*Then, all $e_i = 0$, i.e., $\mathbb{H} = \{(h_i, 1) \mid i\}$, and $\{h_i \mid i\} \sqcup Y$ is a part of an r.s.p. for $R$. Moreover, we have $\mathbb{I} = \mathrm{G}(\mathbb{H})$ in (1), and $\mathbb{I} = \mathrm{G}\left(\mathbb{H} \cup \{(Y^{\lceil \frac{i}{d} \alpha \rceil}, \frac{im}{d}) \mid 1 \leq i \leq d\}\right)$ in (2), where $d = \mathrm{LCM}\{\alpha_j \mid j\}$.*

Proof. We derive the contradiction assuming $e_i > 0$ for some $i$. First consider the case (2). By taking the quotient by $(h_i \mid e_i = 0)$ if necessary, we may assume $e_i > 0$ for any $i$. Set $e = \min\{e_i \mid i\} > 0$, $Y_0 = \prod_{\alpha_j > 0} y_j$ and $J = I(\mathbb{H}) + (Y_0) \subset \mathfrak{m}$. Note that $\mathbb{I}_a + (Y_0) = J$ for any $0 < a \leq p^e$, since

$$J = (h_i \mid e_i \geq e) + (Y_0) \subset \mathbb{I}_{p^e} + (Y_0) \subset \mathbb{I}_a + (Y_0) \subset I(\mathbb{H}) + (Y_0) = J.$$

Thus $\mathrm{Diff}_Y^{\leq p^e-1}(J) = J$. By Proposition 2.5.1, $J$ is expressed as $J = \left(Y^{\beta_\ell} f_\ell^{p^e} \mid \ell\right)$. Since $J + (y_j \mid j) = I(\mathbb{H}) + (y_j \mid j) \not\subset \mathfrak{m}^{p^e+1} + (y_j \mid j)$, some $\ell$ satisfies $f_\ell \notin (y_j \mid j) + \mathfrak{m}^2$ and $f_\ell^{p^e} \in J$. Note that $Y_0^{\max\{\alpha_i \mid i\}} \in (Y^\alpha) \subset \mathbb{I}_{\min\{m,p^e\}}$. By setting

$\varepsilon = \min\{m, p^e\}/\max\{\alpha_i \mid i\} \in \mathbb{R}_{>0}$, we have $\mathbb{I}_a = J$ for any $0 < a \leq \varepsilon$. Thus $f_\ell \in \sqrt{J} = J \subset (y_j \mid j) + \mathfrak{m}^{p^e} \subset (y_j \mid j) + \mathfrak{m}^2$, a contradiction.

We have seen $e_i = 0$ for any $i$ in (2). The proof for (1) is identical to this argument by replacing $Y_0$ with 0. The proof for the "Moreover" part is easily shown by considering the expansion and the partial differential operators with respect to the r.s.p. for $R$ containing $\{h_i \mid i\} \sqcup Y$. □

Note that (1) guarantees the nonsingularity of the support when $\mu^\sim(\mathbb{I}) = \infty$, and (2) allow us to apply the combinatorial argument as in the classical algorithm when $\mu^\sim(\mathbb{I}) = 0$. Thus we have the ideal situation in the last modification in arbitrary dimension *provided* we are allowed to use the $\mathfrak{RD}_E$-saturation.

However, the life is not so easy. In fact, if we utilize the $\mathfrak{RD}_E$-saturation, the invariant may increase after blow up, as illustrated by the following example.

Example 2.5.3. Assume $\mathrm{char}(k) = 2$. Let $\mathbb{I} = \mathrm{G}(\{(x^2 + y^5, 2)\})$ over $R = k[x, y]_{(x,y)}$. Recall $\mathfrak{RD}(\mathbb{I}) = \mathrm{G}\left(\{(x^2 + y^5, 2), (x, 5/8), (y, 1/4)\}\right)$, $\mathbb{H} = \{(x^2+y^5, 2)\}$, $\sigma = (2, 1, 1, \ldots)$ and $\mu^\sim = 8/5$. Now blow up at the origin and look in $y$-chart, namely, $x \mapsto ex$, $y \mapsto e$ with the exceptional variable $e$. Then, the transformation of $\mathbb{I}$ is $\mathbb{I}^\vee = \mathrm{G}(\{(x^2 + e^3, 2)\})$ with exceptional divisor $E = \{\mathrm{V}(e)\}$. Therefore we have $\mathfrak{RD}_E(\mathbb{I}^\vee) = \mathrm{G}(\{(x^2 + e^3, 2), (e, 1/3), (x, 1/2)\})$, $\mathbb{H} = \{(x^2 + e^3, 2)\}$, $\sigma = (2, 1, 1, \ldots)$ and $\mu^\sim = 2 > 8/5$. Thus $\mu^\sim$ increases after blow up.

In order to establish the non-increasing resolution invariant compatible with $\mathfrak{RD}_E$-saturation, the author is trying to construct a new algorithm. See §2.4(3)(ii).

Now we discuss the second way to analyze the monomial case where we only assume the idealistic filtration is relatively $\mathfrak{D}$-saturated (for the definition, see below). For the reason to use the notion of being relatively $\mathfrak{D}$-saturated, which is weaker than the notion of being $\mathfrak{D}$-saturated, see the explanation of the construction of the algorithm in the next paragraph.

Definition 2.5.4. An idealistic filtration $\mathbb{I}$ is called relatively $\mathfrak{D}$-saturated if there exist an r.s.p. $X \subset R$ for $R$ and a subset $Z \subset X$ such that $\mathrm{L}(\mathfrak{D}(\mathbb{I})) = k[\overline{z_i}^{p^{e_i}} \mid i]$ and $\mathbb{I}$ is $\{\partial_{Z^I} \mid I\}$-saturated, i.e., $f \in \mathbb{I}_a \Rightarrow \partial_{Z^I} f \in \mathbb{I}_{a-|I|}$ for all $I$. Note that $\{\partial_{Z^I} \mid I\} \subset \{\partial_{X^J} \mid J\}$ depends not only $Z$ but also $X$.

Remark 2.5.5. The notion of being relatively $\mathfrak{D}$-saturated can be traced back to the construction of the coefficient ideal in the classical algorithm, where not all the differential operators, but only the partial ones with respect to the variable defining a hypersurface of maximal contact appear (cf. [**4**]). However, our direct reference of this notion appears in Villamayor's projection method [**5**], where this notion has a more transparent interpretation. Namely, in their framework, they consider a projection which is transversal to the tangent cone $\mathrm{V}(z_i)$ of $\mathrm{V}(h_i)$ for each elements $h_i$ of an LGS of $\mathfrak{D}(\mathbb{I})$. Then, roughly speaking, the composition of them is a projection $\pi\colon M = \operatorname{Spec} k[X] \to M' = \operatorname{Spec} k[X \setminus Z]$ for some r.s.p. $X$ for $R$. Under this setting, the differential operators $\{\partial_{Z^I} \mid I\}$ is interpreted as the differential operators on $M$ which kill all the functions $\pi^*\mathcal{O}_{M'}$ coming from $M'$, i.e., the *relative* differential operators with respect to $\pi$.

In [**18**], we construct the algorithm where the resolution invariant does not increase after blow up. In this algorithm, we introduced the following construction after blow up, to assure that the condition of being relatively $\mathfrak{D}$-saturated is always satisfied.

- If the resolution invariant up to the previous basic unit or the the invariant $\sigma$ of the $\mathfrak{D}$-saturation of the modification has decreased after blow up, then take the full $\mathfrak{D}$-saturation of the transformation.
- Otherwise, i.e., if none of the above values decreases after blow up, just take the transformation without any saturation, instead of taking its $\mathfrak{D}$-saturation.

We can choose $Z = \{z_i \mid i\}$ in Definition 2.5.4 as a part of a generators of the ideal which defines a nonsingular center of blow up. If $\sigma$ is unchanged, the transformation of an LGS is again an LGS, and thus, roughly speaking, $\{z_i \mid i\}$ survives after blow up. Then the transformation of the variable after blow up $\pi$ is $\pi^*(z_i) = ez_i$, where $e$ is the exceptional variable with respect to the exceptional divisor of this blow up. This observation guarantees that the condition of being relatively $\mathfrak{D}$-saturated persists to hold. For example, the following calculation shows that the condition of being $\{\partial_{z_i^\ell} \mid \ell \in \mathbb{Z}_{\geq 0}\}$-saturated is preserved:

$$\begin{aligned} \partial_{z_i^\ell}((f,a)^\vee) &= \partial_{z_i^\ell}(e^{-\lfloor a \rfloor}\pi^* f, a) = (e^{-\lfloor a \rfloor}\partial_{z_i^\ell}(\pi^*(f)), a-\ell) \\ &= (e^{\ell-\lfloor a \rfloor}\pi^*(\partial_{z_i^\ell} f), a-\ell) = (\partial_{z_i^\ell} f, a-\ell)^\vee = (\partial_{z_i^\ell}(f,a))^\vee. \end{aligned}$$

We should mention, in contrast to the above calculation, that the operation of taking the transformation and that of taking the full $\mathfrak{D}$-saturation do not commute.

In the case of $\mu^\sim = \infty$, we can show that in fact the last modification is $\mathfrak{D}$-saturated and we can apply the nonsingularity principle. However, in the monomial case, an LGS may define a singular subvariety and hence further analysis is needed. For a surface in a three dimensional ambient space (in fact, with some additional argument, also a surface embedded in an ambient space of arbitrary dimension), we succeed in working out the algorithm according to the IFP by introducing an additional invariant for the monomial case [**18**]. We do not explain the details of this additional invariant here, but it should be mentioned that it is a simplification of Benito-Villamayor's analysis [**5**] of their monomial case to achieve the embedded resolution of surfaces in positive characteristic.

THEOREM 2.5.6. *The IFP with the relative differential saturation gives an algorithm for embedded resolution of surfaces in arbitrary characteristic.*

REMARK 2.5.7. We should mention that the embedded resolution of surfaces (over an algebraically closed field) is already known. The proof in the case of a three dimensional ambient space is given by Abhyankar [**1**], and also, for an excellent surface, by Hironaka's lectures at Bowdoin [**14**]. There are expository accounts on them with some simplification, by Cutkosky [**8**] for Abhyankar's work, and by Hauser [**10**] for Hironaka's work. The proof in the case of a higher dimensional ambient space is given by Cossart-Jannsen-Saito [**6**] and later by Benito-Villamayor [**5**]. Cossart-Jannsen-Saito generalizes Hironaka's method, which gives rise to the functorial embedded resolution of excellent surfaces. The embedded resolution of surfaces by Villamayor's projection method, or by the IFP, is expected as the first step of the general projects toward the embedded resolution of arbitrary dimensional varieties in positive characteristic.

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Clay Mathematics Proceedings
Volume **20**, 2014

# Algebraic Approaches to FlipIt

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Abstract. This paper discusses a simple application of mathematics to a one person game which is, under various names, available on hand held game consoles, mobile phones, and similar devices. A linear algebra approach is presented in order to show properties of this puzzle. Using some more algebra, the question whether the special case where the game is played on a toric grid has a unique solution is briefly discussed.

## Introduction

Various forms (in some cases merely describing special cases) of the game described in this paper exist under names such as Button Madness, Fiver, FlipIt, Lights Out, Magic Square, XL-25, Token Flip, and Orbix. The game can be described as follows. Take a finite, simple, undirected graph $\Gamma$. By definition, this is a finite nonempty set $v(\Gamma)$, and a finite set $e(\Gamma) \subset 2^{v(\Gamma)}$ (the power set of $v(\Gamma)$), such that every element of $e(\Gamma)$ has cardinality 2. The elements of $v(\Gamma)$ are called the vertices of $\Gamma$, and the elements of $e(\Gamma)$ are called the edges. For $a, b \in v(\Gamma)$, $b \in v(\Gamma)$ is called a neighbour of $a$ if $\{a, b\} \in e(\Gamma)$.

The FlipIt game is now played as follows. The player is supposed to select a subset $S$ of $v(\Gamma)$ in such a way, that for every $a \in v(\Gamma)$ we have

$$\#\left(S \cap (\{a\} \cup \{b \in v(\Gamma) \; : \; \{a, b\} \in e(\Gamma)\})\right) \equiv 1 \bmod 2.$$

An alternative way to describe this, begins by giving every vertex in the graph $\Gamma$ the value 0 ("lights out"). The player consecutively selects vertices

$$v_1, v_2, \ldots, v_f$$

and selecting $v_j$ results in switching the value of $v_j$ and of each of its neighbours. So value 0 now becomes 1, and vice versa. Clearly, at the end of the sequence, a vertex $v$ will have value 1 precisely when it happened an odd number of times that either it was selected, or one of its neighbours was selected. Now the aim of this game is to find a sequence $(v_j)_{1 \leq j \leq f}$ of vertices such that the value of each element of $v(\Gamma)$ is changed to 1.

The main result on the FlipIt game is that such a sequence exists:

2010 *Mathematics Subject Classification.* Primary 05C57, 15A06; Secondary 05C50, 91A43, 91A46.

*Key words and phrases.* Graphs, symmetric matrices, linear algebra, toric grid.

THEOREM 0.1. *For every finite, simple, undirected graph $\Gamma$, the FlipIt game admits a solution.*

To our knowledge, no constructive proof of this result is known. In this note we present a non-constructive proof using only basic linear algebra over the field $\mathbb{F}_2$ consisting of precisely two elements. The properties of this field enter in an interesting way: over $\mathbb{F}_2$ – and only over $\mathbb{F}_2$ –, every homogeneous quadratic function is linear.

In a course at the Clay Mathematics Institute Summer School in Obergurgl (2012) by the first author of this paper, the main topic was a game-theoretic approach to the problem of resolution of singularities (see [**S**]). As an aside, several other games with a mathematical flavor were discussed, and FlipIt was one of them. The proof in this paper (which the first author learned from the webpage of the second author) was presented there.

The second author presented the same result in a lecture [**T**] aimed at (partly non-mathematical) friends and colleagues of the Groningen operator theorist Henk de Snoo, on the occasion of Henk's retirement (2010). With his son he published an expository note [**TT**] (also descibing constructive solutions for particular types of graphs) in the Dutch mathematical journal for high school students *Pythagoras*. Moreover he supervised two theses on the subject: a bachelor's thesis [**J**] dealing with uniqueness of solutions for particular types of graphs, and a master's thesis [**H**] investigating a technique from numerical linear algebra (notably the Block-Lanczos method) to find a solution. During the Clay Summer School mentioned above, an alternative, graph theoretic proof for the existence of solutions was presented by one of the participants.

We are very grateful to Norbert Pintye from Budapest, another participant of the summer school, who provided us with the following background on the FlipIt game. We only slightly supplemented his original email message with additional details.

> The game is due to a Hungarian research psychologist and popular science author László Mérő [**M1**], who not only invented the game in the early 1980s, but created it in reality, called XL-25 [**XL**]. It was presented at the International Game Expo, London, 1983. The first proof [**M2**] of the theorem - a purely graph theoretical one - is also due to him, and was published at the Mathematical and Physical Journal for Secondary Schools (KöMaL) in 1986.
>
> The first proof using linear algebra is due to Klaus Sutner [**Su**]. The one presented below was also found (much earlier than we did) by the Eindhoven problem solvers group O.P. Lossers [**L**]; Theorem 0.1 above was actually proposed by Uri Peled from Chicago as a problem in the Amer. Math. Monthly in 1992. Including the one by the proposer, the Monthly received 42 solutions for it. The problem (and solutions to it) were known in Eindhoven before: in 1987, the Eindhoven mathematicians A. Blokhuis and H.A. Wilbrink proposed the linear algebra formulation in the problem session of the Dutch journal *Nieuw Archief voor Wiskunde*, see [**BW**]. The solution found by the proposers appeared 4 years later [**BW2**]; according to the journal, three others also solved the problem.

A generalisation of the problem has been described in [**BR**].

The Eindhoven mathematician Andries Brouwer [**Br**] collected a lot of information on the problem, including various links and references.

## 1. Linear algebra

Let $\Gamma$ be a finite, simple, undirected graph. We enumerate the vertices as

$$v(\Gamma) = \{1, 2, \ldots, n\}\,.$$

It is clear from the description of the FlipIt game, that any permutation of the sequence $(v_j)_{1\leq j\leq f}$ of selected vertices results in the same set of vertices switching from 0 to 1. Moreover, if a vertex appears twice in the sequence, the result is the same as when two appearances of this particular vertex are removed from the sequence. In other words, the effect depends only on the unordered set

$$S := \{v \;:\; v \text{ appears an odd number of times in } (v_j)_{1\leq j\leq f}\}\,.$$

Such a set is described by an element $x = (\xi_1, \ldots, \xi_n) \in \mathbb{F}_2^n$, with $\xi_j = 1 \Leftrightarrow j \in S$. The result of selecting precisely the vertices in $S$, or equivalently, the indices where $x$ has a nonzero coordinate, is also described by an element $r \in \mathbb{F}_2^n$: its $i$th coordinate is nonzero precisely when vertex $i$ has switched from 0 to 1 after selecting all vertices in the set $S$.

In this way, FlipIt on the graph $\Gamma$ may be descibed as a map

$$\varphi \;:\; \{\text{subsets of } v(\Gamma)\} = \mathbb{F}_2^n \longrightarrow \{\text{possible results}\} \subseteq \mathbb{F}_2^n,$$

given by $\varphi(x) = r$. The question whether the game admits a solution translates into the existence of a vector $x \in \mathbb{F}_2^n$ such that $\varphi(x) = \mathbf{1}$, the vector with all coordinates equal to 1. Below we will prove Theorem 0.1, i.e., we claim that such a vector $x$ exists.

LEMMA 1.1. *For a graph $\Gamma$ as above, the associated map $\varphi : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is linear over $\mathbb{F}_2$. With respect to the standard basis of $\mathbb{F}_2^n$, it is given by a symmetric matrix $A = (a_{i,j})$.*

PROOF. From the description of FlipIt, if $x, y \in \mathbb{F}_2^n$ correspond to sets of vertices $S_x = \{v_1, \ldots, v_s\}$ and $S_y = \{w_1, \ldots w_t\}$, respectively, then $x + y$ corresponds to $S_x \cup S_y \setminus (S_x \cap S_y)$. A vertex $v$ is switched by selecting one of $S_x, S_y$ and then the other, precisely when it is switched by only one of them. This shows $\varphi(x + y) = \varphi(x) + \varphi(y)$, implying that $\varphi$ is linear over $\mathbb{F}_2$.

The $i$th column $(a_{1,i}, a_{2,i}, \ldots, a_{n,i})^t$ of the matrix $A$ shows the effect of only selecting the vertex $i$. The result of this is that $i$ and all its neighbours in $\Gamma$ switch from 0 to 1. Hence

$$a_{j,i} = 1 \Leftrightarrow i = j \text{ or } \{i, j\} \in e(\Gamma) \Leftrightarrow a_{i,j} = 1$$

which proves that $A$ is symmetric. □

Note that the diagonal $d$ of the matrix $A$ has all its entries equal to 1. Hence Theorem 0.1 is a special case of the following more general result.

THEOREM 1.2. *Suppose $B = (b_{i,j})$ is a symmetric $n \times n$ matrix with coefficients in the finite field $\mathbb{F}_2$. Let $d = (d_{1,1}, d_{2,2}, \ldots, d_{n,n})^t$ be the diagonal of $B$, which we regard as a column vector.*

*Then the equation $Bx = d$ admits a solution $x \in \mathbb{F}_2^n$ (here $x$ is a column vector).*

PROOF. One defines a bilinear form $\mathbb{F}_2^n \times \mathbb{F}_2^n \to \mathbb{F}_2$ by

$$(x, y) \mapsto x \cdot y := \sum \xi_j \eta_j$$

where $x = (\xi_1, \ldots, \xi_n)$ and $y = (\eta_1, \ldots, \eta_n)$. This form is nondegenerate, which means that if $x \cdot y = 0$ for every $y \in \mathbb{F}_2^n$, then $x$ is the zero vector.

Given a nonempty $S \subset \mathbb{F}_2^n$, its orthogonal complement $S^\perp$ is defined by

$$S^\perp := \{x \in \mathbb{F}_2^n \ : \ x \cdot s = 0 \ \ \forall s \in S\}.$$

This is a linear subspace of $\mathbb{F}_2^n$. Since the bilinear form used here is nondegenerate, we have that $\dim S^\perp = n - \dim \operatorname{span}(S)$.

Moreover, for subspaces $V, W \subset \mathbb{F}_2^n$ it is true that

$$V \subset W \Leftrightarrow V^\perp \supset W^\perp$$

and

$$V^{\perp^\perp} = V.$$

We have to prove that $Bx = d$ has a solution, in other words, that

$$\operatorname{span}(d) \subset B(\mathbb{F}_2^n).$$

This is equivalent to showing that

$$B(\mathbb{F}_2^n)^\perp \subset \{d\}^\perp.$$

To show that the latter inclusion indeed holds, let $y \in B(\mathbb{F}_2^n)^\perp$. Then in particular $y \cdot (By) = 0$. Writing $y = (\eta_1, \ldots, \eta_n)^t$, this means

$$0 = \sum_{i,j} b_{i,j} \eta_i \eta_j = \sum_{i=1}^n b_{i,i} \eta_i = d \cdot y,$$

since $B$ is symmetric and we work over $\mathbb{F}_2$. This proves the result. □

REMARK 1.3. The argument given here shows (and uses) that the quadratic map given by $x \mapsto x \cdot (Bx)$ is linear over $\mathbb{F}_2$, and in fact it may also be described as $x \mapsto x \cdot d$. It is clear from this that for the proof it is crucial that we work over $\mathbb{F}_2$: over other fields, quadratic maps are not linear.

Evidently, the above existence proof is nonconstructive. Constructions of solutions, for example in case the graph $\Gamma$ is an $m \times n$ rectangular grid with $m \leq 4$, can be found in [**TT**] and in [**DF+**].

EXERCISE 1.4. Provide a constructive solution on the $n \times n$ square grid.

## 2. Polynomial algebra

In a variant of the game, one may give any possible target $y \in \mathbb{F}_2^n$ and ask for $x$ such that $\varphi(x) = y$. This game is solvable for every $y$ if and only if the linear map $\varphi : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is invertible, and this depends on the graph $\Gamma$. If $\varphi$ is not invertible, then the probability of solvability is a (negative) power of 2, because the image of $\varphi$ is an $\mathbb{F}_2$-vectorspace and therefore its cardinality is a power of 2.

We can compute these probabilities for certain graphs, called toric grids. For integers $u, v \geq 3$, the graph $T_{u,v}$ has vertex set $\{(i, j) \mid 0 \leq i < u, 0 \leq j < v\}$; there is an edge $\{(i_1, j_1), (i_2, j_2)\}$ if and only if ($i_1 = i_2$ and $j_1 - j_2 \equiv \pm 1 \mod v$) or ($j_1 = j_2$ and $i_1 - i_2 \equiv \pm 1 \mod u$). Each subset of $v(T_{u,v})$ corresponds to a unique polynomial in the polynomial algebra $R := \mathbb{F}_2[x, y]/\langle x^u - 1, y^v - 1\rangle$, namely the sum of all $x^i y^j$ with $(i, j)$ in the subset. Stretching the language, we also denote by $\varphi$ the linear function $R \to R$ that maps the polynomial $f = \sum_{(i,j)\in S} x^i y^j$ corresponding to a subset $S$, to the polynomial corresponding to $\varphi(S)$. Then $\varphi$ is equal to multiplication by the polynomial $1 + x + x^{-1} + y + y^{-1} \in R$. Theorem 0.1 applied to the graph $T_{u,v}$ states that $t := \sum_{0\le i<u, 0\le j<v} x^i y^j$ is in the image of $\varphi$; in fact, in this special case we have $\varphi(t) = t$.

THEOREM 2.1. *The ratio of subsets of $v(T_{u,v})$ for which the above variant of FlipIt is solvable equals $2^{-r}$, where $r$ is the dimension of the quotient algebra $R/\langle 1 + x + x^{-1} + y + y^{-1}\rangle$. In particular, the variant is solvable for all subsets if and only if $r = 0$, i.e. $1 + x + x^{-1} + y + y^{-1}$ is invertible in $R$.*

PROOF. The image of $\varphi$ is the principal ideal $I = \langle 1 + x + x^{-1} + y + y^{-1}\rangle$. Assume that its dimension as an $\mathbb{F}_2$-vectorspace is $s$. Then the ratio of elements in the image compared to elements in $R$ is $2^{s-n}$, where $n = uv$. On the other hand, $n - s$ is the dimension of the quotient algebra $R/I$. $\square$

Here is a table showing the numbers $r$ for $u, v \le 18$. We used the computer algebra [**M**] for the calculation. A more extensive table can be found on [**Br**].

| $u, v$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 4 | 2 | 6 | 2 | 4 | 4 | 4 | 2 | 6 | 2 | 4 | 4 | 4 | 2 | 6 |
| 4 | 4 | 0 | 0 | 8 | 0 | 0 | 4 | 0 | 0 | 8 | 0 | 0 | 4 | 0 | 0 | 8 |
| 5 | 2 | 0 | 8 | 2 | 0 | 0 | 2 | 8 | 0 | 2 | 0 | 0 | 10 | 0 | 0 | 2 |
| 6 | 6 | 8 | 2 | 8 | 2 | 8 | 6 | 4 | 2 | 12 | 2 | 4 | 6 | 8 | 2 | 8 |
| 7 | 2 | 0 | 0 | 2 | 0 | 0 | 14 | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 0 | 14 |
| 8 | 4 | 0 | 0 | 8 | 0 | 0 | 4 | 0 | 0 | 16 | 0 | 0 | 4 | 0 | 0 | 8 |
| 9 | 4 | 4 | 2 | 6 | 14 | 4 | 4 | 4 | 2 | 6 | 2 | 16 | 4 | 4 | 2 | 6 |
| 10 | 4 | 0 | 8 | 4 | 0 | 0 | 4 | 16 | 0 | 4 | 0 | 0 | 12 | 0 | 0 | 4 |
| 11 | 2 | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 0 | 2 |
| 12 | 6 | 8 | 2 | 12 | 2 | 16 | 6 | 4 | 2 | 16 | 2 | 4 | 6 | 16 | 2 | 12 |
| 13 | 2 | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 0 | 2 |
| 14 | 4 | 0 | 0 | 4 | 0 | 0 | 16 | 0 | 0 | 4 | 0 | 0 | 4 | 0 | 0 | 28 |
| 15 | 4 | 4 | 10 | 6 | 2 | 4 | 4 | 12 | 2 | 6 | 2 | 4 | 12 | 4 | 18 | 6 |
| 16 | 4 | 0 | 0 | 8 | 0 | 0 | 4 | 0 | 0 | 16 | 0 | 0 | 4 | 0 | 0 | 8 |
| 17 | 2 | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 0 | 18 | 0 | 16 | 2 |
| 18 | 6 | 8 | 2 | 8 | 14 | 8 | 6 | 4 | 2 | 12 | 2 | 28 | 6 | 8 | 2 | 8 |

A simpler special case of the game is obtained by using a chain consisting of $n \ge 3$ vertices. Each vertex $i$ now corresponds to the monomial $x^i \in R_n := \mathbb{F}_2[x]/(x^n + 1)$. In the game, selecting the vertex $i$ corresponds to adding $x^i(x + x^{-1} + 1)$ to the element already obtained. The fact that the game has a solution means that the element $t := \sum_{i=1}^n x^i$ is in the ideal $(x + x^{-1} + 1)R_n$. A different generator of this ideal is $x^2 + x + 1$. So the conclusion is that the game has a unique solution precisely when $\gcd(x^2 + x + 1, x^n + 1) = 1$ in the polynomial ring $\mathbb{F}_2[x]$, which is equivalent to $n \not\equiv 0 \bmod 3$. And in case $3|n$, the 'probability' of the variant of the game described above equals 1/4: indeed, $\#R_n/(x^2 + x + 1) = \#\mathbb{F}_2[x]/(x^2 + x + 1) = 4$.

As the above table suggests, the situation is not as simple for the toric grids $T_{u,v}$. We will now briefly indicate how one can describe in terms of the integers $u, v$, the cardinality of the set of solutions to the FlipIt game.

The polynomial algebra $R = R_{u,v} := \mathbb{F}_2[x,y]/(x^u+1, y^v+1)$ is a finite, commutative ring, hence in particular every element in $R_{u,v}$ is either a unit or a zero-divisor. For $a, b \geq 0$, the kernel of the reduction homomorphism

$$\pi \ : \ R_{2^a u, 2^b v} \longrightarrow R_{u,v}$$

consists of nilpotent elements, hence an element $\alpha \in R_{2^a u, 2^b v}$ is a unit if and only if its image $\pi(\alpha) \in R_{u,v}$ is a unit. Applying this to $1+x+x^{-1}+y+y^{-1}$, the conclusion is that the FlipIt game has a unique solution for $T_{u,v}$ if and only if it has a unique solution for $T_{2^a u, 2^b v}$.

This reduces the problem of unique solvability for toric grids to the case where both $u$ and $v$ are odd. In this case, $R_{u,v}$ is the coordinate ring of the finite set $\mu_u \times \mu_v$ consisting of all pairs $(\alpha, \beta) \in \overline{\mathbb{F}_2} \times \overline{\mathbb{F}_2}$ satisfying $\alpha^u = \beta^v = 1$ (Here $\overline{\mathbb{F}_2}$ denotes a fixed algebraic closure of $\mathbb{F}_2$). Hence $f \in R_{u,v}$ is a unit precisely when $f(\alpha, \beta) \neq 0$ for all $(\alpha, \beta) \in \mu_u \times \mu_v$. Applying this to

$$g := 1+x+x^{-1}+y+y^{-1} \in R_{u,v},$$

a direct consequence is: suppose $3|u$. Take $(\omega, 1) \in \mu_u \times \mu_v$ where $\omega$ is a primitive cube root of 1. Then $g(\omega, 1) = 0$, hence the Flipit game on $T_{u,v}$ has more than one solution whenever $3|uv$.

Similarly, suppose $5|u$ and $5|v$. Taking a primitive 5th root of unity $\zeta$, one has $g(\zeta, \zeta^2) = 0$ hence also in this case, the game has more than one solution.

A general statement in the same spirit is: if the FlipIt game on $T_{u,v}$ has more than one solution, then it also has several solutions on $T_{u',v'}$ whenever $u|u'$ and $v|v'$. This is true since a zero of $g$ in $\mu_u \times \mu_v$ is obviously also a zero of $g$ in the larger set $\mu_{u'} \times \mu_{v'}$.

To say slightly more, suppose $\zeta_n, \zeta_m \in \overline{\mathbb{F}_2}$ have exact order $n$ and $m$ respectively (so they are primitive $n$th and $m$th roots of unity) and suppose $g(\zeta_n, \zeta_m) = 0$. This implies in particular that the degree of the field $\mathbb{F}_2(\zeta_n, \zeta_m)$ over each of the subfields $\mathbb{F}_2(\zeta_m)$ and $\mathbb{F}_2(\zeta_n)$ is at most 2. As a consequence,

$$\frac{[\mathbb{F}_2(\zeta_m) : \mathbb{F}_2]}{[\mathbb{F}_2(\zeta_n) : \mathbb{F}_2]} \in \{\frac{1}{2}, 1, 2\}.$$

Since the extension field of $\mathbb{F}_2$ generated by a primitive $k$th root of unity has degree $\mathrm{ord}(2,k) :=$ the order of 2 mod $k$ in the group $(\mathbb{Z}/k\mathbb{Z})^\times$, one concludes

$$\frac{\mathrm{ord}(2,m)}{\mathrm{ord}(2,n)} \in \{\frac{1}{2}, 1, 2\}.$$

As an example, the toric grid $T_{5,7}$ admits a unique solution. Namely, $g(1, \alpha)$ and $g(\beta, 1)$ are zero only for $\alpha, \beta$ of order 3, which do not occur in $\mu_5$ or $\mu_7$. And $g(\zeta_5, \zeta_7) \neq 0$ since $\mathrm{ord}(2,5)/\mathrm{ord}(2,7) = 4/3$.

As a final remark, for $u, v$ odd, the dimension $r$ of the quotient algebra $R/gR$ introduced above, equals the cardinality of the set

$$V(g) := \{(\alpha, \beta) \in \mu_u \times \mu_v \ : \ g(\alpha, \beta) = 0\}.$$

So the toric grid cases of FlipIt allows a nice interpretation which may be described in terms of elementary algebraic geometry.

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Clay Mathematics Proceedings
Volume **20**, 2014

# Higher Semple-Nash blowups and F-blowups

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Abstract. We give an exposition on the one-step desingularization problem by using higher Semple-Nash blowups and F-blowups.

## 1. Introduction

The most common and successful approach to resolution of singularities is, roughly speaking, to improve singularities step by step with blowups along smooth centers. One needs to choose centers of blowups and prove that singularities are actually improved after blowups, looking at invariants. Such arguments are of algebraic nature rather than geometric. Hironaka's proof of resolution of singularities in characteristic zero is a milestone in this approach [**15**].

An alternative approach with blowups more geometrically constructed were proposed by Semple [**28**], and then independently by Nash; we call them *(classical) Semple-Nash blowups.*[1] Given a variety, the blowup is defined as the parameter space of the tangent spaces at smooth points and their limits at singular points. They asked whether an iteration of Semple-Nash blowups always leads to a smooth variety. If the answer is yes, then we would have a completely geometric and canonical way of desingularizing. Also there is a similar question on *normalized* Semple-Nash blowups instead of Semple-Nash blowups; the normalized Semple-Nash blowup is the Semple-Nash blowup followed by the normalization. Both questions are still open in characteristic zero,[2] although affirmative results for some classes of singularities have been obtained. We refer the reader to Introduction of [**30**] for a historical account and to [**3, 7, 27**] for recent works in the toric case.

This paper concerns another geometric approach proposed by the author in [**36**] and subsequent works. The difference from Semple and Nash's is that instead of iterating blowups, one constructs a series of blowups directly from the given variety, using not only first-order data, but also higher order ones. Each blowup is the parameter space of some geometric objects on the given variety. So far, there exist essentially two different but similar such constructions. One is what we call *higher Semple-Nash blowups*, considered mainly in characteristic zero. The other

2010 *Mathematics Subject Classification.* 14E15, 14B05, 14J17.

[1] The blowups are often called like *Nash blowing-ups.* It seems that it was Lejeune-Jalabert [**23**] who rediscovered Semple's work in this context and pointed out that his construction is the same as Nash's. The author learned about Semple's work from Pedro González Pérez.

[2] In positive characteristic, the Semple-Nash blowup of a singular variety can be trivial (see [**25**]), and one cannot resolve singularities using this method.

is called *F-blowups*, defined only in positive characteristic. As for the former, the $n$-th Semple-Nash blowup is defined as the parameter space of $n$-th infinitesimal neighborhoods of smooth points and their limits as subschemes, constructed as a subscheme of a suitable Hilbert scheme. The first Semple-Nash blowup coincides with the classical Semple-Nash blowup. In the definition of F-blowups, we use infinitesimal neighborhoods defined with Frobenius powers of maximal ideals instead of ordinary powers.

A central question on these blowups is: given a variety, does there exist a smooth variety in the series of blowups? If this is the case, we can resolve singularities in one step. Moreover the resulting smooth variety would keep a more direct link with the original than in other constructions of desingularization. Unfortunately the answer seems to be be generally negative, as far as higher Semple-Nash blowups and F-blowups are concerned. For instance, the author [**40**, Remark 1.5] conjectured that all higher Sepmle-Nash blowups of the $A_3$-singularity of dimension two are singular. As for F-blowups, there are surface singularities whose F-blowups are all singular [**11, 12**]. On the other hand, we obtain resolutions having unexpected interesting features in several special cases: both blowups work well for curve singularities. For F-regular surface singularities, high F-blowups give the minimal resolution. For some three-dimensional Gorenstein singularities, they give a crepant resolution.

Another interesting feature of F-blowups is a relation with the $G$-Hilbert scheme introduced by Ito and Nakamura [**20**]. For a smooth quasi-projective variety $M$ with a faithful action of a finite group $G$, the associated $G$-Hilbert scheme gives a canonical modification of the quotient variety $M/G$. If $G$ has order prime to the characteristic of the base field, then there exists a natural morphism from the $G$-Hilbert scheme to every F-blowup. Moreover this morphism is an isomorphism for sufficiently high F-blowups.

Sections 2 and 3 deal with higher Semple-Nash blowups and F-blowups respectively. Then Section 4 provides some open problems.

**Convention.** For simplicity, we will work over an algebraically closed field $k$. A *variety* means a separated integral scheme of finite type over $k$. *Points* of a variety mean closed points. Most results in this paper can be generalized to perfect base fields. A *singularity* often means a germ of variety.

## 2. Higher Semple-Nash blowups

In this section, we suppose that $k$ has characteristic zero.

**2.1. Classical Semple-Nash blowups.** Let $X \subset \mathbb{A}_k^n$ be a $d$-dimensional affine variety with the smooth locus $X_{\mathrm{sm}}$. For each $x \in X_{\mathrm{sm}}$, the tangent space $T_xX$ is identified with a $d$-dimensional linear subspace of $\mathbb{A}_k^n$ and corresponds to a point of the Grassmaniann $G(d,n)$: the point is denoted by $[T_xX]$. The map

$$\gamma : X_{\mathrm{sm}} \to G(d,n),\ x \mapsto [T_xX]$$

is called the *Gauss map.* The classical Semple-Nash blowup of $X$ is defined as the graph closure of $\gamma$. More precisely,

Definition 2.1. The *classical Semple-Nash blowup* of $X$, denoted $\mathrm{SNB}(X)$, is defined to be the Zariski closure of the locally closed subset $\{(x,[T_xX]) \mid x \in X_{\mathrm{sm}}\}$ of $X \times G(d,n)$.

The projection $\mathrm{SNB}(X) \to X$ is projective and birational. Hence this is actually isomorphic to the blowup with respect to some ideal of $\mathcal{O}_X$ (see [**14**, Ch. II, Th. 7.17]). The blowup is, in fact, independent of the embedding of $X$ into $\mathbb{A}_k^n$. Moreover, we can define the classical Semple-Nash blowup of a non-affine variety by gluing those of affine charts.

THEOREM 2.2 ([**25**]). *The blowup* $\mathrm{SNB}(X) \to X$ *is an isomorphism if and only if* $X$ *is smooth.*

This theorem implies in particular that curve singularities can be resolved by iteration of Semple-Nash blowups. Let us denote by $\mathrm{SNB}^n(X)$ the $n$-iterated Semple-Nash blowup of a variety $X$.

COROLLARY 2.3. *If* $X$ *has dimension one, for* $n \gg 0$ *the variety* $\mathrm{SNB}^n(X)$ *is smooth.*

DEFINITION 2.4. The *normalized Semple-Nash blowup* of $X$, denoted $\widetilde{\mathrm{SNB}}(X)$, is the normalization of $\mathrm{SNB}(X)$ endowed with the canonical morphism onto $X$. For $n > 0$ the $n$-iterated normalized Semple-Nash blowup of $X$ is denoted by $\widetilde{\mathrm{SNB}}^n(X)$.

After works of González-Sprinberg [**8**] and Hironaka [**16**], Spivakovsky proved the following result:

THEOREM 2.5 ([**30**]). *If* $X$ *has dimension two, then for* $n \gg 0$ *the variety* $\widetilde{\mathrm{SNB}}^n(X)$ *is smooth.*

**2.2. Definition.** Let $X$ be an irreducible variety of dimension $d$. For an integer $l > 0$, a *length* $l$ *subscheme* of $X$ is a zero-dimensional subscheme of $X$ whose coordinate ring is an $l$-dimensional $k$-vector space. The Hilbert scheme $\mathrm{Hilb}_l(X)$ parameterizes the length $l$ subschemes of $X$. The point of $\mathrm{Hilb}_l(X)$ corresponding to $Z$ is denoted by $[Z]$. There exists a closed subscheme $\mathcal{U}$ of $\mathrm{Hilb}_l(X) \times X$, called the *universal family*, such that the fiber of the projection $\mathcal{U} \to \mathrm{Hilb}_l(X)$ over $[Z]$ is $Z$. The Hilbert scheme has the following universal property: If $\mathcal{Z} \subset T \times X$ is a flat family over a scheme $T$ of length $l$ subschemes of $X$, then there exists a unique morphism $f : T \to X$ such that $\mathcal{Z}$ is the pullback of the universal family $\mathcal{U}$ by $f$. Conversely, given a morphism $f : T \to \mathrm{Hilb}_l(X)$, we obtain a family of length $l$ subschemes of $X$ over $T$ by pulling back $\mathcal{U}$ by $f$. For more details on the Hilbert scheme, see for instance [**13**].

For a point $x \in X$, we denote by $\mathfrak{m}_x \subset \mathcal{O}_X$ its maximal ideal. The *n-th infinitesimal neighborhood* of $x$, denoted $x^{(n)}$, is the closed subscheme of $X$ defined by $\mathfrak{m}_x^{n+1}$, which is zero-dimensional and supported at $x$. Let $X_{\mathrm{sm}}$ denote the smooth locus of $X$. If $x \in X_{\mathrm{sm}}$, then $x^{(n)}$ has length $\binom{n+d}{d}$, hence corresponds to a point $[x^{(n)}]$ of the Hilbert scheme $\mathrm{Hilb}_{\binom{n+d}{d}}(X)$.

DEFINITION 2.6 ([**36**]). The *n-th Semple-Nash blowup* of $X$, denoted $\mathrm{SNB}_n(X)$, is the Zariski closure of $\{[x^{(n)}] \mid x \in X_{\mathrm{sm}}\}$ in $\mathrm{Hilb}_{\binom{n+d}{d}}(X)$.[3]

[3] In [**36**], this blowup was called the $n$-th Nash blowup and defined in a slightly different way: the blowup was defined to be the Zariski closure of $\{(x, [x^{(n)}]) \mid x \in X_{\mathrm{sm}}\}$ in $X \times \mathrm{Hilb}_{\binom{n+d}{d}}(X)$. However as we are working in characteristic zero, from Proposition 1.3 in that paper, the two definitions coincide.

Each point of $\mathrm{SNB}_n(X)$ corresponds to a zero-dimensional subscheme $Z \subset X$, which is a limit of $n$-th infinitesimal neighborhoods of smooth points along some path; the point of $\mathrm{SNB}_n(X)$ is denoted by $[Z]$. Such a subscheme $Z$ satisfies the following properties:

- $Z$ has length $\binom{n+d}{d}$,
- $Z$ is supported at a point, say $x \in X$, and then,
- $Z$ is scheme-theoretically contained in $x^{(n)}$.

For every $n$, there exists a map

$$\pi_n : \mathrm{SNB}_n(X) \to X,\ [Z] \mapsto \mathrm{Supp}(Z).$$

We can show that the map is in fact a projective and birational morphism of varieties. The zeroth Semple-Nash blowup is identical to $X$. The first Semple-Nash blowup is identical to the classical Semple-Nash blowup (Corollary 2.10). Note that $\mathrm{SNB}_{n+1}(X)$ does not generally dominate[4] $\mathrm{SNB}_n(X)$ (see Example 2.15). In particular, $\mathrm{SNB}_{n+1}(X)$ and $\mathrm{SNB}_1(\mathrm{SNB}_n(X))$ do not generally coincide.

### 2.3. Blowups at modules.

Definition 2.7. Let $X$ be a reduced Noetherian scheme and $M$ a coherent $\mathcal{O}_X$-module. A proper birational morphism $f : Y \to X$ is called a *flattening* of $M$ if the pullback of $M$ by $f$ modulo the torsion part, $f^*M/\mathrm{tors}$, is flat. A flattening of $M$ is called *universal* if any other flattening of $M$ factors through it. The universal flattening is also called the *blowup at $M$*. For details, see [**26, 34**].

If $M$ is flat of constant rank, say $r$, on an open dense subset of $X$, then the blowup at $M$ exists and can be constructed as the union of those irreducible components of the Quot scheme $\mathrm{Quot}_r(M)$ that dominate an irreducible component of $X$. As explained in the cited papers, the blowup at a module is actually identical to the blowup at some ideal (sheaf). In the case where $X$ is affine, such an ideal is obtained as the $r$-th Fitting ideal of some modification of $M$. However we cannot generally choose an ideal in a canonical way.

Proposition 2.8 (For instance, see [**8**]). *The classical Semple-Nash blowup* $\mathrm{SNB}(X)$ *of a variety $X$ is identical to the blowup at the module of differentials,* $\Omega_{X/k}$.

Higher Semple-Nash blowups are also described as blowups at certain modules: Given a variety $X$, we put $P^n_X := \mathcal{O}_{X\times X}/I_\Delta^{n+1}$, where $I_\Delta$ is the defining ideal sheaf of the diagonal. The sheaf $P^n_X$, regarded as an $\mathcal{O}_X$-module, is called the *sheaf of principal parts of order $n$.*

Proposition 2.9 ([**36**]). *The $n$-th Semple-Nash blowup* $\mathrm{SNB}_n(X)$ *of a variety $X$ is the blowup at $P^n_X$.*

Corollary 2.10. *For a variety $X$, the first Semple-Nash blowup* $\mathrm{SNB}_1(X)$ *is identical to the classical Semple-Nash blowup* $\mathrm{SNB}(X)$.

Proof. Since the inclusion map $\mathcal{O}_X \hookrightarrow P^n_X$ gives a section of the quotient map

$$P^n_X \to \mathcal{O}_X = P^n_X/(I_\Delta/I_\Delta^{n+1}),$$

[4]For two proper birational morphisms of varieties $Y \to X$ and $Y' \to X$, we say that $Y$ *dominates* $Y'$ if the natural birational map $Y \dashrightarrow Y'$ is defined on the entire $Y$.

the sheaf $P^n_X$ decomposes as $\mathcal{O}_X \oplus (I_\Delta/I_\Delta^{n+1})$. Since the first summand $\mathcal{O}_X$ does not affect the blowup, $\mathrm{SNB}_n(X)$ is also the blowup at $I_\Delta/I_\Delta^{n+1}$. For $n = 1$, this shows the corollary. □

We may use the proposition as the definition of higher Semple-Nash blowups. Sometimes this is more useful and also valid in a more general setting. For instance, suppose that $R$ is a local complete Noetherian $k$-algebra and that it is reduced and of pure dimension. Then the $n$-th Semple-Nash blowup of $X = \operatorname{Spec} R$ is defined as the blowup at the *complete* module of principal parts, $R\widehat{\otimes}R/I_\Delta^{n+1}$. Then, for a point $x$ of a variety $X$, if we put $\widehat{X} = \operatorname{Spec}\widehat{\mathcal{O}_{X,x}}$, then we have $\mathrm{SNB}_n(X)\times_X \widehat{X} = \mathrm{SNB}_n(\widehat{X})$.

**2.4. Separation of analytic branches.** For a point $x$ of a variety $X$, let $Y := \operatorname{Spec}\widehat{\mathcal{O}_{X,x}}$ and $Y_i$, $i = 1,\dots,l$, its irreducible components. Then

$$\mathrm{SNB}_n(Y) = \mathrm{SNB}_n(X) \times_X Y = \bigcup_{i=1}^{l} \mathrm{SNB}_n(Y_i).$$

If $\mathrm{SNB}_n(Y_i)$, $i = 1,\dots,l$, are mutually disjoint, then we say that *the $n$-th Semple-Nash blowup separates the analytic branches at $x$.*

Proposition 2.11 ([**36**]). *Let $X$ be a variety. Then, for $n \gg 0$, the $n$-th Semple-Nash blowup separates the analytic branches at every point.*

Sketch of the proof. Suppose that for $i \neq j$, $\mathrm{SNB}_n(Y_i)$ and $\mathrm{SNB}_n(Y_j)$ intersect, say at a closed point $[Z] \in \mathrm{SNB}_n(Y_i) \cap \mathrm{SNB}_n(Y_j)$. Then the subscheme $Z$ of $X$ is contained in the zero-dimensional scheme $x^{(n)} \cap Y_i \cap Y_j$, whose length is the value of the Hilbert-Samuel function of $Y_i \cap Y_j$ at $n$. On the other hand, if $X$ has dimension $d$, then $Z$ has length $\binom{n+d}{d}$ by construction, the value of the Hilbert-Samuel function of $k[[x_1,\dots,x_d]]$ at $n$. Since $\dim Y_i \cap Y_j < d$, for $n \gg 0$, we obtain the contradiction

$$\operatorname{length}(Z) \le \operatorname{length}(x^{(n)} \cap Y_i \cap Y_j) < \binom{n+d}{d} = \operatorname{length}(Z).$$

This proves the separation of the branches $Y_i$ and $Y_j$. A simultaneous separation at all points follows from the upper semi-continuity of the Hilbert-Samuel function [**4**]. □

**2.5. Resolution of curve singularities.**

Theorem 2.12 ([**36**]). *Let $X$ be a one-dimensional variety. Then for $n \gg 0$, $\mathrm{SNB}_n(X)$ is smooth.*

Thus our dream of resolution in one step is realized in dimension one in characteristic zero. For a similar result in positive characteristic, see Section 3.3. Of course, the normalization also resolves curve singularities in one step. However, our construction is more geometric and moduli-theoretic.

The theorem follows from a more precise result. Let $\nu : \tilde{X} \to X$ be the normalization of a variety $X$. Then the *conductor ideal sheaf* of $X$ is defined as the annihilator of $\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X$. The *conductor subscheme* $C$ of $X$ is the closed subscheme defined by the conductor ideal sheaf, which is supported along the non-normal locus of $X$.

THEOREM 2.13. *Let $X$ be a one-dimensional variety. Let $[Z] \in \mathrm{SNB}_n(X)$ correspond to a zero-dimensional subscheme $Z \subset X$. If $Z$ is not scheme-theoretically contained in $C$, then $\mathrm{SNB}_n(X)$ is smooth at $[Z]$.*

Theorem 2.12 obviously follows from Theorem 2.13. In turn, Theorem 2.13 follows from a more precise result for an analytically irreducible singularity. In this case, we can completely determine for which $n$, $\mathrm{SNB}_n(X)$ is regular. So put $X = \operatorname{Spec} R$, where $R \subset k[[x]]$ is a $k$-subalgebra such that $k[[x]]/R$ is a finite-dimensional $k$-vector space. We define a set $S$ of non-negative integers as follows: $s \in S$ if and only if there exists a power series $f$ of order $s$ in $R$. Then $S$ becomes a numerical monoid. Namely, it contains zero, is closed under addition, and the complement $\mathbb{N} \setminus S$ is a finite set. We label its elements as

$$0 = s_{-1} < s_0 < s_1 < \cdots .$$

THEOREM 2.14. *For $X = \operatorname{Spec} R$ as above, $\mathrm{SNB}_n(X)$ is regular if and only if $s_n - 1 \in S$. In particular, for $n \gg 0$, $\mathrm{SNB}_n(X)$ is regular.*

OUTLINE OF THE PROOF. Let $\tilde{X}$ be the normalization of $X$. The natural morphism

$$\phi_n : \tilde{X} \to \mathrm{SNB}_n(X) \subset \mathrm{Hilb}_{n+1}(X)$$

corresponds to a family of subschemes of $X$ over $\tilde{X}$. Restricting the family to the first infinitesimal neighborhood $\operatorname{Spec} k[x]/(x^2)$ of the closed point of $\tilde{X} = \operatorname{Spec} k[[x]]$, we obtain a first-order deformation of a subscheme of $X$. The morphism $\phi_n$ is an isomorphism if and only if the composition

$$\operatorname{Spec} k[x]/(x^2) \hookrightarrow \tilde{X} \xrightarrow{\phi_n} \mathrm{SNB}_n(X)$$

is a closed immersion. This is then equivalent to that the first-order deformation is non-trivial. Analyzing the deformation in detail, we can determine whether $\phi_n$ is an isomorphism, and get the theorem. □

EXAMPLE 2.15. For $X = \operatorname{Spec} k[[x^5, x^7]]$, the scheme $\mathrm{SNB}_n(X)$ is non-normal only when $n = 0, 1, 2, 3, 4, 6, 7, 11$. This shows that in general there does not exist a morphism $\mathrm{SNB}_{n+1}(X) \to \mathrm{SNB}_n(X)$.

REMARK 2.16. The author has computed higher Semple-Nash blowups of a few surface singularities, using a computer. The computation seems to indicate that higher Semple-Nash blowups of the $A_3$-singularity form a non-stabilizing sequence of singular varieties, as stated in [**40**, Remark 1.5]. In particular, this is probably a counterexample for the one-step resolution by a higher Semple-Nash blowup.

**2.6. Flag higher Semple-Nash blowups.** A slight change of the definition provides blowups with nicer properties than higher Semple-Nash blowups.

DEFINITION 2.17 ([**37**]). Let $X$ be a variety. The *$n$-th flag Semple-Nash blowup* of $X$, denoted $\mathrm{fSNB}_n(X)$, is the Zariski closure of

$$\{([x], [x^{(1)}], \dots, [x^{(n)}]) \mid x \in X_{\mathrm{sm}}\}$$

in $\prod_{m=0}^{n} \mathrm{Hilb}_{\binom{m+d}{d}}(X)$.

The variety $\mathrm{fSNB}_n(X)$ is isomorphic to the irreducible component of

$$\mathrm{SNB}_0(X) \times_X \mathrm{SNB}_1(X) \times_X \cdots \times_X \mathrm{SNB}_n(X)$$

dominating $X$. The flag higher Semple-Nash blowup has the following properties, which the higher Semple-Nash blowup does not have:

- For varieties $X$ and $Y$, $\mathrm{fSNB}_n(X) \times \mathrm{fSNB}_n(Y) \cong \mathrm{fSNB}_n(X \times Y)$.
- For a smooth morphism $Y \to X$, $\mathrm{fSNB}_n(Y) \cong Y \times_X \mathrm{fSNB}_n(X)$.
- For $m \geq n$, $\mathrm{fSNB}_m(X)$ dominates $\mathrm{fSNB}_n(X)$.

In spite of these, however, the flag construction seem not to help in desingularization of higher dimensional singularities such as the $A_3$-singularity (Remark 2.16).

## 3. F-blowups

In this section, we will work over an algebraically closed field $k$ of characteristic $p > 0$.

**3.1. Definition.** F-blowups are a variant of higher Semple-Nash blowups in positive characteristic. In its definition, we just use Frobenius powers of maximal ideals instead of ordinary powers. This slight change in the definition makes a big difference in the behavior of blowups especially in higher dimensions. As expected from the name and construction, the blowup has a close relation with F-singularities, more precisely, with various classes of singularities defined with Frobenius maps.

Let $X$ be a variety and $x$ a point on this variety with the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_X$. For $e \geq 0$, the *e-th Frobenius power* of $\mathfrak{m}_x$, denoted $\mathfrak{m}_x^{[p^e]}$, is the ideal (sheaf) generated by $f^{p^e}$, $f \in \mathfrak{m}_x$ (the Frobenius power is defined for any ideal in a commutative ring of positive characteristic). We denote by $x^{[p^e]}$ the closed subscheme of $X$ defined by $\mathfrak{m}_x^{[p^e]}$. This is a zero-dimensional subscheme supported at $x$. Let $F : X \to X$ be the Frobenius morphism: this is the identity map of the underlying topological space and corresponds to the map of sheaves of rings $F^* : \mathcal{O}_X \to \mathcal{O}_X$, $f \mapsto f^p$. We denote by $F^e$ the $e$-times iteration of $F$. Then $x^{[p^e]}$ is equal to the scheme-theoretic preimage $(F^e)^{-1}(x)$ of $x$ by $F^e$. If $x \in X_{\mathrm{sm}}$ and $d := \dim X$, then $x^{[p^e]}$ has length $p^{ed}$ and corresponds to a point of $\mathrm{Hilb}_{p^{ed}}(X)$, denoted by $[x^{[p^e]}]$.

DEFINITION 3.1 ([**40**]). The *e-th F-blowup* of a $d$-dimensional variety $X$ is the Zariski closure of $\{[x^{[p^e]}] \mid x \in X_{\mathrm{sm}}\}$ in $\mathrm{Hilb}_{p^{ed}}(X)$. It is denoted by $\mathrm{FB}_e(X)$.

Like in the case of higher Semple-Nash blowups, for each $e$ there exists a projective birational morphism

$$\mathrm{FB}_e(X) \to X,\ [Z] \mapsto \mathrm{Supp}(Z).$$

**3.2. Basic properties.**

PROPOSITION 3.2. *The e-th F-blowup of $X$ is the blowup at $F^e_*\mathcal{O}_X$.*

From the definition, the blowup at a module $M$ is trivial if and only if the quotient module $M/\mathrm{tors}$ modulo the torsions is flat. By Kunz's theorem [**22**], for $e > 0$, the module $F^e_*\mathcal{O}_X$ is flat exactly over $X_{\mathrm{sm}}$. For every $e > 0$, since $F^e_*\mathcal{O}_X$ is torsion-free, the $e$-th F-blowup is trivial if and only if $X$ is smooth. Again, using the proposition, we can generalize F-blowups to the complete local case.

Like flag higher Semple-Nash blowups, F-blowups have nice functorial properties:

PROPOSITION 3.3. *F-blowups are compatible with smooth morphisms and products.*

**3.3. Separation of analytic branches and resolution of curve singularities.** In a similar way with the proof of Proposition 2.11, we can prove:

PROPOSITION 3.4 ([**40**]). *Let $X$ be a variety and $x \in X$. For $e \gg 0$, the $e$-th F-blowup separates analytic branches at $x$.*

To obtain a simultaneous separation of analytic branches at all points of $X$, we need that Hilbert-Kunz functions[5] associated to points of $X$ are bounded from above, but the author does not know whether this holds.

The proof of the following theorem is similar (even easier) to the one of Theorem 2.12:

THEOREM 3.5 ([**40**]). *Let $X$ be a one-dimensional variety. For $e \gg 0$, $\mathrm{FB}_e(X)$ is smooth.*

**3.4. The monotonicity of the F-blowup sequence.** We say that the F-blowup sequence of $X$ is *monotone* if for every $e$, $\mathrm{FB}_{e+1}(X)$ dominates $\mathrm{FB}_e(X)$. A simple sufficient condition for the monotonicity is the F-purity. A variety $X$ is called *F-pure* if the $\mathcal{O}_X$-linear map $\mathcal{O}_X \to F_*\mathcal{O}_X$ locally splits (see [**18**]).

PROPOSITION 3.6 ([**38**]). *If $X$ is F-pure, then its F-blowup sequence is monotone.*

We can refine this as follows:

PROPOSITION 3.7 ([**38**]). *Let $[Z] \in \mathrm{FB}_{e+1}(X)$. If $[F(Z)] \in \mathrm{FB}_e(X)$, where $F(Z)$ is the scheme-theoretic image of $Z$ by the Frobenius morphism, then the birational map $\mathrm{FB}_{e+1}(X) \dashrightarrow \mathrm{FB}_e(X)$ is defined at $[Z]$ with the image $[F(Z)]$.*

Note that if $X$ is F-pure, then the assumption of Proposition 3.7 holds for every $[Z] \in \mathrm{FB}_{e+1}(X)$. Thus this proposition implies the preceding one.

**3.5. Tame quotient singularities and the $G$-Hilbert scheme.** Let $G$ be a finite group with $p \nmid \sharp G$ and let $M$ be a smooth quasi-projective variety endowed with a faithful $G$-action.

DEFINITION 3.8 ([**20**]). The *$G$-Hilbert scheme* of $M$, denoted by $\mathrm{Hilb}^G(M)$, is the closure of the locus of those points of $\mathrm{Hilb}_{\sharp G}(M)$ corresponding to free $G$-orbits.

Let $X := M/G$ be the quotient variety, which is regarded as the parameter space of all orbits. If $[Z] \in \mathrm{Hilb}^G(M)$ corresponds to a subscheme $Z \subset M$, then the support of $Z$ is a $G$-orbit. Therefore we have a map

$$\mathrm{Hilb}^G(M) \to X,\ [Z] \mapsto [\mathrm{Supp}(Z)].$$

Actually this is projective and birational. An important fact is that the $G$-Hilbert scheme gives the minimal resolution of $X$ in dimension two, and it is a crepant[6] resolution of $X$ in dimension three if $X$ is Gorenstein ([**20, 21, 19, 24, 5, 40**]). Surprisingly the $G$-Hilbert scheme coincides with high F-blowups of $X$:

[5]The Hilbert-Kunz function for $x \in X$ associates to a non-negative integer $e$ the length of the subscheme $x^{[p^e]} \subset X$.

[6]A resolution $f : Y \to X$ is called *crepant* if $f^*\omega_X = \omega_Y$.

THEOREM 3.9 ([**40, 32**]). *For each* $e \geq 0$, *there exists a morphism* $\phi_e : \mathrm{Hilb}^G(M) \to \mathrm{FB}_e(X)$ *compatible with morphisms to* $X$. *This sends* $[Z]$ *to* $[((F^e)^{-1}Z)/G]$. *Moreover for* $e \gg 0$, $\phi_e$ *is an isomorphism.*

Let $\pi : M \to X$ be the quotient map. The $G$-Hilbert scheme is the blowup at $\pi_*\mathcal{O}_M$ and the $e$-th F-blowup is the blowup at $F^e_*\mathcal{O}_X$. The last assertion of the theorem is based on the fact that $\pi_*\mathcal{O}_M$ and $F^e_*\mathcal{O}_X$ ($e \gg 0$) are locally equivalent in the following sense: at each point $x \in X$, there are finitely many $\hat{\mathcal{O}}_{X,x}$-modules $M_i$ (called modules of covariants) such that the completions of both $\pi_*\mathcal{O}_M$ and $F^e_*\mathcal{O}_X$ ($e \gg 0$) are of the form $\bigoplus M_i^{\oplus a_i}$, $a_i > 0$. The isomorphism $\phi_e$ can be explained also from the viewpoint of non-commutative resolution as in [**32**].

COROLLARY 3.10. *For* $X = M/G$ *as above, its F-blowup sequence is monotone and stabilizes.*

PROOF. Since $\mathcal{O}_X$-linear maps $\mathcal{O}_X \to \pi_*\mathcal{O}_M$ and $\pi_*\mathcal{O}_M \to \pi_*F_*\mathcal{O}_M$ locally split, their composition $\mathcal{O}_X \to \pi_*F_*\mathcal{O}_M$ also locally splits. Since the last map factors also as $\mathcal{O}_X \to F_*\mathcal{O}_X \to \pi_*F_*\mathcal{O}_M$, the map $\mathcal{O}_X \to F_*\mathcal{O}_X$ also locally splits, and $X$ is F-pure. Therefore the F-blowup sequence of $X$ is monotone. The stability is obvious. □

COROLLARY 3.11. *Suppose that a variety* $X$ *has only tame*[7] *quotient singularities.*

(1) If $X$ is two-dimensional, then for $e \gg 0$, $\mathrm{FB}_e(X)$ is the minimal resolution of $X$.
(2) If $X$ is Gorenstein and of dimension at most 3, then for $e \gg 0$, $\mathrm{FB}_e(X)$ is a crepant resolution of $X$.

The isomorphism $\phi_e$ implies a negative result on F-blowups as well: an example of Craw, Maclagan and Thomas [**6**] says that the $G$-Hilbert scheme and hence high F-blowups of $X$ can be non-normal. The example is six-dimensional.

**3.6. Toric singularities.** Let $A \subset M = \mathbb{Z}^d$ be a finitely generated submonoid. Let $k[A] = k[x^a \mid a \in A]$ be the corresponding monoid algebra and $X := \operatorname{Spec} k[A]$ the corresponding (not necessarily normal) affine toric variety. Suppose that $A$ generates $M$ as a group and that the cone $A_{\mathbb{R}} \subset M_{\mathbb{R}} = \mathbb{R}^d$ spanned by $A$ has a vertex at the origin, as we can reduce the study of any toric singularity to this case.

The F-blowups are described by using Gröbner fans. Let $A^\vee_{\mathbb{R}} \subset N_{\mathbb{R}} := M^\vee_{\mathbb{R}}$ be the dual cone of $A_{\mathbb{R}}$. For each element $w \in A^\vee_{\mathbb{R}}$, we have an $\mathbb{R}_{\geq 0}$-grading of $k[A]$. For any ideal $I \subset k[A]$, we have its initial ideal $in_w(I)$ with respect to this grading. The *Gröbner fan* of $I$ is the fan $\Delta$ in $N_{\mathbb{R}}$ such that

- the support of $\Delta$ is $A_{\mathbb{R}}$, and
- $in_w(I) = in_{w'}(I)$ if and only if $w$ and $w'$ lie in the relative interior of the same cone in $\Delta$.

See [**31**] for details on Gröbner fans.

Consider the ideal $J := \langle x_1 - 1, \ldots, x_d - 1\rangle_{k[x_1^{\pm}, \ldots, x_d^{\pm}]} \cap k[A]$, where we identify the Laurent polynomial ring $k[x_1^{\pm}, \ldots, x_d^{\pm}]$ with $k[M]$. This is the defining ideal

[7] A quotient singularity is called *tame* if the relevant finite group has order prime to the characteristic of the base field.

of the unit point of the torus $T = \operatorname{Spec} k[M]$ as a point of $X$. The ideal is also described as $J = \left\langle x^a - x^b \mid a, b \in A \right\rangle$. In this case, for every $e \geq 0$, we have

$$J^{[p^e]} = \left\langle x^a - x^b \mid a, b \in A \text{ and } p^e \text{ divides } a - b \right\rangle.$$

THEOREM 3.12 ([**40**]). *The fan corresponding to the normalization of* $\mathrm{FB}_e(X)$ *is the Gröbner fan of* $J^{[p^e]}$.

In [**40**], coordinate rings of affine charts of $\mathrm{FB}_e(X)$ are also described and this gives a complete description of $\mathrm{FB}_e(X)$. Using it, we can show:

COROLLARY 3.13. *The F-blowup sequence of X is bounded, that is, there exists a proper birational morphism $Y \to X$ factoring through every* $\mathrm{FB}_e(X)$. *If X is normal, then the sequence is monotone and stabilizes.*

**3.7. Singularities of finite F-representation type.** We saw that the F-blowup sequence for normal toric singularities and tame quotient singularities is monotone and stabilizes. This can be generalized as follows.

DEFINITION 3.14 ([**29**]). An affine $k$-scheme $X$ is *of finite F-representation type* if there exist finitely many indecomposable $\mathcal{O}_X$-modules $M_1, \dots, M_l$ such that for every $e \geq 0$, $F^e_* \mathcal{O}_X \cong \bigoplus_{i=1}^l M_i^{\oplus a_i}$, $a_i \geq 0$.

Normal toric singularities and tame quotient singularities are F-pure and of finite F-representation type.

PROPOSITION 3.15. *Suppose that X is an affine variety of finite F-representation type. Then for $M_i$, $i = 1, \dots, l$, as above, the blowup at $\bigoplus_{i=1}^l M_i$ dominates all F-blowups of X. Moreover, if X is F-pure, then the F-blowup sequence stabilizes.*

**3.8. F-regular surface singularities.** The author [**40**] proved that for a normal toric surface singularity, high F-blowups are the minimal resolution. It was the starting point of his study of F-blowups. The same assertion was then proved for 2-dimensional tame quotient singularities by Toda and the author [**32**] and F-rational double points by Hara and Sawada [**11**]. Generalizing all these, Hara [**10**] finally proved the assertion for F-regular surface singularities.

For a domain $R$ of characteristic $p > 0$, we put $R^{1/p^e} := \{f^{1/p^e} \mid f \in R\}$, considered as a subset of the algebraic closure of the quotient field of $R$. The inclusion map $R \hookrightarrow R^{1/p^e}$ is isomorphic to the $e$-iterate Frobenius map $R \to R$, $f \mapsto f^{p^e}$.

DEFINITION 3.16 ([**17**]). Let $R$ be an F-finite domain, that is, $R^{1/p}$ is a finitely generated $R$-module. We say that $R$ is *F-regular*[8] if for every $0 \neq c \in R$, there exists $e > 0$ such that the $R$-linear map $R \to R^{1/p^e}$ defined by $1 \mapsto c^{1/p^e}$ splits. An affine integral scheme is called *F-regular* if the corresponding ring has this property.

THEOREM 3.17 ([**10**]). *For an F-regular surface singularity X and for $e \gg 0$,* $\mathrm{FB}_e(X)$ *is the minimal resolution.*

To prove this, Hara proved that an F-regular surface singularity $X$ always admits a finite cover $\pi : Y \to X$ with $Y$ regular. Using this property and the F-regularity of $X$, he showed that every indecomposable reflexive $\mathcal{O}_X$-module appears as a direct summand of $F^e_* \mathcal{O}_X$ for $e \gg 0$. The theorem is then proved with the aid of Wunram's correspondence [**35**] between some of indecomposable reflexive modules and irreducible exceptional curves on the minimal resolution.

[8]In the literature, this is usually called *strongly* F-regular.

**3.9. Non-F-regular surface singularities.** In their study of Frobenius sandwich singularities, Hara and Sawada [**11**] first gave an example of surface singularities for which high F-blowups are not the minimal resolution. Then they and the author [**12, 9**] studied F-blowups of non-F-regular normal surface singularities, especially non-F-regular rational double points and simple elliptic singularities.

3.9.1. *Non-F-regular rational double points.* Non-F-regular rational double points exist only in characteristics 2, 3 and 5. To study F-blowups of such singularities, we first need the following result:

PROPOSITION 3.18 ([**12**]). *Every F-blowup of a rational surface singularity is normal and dominated by the minimal resolution. In other words, it is obtained by contracting some exceptional curves on the minimal resolution.*

COROLLARY 3.19. *The minimal resolution of a rational surface singularity is obtained by an iteration of F-blowups.*

By Proposition 3.18, an F-blowup of a rational surface singularity is determined by which exceptional curves it has. A result of Artin and Verdier [**2**] says that in the case of rational double points, the exceptional curves on the minimal resolution are in one-to-one correspondence with non-trivial indecomposable reflexive $\mathcal{O}_X$-modules. Then the exceptional curves appearing on the $e$-th F-blowup are the ones corresponding to indecomposable direct summands of $F^e_*\mathcal{O}_X$. In [**12**], the direct summands of $F^e_*\mathcal{O}_X$ are determined for many non-F-regular rational double points. As a consequence, F-blowups of these singularities were determined. Results are summarized as follows: *for many non-F-regular rational double points, no F-blowup gives the minimal resolution. However, there are a few exceptions. With Artin's notation* [**1**]*, for $D_4^1$ and $D_5^1$-singularities in characteristic two, high F-blowups are the minimal resolution.*

3.9.2. *Simple elliptic singularities.* A simple elliptic singularity is a normal surface singularity whose minimal resolution has a smooth elliptic curve as its exceptional set. Completing a partial result in [**12**], Hara proved the following:

THEOREM 3.20 ([**9**]). *Let $X$ be a simple elliptic singularity, $\tilde{X} \to X$ its minimal resolution and $E \subset \tilde{X}$ the exceptional elliptic curve. We denote by $E^2$ the self-intersection number of $E$.*

(1) If $-E^2$ is not a power of $p$, then every F-blowup of $X$ (except the 0-th) is the minimal resolution $\tilde{X}$ of $X$.
(2) If $-E^2$ is a power of $p$ and if $X$ is F-pure, then for $e \gg 0$, the $e$-th F-blowup of $X$ is the blowup of $\tilde{X}$ at all the $p^e$-torsion points of $E$ (except the trivial case when $-E^2 = 1$).
(3) If $-E^2$ is a power of $p$ and if $X$ is not $F$-pure, then for $e \gg 0$, the $e$-th F-blowup of $X$ is the blowup of $\tilde{X}$ at a non-radical ideal and has only an $A_n$-singularity, where $n = p^e - 1$ if $E^2 = -1$ and $n = p^e - 2$ otherwise.

Thus F-blowups of a simple elliptic singularity are not generally dominated by the minimal resolution. Their behavior depends on the value of $E^2$ and whether the singularity is F-pure. However, high enough F-blowups are always normal and eventually improve singularities in some sense, (note that for low $e$'s, F-blowups of a simple elliptic singularity can be non-normal [**12**]).

**3.10. Crepant resolutions.** As we saw, for Gorenstein tame quotient singularities of dimension $\leq 3$, high F-blowups are a crepant resolution. This can be generalized to non-quotient singularities as follows:

THEOREM 3.21. *Let $X$ be a Gorenstein F-regular singularity of dimension $\leq 3$ and let $\phi : Y \to X$ be a finite covering. Suppose that $Y$ is regular and that the endomorphism ring $\mathrm{End}(\phi_* \mathcal{O}_Y)$ of the $\mathcal{O}_X$-module $\phi_* \mathcal{O}_Y$ is Cohen-Macaualy as an $\mathcal{O}_X$-module. Then for $e \gg 0$, $\mathrm{FB}_e(X)$ is a crepant resolution of $X$.*

PROOF. By [**39**], the ring $\mathrm{End}(\phi_* \mathcal{O}_Y)$ is a so-called non-commutative crepant resolution. In the same paper, it was shown that for large $e$, the ring $\mathrm{End}(F^e_* \mathcal{O}_X)$ is Morita equivalent to $\mathrm{End}(\phi_* \mathcal{O}_Y)$ and it is also a non-commutative crepant resolution. On the other hand, the $e$-th F-blowup is the moduli space of stable modules over $\mathrm{End}(F^e_* \mathcal{O}_X)$ with respect to some stability. Then the theorem follows from Van den Bergh's result [**33**]. □

## 4. Open problems

In this section, we collect some open problems related to higher Semple-Nash blowups and F-blowups, along with a few comments on each.

PROBLEM 4.1. For which surfaces do F-blowups give the minimal resolution?

Besides F-regular surface singularities, this is the case for some of simple elliptic singularities. What other classes of surface singularities share this nice property?

PROBLEM 4.2. Are high F-blowups of a normal surface singularity normal?

There is no reason to expect this, while there is no known counter-example.

PROBLEM 4.3. How do higher Semple-Nash blowups and F-blowups affect invariants of singularities?

Even if high F-blowups are not smooth, they might improve singularities as suggested by Theorem 3.20. To make it rigorous, we need to study how invariants of singularities change after blowups. We may ask the same question for higher Semple-Nash blowups. If this problem is affirmatively solved, these blowups might be applied to the study of algorithmic resolution of singularities.

PROBLEM 4.4. Construct one-step canonical resolutions for more general singularities, for instance, normal toric singularities.

As we saw in this paper, some classes of singularities such as curve singularities and F-regular surface singularities can be resolved in one step. Is there any different construction of one-step resolution which is valid for other classes?

PROBLEM 4.5. Study higher Semple-Nash blowups or F-blowups for algebraic singular foliations.

In fact, the author first defined higher Semple-Nash blowups for foliated varieties. Though he did not get any positive result, this blowup operation seems to be even more natural for singularities of foliations.

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