On the Number of Prime Numbers less than a Given Quantity.
(Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.)

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I believe that I can best convey my thanks for the honour which the Academy has to some degree conferred on me, through my admission as one of its correspondents, if I speedily make use of the permission thereby received to communicate an investigation into the accumulation of the prime numbers; a topic which perhaps seems not wholly unworthy of such a communication, given the interest which *Gauss* and *Dirichlet* have themselves shown in it over a lengthy period.

For this investigation my point of departure is provided by the observation of *Euler* that the product

$$\prod \frac{1}{1-\dfrac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

if one substitutes for p all prime numbers, and for n all whole numbers. The function of the complex variable s which is represented by these two expressions, wherever they converge, I denote by $\zeta(s)$. Both expressions converge only when the real part of s is greater than 1; at the same time an expression for the function can easily be found which always remains valid. On making use of the equation

$$\int_0^\infty e^{-nx} x^{s-1}\, dx = \frac{\Pi(s-1)}{n^s}$$

one first sees that

$$\Pi(s-1)\zeta(s) = \int\limits_0^\infty \frac{x^{s-1}\,dx}{e^x-1}.$$

If one now considers the integral

$$\int \frac{(-x)^{s-1}\,dx}{e^x-1}$$

from $+\infty$ to $+\infty$ taken in a positive sense around a domain which includes the value 0 but no other point of discontinuity of the integrand in its interior, then this is easily seen to be equal to

$$(e^{-\pi s i} - e^{\pi s i}) \int\limits_0^\infty \frac{x^{s-1}\,dx}{e^x-1},$$

provided that, in the many-valued function $(-x)^{s-1} = e^{(s-1)\log(-x)}$, the logarithm of $-x$ is determined so as to be real when x is negative. Hence