so that

$$\xi(t) = \tfrac{1}{2} - (tt + \tfrac{1}{4}) \int_1^\infty \psi(x) x^{-\frac{3}{4}} \cos(\tfrac{1}{2} t \log x)\, dx$$

or, in addition,

$$\xi(t) = 4 \int_1^\infty \frac{d(x^{\frac{3}{2}} \psi'(x))}{dx} x^{-\frac{1}{4}} \cos(\tfrac{1}{2} t \log x)\, dx.$$

This function is finite for all finite values of t, and allows itself to be developed in powers of tt as a very rapidly converging series. Since, for a value of s whose real part is greater than 1, $\log \zeta(s) = -\sum \log(1 - p^{-s})$ remains finite, and since the same holds for the logarithms of the other factors of $\xi(t)$, it follows that the function $\xi(t)$ can only vanish if the imaginary part of t lies between $\frac{1}{2}i$ and $-\frac{1}{2}i$. The number of roots of $\xi(t) = 0$, whose real parts lie between 0 and T is approximately

$$= \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi};$$

because the integral $\int d \log \xi(t)$, taken in a positive sense around the region consisting of the values of t whose imaginary parts lie between $\frac{1}{2}i$ and $-\frac{1}{2}i$ and whose real parts lie between 0 and T, is (up to a fraction of the order of magnitude of the quantity $\frac{1}{T}$) equal to $\left(T \log \dfrac{T}{2\pi} - T\right) i$; this integral however is equal to the number of roots of $\xi(t) = 0$ lying within in this region, multiplied by $2\pi i$. One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation.

If one denotes by α all the roots of the equation $\xi(\alpha) = 0$, one can express $\log \xi(t)$ as

$$\sum \log \left(1 - \frac{tt}{\alpha\alpha}\right) + \log \xi(0);$$

for, since the density of the roots of the quantity t grows with t only as $\log \dfrac{t}{2\pi}$, it follows that this expression converges and becomes for an infinite t only infinite as $t \log t$; thus it differs from $\log \xi(t)$ by a function of tt, that for a finite t remains continuous and finite and, when divided by tt, becomes infinitely small for infinite t. This difference is consequently a constant, whose value can be determined through setting $t = 0$.

With the assistance of these methods, the number of prime numbers that are smaller than x can now be determined.

Let $F(x)$ be equal to this number when x is not exactly equal to a prime number; but let it be greater by $\frac{1}{2}$ when x is a prime number, so that, for any x at which there is a jump in the value in $F(x)$,