ON THE TRANSFER OF DISTRIBUTIONS: WEIGHTED ORBITAL INTEGRALS

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Introduction

Let G be a connected reductive group over a local field of characteristic 0. The trace formula leads directly to the study of a certain family of distributions on $G(F)$. An important problem is to understand how these distributions change as G varies. A satisfactory solution of the problem would allow one to compare fundamental spectral data in different trace formulas, and would go a long way towards establishing new reciprocity laws between automorphic representations. In a paper [8], we stated a conjecture on the comparison of these distributions on different groups. The purpose of this paper is to lay the foundations for a general comparison of trace formulas. In the process, we shall obtain three pieces of evidence for the conjecture.

The distributions in question come from weighted orbital integrals

$$J_M(\gamma, f) \ = \ |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x) v_M(x) dx \, , \qquad \gamma \in M(F) \cap G_{\mathrm{reg}}, \ f \in \mathcal{C}(G),$$

in which M is a Levi subgroup of G. These are the terms on the geometric side of the local trace formula [4]. They are also the primary local terms on the geometric side of the global trace formula. We are speaking of the basic non-invariant trace formulas, whose individual terms change if f is replaced by a $G(F)$-conjugate. (The noninvariance is a general consequence of the truncation operations used in the derivation of the trace formulas; in $J_M(\gamma, f)$ it arises from the weight factor $v_M(x)$.) However, the original trace formulas have been refined so as to make their individual terms invariant [2], [5]. The process replaces $J_M(\gamma, f)$ by a natural invariant distribution

$$I_M(\gamma, f) \ = \ I_M^G(\gamma, f) \ = \ J_M(\gamma, f) - \sum_{L \neq G} \widehat{I}_M^L\big(\gamma, \phi_L(f)\big) \, ,$$

with correction terms in the sum constructed from spectral analogues of weighted orbital integrals. It actually is the invariant distributions $I_M(\gamma, f)$ whose transfer properties we seek.

In §3 we shall state Conjecture 3.3, an expanded version of the main transfer conjecture of [8]. It takes the form of two conjectural identities, satisfied by two new families of distributions $I^{\mathcal{E}}_M(\gamma, f)$ and $S^G_M(M', \delta', f)$. Each of the new distributions is defined inductively from stable distributions on endoscopic groups $G'(F)$. We recall that the endoscopic groups $\{G'\}$ for G are quasisplit groups introduced by Langlands to measure the difference between conjugacy and stable conjugacy in $G(F)$. The distributions $S^G_M(M', \delta', f)$ are defined only when G is quasisplit. Conjecture 3.3 asserts that they often vanish, and are always stable. They are to be regarded as stable analogues of the original distributions. The distributions $I^{\mathcal{E}}_M(\gamma, f)$ can be regarded as "endoscopic" analogues of the original ones, since they ultimately come from distributions $I^{G_1}_{M_1}(\gamma_1, f_1)$ on groups $G_1(F)$ obtained from G by a succession of endoscopic groups. In this case, Conjecture 3.3 asserts that $I^{\mathcal{E}}_M(\gamma, f)$ equals $I_M(\gamma, f)$.

The original distributions $I_M(\gamma)$ can be extended to a product of several copies of G. In this compound form, they satisfy a well known splitting formula. If γ belongs to a proper Levi subgroup of $M(F)$, $I_M(\gamma, f)$ also satisfies a descent formula. The splitting and descent formulas together reduce the study of the compound distributions to the special case of the simple ones, in which γ is elliptic in the corresponding Levi subgroup. One of our main tasks will be to extend these results to the new distributions. After some preparation in §4, we shall construct compound forms of $S^G_M(M', \delta')$ and $I^{\mathcal{E}}_M(\gamma)$ in §5. We shall then establish splitting formulas for the new distributions in §6, and descent formulas in §7. The splitting and descent formulas for $I^{\mathcal{E}}_M(\gamma)$ will be seen to be identical to those for $I_M(\gamma)$. The formulas for $S^G_M(M', \delta')$ will also be of the same general form, apart from the introduction of some new coefficients. We recall that it is actually the compound form of the distributions $I_M(\gamma)$ that occurs in the trace formulas. We would expect the same to be true of the new distributions. The fact that they all have similar splitting and descent properties is strong circumstantial evidence for Conjecture 3.3.

We shall find evidence of a different sort in §8. In this section, we shall establish two local vanishing theorems. These theorems, which apply to the case that G is not quasisplit, generalize results for inner forms of $GL(n)$ [3] that were required for base change [9]. They are purely combinatorial. However, they are more subtle than the results in [3]. Unlike the special case of $GL(n)$, the general vanishing theorems depend on the unexpected cancellation of various terms. The cancellation will in fact be forced on us by some internal signs in the Langlands-Shelstad transfer factors.

In §9, we shall apply one of the vanishing theorems to the local trace formula. Recall that the geometric side of the local trace formula is an expansion of a certain distribution

$$I(f)\,, \qquad\qquad f \in \mathcal{C}(G) \times \mathcal{C}(G),$$

on $G(F) \times G(F)$ in terms of the compound distributions $I_M(\gamma, f)$. In §9, we shall stabilize this distribution. More precisely, we shall construct distributions $S^G(f)$ and $I^{\mathcal{E}}(f)$ from $I(f)$, and we shall establish expansions for $S^G(f)$ and $I^{\mathcal{E}}(f)$ in terms of the compound distributions $S^G_M(M', \delta', f)$ and $I^{\mathcal{E}}_M(\gamma, f)$. Conjecture 3.3 would then imply that $S^G(f)$ is stable and that $I^{\mathcal{E}}(f)$ equals $I(f)$. The construction is a local generalization of results in [9] on the comparison of global trace formulas related to $GL(n)$. One of the goals of the global comparison in [9, Chapter 2] was to deduce Conjecture 3.3 in the special case of inner forms of $GL(n)$. The results in §9 here can thus be regarded as further evidence for the conjecture.

We shall conclude the paper in §10 with an application of the stabilization of §9. We shall establish Conjecture 3.3 for cuspidal functions on $G(F)$, subject in the p-adic case to the fundamental lemma. The result holds unconditionally for real groups, and also for p-adic inner forms of the groups $SL(n)$, $Sp(4)$ and $GSp(4)$. The proof is a simple illustration of how a comparison of trace formulas can lead to information on the conjecture.

The results of this paper will sometimes appear more complicated than we have indicated in the introduction. There are three technical reasons for this. We shall briefly discuss each of them in turn.

The first is peculiar to the case $F = \mathbb{R}$, and has origins in properties of the Galois cohomology of G. Vogan observed some years ago that missing elements in the L-packets of Shelstad [16] could be recovered by treating several groups simultaneously. The idea played an important role in the volume [1]. When Kottwitz learned of Vogan's idea, he saw how to apply it to the Langlands-Shelstad transfer factors. In this paper, the vanishing theorems we have described also require that we treat more than one real group at a time. This forces us to take G to be a disjoint union $\coprod_{\alpha} G_\alpha$ of several connected groups. We shall introduce objects of this sort in §1, and we shall endow them with extra structure, following the suggestions of Kottwitz. In §2 we shall see that a special case of the objects of §1 provides a natural domain for the transfer factors. This special case, which we shall call a K-group, will be the setting for the results of the rest of the paper.

A second technical concern is related to the splitting formulas. For reasons of induction, the construction of the compound versions of $S_M^G(M', \delta', f)$ and $I_M^{\mathcal{E}}(\gamma, f)$ in §5 requires that the original compound distributions be defined in considerably greater generality. In §4 we shall define distributions $I_M(\gamma)$ on a product of groups G_v, or rather K-groups, that are only distantly related to G. The original objects (F, G, M) will be retained only to index the weight factor $v_M(x)$. To describe the general class of K-groups G_v we can use, we shall introduce the notion of a satellite of (F, G, M). The compound distributions and splitting formulas will then exist for any finite set $\{(F_v, G_v, M_v)\}$ of satellites of (F, G, M).

The induction hypotheses themselves are a factor that tends to complicate the discussion. The definition of even the basic distributions of §3 is inductive. It requires that the stability part of Conjecture 3.3 hold for proper endoscopic groups of G. Since we cannot establish the conjecture in this paper, we shall have to treat such hypotheses with care. We shall carry them throughout the paper, in a form that is suitable for the general comparison of trace formulas. We shall study the stabilization of the global trace formula in a future paper. At that point we will be able to resolve the induction hypotheses taken on in this paper.

1. Multiple groups

Throughout the paper, F will be a field of characteristic 0. Suppose for a moment that G is a connected reductive algebraic group over F. Following standard notation, we write $Z(G)$ for the center of G, G_{sc} for the simply connected cover of the derived group of G, and G_{ad} for the adjoint group of G. Suppose that $M \supset Z(G)$ is an algebraic subgroup of G over F. We shall generally write M_{sc} for the preimage of M in G_{sc} and M_{ad} for the image of M in G_{ad}. Usually M will be a Levi subgroup of G, by which we mean an F-rational Levi component of a parabolic subgroup of G over F. In this case, M_{sc} is a Levi subgroup of G_{sc}, and M_{ad} is a Levi subgroup of G_{ad}.

It will be useful in this paper to work with several groups simultaneously. We shall do so by letting G stand for an algebraic variety whose connected components are reductive algebraic groups over F. We shall write $\pi_0(G)$ for the set of connected components of G, as usual, but we shall generally treat $\pi_0(G)$ as a set of indices for the components in question. We assume that for every pair $\alpha, \beta \in \pi_0(G)$, the connected reductive groups G_α and G_β are isomorphic over an algebraic closure $\overline{F}$ of F.

Given G, we consider families of objects

$$(\psi, u) \;=\; \big\{(\psi_{\alpha\beta}, u_{\alpha\beta}) : \;\alpha, \beta \in \pi_0(G)\big\}\,,$$

where $\psi_{\alpha\beta}\colon G_\beta \to G_\alpha$ is an isomorphism over $\overline{F}$, and $u_{\alpha\beta}\colon \Gamma \to G_{\alpha,\mathrm{sc}}$ is a locally constant function from the Galois group $\Gamma = \mathrm{Gal}(\overline{F}/F)$ to $G_{\alpha,\mathrm{sc}}$. We require that a family satisfy the compatibility conditions

(i) $\quad \psi_{\alpha\beta}\tau(\psi_{\alpha\beta})^{-1} = \mathrm{Int}\big(u_{\alpha\beta}(\tau)\big)$,

(ii) $\quad \psi_{\alpha\gamma} = \psi_{\alpha\beta}\psi_{\beta\gamma}$,

and

(iii) $\quad u_{\alpha\gamma}(\tau) = \psi_{\alpha\beta,\mathrm{sc}}\big(u_{\beta\gamma}(\tau)\big)u_{\alpha\beta}(\tau)$, for any $\alpha, \beta, \gamma \in \pi_0(G)$ and $\tau \in \Gamma$. Observe from (ii) that $\psi_{\alpha\alpha} = 1$, and from (iii) that $u_{\alpha\alpha}(\tau) = 1$. Notice also that if α is fixed, there are no constraints on the choice of

$$\{(\psi_{\alpha\beta}, u_{\alpha\beta}) : \ \beta \in \pi_0(G)\}$$

other than (i). The entire family is then uniquely determined by this subset and the conditions (ii) and (iii). We shall say that two such families (ψ, u) and (ψ', u') are *equivalent* if there are elements $g_{\alpha\beta} \in G_{\alpha,\mathrm{sc}}$ such that

$$\psi'_{\alpha\beta} \ = \ \mathrm{Int}(g_{\alpha\beta})\psi_{\alpha\beta} \,,$$

and

$$u'_{\alpha\beta}(\tau) \ = \ g_{\alpha\beta} u_{\alpha\beta}(\tau)\tau(g_{\alpha\beta})^{-1} \,,$$

for any $\alpha, \beta \in \pi_0(G)$ and $\tau \in \Gamma$. It is easy to see that if such a set of elements exists, it satisfies

$$g_{\alpha\gamma} \ = \ g_{\alpha\beta}\psi_{\alpha\beta}(g_{\beta\gamma}) \,, \qquad \alpha, \beta, \gamma \in \pi_0(G).$$

In particular, it is determined by the subset

$$\{g_{\alpha\beta} : \ \beta \in \pi_0(G)\}$$

attached to any α. The equivalence classes $\{(\psi, u)\}$ may therefore be identified with orbits of the group $G_\alpha^{|\pi_0(G)|-1}$.

We define a *multiple group* over F to be a variety G as above, together with an equivalence class of objects $\{(\psi, u)\}$. We shall call a representative (ψ, u) from the equivalence class a *frame* for G. Observe that if I is any subset of $\pi_0(G)$, we obtain a multiple group G^I from G by deleting the components in the complement of I in $\pi_0(G)$ (as well as the superfluous pairs from each frame). We shall say that a morphism θ: $G \to G_1$ of multiple groups over F is an *F-homomorphism* if it has the following two properties:

(i) For any $\alpha \in \pi_0(G)$, and for $\alpha_1 = \theta(\alpha)$ the image of α in $\pi_0(G)$, the restriction $\theta_\alpha \colon G_\alpha \to G_{1,\alpha_1}$ is an F-homomorphism of connected algebraic groups.

(ii) There are frames (ψ, u) and (ψ_1, u_1) for G and G_1 that are θ-compatible, in the sense that $\theta_\alpha \circ \psi_{\alpha\beta} = \psi_{1,\alpha_1\beta_1} \circ \theta_\beta$ and $u_{1,\alpha_1\beta_1} = \theta_{\alpha,\mathrm{sc}}(u_{\alpha\beta})$, for any $\alpha, \beta \in \pi_0(G)$.

We shall call an invertible F-homomorphism an *F-isomorphism.*

There is at least one example that will be familiar. The transfer factors in [15, §3] for a connected reductive group G over a local field F are defined in terms of a quasisplit inner twist $\psi \colon G \to G^*$ of G, and also a function $u \colon \Gamma \to G^*_{\mathrm{sc}}$ such that

$$\psi \tau(\psi)^{-1} \ = \ \mathrm{Int}\big(u(\tau)\big)\,, \qquad\qquad \tau \in \Gamma.$$

The transfer factors depend only on the G^*-conjugacy class of (ψ, u), so the underlying structure can be regarded as a multiple group with two components. More generally, suppose that G is an arbitrary multiple group. By a *quasisplit inner twist* of G, we mean an embedding of G into a multiple group $G \coprod G^*$, with G^* quasisplit over F. A frame for $G \coprod G^*$ then includes a family of isomorphisms $\psi_\alpha \colon G_\alpha \to G^*$, as well as a corresponding family of functions $u_\alpha \colon \Gamma \to G^*_{\alpha,\mathrm{sc}}$.

Let G be a fixed multiple group. If M_α is a Levi subgroup of G_α, for $\alpha \in \pi_0(G)$, let $I(M_\alpha)$ denote the set of $\beta \in \pi_0(G)$ such that $M_\beta = \psi_{\beta\alpha}(M_\alpha)$ is a Levi subgroup of G_β (and is in particular defined over F), for some frame (ψ, u). We define a *Levi subgroup* of G to be a multiple group M over F, together with an embedding $\pi_0(M) \subset \pi_0(G)$, with the following properties:

(i) For any $\alpha \in \pi_0(M)$, M_α is a Levi subgroup of G_α such that $\pi_0(M)$ equals $I(M_\alpha)$. In particular, the restricted multiple group $G^M = G^{I(M_\alpha)}$ is independent of α.

(ii) The injection $M \subset G$ defined by the embeddings $M_\alpha \subset G_\alpha$ of components $\alpha \in \pi_0(M)$ is an F-homomorphism of multiple groups.

It is clear that if L is a Levi subgroup of G and M is a Levi subgroup of L, then M is a Levi subgroup of G.

Levi subgroups of G are easy to construct. We claim that a Levi subgroup M_α of a connected component G_α can be embedded in a Levi subgroup M of G. By assumption, there is a frame (ψ, u) for G such that for any $\beta \in I(M_\alpha)$, $u_{\alpha\beta}(\tau)$ takes values in the subgroup $M_{\alpha,\mathrm{sc}}$ of $G_{\alpha,\mathrm{sc}}$. It is easy to see that the derived group $(M_{\alpha,\mathrm{sc}})_{\mathrm{der}}$ is simply connected, and that the quotient $M_{\alpha,\mathrm{sc}}/(M_{\alpha,\mathrm{sc}})_{\mathrm{der}}$ is an induced torus. It follows from Shapiro's lemma that $H^1\big(F, (M_{\alpha,\mathrm{sc}})_{\mathrm{der}}\big)$ maps surjectively onto $H^1(F, M_{\alpha,\mathrm{sc}})$. The frame (ψ, u) may therefore be chosen so that $u_{\alpha\beta}(\tau)$ takes values in the subgroup $(M_{\alpha,\mathrm{sc}})_{\mathrm{der}}$ of $M_{\alpha,\mathrm{sc}}$. Such a frame can then be used to construct a multiple group M from M_α, such that $\pi_0(M) = I(M_\alpha)$, and such that M is a Levi subgroup of G. This was the claim.

Suppose that M is a Levi subgroup of G. For each $\alpha \in \pi_0(M)$, we can form the usual real vector space

$$\mathfrak{a}_{M_\alpha} = \mathrm{Hom}\big(X(M_\alpha)_F, \mathbb{R}\big).$$

Any frame for M gives a compatible family of linear isomorphisms $\mathfrak{a}_{M_\beta} \xrightarrow{\sim} \mathfrak{a}_{M_\alpha}$, for $\alpha, \beta \in \pi_0(M)$, which are independent of the choice of frame. We can therefore define a vector space $\mathfrak{a}_M = \varprojlim_\alpha \mathfrak{a}_{M_\alpha}$ which is canonically isomorphic to each space $\mathfrak{a}_{M_\alpha}$. We can also form a Weyl group

$$W(M) = W^G(M) = \varprojlim_\alpha W^{G_\alpha}(M_\alpha) = \varprojlim_\alpha \mathrm{Norm}_{G_\alpha}(M_\alpha)/M_\alpha,$$

that operates on $\mathfrak{a}_M$. We shall sometimes assume implicitly that $\mathfrak{a}_M$ has been equipped with a $W(M)$-invariant Euclidean inner product, together with the corresponding Haar measure.

By a *parabolic subgroup* of G over F with Levi component M, we mean a variety P with $\pi_0(P) = \pi_0(M)$, such that for each $\alpha \in \pi_0(M)$, P_α is a parabolic subgroup of G_α over F with Levi component M_α, and such that the chambers

$$\big\{\mathfrak{a}^+_{P_\alpha} \subset \mathfrak{a}_{M_\alpha} : \ \alpha \in \pi_0(P)\big\}$$

coincide under the isomorphisms above. The space $\mathfrak{a}_M$ obviously inherits chambers from any $\mathfrak{a}_{M_\alpha}$. We see that there is a bijection $P \to \mathfrak{a}_P^+$ from the set $\mathcal{P}(M)$ of parabolic subgroups of G over F with Levi component M, and the set of chambers in $\mathfrak{a}_M$.

The space $\mathfrak{a}_M$ also inherits a stratification from any of the spaces $\mathfrak{a}_{M_\alpha}$. If G is connected, the strata in $\mathfrak{a}_M$ are bijective with the set $\mathcal{L}(M)$ of Levi subgroups of G that contain M. If G is not connected, there can be infinitely many such Levi subgroups. To sidestep this difficulty, let us say that two Levi subgroups L_1 and L_2 of G are *M-equivalent* if they both contain M, and if the restricted multiple groups L_1^M and L_2^M are the same. We then define $\mathcal{L}(M)$ to be the set of M-equivalence classes of Levi subgroups of G. The set $\mathcal{L}(M)$ is canonically bijective with any of the finite sets $\mathcal{L}(M_\alpha)$. We define M-equivalence of parabolic subgroups of G over F in the same way, and we write $\mathcal{F}(M)$ for the set of such equivalence classes. A class Q in $\mathcal{F}(M)$ then has a unique M-equivalence class M_Q of Levi components. Any $L \in \mathcal{L}(M)$ therefore determines a subset $\mathcal{P}(L)$ of $\mathcal{F}(M)$. It is clear that $\mathcal{F}(M)$ is the disjoint union over $L \in \mathcal{L}(M)$ of the set $\mathcal{P}(L)$. As in the connected case, we have a stratification

$$\mathfrak{a}_L \subset \mathfrak{a}_M \,, \qquad\qquad L \in \mathcal{L}(M),$$

of $\mathfrak{a}_M$ by a finite set of subspaces indexed by $\mathcal{L}(M)$, and a partition

$$\mathfrak{a}_M \;=\; \coprod_{Q \in \mathcal{F}(M)} \mathfrak{a}_Q^+$$

of $\mathfrak{a}_M$ into cones indexed by $\mathcal{F}(M)$.

Suppose that G^* is a quasisplit inner twist of G, which we recall is an embedding of G into a multiple group $G \coprod G^*$. By a Levi subgroup M^* of G^* *corresponding* to M, we shall mean a Levi subgroup of $G \coprod G^*$ of the form $M \coprod M^*$. It is easy to see that for any M, such M^* exists. The elements in $\mathcal{L}(M \coprod M^*)$ then determine a bijection $L \to L^*$ from $\mathcal{L}(M)$ onto $L(M^*)$. Similarly, there are bijections $P \to P^*$ and $Q \to Q^*$ from $\mathcal{P}(M)$ and $\mathcal{F}(M)$ onto $\mathcal{P}(M^*)$ and $\mathcal{F}(M^*)$ respectively.

Recall that any group G_α, $\alpha \in \pi_0(G)$, has a canonical based root datum $\Psi(G_\alpha)$ [13, pp. 614–615], that comes with an action of Γ. For $\alpha, \beta \in \pi_0(G)$, the isomorphism $\psi_{\alpha\beta}$: $G_\beta \to G_\alpha$ attached to any frame determines a Γ-bijection from $\Psi(G_\beta)$ to $\Psi(G_\alpha)$ that is independent of the choice of frame. We can therefore attach a canonical based root datum

$$\Psi(G) \; = \; (X_G, \Delta_G, X_G^\vee, \Delta_G^\vee)$$

to G, together with a canonical Γ-bijection from $\Psi(G)$ to each $\Psi(G_\alpha)$. We can also define a common dual group $\widehat{G}$ for each G_α. Then $\widehat{G}$ is a complex connected reductive group, with an L-action of Γ, and a Γ-bijection from the dual based root datum $\Psi(G)^\vee$ of $\Psi(G)$ to the canonical based root datum

$$\Psi(\widehat{G}) \; = \; (X_{\widehat{G}}, \Delta_{\widehat{G}}, X_{\widehat{G}}^\vee, \Delta_{\widehat{G}}^\vee)$$

of $\widehat{G}$.

Suppose that M is a Levi subgroup of G. The choice of an element $P \in \mathcal{P}(M)$ determines a Γ-invariant subset Δ_P of Δ_G, and a Γ-bijection from

$$(X_G, \Delta_M, X_G^\vee, \Delta_M^\vee)\,, \qquad\qquad \Delta_M = \Delta_G - \Delta_P,$$

to the canonical based root datum $\Psi(M)$ for M. By a *Levi subgroup* of $\widehat{G}$, we shall mean a Γ-stable Levi component of a Γ-stable parabolic subgroup of $\widehat{G}$. For any such group $\widehat{M}$, we shall write $\mathcal{P}(\widehat{M})$, $\mathcal{L}(\widehat{M})$ and $\mathcal{F}(\widehat{M})$ as above, with the understanding that the sets contain only Γ-stable elements. The choice of an element $\widehat{P} \in \mathcal{P}(\widehat{M})$ determines a Γ-invariant subset $\Delta_{\widehat{P}}$ of $\Delta_{\widehat{G}}$, and a Γ-bijection from

$$(X_{\widehat{G}}, \Delta_{\widehat{M}}, X_{\widehat{G}}^\vee, \Delta_{\widehat{M}}^\vee)\,, \qquad\qquad \Delta_{\widehat{M}} = \Delta_{\widehat{G}} - \Delta_{\widehat{P}},$$

to the canonical based root datum $\Psi(\widehat{M})$ for $\widehat{M}$. We shall say that P and $\widehat{P}$ are dual if $\Delta_P^\vee$ maps to $\Delta_{\widehat{P}}$ under the canonical bijection of $\Psi(G)^\vee$ with $\Psi(\widehat{G})$. If this is so, $\widehat{M}$ is a dual group of M. For a given M, a *dual Levi subgroup* $\widehat{M} \subset \widehat{G}$ for M will mean a Levi subgroup $\widehat{M}$ of $\widehat{G}$ that is also a dual group of M, relative to some choice of dual elements $P \in \mathcal{P}(M)$ and $\widehat{P} \in \mathcal{P}(\widehat{M})$. For any such $\widehat{M}$, there is a canonical bijection $L \to \widehat{L}$ from $\mathcal{L}(M)$ to $\mathcal{L}(\widehat{M})$ with the property that each $\widehat{L}$ is a dual Levi subgroup for L.

Lemma 1.1. *Suppose that M is a Levi subgroup of G, and that $\widehat{M} \subset \widehat{G}$ is a dual Levi subgroup. Then*

$$Z(\widehat{M})^{\Gamma} \;=\; Z(\widehat{G})^{\Gamma}\big(Z(\widehat{M})^{\Gamma}\big)^{0}.$$

Proof. As usual, $Z(\widehat{M})^{\Gamma}$ stands for the subgroup of invariants of Γ in $Z(\widehat{M})$. The quotient $Z(\widehat{M})^{\Gamma}/Z(\widehat{G})^{\Gamma}$ is a complex diagonalizable group. It is enough to show that this group is connected. Equivalently, it suffices to show that the finitely generated abelian group $X\big(Z(\widehat{M})^{\Gamma}/Z(\widehat{G})^{\Gamma}\big)$ of rational characters of $Z(\widehat{M})^{\Gamma}/Z(\widehat{G})^{\Gamma}$ is free.

Let $Y_{\widehat{G}}$ be the root lattice for $\widehat{G}$, the subgroup of $X_{\widehat{G}}$ generated by $\Delta_{\widehat{G}}$. Then $X\big(Z(\widehat{G})\big)$ is Γ-isomorphic to $X_{\widehat{G}}/Y_{\widehat{G}}$. Similarly, $X\big(Z(\widehat{M})\big)$ is Γ-isomorphic to $X_{\widehat{G}}/Y_{\widehat{M}}$, where $Y_{\widehat{M}}$ is the subgroup of $X_{\widehat{G}}$ generated by $\Delta_{\widehat{M}}$. Now, for any finitely generated abelian group X with an action of Γ, we have the group $X_{\Gamma} = X/X^{0}$ of covariants, in which X^{0} is the subgroup generated by

$$\{\tau\lambda - \lambda : \; \lambda \in X, \; \tau \in \Gamma\}.$$

Then

$$X\big(Z(\widehat{M})^{\Gamma}\big) \;\cong\; X\big(Z(\widehat{M})\big)_{\Gamma} \;=\; X\big(Z(\widehat{M})\big)/X\big(Z(\widehat{M})\big)^{0} \;\cong\; X_{\widehat{G}}/X^{0}_{\widehat{G}} + Y_{\widehat{M}}.$$

Since $X\big(Z(\widehat{M})^{\Gamma}/Z(\widehat{G})^{\Gamma}\big)$ is the subgroup of elements in $X\big(Z(\widehat{M})^{\Gamma}\big)$ which map to 0 in $X\big(Z(\widehat{G})^{\Gamma}\big)$, we have

$$X\big(Z(\widehat{M})^{\Gamma}/Z(\widehat{G})^{\Gamma}\big) \;\cong\; Y_{\widehat{G}}/Y_{\widehat{G}} \cap (X^{0}_{\widehat{G}} + Y_{\widehat{M}}) \;=\; Y_{\widehat{G}}/Y_{\widehat{M}} + (Y_{\widehat{G}} \cap X^{0}_{\widehat{G}}).$$

The action of Γ on $\Delta_{\widehat{G}}$ factors through a normal subgroup Γ_1 of finite index in Γ. Any element λ in $Y_{\widehat{G}} \cap X^{0}_{\widehat{G}}$ satisfies

$$N_1\lambda \;=\; \sum_{\tau \in \Gamma/\Gamma_1} \tau\lambda \;=\; 0.$$

Since Γ/Γ_1 acts by permutation on the basis $\Delta_{\widehat{G}}$ of $Y_{\widehat{G}}$, a variant of Shapiro's lemma tells us that any element $\lambda \in Y_{\widehat{G}}$ with $N_1\lambda = 0$ actually belongs to $Y^0_{\widehat{G}}$. Therefore

$$X\big(Z(\widehat{M})^\Gamma / Z(\widehat{G})^\Gamma\big) \;\cong\; Y_{\widehat{G}}/Y_{\widehat{M}} + Y^0_{\widehat{G}}.$$

By considering the different Γ-orbits in the basis $\Delta_{\widehat{G}}$, we see easily that the quotient on the right is a free abelian group. The lemma follows. $\square$

2. K-groups and transfer factors

Suppose that F is a local field. If G is a connected reductive group over F, Kottwitz [14, §1] defines a morphism of pointed sets

$$H^1(F,G) \longrightarrow \pi_0\big(Z(\widehat{G})^\Gamma\big)^*,$$

that we shall take the liberty of denoting by K_G. (We have written $\pi_0\big(Z(\widehat{G})^\Gamma\big)^*$ for the finite abelian group of characters on $\pi_0\big(Z(\widehat{G})^\Gamma\big)$.) The morphism is functorial relative to a Levi subgroup $M \subset G$, in the sense that the diagram

$$\begin{array}{ccc} H^1(F,M) & \longrightarrow & H^1(F,G) \\ {\scriptstyle K_M}\big\downarrow & & {\scriptstyle K_G}\big\downarrow \\ \pi_0\big(Z(\widehat{M})^\Gamma\big)^* & \longrightarrow & \pi_0\big(Z(\widehat{G})^\Gamma\big)^* \end{array}$$

is commutative. This is a special case of [14, Lemma 4.3], and follows also from the general results of [10]. The horizontal arrows in the diagram are both injective. This is well known in the case of the map $H^1(F,M) \to H^1(F,G)$, and follows for the lower horizontal arrow from Lemma 1.1.

Assume now that G is a multiple group over F, as in §1. Then there is a map

$$K_{G_\alpha} :\ H^1(F,G_\alpha) \longrightarrow \pi_0\big(Z(\widehat{G})^\Gamma\big)^*$$

for each $\alpha \in \pi_0(G)$. We shall say that G is a *K-multiple group*, or simply a *K-group*, if the functions $u_{\alpha\beta}\colon \Gamma \to G_{\alpha,\mathrm{sc}}$ attached to any frame are 1-cocycles, and the corresponding sequences

$$\{1\} \longrightarrow \big\{u_{\alpha\beta} :\ \beta \in \pi_0(G)\big\} \longrightarrow H^1(F,G_\alpha) \xrightarrow{K_{G_\alpha}} \pi_0\big(Z(\widehat{G})^\Gamma\big)^*, \qquad \alpha \in \pi_0(G),$$

of pointed sets are exact. In other words, the map that sends $u_{\alpha\beta}$ to its image in $H^1(F,G_\alpha)$ is a bijection from $\{u_{\alpha\beta} :\ \beta \in \pi_0(G)\}$ onto the subset of elements in $H^1(F,G_\alpha)$ whose image under K_{G_α} is the trivial character on $\pi_0\big(Z(\widehat{G})^\Gamma\big)$. If F is p-adic, K_{G_α} is a bijection [14, Theorem 1.2]. A K-group in this case is therefore just a connected group. If $F = \mathbb{R}$, the kernel of K_{G_α} is the image of $H^1(F, G_{\alpha,\mathrm{sc}})$ in $H^1(F, G_\alpha)$ [14, Theorem 1.2]. The number of components of a K-group over $\mathbb{R}$ therefore equals the number of classes in this image.

Suppose that G is a K-group over F, and that M is a Levi subgroup of G. Then for any α in the subset $\pi_0(M)$ of $\pi_0(G)$, there is associated a Levi subgroup M_α of G_α. A priori, M is just a multiple group. However, one sees easily from the commutative diagram for $M_\alpha \subset G_\alpha$ and the injectivity of $H^1(F, M_\alpha) \to H^1(F, G_\alpha)$ that M is in fact a K-group.

For the rest of this section, G will be a fixed K-group over F. We shall say that G is an *inner K-form* of a quasisplit group G^* if G^* is a quasisplit inner twist of G (in the sense of §1). We fix such a G^*. Then for any α, the projection of u_α onto G^*_{ad} is a 1-cocycle. The corresponding image $\overline{u}_{\alpha,\mathrm{ad}}$ of u_α in $H^1(F, G^*_{\mathrm{ad}})$ is the class which determines G_α as an inner twist of G^*. Set

$$\widehat{Z}_{\mathrm{sc}} \;=\; Z(\widehat{G}_{\mathrm{sc}}) \;\cong\; X_{G^*_{\mathrm{sc}}}/X_{G^*_{\mathrm{ad}}}\,. \tag{2.1}$$

Then

$$\zeta_G \;=\; K_{G^*_{\mathrm{ad}}}(\overline{u}_{\alpha,\mathrm{ad}}) \tag{2.2}$$

is a character on the finite group $\widehat{Z}^\Gamma_{\mathrm{sc}}$ that depends only on G. It follows easily from [14, Theorem 1.2] that $\alpha \to \overline{u}_{\alpha,\mathrm{ad}}$ is a surjective map from $\pi_0(G)$ onto the preimage $K^{-1}_{G^*_{\mathrm{ad}}}(\zeta_G)$ of ζ_G in $H^1(F, G^*_{\mathrm{ad}})$.

We will need to know that K-groups have minimal Levi subgroups with properties similar to those in the connected case. Any $\alpha \in \Delta_G$ determines a fundamental dominant weight $\varpi_\alpha \in X_{G^*_{\mathrm{sc}}}$, whose Γ-orbit matches the Γ-orbit of α in Δ_G. Let ϖ^Γ_α be the sum of the elements in the Γ-orbit of ϖ_α. Then ϖ^Γ_α is a Γ-invariant element in $X_{G^*_{\mathrm{sc}}}$ that depends only on the Γ-orbit of α. Let $z_\alpha \in \widehat{Z}^\Gamma_{\mathrm{sc}}$ be the image of ϖ^Γ_α under the composition of the maps

$$X^\Gamma_{G^*_{\mathrm{sc}}} \;\to\; X^\Gamma_{G^*_{\mathrm{sc}}}/X^\Gamma_{G^*_{\mathrm{ad}}} \;\hookrightarrow\; (X_{G^*_{\mathrm{sc}}}/X_{G^*_{\mathrm{ad}}})^\Gamma \;\cong\; \widehat{Z}^\Gamma_{\mathrm{sc}}\,.$$

We obtain in this way a map $\alpha \to z_\alpha$ from the set Δ_G/Γ of Γ-orbits in Δ_G to $\widehat{Z}^\Gamma_{\mathrm{sc}}$.

Lemma 2.1. *Suppose that Δ is a Γ-stable subset of Δ_G. Then there is a parabolic subgroup P of G over F with $\Delta_P = \Delta$ if and only if Δ is contained in the subset*

$$\Delta_0 \;=\; \big\{\alpha \in \Delta_G :\; \zeta_G(z_\alpha) = 1\big\}$$

of Δ_G.

Proof. Since Δ is Γ-stable, there is a parabolic subgroup P^* of the quasisplit group G^* over F such that $\Delta_{P^*} = \Delta$. Let $\widehat{P} \subset \widehat{G}$ be a dual parabolic subgroup, and let $M^* \subset P^*$ and $\widehat{M} \subset \widehat{P}$ be Γ-stable Levi components. We shall investigate the subgroup

$$\widehat{Z}_{\mathrm{sc}}^{\Gamma} \cap \big(Z(\widehat{M}_{\mathrm{sc}})^{\Gamma}\big)^0$$

of $\widehat{Z}_{\mathrm{sc}}^{\Gamma}$.

The lattice $X = X_{G^*_{\mathrm{ad}}}$ has a Γ-stable basis $\Delta_{G^*_{\mathrm{ad}}}$, while $\widetilde{X} = X_{G^*_{\mathrm{sc}}}$ has a Γ-stable basis composed of the fundamental dominant weights. By a simple application of Shapiro's lemma, the map

$$H^1(\Gamma, X) \longrightarrow H^1(\Gamma, \widetilde{X})$$

is injective. It follows from the exact sequence of cohomology that

$$\widetilde{X}^{\Gamma}/X^{\Gamma} \;=\; (\widetilde{X}/X)^{\Gamma} \;\cong\; \widehat{Z}_{\mathrm{sc}}^{\Gamma}\,.$$

Now $\{\varpi_\alpha^\Gamma :\; \alpha \in \Delta_G/\Gamma\}$ is a basis of $\widetilde{X}^\Gamma$. Let $\widetilde{X}_\Delta^\Gamma$ be the subgroup of $\widetilde{X}$ spanned by the elements $\{\varpi_\alpha^\Gamma :\; \alpha \in \Delta/\Gamma\}$. We may as well identify $\widetilde{X}^\Gamma/X^\Gamma$ with $\widehat{Z}_{\mathrm{sc}}^\Gamma$. Then $\widetilde{X}_\Delta^\Gamma$ maps onto a subgroup

$$\widetilde{X}_\Delta^\Gamma + X^\Gamma/X^\Gamma \;\cong\; \widetilde{X}_\Delta^\Gamma/\widetilde{X}_\Delta^\Gamma \cap X^\Gamma$$

of $\widehat{Z}_{\mathrm{sc}}^\Gamma$. But $\widetilde{X}_\Delta^\Gamma \cap X^\Gamma$ equals the lattice $X_*\big(\big(Z(\widehat{M}_{\mathrm{sc}})^\Gamma\big)^0\big)$ of one parameter subgroups of $\big(Z(\widehat{M}_{\mathrm{sc}})^\Gamma\big)^0$. It follows easily that $\widetilde{X}_\Delta^\Gamma + X^\Gamma/X^\Gamma$ equals the subgroup $\widehat{Z}_{\mathrm{sc}}^\Gamma = \widehat{Z}_{\mathrm{sc}}^\Gamma \cap \big(Z(\widehat{M}_{\mathrm{sc}})^\Gamma\big)^0$ of $\widehat{Z}_{\mathrm{sc}}^\Gamma$. This subgroup is clearly generated by $\{z_\alpha : \alpha \in \Delta/\Gamma\}$. It follows that $\widehat{Z}_{\mathrm{sc}}^{\Gamma} \cap \big(Z(\widehat{M}_{\mathrm{sc}})^{\Gamma}\big)^0$ lies in the kernel of ζ_G if and only if Δ is contained in Δ_0. If this is so, ζ_G determines a character ζ_G^M on the group

$$\pi_0\big(Z(\widehat{M}_{\mathrm{sc}})^{\Gamma}\big) \;\cong\; \widehat{Z}_{\mathrm{sc}}^{\Gamma}/\widehat{Z}_{\mathrm{sc}}^{\Gamma} \cap \big(Z(\widehat{M}_{\mathrm{sc}})^{\Gamma}\big)^0 ,$$

since by Lemma 1.1, $Z(\widehat{M}_{\mathrm{sc}})^{\Gamma}$ equals $\widehat{Z}_{\mathrm{sc}}^{\Gamma}\big(Z(\widehat{M}_{\mathrm{sc}})^{\Gamma}\big)^0$.

Assume first that Δ is contained in Δ_0. The character ζ_G^M on $\pi_0\big(Z(\widehat{M}_{\mathrm{sc}})^{\Gamma}\big)$ is then defined. We note that the group $\widehat{M}_{\mathrm{sc}}$ is dual to the Levi subgroup M_{ad}^* of G_{ad}^*. It is a consequence of [14, Theorem 1.2] that ζ_G^M lies in the image of the map $K_M = K_{M_{\mathrm{ad}}^*}$. Indeed, if F is a p-adic field, K_M is bijective. If $F = \mathbb{R}$, the image of K_M is the set of characters that vanish on the image of the norm map from $\pi_0\big(Z(\widehat{M}_{\mathrm{sc}})\big)$ to $\pi_0\big(Z(\widehat{M}_{\mathrm{sc}})^{\Gamma}\big)$. But ζ_G is in the image of the map $K = K_{G_{\mathrm{ad}}^*}$, and must therefore vanish on the image of the norm map from $\widehat{Z}_{\mathrm{sc}}$ to $\widehat{Z}_{\mathrm{sc}}^{\Gamma}$. It follows easily that ζ_G^M is in the image of K_M in this case as well. Let u_M be a 1-cocycle from Γ to M_{ad}^* whose image in $H^1(F, M_{\mathrm{ad}}^*)$ is mapped by K_M to ζ_G^M, and let u be the image of u_M in G_{ad}^*. Then u is a 1-cocycle of Γ in G_{ad}^* that we can use to construct an inner twist G_1 of G^*. The corresponding inner twists of P^* and M^* by u_M give a parabolic subgroup P_1 and a Levi subgroup M_1 of G_1. Observe that K maps the image $\overline{u}$ of u in $H^1(F, G_{\mathrm{ad}}^*)$ to ζ_G. Since $\alpha \to \overline{u}_{\alpha,\mathrm{ad}}$ is a surjective map from $\pi_0(G)$ onto the preimage under K of ζ_G in $H^1(F, G_{\mathrm{ad}}^*)$, there is an α such that $\overline{u}_{\alpha,\mathrm{ad}} = \overline{u}$. Therefore G_1 is isomorphic to G_α over F. Choose such an isomorphism, and let P_α and M_α be the corresponding images of P_1 and M_1. It follows from the definitions of §1 that P_α may be embedded in a parabolic subgroup P of G over F, with a Levi component M that extends M_α. The construction clearly has the property that $\Delta_P = \Delta$.

Conversely, suppose that there is a parabolic subgroup P of G with $\Delta_P = \Delta$. Let M be a rational Levi component of P, with dual Levi subgroup $\widehat{M} \subset \widehat{G}$. Then for any $\alpha \in \pi_0(M)$, $(G_\alpha, P_\alpha, M_\alpha)$ is an inner twist of (G^*, P^*, M^*). In particular, the class $\overline{u}_{\alpha,\mathrm{ad}} \in H^1(F, G_{\mathrm{ad}}^*)$ that determines G_α as an inner twist of G^* must be the image of a class in $H^1(F, M_{\mathrm{ad}}^*)$. Since $K_{G_{\mathrm{ad}}^*}$ is functorial relative to the embedding $M_{\mathrm{ad}}^* \subset G_{\mathrm{ad}}^*$, the character ζ_G on $\widehat{Z}_{\mathrm{sc}}^{\Gamma}$ is the image of a character ζ_G^M on $\pi_0\bigl(Z(\widehat{M}_{\mathrm{sc}})^{\Gamma}\bigr)$. Therefore, ζ_G is trivial on the intersection of $\widehat{Z}_{\mathrm{sc}}^{\Gamma}$ with $\bigl(Z(\widehat{M}_{\mathrm{sc}})^{\Gamma}\bigr)^0$. As we have seen, this is equivalent to the requirement that Δ is contained in Δ_0. The lemma is proved. □

Corollary 2.2. *Suppose that R is a Levi subgroup of G^*, with a dual Levi subgroup $\widehat{R} \subset \widehat{G}$. Then R corresponds to a Levi subgroup M of G (with dual Levi subgroup $\widehat{M} = \widehat{R}$) if and only if ζ_G is trivial on the subgroup*

$$\widehat{Z}_{\mathrm{sc}}^{\Gamma} \cap \bigl(Z(\widehat{R}_{\mathrm{sc}})^{\Gamma}\bigr)^0 \;=\; \widehat{Z}_{\mathrm{sc}}^{\Gamma} \cap \bigl(Z(\widehat{M}_{\mathrm{sc}})^{\Gamma}\bigr)^0$$

of $\widehat{Z}_{\mathrm{sc}}^{\Gamma}$.

Proof. During the proof of the lemma we established that ζ_G is trivial on $\widehat{Z}_{\mathrm{sc}}^{\Gamma} \cap \bigl(Z(\widehat{R}_{\mathrm{sc}})^{\Gamma}\bigr)^0$ if and only if the complement in Δ_G of the Γ-invariant subset Δ_R is contained in Δ_0. The corollary follows. □

If M is a Levi subgroup of G, with dual Levi subgroup $\widehat{M} \subset \widehat{G}$, we have seen that ζ_G is the pullback of a character ζ_G^M on $\pi_0\bigl(Z(\widehat{M}_{\mathrm{sc}})^{\Gamma}\bigr)$. We shall be particularly concerned with the case that M is *minimal*. This means that for any $P \in \mathcal{P}(M)$, Δ_P equals the set Δ_0 of Lemma 2.1. In this case we shall write $M_0 = M$, and we set

$$\zeta_G^0 \;=\; \zeta_G^{M_0} \;=\; \zeta_G^M\,.$$

The character ζ_G^0 on $\pi_0\bigl(Z(\widehat{M}_{0,\mathrm{sc}})^{\Gamma}\bigr)$ will play an important role in the vanishing theorems of §8.

For many purposes, G can be treated as if it were a connected group over the local field F. In particular, the notation and terminology of [7, §1–§2], which we shall sometimes adopt without comment, extends to the K-group G. Thus

$$\Gamma(G) \;=\; \coprod_{\alpha\in\pi_0(G)} \Gamma(G_\alpha) \;=\; \coprod_{\alpha} \Gamma_{\mathrm{reg}}\bigl(G_\alpha(F)\bigr)$$

is the disjoint union of the sets of strongly regular conjugacy classes in the groups $G_\alpha(F)$, and

$$\Gamma_{\mathrm{ell}}(G) \;=\; \coprod_{\alpha\in\pi_0(G)} \Gamma_{\mathrm{ell}}(G_\alpha) \;=\; \coprod_{\alpha} \Gamma_{\mathrm{reg,ell}}\big(G_\alpha(F)\big)$$

is the corresponding set of elliptic classes. If γ lies in $\Gamma(G_\alpha)$, we write G_γ for the centralizer $G_{\alpha,\gamma}$ in G_α of (some representative of) γ. Stable conjugacy classes in $G(F)$ also make sense. We define classes γ_1 and γ_2 in $\Gamma(G)$, with $\gamma_i \in \Gamma(G_{\alpha_i})$ for $i = 1, 2$, to be *stably conjugate* if $\psi_{\alpha_1\alpha_2}(\gamma_2)$ is conjugate in $G_{\alpha_1}(\overline{F})$ to γ_1, for any frame (ψ, u). We can then write $\Sigma(G) = \Sigma_{\mathrm{reg}}\big(G(F)\big)$ for the set of strongly regular stable conjugacy classes in $G(F)$. There is a canonical injection $\delta \to \delta^*$ from $\Sigma(G)$ to the set $\Sigma(G^*) = \Sigma_{\mathrm{reg}}\big(G^*(F)\big)$ of strongly regular stable classes in our quasisplit inner twist $G^*(F)$.

An endoscopic datum for G is defined entirely in terms of the dual group $\widehat{G}$, and is therefore no different from the case of connected G. As in [7], $\mathcal{E}(G)$ will stand for the set of isomorphism classes of endoscopic data for G that are *relevant* to G. An element in $\mathcal{E}(G)$ is therefore the image of some elliptic endoscopic datum $M' \in \mathcal{E}_{\mathrm{ell}}(M)$, for a Levi subgroup M of G and a dual Levi subgroup $\widehat{M}$ of $\widehat{G}$. The set $\mathcal{E}(G)$ embeds into the larger set $\mathcal{E}(G^*)$, which we identify with the collection of all isomorphism classes of endoscopic data for G. For each $G' \in \mathcal{E}(G^*)$, we fix a central extension

$$1 \longrightarrow \widetilde{Z}' \longrightarrow \widetilde{G}' \longrightarrow G' \longrightarrow 1$$

of G' by a central induced torus $\widetilde{Z}'$ and an L-embedding $\widetilde{\xi}'\colon \mathcal{G}' \to {}^L\widetilde{G}'$, as in [7, §2]. Finally, we define the sets

$$\widetilde{\Gamma}^{\mathcal{E}}_{\mathrm{ell}}(G) \;=\; \coprod_{G'} \big(\Sigma_{G,\mathrm{ell}}(\widetilde{G}')/\mathrm{Out}_G(G')\big)$$

and

$$\widetilde{\Gamma}^{\mathcal{E}}(G) \;=\; \coprod_{\{M\}} \big(\widetilde{\Gamma}^{\mathcal{E}}_{G,\mathrm{ell}}(M)/W(M)\big)\,,$$

as well as their corresponding quotients $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G)$ and $\Gamma^{\mathcal{E}}(G)$, by copying the construction of [7, §2]. The index G' here ranges over the elliptic endoscopic data $\mathcal{E}_{\mathrm{ell}}(G)$, while $\{M\}$

ranges over the orbits of $W(M_0)$ in the set $\mathcal{L}^G(M_0)$, for a fixed minimal Levi subgroup M_0 of G. By definition, $\Sigma_{G,\mathrm{ell}}(\widetilde{G}')$ is the subset of elements $\delta' \in \Sigma_G(\widetilde{G}')$ that are elliptic, in the sense that the centralizer $\widetilde{G}'_{\delta'}$ of (a representative of) δ' is anisotropic over F, modulo the center of $\widetilde{G}'$. In particular, $\widetilde{\Gamma}^{\mathcal{E}}_{\mathrm{ell}}(G)$ could be empty if $F = \mathbb{R}$.

A K-group is a natural domain for the transfer factors of [15]. This observation is due to Kottwitz. We shall consider first the case that $\widetilde{G}' = G'$, for a given $G' \in \mathcal{E}(G^*)$. Suppose that $\gamma, \overline{\gamma} \in \Gamma(G)$ and $\delta', \overline{\delta}' \in \Sigma_G(G')$. Then $\gamma \in \Gamma(G_\alpha)$ and $\overline{\gamma} \in \Gamma(G_{\overline{\alpha}})$, for indices $\alpha, \overline{\alpha} \in \pi_0(G)$. The relative transfer factor

$$\Delta(\delta', \gamma; \overline{\delta}', \overline{\gamma}) \;=\; \Delta_G(\delta', \gamma; \overline{\delta}', \overline{\gamma})$$

is defined to be 0 unless δ' and $\overline{\delta}'$ are images of γ and $\overline{\gamma}$ (in the sense of [15, (1.3)]), in which case $\Delta(\delta', \gamma; \overline{\delta}', \overline{\gamma})$ is defined as a product of four terms

$$\frac{\Delta_I(\delta', \gamma)}{\Delta_I(\overline{\delta}', \overline{\gamma})} \cdot \frac{\Delta_{II}(\delta', \gamma)}{\Delta_{II}(\overline{\delta}', \overline{\gamma})} \cdot \frac{\Delta_2(\delta', \gamma)}{\Delta_2(\overline{\delta}', \overline{\gamma})} \cdot \Delta_1(\delta', \gamma; \overline{\delta}', \overline{\gamma}) \,, \tag{2.3}$$

as in [15, (3.7)]. The first three terms are defined for the pairs (G_α, G') and $(G_{\overline{\alpha}}, G')$, as in the sections (3.2), (3.3) and (3.5) of [15]. For the only truly relative term $\Delta_1(\delta', \gamma; \overline{\delta}', \overline{\gamma})$, we have to copy the construction in [15, (3.4)].

Letting γ, $\overline{\gamma}$, δ' and $\overline{\delta}'$ stand for representatives within the given conjugacy and stable conjugacy classes, we define T' and $\overline{T}'$ to be the centralizers of δ' and $\overline{\delta}'$ in G'. We have already fixed a quasisplit inner twist G^* of G. Choose admissible embeddings $T' \to T$ and $\overline{T}' \to \overline{T}$ of T' and $\overline{T}'$ into maximal tori of G^* [15, (1.3)], and let $\gamma^* \in T(F)$ and $\overline{\gamma}^* \in \overline{T}(F)$ be the corresponding images of δ' and $\overline{\delta}'$. If (ψ, u) is a fixed frame for $G \coprod G^*$, and τ belongs to Γ, we set

$$v(\tau) \;=\; h u_\alpha(\tau) \tau(h)^{-1} \qquad \text{and} \qquad \overline{v}(\tau) \;=\; \overline{h} u_{\overline{\alpha}}(\tau) \tau(\overline{h})^{-1} \,,$$

for elements h and $\overline{h}$ in G^*_{sc} such that

$$h \psi_\alpha(\gamma) h^{-1} \;=\; \gamma^* \qquad \text{and} \qquad \overline{h} \psi_{\overline{\alpha}}(\overline{\gamma}) \overline{h}^{-1} \;=\; \overline{\gamma}^* \,.$$

Now

$$u_{\overline{\alpha}}(\tau) \;=\; \psi_{\alpha,\mathrm{sc}}\big(u_{\alpha\overline{\alpha}}(\tau)\big)u_\alpha(\tau)\,,$$

with $u_{\alpha\overline{\alpha}}(\tau)$ being a 1-cocycle from T to $G_{\alpha,\mathrm{sc}}$. It follows that

$$\partial v \;=\; \partial u_\alpha \;=\; \partial u_{\overline{\alpha}} \;=\; \partial\overline{v}\,,$$

each coboundary taking values in the center Z^*_{sc} of G^*_{sc}. Then

$$\tau \longrightarrow \big(v(\tau)^{-1}, \overline{v}(\tau)\big)$$

is a 1-cocycle with values in the torus

$$U \;=\; T_{\mathrm{sc}} \times \overline{T}_{\mathrm{sc}}/\big\{(z^{-1}, z):\; z \in Z^*_{\mathrm{sc}}\big\}\,,$$

and defines a class

$$\mathrm{inv}\Big(\frac{\delta', \gamma}{\overline{\delta}', \overline{\gamma}}\Big)$$

in $H^1(T, U)$. On the other hand, G' really stands for an endoscopic datum, composed of four objects $(G', s', \mathcal{G}', \xi')$. As in [15, p. 246], s' determines an element s'_U in $\pi_0(\widehat{U}^\Gamma)$, for the dual torus

$$\widehat{U} \;=\; \widehat{T}_{\mathrm{sc}} \times \widehat{\overline{T}}_{\mathrm{sc}}/\big\{(z, z):\; z \in \widehat{Z}_{\mathrm{sc}}\big\}\,.$$

The fourth term in the product (2.3) is defined as the Tate-Nakayama pairing

$$\Delta_1(\delta', \gamma; \overline{\delta}', \overline{\gamma}) \;=\; \Big\langle \mathrm{inv}\Big(\frac{\delta', \gamma}{\overline{\delta}', \overline{\gamma}}\Big),\, s'_U \Big\rangle\,. \tag{2.4}$$

Having defined the transfer factors by the product (2.3) if $\widetilde{G}' = G'$, we treat the general case as in [15]. Let

$$1 \longrightarrow \widetilde{Z} \longrightarrow \widetilde{G} \xrightarrow{\;r\;} G \longrightarrow 1$$

be a z-extension of G by an induced torus $\widetilde{Z}$. By this we mean a multiple group $\widetilde{G}$ with $\pi_0(\widetilde{G}) = \pi_0(G)$, such that $\widetilde{G}_\alpha$ is a z-extension [12, §1] of G_α by $\widetilde{Z}$ for each $\alpha \in \pi_0(G)$, and such that for any frame (ψ, u) for G, there is a corresponding frame $(\widetilde{\psi}, \widetilde{u})$ for $\widetilde{G}$ with $r_\alpha \widetilde{\psi}_{\alpha\beta} = \psi_{\alpha\beta} r_\beta$ and $\widetilde{u}_{\alpha\beta} = u_{\alpha\beta}$, for each α, β. (The groups $\widetilde{G}_{\alpha,\mathrm{sc}}$ and $G_{\alpha,\mathrm{sc}}$ are equal, so the last condition makes sense.) It follows easily from the triviality of $H^1(F, \widetilde{Z})$ that $\widetilde{G}$ is also a K-group. Now any element $G' \in \mathcal{E}(G)$ determines an element $\widetilde{G}' \in \mathcal{E}(\widetilde{G})$ for which $\widetilde{\mathcal{G}}'$ is L-isomorphic to ${}^L\widetilde{G}'$. Suppose that (δ', γ) and $(\overline{\delta}', \overline{\gamma})$ are two pairs in $\Sigma_G(\widetilde{G}') \times \Gamma(G)$. The relative transfer factor $\Delta(\delta', \gamma; \overline{\delta}', \overline{\gamma})$ is then given by the definition in [15, (4.4)]. That is, $\Delta(\delta', \gamma; \overline{\delta}', \overline{\gamma})$ equals 0 unless δ' and $\overline{\delta}'$ are images of elements $\widetilde{\gamma}$ and $\widetilde{\overline{\gamma}}$ in $\Gamma(\widetilde{G})$ that map to γ and $\overline{\gamma}$ respectively, in which case

$$\Delta(\delta', \gamma; \overline{\delta}', \overline{\gamma}) \ = \ \Delta(\delta', \widetilde{\gamma}; \overline{\delta}', \widetilde{\overline{\gamma}}) \, .$$

Given the relative transfer factors, we define absolute transfer factors as in [15, (3.7)] by treating $(\overline{\delta}', \overline{\gamma})$ as a base point. We fix $(\overline{\delta}', \overline{\gamma})$ such that $\overline{\delta}'$ is an image of $\overline{\gamma}$, and we arbitrarily assign $\Delta(\overline{\delta}', \overline{\gamma})$ any fixed complex value, that we can assume has absolute value 1. We then define the absolute transfer factor by setting

$$\Delta(\delta', \gamma) \ = \ \Delta(\delta', \gamma; \overline{\delta}', \overline{\gamma}) \Delta(\overline{\delta}', \overline{\gamma}) \, , \qquad \delta' \in \Sigma_G(\widetilde{G}'), \ \gamma \in \Gamma(G).$$

It depends only on the image of δ' in the set $\Gamma^{\mathcal{E}}(G)$. We also define the adjoint transfer factor

$$\Delta(\gamma, \delta') \ = \ |\mathcal{K}_\gamma|^{-1} \overline{\Delta(\delta', \gamma)} \, , \qquad (\delta', \gamma) \in \widetilde{\Gamma}^{\mathcal{E}}(G) \times \Gamma(G), \tag{2.5}$$

where

$$\mathcal{K}_\gamma \ = \ \mathcal{K}(G_\gamma) \ = \ \pi_0\big(\widehat{G}_\gamma^\Gamma / Z(\widehat{G})^\Gamma\big) \, ,$$

as in [7, §1].

The next lemma contains adjoint relations that are one of the main reasons for introducing K-groups. It is a generalization of a result (Lemma 2.2) from [7] that applied only to the case of p-adic F.

Lemma 2.3. *The transfer factors satisfy*

$$\sum_{\delta' \in \Gamma^{\mathcal{E}}(G)} \Delta(\gamma, \delta')\Delta(\delta', \gamma_1) \;=\; \delta(\gamma, \gamma_1)\,, \qquad \gamma, \gamma_1 \in \Gamma(G), \tag{2.6}$$

and

$$\sum_{\gamma \in \Gamma(G)} \Delta(\delta', \gamma)\Delta(\gamma, \delta'_1) \;=\; \widetilde{\delta}(\delta', \delta'_1)\,, \qquad \delta', \delta'_1 \in \widetilde{\Gamma}^{\mathcal{E}}(G), \tag{2.7}$$

for Kronecker delta functions $\delta(\cdot, \cdot)$ and $\widetilde{\delta}(\cdot, \cdot)$ defined as in [7, p. 516].

Proof. By the discussion at the end of [7, §2], it is enough to consider the case that the fixed points (γ, γ_1) in (2.6) and (δ_1, δ'_1) in (2.7) are elliptic. The sums on the left hand sides of (2.6) and (2.7) may then be taken over the sets $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G)$ and $\Gamma_{\mathrm{ell}}(G)$. We can also assume that $\widetilde{G}' = G'$ for each $G' \in \mathcal{E}(G)$. The relative transfer factors are then defined as in (2.3) as a product of four terms.

The proof is essentially that of [7, Lemma 2.2]. Since $\Delta(\overline{\delta}', \overline{\gamma})$ has absolute value 1, the summand in (2.6) is independent of the base point $(\overline{\delta}', \overline{\gamma})$. It reduces to

$$\Delta(\gamma, \delta')\Delta(\delta', \gamma_1) \;=\; |\mathcal{K}_\gamma|^{-1}\Delta_1(\delta', \gamma_1; \delta', \gamma)\,,$$

as in [7, p. 517]. Because δ' occurs in both of the first and third arguments of the factor Δ_1, there is just one admissible embedding $T' \to T$ to account for. The factor simplifies to

$$\Delta_1(\delta', \gamma_1; \delta', \gamma) \;=\; \Big\langle \mathrm{inv}\Big(\frac{\delta', \gamma_1}{\delta', \gamma}\Big),\, s'_U \Big\rangle \;=\; \big\langle \mu_T(\gamma, \gamma_1),\, s_T \big\rangle\,,$$

where $\mu_T(\gamma, \gamma_1)$ is the class of the cocycle

$$\tau \longrightarrow v_1(\tau)^{-1} v(\tau)\,, \qquad \tau \in \mathrm{Gal}(\overline{F}/F),$$

in $H^1(F, T_{\mathrm{sc}})$, and $s_T = s_T(\delta')$ is the element in

$$\mathcal{K}(T) \;=\; \pi_0\big(\widehat{T}^\Gamma / Z(\widehat{G})^\Gamma\big)$$

defined in [15, p. 241]. (The functions $v(\tau)$ and $v_1(\tau)$ are constructed from the pairs (δ', γ) and (δ', γ_1) as above.) The pairing depends only on the image $\overline{\mu}_T(\gamma, \gamma_1)$ of $\mu_T(\gamma, \gamma_1)$ in $H^1(F, T)$.

Suppose that γ_1 is fixed and that δ' is a fixed image of γ_1. Then $\gamma \to \overline{\mu}_T(\gamma, \gamma_1)$ is a bijection from the set of conjugacy classes in the stable conjugacy class of γ_1, which is the same as the set of γ of which δ' is an image, onto the image $\mathcal{E}(T)$ of $H^1(F, T_{\mathrm{sc}})$ in $H^1(F, T)$. (This is the assertion that relies on G being a K-group; if G were a connected group, the image would be only a subset $\mathcal{D}(T)$ of $\mathcal{E}(T)$.) By Tate-Nakayama duality, $\mathcal{E}(T)$ is isomorphic to the group $\mathcal{K}(T)^*$ of characters on $\mathcal{K}(T)$. On the other hand, suppose that γ^* is a fixed G^*-regular point in $T(F)$. If $\delta' \in \Sigma_G(G')$ is an image of γ^*, and T' is the centralizer of (a representative of) δ' in G', let $T' \to T$ be the admissible embedding that maps δ' to γ^*. It is this embedding that determines the point $s_T(\delta')$ in $\mathcal{K}(T)$. A variant of [14, Lemma 9.7] asserts that $\delta' \to s_T(\delta')$ is a bijection from the set of images $\delta' \in \Gamma^{\mathcal{E}}(G)$ of γ^* onto $\mathcal{K}(T)$. The summations in (2.6) and (2.7) can therefore be taken over finite groups that are duality with each other. Keeping in mind that $\mathcal{K}_\gamma \cong \mathcal{K}(T)$, we deduce the relations (2.6) and (2.7) as in the latter part of the proof of [7, Lemma 2.2]. $\square$

3. The conjectural transfer identity

In the paper [8], we stated a conjectural identity for the behaviour of weighted orbital integrals under transfer. The identity relates two new families of distributions, $S^G_M(\delta)$ and $I^{\mathcal{E}}_M(\gamma)$, that may be regarded as stable and endoscopic analogues of weighted orbital integrals, or rather the invariant distributions attached to weighted orbital integrals. The aim of this paper is to study these new distributions. In this section, we shall generalize the construction to K-groups. We shall also isolate the inductive definitions from the conjecture, in order to be able to work with the distributions without having proved the conjecture.

In this section, G will continue to be a K-group over the local field F. As in §2, an induced torus over F is understood to be a product of tori of the form $\mathrm{Res}_{E/F}(\mathbb{G}_m)$, for finite extensions E of F. A central induced torus in G will mean an induced torus Z over F, together with embeddings

$$Z \xrightarrow{\sim} Z_\alpha \ \subset \ Z(G_\alpha)\,, \qquad\qquad \alpha \in \pi_0(G),$$

over F that are compatible with the isomorphisms $\psi_{\alpha\beta}$: $G_\beta \to G_\alpha$. For purposes of induction, it is convenient to fix such a Z, and also a character ζ on $Z(F)$. For each α, ζ determines a character ζ_α on the central induced torus $Z_\alpha(F)$ in $G_\alpha(F)$. Having fixed Z and ζ, we can define spaces of ζ^{-1}-equivariant functions on $G(F)$. We will be dealing exclusively with tempered distributions in this paper, so we may as well work with the full Schwartz space. We set

$$\mathcal{C}(G,\zeta) \ = \bigoplus_{\alpha\in\pi_0(G)} \mathcal{C}(G_\alpha,\zeta_\alpha)\,, \tag{3.1}$$

where $\mathcal{C}(G_\alpha,\zeta_\alpha)$ is the Schwartz space of ζ_α^{-1}-equivariant functions on $G_\alpha(F)$. Any element in $\mathcal{C}(G,\zeta)$ can then be regarded as a function on $G(F)$.

The construction begins with weighted orbital integrals and their corresponding invariant distributions. The extension of these objects to functions $f = \bigoplus\limits_\alpha f_\alpha$ in the larger space (3.1) is purely a matter of notation. We fix a Levi subgroup M of G over F. If γ is an element in the set $\Gamma_G(M) = \Gamma_{G\text{-reg}}\big(M(F)\big)$, let $\alpha \in \pi_0(M)$ be the index such that γ belongs to $\Gamma_G(M_\alpha)$. We define the weighted orbital integral of f at γ simply by

$$J_M(\gamma, f) \ = \ J_M(\gamma, f_\alpha)\,,$$

where $J_M(\gamma, f_\alpha)$ is the weighted orbital integral on $G_\alpha(F)$ described in, for example, [8, §3]. Similarly, we set

$$I_M(\gamma, f) \ = \ I_M(\gamma, f_\alpha)\,,$$

where $I_M(\gamma, f_\alpha)$ is the invariant distribution on $G_\alpha(F)$ defined also in [8, §3]. Recall that $I_M(\gamma, f_\alpha)$ is obtained from $J_M(\gamma, f_\alpha)$ by adding some correction terms built out of weighted characters. We assign these weighted characters the canonical normalization defined in [8, §2].

The Langlands-Shelstad transfer mappings extend to K-groups in an equally simple fashion. We fix a quasisplit inner twist G^* of G, and a Levi subgroup M^* of G^* corresponding to M. Then Z and ζ determine corresponding objects Z^* and ζ^* for G^*. Suppose that G' is an element in $\mathcal{E}(G)$, with central extension $\widetilde{G}'$ as in §2. The transfer map

$$f \longrightarrow f'(\delta') \ = \sum_{\gamma \in \Gamma(G)} \Delta_G(\delta', \gamma) f_G(\gamma)\,, \qquad \delta' \in \Sigma_G(\widetilde{G}'),$$

goes from functions $f \in \mathcal{C}(G, \zeta)$ to functions $f' = f^{G'}$ on $\Sigma_G(\widetilde{G}')$. As in [7], $f_G(\gamma)$ denotes the invariant orbital integral $J_G(\gamma, f) = I_G(\gamma, f)$. If f equals $\bigoplus\limits_\alpha f_\alpha$, f' obviously equals $\sum\limits_\alpha f'_\alpha$. The point of having transfer factors for K-groups is that f' depends only on the one base point $(\overline{\delta}', \overline{\gamma})$, rather than a base point for each G_α. Now the extension $\widetilde{G}'$ comes with a central induced torus $\widetilde{Z}'Z$ in $\widetilde{G}'$ and a character $\widetilde{\zeta}'\zeta$ on $(\widetilde{Z}'Z)(F)$ [7, p. 529]. The Langlands-Shelstad transfer conjecture, applied to each of the groups G_α, asserts that f' belongs to the space $S\mathcal{I}(\widetilde{G}', \widetilde{\zeta}'\zeta)$ of stable orbital integrals of functions in $\mathcal{C}(\widetilde{G}', \widetilde{\zeta}'\zeta)$.

We shall say that the K-group G is *quasisplit* if it has a connected component that is quasisplit (over F). It is easy to see that this property holds for G if and only if it also holds for the Levi subgroup M, so we shall sometimes speak of the triplet (F, G, M) being quasi-split. We note that G is quasisplit if and only if the associated character ζ_G on $\widehat{Z}_{\mathrm{sc}}^{\Gamma}$ is trivial. If G is quasisplit, the canonical injection of $\Sigma(G)$ into $\Sigma(G^*)$ is a bijection, and we can identify the two sets. In particular, we can identify the stable orbital integral

$$f^G(\delta) \;=\; \sum_{\gamma\in\delta} f_G(\gamma)\,, \qquad\qquad \delta\in\Sigma(G),$$

with the stable transfer map $f \to f^* = f^{G^*}$ from $\mathcal{C}(G,\zeta)$ to $S\mathcal{I}(G^*,\zeta^*)$. This map is surjective. In general, we shall say that a ζ-equivariant distribution S on $G(F)$ is *stable* if its value at any f depends only on the image f^*. In the case that G is quasisplit, there is a unique linear form $\widehat{S}$ on $S\mathcal{I}(G^*,\zeta^*)$ attached to any stable distribution S such that

$$\widehat{S}(f^*) \;=\; S(f)\,, \qquad\qquad f\in\mathcal{C}(G,\zeta).$$

Next, we recall the set $\mathcal{E}_{M'}(G)$, introduced for connected groups in [8]. The symbol M' represents an elliptic endoscopic datum $(M', \mathcal{M}', s'_M, \xi'_M)$ for M, with $\mathcal{M}'$ being an L-subgroup of ${}^LM$ and ξ'_M the identity embedding of $\mathcal{M}'$ into ${}^LM$. We fix a Levi subgroup $\widehat{M}$ of $\widehat{G}$ that is dual to M. In this paper, we shall define $\mathcal{E}_{M'}(G)$ to be the family of endoscopic data $G' = (G', \mathcal{G}', s', \xi')$ for G, taken modulo translation of s' by $Z(\widehat{G})^{\Gamma}$ (rather than the full equivalence relation defined by isomorphisms of endoscopic data), in which s' lies in $s'_M Z(\widehat{M})^{\Gamma}$, $\widehat{G}'$ is the connected centralizer of s' in $\widehat{G}$, $\mathcal{G}'$ equals $\mathcal{M}'\widehat{G}'$, and ξ' is the identity embedding of $\mathcal{G}'$ into ${}^LG$. For any $G'\in\mathcal{E}_{M'}(G)$, the dual group $\widehat{M}'$ of M' comes with the structure of a Levi subgroup of $\widehat{G}'$. The group M' has an embedding $M'\subset G'$ for which $\widehat{M}'\subset\widehat{G}'$ is a dual Levi subgroup, but this is determined only up to $G'(F)$-conjugacy. We fix such an embedding for each G', thereby identifying M' with a Levi subgroup of G'. Any objects we construct will later be seen to depend only on the $G'(F)$-orbit of M'. Let $\widetilde{M}'$ be a fixed central extension of M' by an induced torus $\widetilde{Z}'$, with the properties of

[7, Lemma 2.1]. Then for any $G' \in \mathcal{E}_{M'}(G)$, we have a central extension $\widetilde{G}'$ of G' by $\widetilde{Z}'$ with the same properties, that contains $\widetilde{M}'$ as a Levi subgroup. Finally, we have a simple coefficient

$$\iota_{M'}(G, G') = \left|Z(\widehat{M}')^{\Gamma}/Z(\widehat{M})^{\Gamma}\right| \left|Z(\widehat{G}')^{\Gamma}/Z(\widehat{G})^{\Gamma}\right|^{-1} \tag{3.2}$$

attached to any $G' \in \mathcal{E}_{M'}(G)$. We have not required the elements G' in $\mathcal{E}_{M'}(G)$ to be elliptic, as we did in [8]. However, if G' is not elliptic, $\iota_{M'}(G, G')$ vanishes.

It is sometimes necessary to treat the case that G is quasisplit separately. To this end, we write

$$\epsilon(G) = \begin{cases} 1, & \text{if } G \text{ is quasisplit,} \\ 0, & \text{otherwise.} \end{cases}$$

We also define a subset of $\mathcal{E}_{M'}(G)$ by

$$\mathcal{E}^0_{M'}(G) = \begin{cases} \mathcal{E}_{M'}(G) - \{G^*\}, & \text{if } G \text{ is quasisplit,} \\ \mathcal{E}_{M'}(G), & \text{otherwise.} \end{cases}$$

Observe that $\mathcal{E}_{M'}(G)$ contains G^* if and only if s'_M lies in $Z(\widehat{M})^{\Gamma}$, or equivalently, if and only if the endoscopic datum M' for M is isomorphic to M^*. In particular, $\mathcal{E}^0_{M'}(G)$ could equal $\mathcal{E}_{M'}(G)$ even if G is quasisplit.

We shall now construct new distributions $S^G_M(M', \delta', f)$ and $I^{\mathcal{E}}_M(\gamma, f)$ from the invariant distributions $I_M(\gamma, f)$ described above. As in [8, §4], the basic definitions of this section will apply only to classes δ' and γ which are elliptic for M. We first set

$$I_M(\delta', f) = \sum_{\gamma \in \Gamma_{G,\mathrm{ell}}(M)} \Delta_M(\delta', \gamma) I_M(\gamma, f)\,, \tag{3.3}$$

for any $\delta' \in \widetilde{\Gamma}^{\mathcal{E}}_{G,\mathrm{ell}}(M)$. Since δ' is elliptic, we note that there is a *unique* $M' \in \mathcal{E}_{\mathrm{ell}}(M)$ such that δ' is the image of an element in $\Sigma_{G,\mathrm{ell}}(\widetilde{M}')$. The rest of the definition is inductive. We assume inductively that for any $M' \in \mathcal{E}_{\mathrm{ell}}(M)$, $\delta' \in \Sigma_{G,\mathrm{ell}}(\widetilde{M}')$ and $G' \in \mathcal{E}^0_{M'}(G)$, we have defined a linear form $\widehat{S}^{\widetilde{G}'}_{\widetilde{M}'}(\delta')$ on $S\mathcal{I}(\widetilde{G}', \widetilde{\delta}'\delta)$. We also assume that the Langlands-Shelstad transfer conjecture holds for each G'. Then $\widehat{S}^{\widetilde{G}'}_{\widetilde{M}'}(\delta', f')$ makes sense for any

$f \in \mathcal{C}(G,\zeta)$. With these assumptions, we construct our distributions as follows. In the case that $\epsilon(G) = 1$, so that G is quasisplit, we define

$$S_M^G(M',\delta',f) \;=\; I_M(\delta',f) - \sum_{G'\in\mathcal{E}^0_{M'}(G)} \iota_{M'}(G,G')\widehat{S}^{\widetilde{G}'}_{\widetilde{M}'}(\delta',f') \,. \tag{3.4}$$

In the general case, we define

$$I_M^{\mathcal{E}}(\delta',f) \;=\; \sum_{G'\in\mathcal{E}^0_{M'}(G)} \iota_{M'}(G,G')\widehat{S}^{\widetilde{G}'}_{\widetilde{M}'}(\delta',f') + \epsilon(G) S_M^G(M',\delta',f) \,. \tag{3.5}$$

We then set

$$I_M^{\mathcal{E}}(\gamma,f) \;=\; \sum_{\delta'\in\Gamma^{\mathcal{E}}_{G,\mathrm{ell}}(M)} \Delta_M(\gamma,\delta') I_M^{\mathcal{E}}(\delta',f) \,, \tag{3.6}$$

for any $\gamma \in \Gamma_{G,\mathrm{ell}}(M)$. To complete the inductive definition, we would have to show in the special case of G quasisplit and $M' = M^*$, that the distribution

$$S_M^G(\delta,f) \;=\; S_M^G(M^*,\delta^*,f) \,, \qquad \delta \in \Sigma_G(M),$$

is stable. Only then would we have a linear form $\widehat{S}^{G^*}_{M^*}(\delta^*)$ on $S\mathcal{I}(G^*,\zeta^*)$, with

$$\widehat{S}^{G^*}_{M^*}(\delta^*,f^*) \;=\; S_M^G(\delta,f) \,, \qquad f \in \mathcal{C}(G,\zeta),$$

that is the analogue of $\widehat{S}^{G'}_{M'}(\delta')$ for (G^*,M^*).

The definitions require some preliminary observations, that we will summarize as two lemmas. We need to know that $I_M^{\mathcal{E}}(\delta',f)$ depends only on the image of δ' in $\widetilde{\Gamma}^{\mathcal{E}}_{G,\mathrm{ell}}(M)$, in order for the right hand side of (3.6) to make sense. We shall actually show that the individual terms in the definition (3.5) each depend only on the image of δ' in $\widetilde{\Gamma}^{\mathcal{E}}_{G,\mathrm{ell}}(M)$. This includes an assertion that the terms are independent of the representative M' within the equivalence class in $\mathcal{E}_{\mathrm{ell}}(M)$, something that is not immediately clear from the construction. We shall also investigate how the distributions depend on the base points that are implicit in the choice of transfer factors.

The first question will be resolved as a special case of the general behaviour of the distributions under isomorphisms. Suppose that θ is an F-isomorphism from G onto another K-group G_1. For any function f on $G(F)$, we have a corresponding function $(\theta f)(x_1) = f(\theta^{-1}x_1)$ on $G_1(F)$. We also have a bijection $\gamma \to \theta\gamma$ from $\Gamma(G)$ onto $\Gamma(G_1)$. Let $\widehat{\theta}$: $\widehat{G} \to \widehat{G}_1$ be a Γ-isomorphism that is dual to θ, and let

$$
{}^L\theta \ = \ \widehat{\theta} \times \mathrm{Id}_{W_F} : \ {}^LG \ = \ \widehat{G} \rtimes W_F \longrightarrow \widehat{G}_1 \rtimes W_F \ = \ {}^LG\,,
$$

be the corresponding isomorphism of L-groups. Then ${}^L\theta$ maps any endoscopic datum $(G', \mathcal{G}', s', \xi')$ for G to an endoscopic datum $(G'_1, \mathcal{G}'_1, s'_1, \xi'_1)$ for G_1, the isomorphism class of which is independent of the choice of $\widehat{\theta}$. We also obtain an isomorphism
θ': $\widetilde{G}' \to \widetilde{G}'_1$ between quasisplit groups over F, whose orbit under right translation by the group $\mathrm{Aut}_G(G')$ is also independent of the choice of $\widehat{\theta}$. This gives us a bijection

$$
\theta' : \ \Sigma_{G,\mathrm{ell}}(\widetilde{G}')/\mathrm{Out}_G(G') \longrightarrow \Sigma_{G_1,\mathrm{ell}}(\widetilde{G}'_1)/\mathrm{Out}_{G_1}(G'_1)\,.
$$

Putting the endoscopic data together, we obtain a bijection $\theta^{\mathcal{E}} = \coprod_{G'} \theta'$ from $\widetilde{\Gamma}^{\mathcal{E}}_{G,\mathrm{ell}}(G)$ onto $\widetilde{\Gamma}^{\mathcal{E}}_{G_1,\mathrm{ell}}(G_1)$. Of course θ maps M to a Levi subgroup $M_1 = \theta M$ of G_1, so we also obtain bijections from $\Gamma_{G,\mathrm{ell}}(M)$ onto $\Gamma_{G,\mathrm{ell}}(M_1)$ and from $\widetilde{\Gamma}^{\mathcal{E}}_{G,\mathrm{ell}}(M)$ onto $\widetilde{\Gamma}^{\mathcal{E}}_{G,\mathrm{ell}}(M_1)$.

Lemma 3.1. *(i) For any θ, the distributions satisfy*

$$
S^{\theta G}_{\theta M}(\theta M', \theta\delta', \theta f) \ = \ S^G_M(M', \delta', f)
$$

and

$$
I^{\mathcal{E}}_{\theta M}(\theta\gamma, \theta f) \ = \ I^{\mathcal{E}}_M(\gamma, f)\,.
$$

(ii) The distributions $S^G_M(M',\delta',f)$ and $I^{\mathcal{E}}_M(\delta',f)$ depend only on the image of δ' in $\widetilde{\Gamma}^{\mathcal{E}}_{G,\mathrm{ell}}(M)$.

As for the second question, observe that transfer factors have two roles in the definitions. The transfer factor $\Delta_G(\cdot,\cdot)$ for G and G' is implicit in the function f' which occurs in (3.4) and (3.5). The transfer factor $\Delta_M(\cdot,\cdot)$ for M and M' occurs explicitly in (3.3) and (3.6). Since G' lies in $\mathcal{E}_{M'}(G)$, we can choose a common base point. For each M', we fix elements $\overline{\delta}' \in \Sigma_G(\widetilde{M}')$ and $\overline{\gamma} \in \Gamma_G(M)$ such that $\overline{\delta}'$ is an image of $\overline{\gamma}$ (relative to M). Then $(\overline{\delta}',\overline{\gamma})$ can serve as a base point for both $\Delta_G(\cdot,\cdot)$ and $\Delta_M(\cdot,\cdot)$, since M' is a Levi subgroup of G'. We take the preassigned values $\Delta_M(\overline{\delta}',\overline{\gamma})$ and $\Delta_G(\overline{\delta}',\overline{\gamma})$ (of absolute value 1) to be equal.

Lemma 3.2. *The distribution $I^{\mathcal{E}}_M(\gamma,f)$ is independent of the choice of base points. If G is quasisplit and δ belongs to $\Sigma_G(M)$, $S^G_M(\delta,f)$ is the also independent of the base point.*

Proof of Lemmas 3.1 and 3.2. We assume inductively that the lemmas hold if G is replaced by any group $\widetilde{G}'$, with $G' \in \mathcal{E}^0_{M'}(G)$. With this induction hypothesis, it is easy to establish the second part of Lemma 3.1. Any element $\theta' \in \mathrm{Out}_M(M')$ can be extended to an outer automorphism of $\widetilde{G}'$ that lies in $\mathrm{Out}_G(G')$. Applying Lemma 3.1(i) inductively to $\widetilde{G}'$, we obtain

$$\widehat{S}^{\widetilde{G}'}_{\widetilde{M}'}(\delta',f') \;=\; \widehat{S}^{\theta'\widetilde{G}'}_{\theta'\widetilde{M}'}(\theta'\delta',\theta'f') \;=\; \widehat{S}^{\widetilde{G}'}_{\widetilde{M}'}(\theta'\delta',f'),$$

since $\theta'G' = G'$, $\theta'M' = M'$ and $\theta'f' = f'$. It follows from the definition (3.3) and the basic properties of the transfer factor $\Delta_M(\delta',\gamma)$ that $I_M(\delta',f)$ depends only on the image of δ' in $\widetilde{\Gamma}^{\mathcal{E}}_{G,\mathrm{ell}}(M)$. From (3.4) and (3.5), we conclude that $S^G_M(M',\delta',f)$ and $I^{\mathcal{E}}_M(\delta',f)$ also depend only on the image of δ' in $\widetilde{\Gamma}^{\mathcal{E}}_{G,\mathrm{ell}}(M)$. We have established the assertion (ii) of Lemma 3.1, and in particular, that the definition (3.6) makes sense.

We take care of Lemma 3.2 next. Suppose that the base point $(\overline{\delta}',\overline{\gamma})$ is replaced by a second point $(\overline{\delta}'_1,\overline{\gamma}_1)$ in $\Sigma_G(\widetilde{M}') \times \Gamma_G(M)$, with $\overline{\delta}'_1$ being an image of $\overline{\gamma}_1$ (relative to M). The transfer factor for G and $G' \in \mathcal{E}^0_{M'}(G)$ has then to be multiplied by the factor

$\Delta_G(\overline{\delta}', \overline{\gamma}; \overline{\delta}'_1, \overline{\gamma}_1)$, in view of [15, Lemma 4.1.A]. The same goes for the image of the transfer map $f \to f'$. Replacing G by a z-extension if necessary, we can assume that $\widetilde{G}' = G'$. The relative transfer factors are then defined by a product (2.3). An inspection of the four terms in the product reveals that $\Delta_G(\overline{\delta}', \overline{\gamma}; \overline{\delta}'_1, \overline{\gamma}_1)$ equals $\Delta_M(\overline{\delta}', \overline{\gamma}; \overline{\delta}'_1, \overline{\gamma}_1)$, and that this number has absolute value 1. If G is quasisplit, we see from (3.4) (together with (3.3)) that the change of base points transforms $S^G_M(M', \delta', f)$ by the factor $\Delta_M(\overline{\delta}', \overline{\gamma}, \overline{\delta}'_1, \overline{\gamma}_1)$. But if $M' = M^*$, the absolute transfer factor for M and M^* is constant. The relative transfer factor then equals 1, and $S^G_M(M', \delta', f)$ does not change. To deal with $I^{\mathcal{E}}_M(\gamma, f)$, observe that the change of base point has the effect of multiplying the adjoint transfer factor $\Delta_M(\gamma, \delta')$ by the complex conjugate of $\Delta_M(\overline{\delta}', \overline{\gamma}; \overline{\delta}'_1, \overline{\gamma}'_1)$. This cancels the effect of the change on f'. The invariance of $I^{\mathcal{E}}_M(\gamma, f)$ under the change follows from (3.5) and (3.6). We have proved Lemma 3.2.

Consider finally the remaining part (i) of Lemma 3.1. It is implicit in the first assertion of (i) that the respective base points $(\overline{\delta}', \overline{\gamma})$ and $(\overline{\delta}'_1, \overline{\gamma}_1)$ for M' and $M'_1 = \theta M'$ satisfy $\overline{\delta}'_1 = \theta' \overline{\delta}'$ and $\overline{\gamma}_1 = \theta \overline{\gamma}$. Of course this is a restriction only in the case $M' \neq M^*$ not covered by Lemma 3.2. With the base points so related, it is easy to see that

$$\Delta_{\theta G}(\theta' \delta', \theta \gamma) \;=\; \Delta_G(\delta', \gamma)\,. \tag{3.7}$$

We leave the reader to check this point, which follows from an inspection of the various factors in the product (2.3). From (3.7), we see immediately then $(\theta f)' = \theta' f'$. The analogue of (3.7) for M is of course also valid. Combined with [8, Lemma 3.3] and the definition (3.3), it yields an identity

$$I_{\theta M}(\theta' \delta', \theta f) \;=\; I_M(\delta', f)\,.$$

Applying this identity in turn to the definition (3.4), we conclude from our induction hypothesis that

$$
\begin{aligned}
&S^{\theta G}_{\theta M}(\theta' M', \theta'\delta', \theta f)\\
&= I_{\theta M}(\theta'\delta', \theta f) - \sum_{G' \in \mathcal{E}^0_{M'}(G)} \iota_{\theta' M'}(\theta G, \theta' G') \widehat{S}^{\theta' \widetilde{G}'}_{\theta' \widetilde{M}'}\big(\theta'\delta', (\theta f)'\big)\\
&= I_M(\delta', f) - \sum_{G'} \iota_{M'}(G, G') \widehat{S}^{\widetilde{G}'}_{\widetilde{M}'}(\delta', f')\\
&= S^G_M(M', \delta', f)\,.
\end{aligned}
$$

This is the first assertion of Lemma 3.1(i). The proof of the second assertion is similar. □

We can now state our main conjecture. It includes the stability assertion required to complete the inductive definitions.

Conjecture 3.3. *(a) If G is arbitrary,*

$$
I^{\mathcal{E}}_M(\gamma, f) \;=\; I_M(\gamma, f)\,, \qquad\qquad \gamma \in \Gamma_{G,\mathrm{ell}}(M).
$$

(b) Suppose that G is quasisplit and that δ' belongs to $\Sigma_{G,\mathrm{ell}}(\widetilde{M}')$. Then the distribution

$$
f \longrightarrow S^G_M(M', \delta', f)
$$

vanishes unless $M' = M^*$, in which case it is stable.

Remarks. 1. If G is quasisplit, the assertion (a) is a consequence of the definitions. Indeed, (3.4) and (3.5) imply that $I^{\mathcal{E}}_M(\delta', f) = I_M(\delta', f)$, and if we combine this with (3.3), (3.6) and the adjoint relation (2.7), we see that $I^{\mathcal{E}}_M(\gamma, f) = I_M(\gamma, f)$. This identity could in fact be used in place of (3.4) in the original definition. On the other hand, if G is not quasisplit, the assertion seems to be quite hard.

2. We gave a similar conjecture in [8] that included a special case of the definitions (3.3)–(3.6) in its statement. The conjecture here is more general. It includes an implicit vanishing assertion in (a) (that applies to nonconnected K-groups) and an explicit vanishing assertion in (b), neither of which was a part of the conjecture in [8].

3. Suppose that G is an inner form of $GL(n)$. Then G is quasisplit if and only if G is F-isomorphic to $GL(n)$. Since stable conjugacy is the same as conjugacy in this case, part (b) of the conjecture is trivial. Part (a) is by no means trivial, but has been proved. It was established by global methods as one of the main results [9, Theorem A(i), p. 108] of Chapter 2 of [9].

4. A generalization of weighted orbital integrals

Weighted orbital integrals and their associated invariant distributions can be defined on a product of several copies of G. In this form, they exhibit important splitting properties [2, §11], [3, §9]. In order to formulate a splitting property for the distributions $I_M^{\mathcal{E}}(\gamma)$ and $S_M^G(M',\delta')$, however, we have to consider a generalization. In this section we shall define weighted orbital integrals and corresponding invariant distributions on products of groups which can be distinct. In the next section we will see how to generalize the distributions $I_M^{\mathcal{E}}(\gamma)$ and $S_M^G(M',\delta')$ to the same setting.

We have been working with a triplet (F,G,M). In this section, (F,G,M) will be demoted to the role of a label, that can satisfy more general conditions. We assume that the field is arbitrary (of characteristic 0), that G is any multiple group over F, and that M is a Levi subgroup of G. Consider another such triplet (F_1,G_1,M_1), together with a linear isometric embedding $\mathfrak{a}_M \subset \mathfrak{a}_{M_1}$. Let us say that (F_1,G_1,M_1) is a *satellite* of (F,G,M) if it satisfies the following two conditions.

(i) For any $Q \in \mathcal{F}(M)$, the cone $\mathfrak{a}_Q^+$ in $\mathfrak{a}_M$ is contained in a cone $\mathfrak{a}_{Q_1}^+$ in $\mathfrak{a}_{M_1}$, for some $Q_1 \in \mathcal{F}(M_1)$.

(ii) The only element $L_1 \in \mathcal{L}(M_1)$ with $\mathfrak{a}_{L_1} \supset \mathfrak{a}_M$ is M_1 itself.

The first condition provides a map $Q \to Q_1$ from $\mathcal{F}(M)$ to $\mathcal{F}(M_1)$. It also determines a map $L \to L_1$ from $\mathcal{L}(M)$ to $\mathcal{L}(M_1)$ with the property that if $Q \in \mathcal{P}(L)$, then $Q_1 \in \mathcal{P}(L_1)$. In other words, L_1 is the maximal element in $\mathcal{L}(M_1)$ such that $\mathfrak{a}_{L_1}$ contains $\mathfrak{a}_L$. The second condition asserts that M_1 is the image of M under this map.

The examples we have in mind come from endoscopic groups. If G' belongs to the set $\mathcal{E}_{M'}(G)$ defined in §3, then (F,G',M') is a satellite of (F,G,M). We are dealing with a transitive relation; if (F_2,G_2,M_2) is a satellite of (F_1,G_1,M_1) and (F_1,G_1,M_1) is a satellite of (F,G,M), then (F_2,G_2,M_2) is a satellite of (F,G,M). We can therefore construct satellites of (F,G,M) from chains of successive endoscopic data. There are of course other examples. If G_1 and M_1 are extensions of G and M to a field F_1 which contains

F, (F_1, G_1, M_1) is a satellite of (F, G, M). If (F_1, G_1, M_1) is a satellite of (F, G, M) and $L \in \mathcal{L}(M)$, then (F_1, G_1, L_1) is a satellite of (F, G, L), and (F_1, L_1, M_1) is a satellite of (F, L, M). Finally, the triplet $(F_1, G_1, M_1) = (F, G \times G, M \times M)$, with $\mathfrak{a}_M$ embedded diagonally in $\mathfrak{a}_M \oplus \mathfrak{a}_M$, is a satellite of (F, G, M).

If G is connected, the notion of a (G, M)-family of functions [2, §6] depends only on the space $i\mathfrak{a}_M^*$ and the chambers $\{\mathfrak{a}_P^+ : \ P \in \mathcal{P}(M)\}$. The notion therefore makes sense for our general triplet (F, G, M). Suppose that (F_1, G_1, M_1) is a satellite of (F, G, M), and that

$$c_{P_1}(\lambda_1)\,, \qquad\qquad P_1 \in \mathcal{P}(M_1),\ \lambda_1 \in i\mathfrak{a}_{M_1}^*,$$

is a (G_1, M_1)-family. Then if Q_1 is any element in $\mathcal{F}(M_1)$, the function

$$c_{Q_1}(\lambda_1) \ = \ c_{P_1}(\lambda_1)\,, \qquad\qquad P_1 \subset Q_1,\ \lambda_1 \in i\mathfrak{a}_{Q_1}^*,$$

on $i\mathfrak{a}_{Q_1}^*$ is independent of $P_1 \in \mathcal{P}(M_1)$. If Q belongs to $\mathcal{F}(M)$ and λ lies in $i\mathfrak{a}_M^*$, we define

$$c_Q(\lambda) \ = \ c_{Q_1}(\lambda)\,,$$

where Q_1 is the satellite image of Q. Then

$$c_P(\lambda)\,, \qquad\qquad P \in \mathcal{P}(M),\ \lambda \in i\mathfrak{a}_M^*, \tag{4.1}$$

is a (G, M)-family of functions. It gives rise to the smooth function

$$c_M(\lambda) \ = \sum_{P \in \mathcal{P}(M)} c_P(\lambda)\theta_P(\lambda)^{-1}$$

of $\lambda \in i\mathfrak{a}_M^*$ [2, Lemma 6.2]. This function in turn has an expansion in terms of corresponding functions attached to the original (G_1, M_1)-family.

To state the expansion at the appropriate level of generality, we take a Levi subgroup R_1 of M_1, and we assume that $\{c_P(\lambda_1)\}$ comes from a (G_1, R_1)-family

$$c_{S_1}(\nu_1)\,, \qquad\qquad S_1 \in \mathcal{P}(R_1),\ \nu_1 \in i\mathfrak{a}_{R_1}^*.$$

That is,

$$c_{P_1}(\lambda_1) \; = \; c_{S_1}(\lambda_1)\,, \qquad\qquad P_1 \in \mathcal{P}(M_1),\; S_1 \subset P_1,\; \lambda_1 \in i\mathfrak{a}^*_{M_1}.$$

Then $\mathfrak{a}_M$ is a subspace of $\mathfrak{a}_{R_1}$, whose orthogonal complement we denote by $\mathfrak{a}^M_{R_1}$. If L_1 belongs to $\mathcal{L}(R_1)$, we have a map

$$\mathfrak{a}^M_{R_1} \oplus \mathfrak{a}^{L_1}_{R_1} \longrightarrow \mathfrak{a}^G_{R_1}\,.$$

We can then define a coefficient $d^G_{R_1}(M, L_1)$ as in the special case of [3, §7]. That is, we set $d^G_{R_1}(M, L_1) = 0$ unless the map is an isomorphism, in which case we define $d^G_{R_1}(M, L_1)$ to be the volume in $\mathfrak{a}^G_{R_1}$ of the image of a unit cube in $\mathfrak{a}^M_{R_1} \oplus \mathfrak{a}^{L_1}_{R_1}$. Let ξ be a fixed point in general position in $\mathfrak{a}^M_{R_1}$. If $d^G_{R_1}(M, L_1) \neq 0$, the spaces $\xi + \mathfrak{a}^G_M$ and $\mathfrak{a}^G_{L_1}$ meet in exactly one point. This point lies in a chamber $\mathfrak{a}^+_{Q_1}$ of $\mathfrak{a}_{L_1}$, for a unique $Q_1 \in \mathcal{P}(L_1)$. Thus, ξ determines a section $L_1 \to Q_1$ from

$$\big\{L_1 \in \mathcal{L}(R_1) :\; d^G_{R_1}(M, L_1) \neq 0\big\}$$

to the fibres $\mathcal{P}(L_1)$. (See [3, §7].) In particular, for any $L_1 \in \mathcal{L}(R_1)$, we obtain an (L_1, R_1)-family

$$c^{Q_1}_{S_1 \cap L_1}(\nu_1) \; = \; c_{S_1}(\nu_1)\,, \qquad\qquad S_1 \in \mathcal{P}(R_1),\; S_1 \subset Q_1,\; \nu_1 \in i\mathfrak{a}^*_{R_1}.$$

Lemma 4.1. *We have*

$$c_M(\lambda) \; = \sum_{L_1 \in \mathcal{L}(R_1)} d^G_{R_1}(M, L_1) c^{Q_1}_{R_1}(\lambda)\,.$$

Proof. The assertion is identical to [3, Proposition 7.1], but we are working under slightly different conditions. The setting of [3, §7] applies only to the special case here that the chambers in $\mathfrak{a}_M$ are defined by the intersections of $\mathfrak{a}_M$ with chambers of $\mathfrak{a}_{M_1}$, or equivalently, that the map $P \to P_1$ from $\mathcal{P}(M)$ to $\mathcal{P}(M_1)$ is injective. However, the proof given in the appendix of [3] is not dependent on this constraint.

It is a direct consequence of our definitions that

$$\sum_{L_1\in\mathcal{L}(R_1)} d^G_{R_1}(M,L_1)c^{Q_1}_{R_1}(\lambda) \;=\; \sum_{S_1\in\mathcal{P}(R_1)} c_{S_1}(\lambda)r_{S_1,\xi}(\lambda)\,,$$

where

$$r_{S_1,\xi}(\lambda) \;=\; \sum_{\{L_1\in\mathcal{L}(R_1):\,Q_1\supset S_1\}} d^G_{R_1}(M,L_1)\theta_{S_1\cap L_1}(\lambda)^{-1}\,.$$

This is the analogue of [3, (A.5)]. Almost all of the discussion of the appendix of [3], including Lemma A.1, is aimed at evaluating $r_{S_1,\xi}(\lambda)$. This discussion applies essentially without change to the present situation. We shall just quote the final result. That is, $r_{S_1,\xi}(\lambda)$ vanishes unless S_1 is contained in the image P_1 of a group $P\in\mathcal{P}(M)$, in which case $r_{S_1,\xi}(\lambda)$ is the sum, over all $P\in\mathcal{P}(M)$ which map to P_1, of the functions $\theta_P(\lambda)^{-1}$. Since λ lies in $i\mathfrak{a}^*_M$,

$$c_{S_1}(\lambda) \;=\; c_{P_1}(\lambda) \;=\; c_P(\lambda)\,,$$

for any P that maps to P_1. The lemma follows. □

In order to construct products, we shall take several satellites. Suppose that

$$\big\{(F_v,G_v,M_v):\;v\in V\big\}$$

is a family of satellites of (F,G,M), indexed by a finite set V. Then we have a map

$$P\longrightarrow P_V \;=\; \prod_{v\in V} P_v$$

from $\mathcal{F}(M)$ to $\mathcal{F}(M_V)=\prod\limits_v\mathcal{F}(M_v)$, as well as a map

$$L\longrightarrow L_V \;=\; \prod_{v\in V} L_v$$

from $\mathcal{L}(M)$ to $\mathcal{L}(M_V)=\prod\limits_v\mathcal{L}(M_v)$. (We shall often write $M_V=\prod\limits_v M_v$ and $G_V=\prod\limits_v G_v$.) We assume that for each v, F_v is a local field of characteristic 0. It will also be convenient to assume that the range of the absolute value on $F_V=\prod\limits_v F_v$ is closed in $\mathbb{R}$. This means that either one of the fields F_v is archimedean, or all of the fields have the same residual characteristic [3, §1]. We are really interested in the case that G_v is a K-group over F_v. For this section, however, we may as well assume that each G_v is a general multiple group. (We take for granted the obvious analogues for G_v of the more elementary definitions of §2 and §3.) Suppose that for each $v \in V$, Z_v is a central induced torus over F_v in G_v, and that ζ_v is a character on $Z_v(F_v)$. Then $\zeta_V = \bigotimes_v \zeta_v$ is a character on $Z_V(F_V) = \prod_v Z_v(F_v)$. We shall construct some linear forms on the Schwartz space

$$\mathcal{C}(G_V, \zeta_V) \; = \; \bigotimes_{v \in V} \mathcal{C}(G_v, \zeta_v)$$

of ζ_V^{-1}-equivariant functions on $G_V(F_V) = \prod_v G_v(F_v)$.

If $x_V = \prod_v x_v$ is a point in $G_V(F_V)$, we can form the (G, M)-family of functions

$$v_P(\lambda, x_V) \; = \; \prod_v v_P(\lambda, x_v) \; = \; \prod_v e^{-\lambda(H_{P_v}(x_v))}\,, \qquad\qquad P \in \mathcal{P}(M),$$

of $\lambda \in i\mathfrak{a}_M^*$. It is a product of (G, M)-families of the form (4.1). As usual, we write

$$v_M(x_V) \; = \; \lim_{\lambda \to 0} \sum_{P \in \mathcal{P}(M)} v_P(\lambda, x_V)\theta_P(\lambda)^{-1}$$

for the value at $\lambda = 0$ of the function $v_M(\lambda, x_V)$. Suppose that $\gamma_V = \prod_v \gamma_v$ is a point in $\Gamma_{G_V}(M_V) = \prod_v \Gamma_{G_v}(M_v)$, and that f_V is a function in $\mathcal{C}(G_V, \zeta_V)$. If G_V is connected, we define the weighted orbital integral by the familiar formula

$$J_M(\gamma_V, f_V) \; = \; |D(\gamma_V)|_V^{\frac{1}{2}} \int_{G_{V,\gamma_V}(F_V)\backslash G_V(F_V)} f_V(x_V^{-1}\gamma_V x_V) v_M(x_V) dx_V\,, \tag{4.2}$$

where $G_{V,\gamma_V}(F_V) = \prod_v G_{v,\gamma_v}(F_v)$, and $|D(\gamma_V)|_V = \prod_v |D(\gamma_v)|_v$. We define the weighted orbital integral in general by

$$J_M(\gamma_V, f_V) \; = \; J_M(\gamma_V, f_{V,\alpha_V})\,,$$

where f_{V,α_V} is the component of f_V relative to the connected component α_V of G_V that contains γ_V. Observe that the weight factor $v_M(x_V)$ links the distinct groups $\{G_v\}$ in a nontrivial way. The new distributions are for this reason considerably more general than the weighted orbital integrals of §3. However, they inherit many of the same formal properties. In particular, we can make them invariant by combining them with weighted characters.

To define weighted characters in this context, assume first that G_V is connected. Suppose that $\pi_V = \bigotimes_v \pi_v$ belongs to the set $\Pi_{\mathrm{temp}}(M_V, \zeta_V)$ of (equivalence classes of) irreducible tempered representations of $M_V(F_V)$, with $Z_V(F_V)$-central character ζ_V. Fix $P \in \mathcal{P}(M)$. We can then form the (G, M)-family of (operator valued) functions

$$\mathcal{M}_Q(\lambda, \pi_V, P_V) \;=\; \bigotimes_{v \in V} \mathcal{M}_{Q_v}(\lambda, \pi_v, P_v)\,, \qquad Q \in \mathcal{P}(M),$$

of $\lambda \in i\mathfrak{a}_M^*$, with operators

$$\mathcal{M}_{Q_v}(\lambda, \pi_v, P_v) \;=\; \mu_{Q_v}(\lambda, \pi_v, P_v)\mathcal{J}_{Q_v}(\lambda, \pi_v, P_v)$$

defined as in [8, §2] in terms of Plancherel densities and unnormalized intertwining operators. Again we have a product of (G, M)-families of the form (4.1), from which we obtain the operator

$$\mathcal{M}_M(\pi_V, P_V) \;=\; \lim_{\lambda \to 0} \sum_{Q \in \mathcal{P}(M)} \mathcal{M}_Q(\lambda, \pi_V, P_V)\theta_Q(\lambda)^{-1}\,.$$

The weighted character is then defined in the connected case by

$$J_M(\pi_V, f_V) \;=\; \mathrm{tr}\big(\mathcal{M}_M(\pi_V, P_V)\mathcal{I}_{P_V}(\pi_V, f_V)\big)\,, \qquad f \in \mathcal{C}(G_V, \zeta_V), \tag{4.3}$$

where $\mathcal{I}_{P_V}(\pi_V)$ is the usual induced representation. For the general case, suppose that π_V belongs to the set

$$\Pi_{\mathrm{temp}}(M_V, \zeta_V) \;=\; \coprod_{\alpha_V \in \pi_0(M_V)} \Pi_{\mathrm{temp}}(M_{V,\alpha_V}, \zeta_{V,\alpha_V})\,.$$

The general weighted character is then defined by

$$J_M(\pi_V, f_V) \;=\; J_M(\pi_V, f_{V,\alpha_V})\,,$$

where f_{V,α_V} is the component of f_V relative to the connected component α_V of G_V associated to π_V. Again the weight factor $\mathcal{M}_M(\pi_V, P_V)$, this time operator valued, links the distinct groups $\{G_v\}$ in a nontrivial way.

Once we have the weighted characters, we can define maps

$$\phi_L : \mathcal{C}(G_V, \zeta_V) \longrightarrow \mathcal{I}(L_V, \zeta_V), \qquad L \in \mathcal{L}(M),$$

by

$$\phi_L(f_V, \pi_V) = J_L(\pi_V, f_V), \qquad \pi_V \in \Pi_{\mathrm{temp}}(L_V, \zeta_V).$$

(We are really writing L here for a representative of an M-equivalence class of Levi subgroups. With this understanding, we have the general space

$$\mathcal{I}(L_V, \zeta_V) = \bigoplus_{\alpha_V \in \pi_0(L_V)} \mathcal{I}(L_{V,\alpha_V}, \zeta_{V,\alpha_V}) = \bigoplus_{\alpha_V} \Big(\bigotimes_{v \in V} \mathcal{I}(L_{v,\alpha_v}, \zeta_{v,\alpha_v}) \Big),$$

where $\mathcal{I}(L_{v,\alpha_v}, \zeta_{v,\alpha_v})$ is the ζ_{v,α_v}^{-1}-equivariant version of the invariant Schwartz space $\mathcal{I}(L_{v,\alpha_v})$ discussed, for example, in [8, §3].) We then define invariant distributions

$$I_M(\gamma_V, f_V) = I_M^G(\gamma_V, f_V), \qquad f_V \in \mathcal{C}(G_V, \zeta_V),$$

in the usual inductive fashion by setting

$$J_M(\gamma_V, f_V) = \sum_{L \in \mathcal{L}(M)} \widehat{I}_M^L\big(\gamma_V, \phi_L(f_V)\big). \tag{4.4}$$

Like the special cases in [3], the distributions we have just defined have familiar descent and splitting properties. For the descent formula, we assume that V consists of just one element v. Suppose that (F_1, G_1, M_1) is a satellite of (F, G, M), and that R_1 is a Levi subgroup of M_1 over F_1. The element v in V is to parametrize a satellite (F_v, G_v, R_v) of (F_1, G_1, R_1). We shall write

$$L_1 \longrightarrow L_v = L_{1,v}, \qquad L_1 \in \mathcal{L}(R_1),$$

for the map from $\mathcal{L}(R_1)$ to $\mathcal{L}(R_v)$. This emphasizes the fact that (F_v, G_v, M_v) is also a satellite of (F, G, M).

Proposition 4.2. *Suppose that γ_v lies in $\Gamma_{G_v}(R_v)$ and that f_v belongs to $\mathcal{C}(G_v,\zeta_v)$. Then*

$$I_M(\gamma_v, f_v) \;=\; \sum_{L_1\in\mathcal{L}(R_1)} d^G_{R_1}(M,L_1)\widehat{I}^{L_1}_{R_1}(\gamma_v, f_{v,L_v})\,. \tag{4.5}$$

Proof. This is a variant of [3, Theorem 8.1]. The arguments are identical, with the general descent formula of Lemma 4.1 taking the place of [8, Proposition 7.1]. The map $\phi_L(f_v)$ that goes into the definition (4.4) is given by the normalized weighted characters of [8], rather than the weighted characters that went into the earlier definition [3, (2.1)]. One sees easily from the discussion of [8, §2], however, that the second version of the map has the same formal properties as the first. □

The restriction we imposed on V in Proposition 4.2 was purely for simplicity. We could have taken V to contain several elements, each of which parametrizes a satellite (F_v, G_v, M_v) of (F_1, G_1, M_1). In this generality, the descent formula (4.5) holds as stated, and is proved in a similar way.

For the splitting property, we assume that V is a disjoint union of V_1 and V_2, and that the image of the absolute value on each F_{V_i} is closed in $\mathbb{R}$. To simplify our notation, we shall allow ourselves to write L_i as a subscript, when it is really the image $L_{i,V_i} = \prod_{v\in V_i} L_{i,v}$ of a Levi subgroup $L_i \in \mathcal{L}(M)$ that is called for. We shall also sometimes write $\Gamma(L_{i,V_i})$ (without the subscript G_{V_i}) for the G_{V_i}-regular conjugacy classes in $L_{i,V_i}(F_{V_i})$. Thus, if f_{V_i} belongs to $\mathcal{C}(G_{V_i},\zeta_{V_i})$, f_{V_i,L_i} is the function

$$f_{V_i,L_{i,V_i}}(\gamma_{V_i}) \;=\; I_G(\gamma_{V_i}, f_{V_i})\,, \qquad\qquad \gamma_{V_i}\in\Gamma(L_{i,V_i}),$$

in $\mathcal{I}(L_{i,V_i},\zeta_{V_i})$.

Proposition 4.3. *Suppose that $\gamma_V = (\gamma_{V_1},\gamma_{V_2})$ lies in $\Gamma(M_{V_1})\times\Gamma(M_{V_2})$ and that $f_V = f_{V_1}\times f_{V_2}$ belongs to $\mathcal{C}(G_{V_1},\zeta_{V_1})\times\mathcal{C}(G_{V_2},\zeta_{V_2})$. Then*

$$I_M(\gamma_V, f_V) \;=\; \sum_{L_1,L_2\in\mathcal{L}(M)} d^G_M(L_1,L_2)\widehat{I}^{L_1}_M(\gamma_{V_1}, f_{V_1,L_1})\widehat{I}^{L_2}_M(\gamma_{V_2}, f_{V_2,L_2})\,. \tag{4.6}$$

Proof. This is a straightforward extension of [3, Proposition 9.1]. The methods of [3] carry over directly to the present setting. The map $\phi_L(f_V)$ in (4.4) is normalized differently from the one in [3], but as in the proof of Proposition 4.2, we require only that it have the same formal properties. $\square$

The following lemma is a typical application of the splitting and descent formulas.

Lemma 4.4. *The distribution $I_M(\gamma_V, f_V)$ vanishes identically unless the space*

$$\mathfrak{a}_{M,G_V} \;=\; \bigcap_{v\in V} (\mathfrak{a}_M \cap \mathfrak{a}_{G_v})$$

equals $\mathfrak{a}_G$.

Proof. The space obviously contains $\mathfrak{a}_G$. Assume that it contains $\mathfrak{a}_G$ properly. We have to show that $I_M(\gamma_V, f_V)$ vanishes.

Consider first the case that V consists of one element v. Then we can apply the descent formula (4.5), with $(F_1, G_1, M_1) = (F_v, G_v, M_v)$, and $R_1 = M_1$. It is clear that for any $L_1 \in \mathcal{L}(M_1)$, the space $\mathfrak{a}^M_{M_1} \oplus \mathfrak{a}^{L_1}_{M_1}$ is orthogonal to the space $\mathfrak{a}_{M,G_v} = \mathfrak{a}_M \cap \mathfrak{a}_{G_1}$, and is therefore a proper subspace of $\mathfrak{a}^G_{M_1}$. It follows that each of the coefficients $d^G_{M_1}(M, L_1)$ that occur in (4.5) vanishes. Therefore $I_M(\gamma_V, f_V) = 0$.

In the general case, we argue by induction on $|V|$. We may assume that V is a disjoint union of proper subsets V_1 and V_2, to which we can apply the proposition. Suppose that $I_M(\gamma_V, f_V) \neq 0$. Then that is a pair $L_1, L_2 \in \mathcal{L}(M)$ for which the corresponding term in (4.6) does not vanish. The nonvanishing of $d^G_M(L_1, L_2)$ tells us that $\mathfrak{a}_{L_1} \cap \mathfrak{a}_{L_2} = \mathfrak{a}_G$, while our induction assumption applied to $\widehat{I}^{L_i}_M(\gamma_{V_i})$ tells us that for $i = 1, 2$, the space $\mathfrak{a}_{M,L_{i,V_i}}$ equals $\mathfrak{a}_{L_i}$. We see that

$$\begin{aligned}
\mathfrak{a}_{M,G_V} \;&=\; \bigcap_{v\in V} (\mathfrak{a}_M \cap \mathfrak{a}_{G_v}) \;\subseteq\; \bigcap_{i=1}^{2} \Big(\bigcap_{v\in V_i} (\mathfrak{a}_M \cap \mathfrak{a}_{L_{i,v}}) \Big) \\
&=\; \mathfrak{a}_{M,L_{1,V_1}} \cap \mathfrak{a}_{M,L_{2,V_2}} \;=\; \mathfrak{a}_{L_1} \cap \mathfrak{a}_{L_2} \;=\; \mathfrak{a}_G \,.
\end{aligned}$$

This contradicts our original assumption on $\mathfrak{a}_{M,G_V}$. Therefore $I_M(\gamma_V, f_V) = 0$ in general.

$\square$

There are two examples of the general constructions of this section that we should always have in mind. For the first, F is a local field and $V = \{v_1, v_2\}$ contains two elements. In this case we take

$$(F_v, G_v, M_v) \ = \ (F, G, M)$$

for each $v \in V$. We define the embedding

$$\mathfrak{a}_M \ \hookrightarrow \ \mathfrak{a}_{M_v} \ = \ \mathfrak{a}_M$$

to be the identity if $v = v_1$, and to be (-1) times the identity if $v = v_2$. If $x_V = (x_1, x_2)$ is a point in $G_V(F_V) = G(F) \times G(F)$, then

$$v_P(\lambda, x_V) \ = \ e^{-\lambda(H_P(x_1))} e^{\lambda(H_{\overline{P}}(x_2))}, \qquad\qquad P \in \mathcal{P}(M).$$

If G is connected, this is essentially the (G, M)-family of §12 of [4]. To match the definition [4, (12.1)], we actually have to replace λ by $-\lambda$ and P by $\overline{P}$. However, the effects of these two substitutions cancel when we form the function $v_M(x_V)$. It follows that if $\gamma_V = (\gamma, \gamma)$ is the diagonal image of an element $\gamma \in \Gamma_G(M)$, the distribution $J_M(\gamma_V, f_V)$ in (4.2) equals the one defined in [4, (12.2)]. It is the main term [4, (12.9)] on the geometric side of the noninvariant local trace formula. The distribution $I_M(\gamma_V, f_V)$ defined by (4.4) plays the same role the corresponding invariant trace formula.

For the other example, F is a global field and V is a finite set of valuations on F. In this case, we would want G to be equipped with a family of F_v-homomorphisms
θ_v: $G \to G_v$ such that the component maps $\theta_{v,\alpha}$: $G_\alpha \to G_{v,\alpha_v}$ are isomorphisms over F_v, and such that the product map

$$\theta_V \ = \ \prod_v \theta_v : \ G(F_V) \longrightarrow \prod_v G_v(F_v)$$

is surjective. (The global results in [14, §2] suggest the notion of a global K-group, that should come with F_v-homomorphisms θ_v: $G \to G_v$ onto local K-groups. We shall not pursue the idea here.) We would of course also require that $\theta_v(M) = M_v$, so that we could then take $\mathfrak{a}_M \subset \mathfrak{a}_{M_v}$ to be the canonical embedding. This is the setting of the global trace formula, at least in the case that the multiple groups are all connected. If γ_V is the diagonal image in $\Gamma_{G_V}(M_V)$ of a rational element $\gamma \in \Gamma_G(M)$, and V is sufficiently large, the distribution $J_M(\gamma_V, f_V)$ in (4.2) is one of the main terms on the geometric side of the noninvariant global trace formula. The distribution $I_M(\gamma_V, f_V)$ in (4.4) plays the same role in the corresponding invariant trace formula.

5. The corresponding endoscopic construction

The next step is to construct endoscopic and stable analogues of the general distributions defined in the last section. In order to do so, we must impose more structure on the underlying data.

We consider triplets (F, G, M) as in §4, but we assume from now on that (F, G, M) is equipped with the structure of a dual Levi subgroup $\widehat{M} \subset \widehat{G}$ for $M \subset G$. We shall say that a satellite (F_1, G_1, M_1) of (F, G, M) is an *L-satellite* if it comes with an embedding of $\Gamma_1 = \mathrm{Gal}(\overline{F}_1/F_1)$ into Γ, and with embeddings $Z(\widehat{M}) \subset Z(\widehat{M}_1)$ and $Z(\widehat{G}) \subset Z(\widehat{G}_1)$ that are compatible with each other and with the actions of Γ and Γ_1. We require also that the embedding $\big(Z(\widehat{M})^\Gamma\big)^0$ into $\big(Z(\widehat{M}_1)^{\Gamma_1}\big)^0$ be dual to the satellite embedding $\mathfrak{a}_M \subset \mathfrak{a}_{M_1}$ of §4. The purpose of this extra structure is to provide compatible embeddings of $Z(\widehat{M})^\Gamma$ and $Z(\widehat{G})^\Gamma$ into $Z(\widehat{M}_1)^{\Gamma_1}$ and $Z(\widehat{G}_1)^{\Gamma_1}$ respectively. The notion of an L-satellite is modelled on the example that $(F_1, G_1, M_1) = (F, G', M')$, where G' is an endoscopic datum in $\mathcal{E}_{M'}(G)$. In fact all of the examples of satellites given in §4 have the natural structure of L-satellites.

Given (F, G, M), we choose a finite set

$$\big\{(F_v, G_v, M_v) :\ v \in V\big\}$$

as in §4. We assume from now on that each (F_v, G_v, M_v) is an L-satellite of (F, G, M). We also assume that each G_v is actually a K-group. For every v, we fix a quasisplit inner twist G_v^* of G_v and a Levi subgroup M_v^* of G_v^* corresponding to M_v.

Suppose that

$$M'_V \ = \prod_{v \in V} M'_v\,, \qquad\qquad M'_v \in \mathcal{E}_{\mathrm{ell}}(M_v),$$

is an equivalence class of elliptic endoscopic data for M_V. We fix a representative $(M'_V, \mathcal{M}'_V, s'_{M_V}, \xi'_{M_V})$ within the equivalence class so that the group $\mathcal{M}'_V = \prod_v \mathcal{M}'_v$ is actually a subgroup of ${}^L M_V = \prod_v {}^L M_v$, and so that the L-embedding ξ'_{M_V} is the identity. The semisimple element $s'_{M_V} = \prod_v s'_{M_v}$ belongs to $\widehat{M}_V = \prod_v \widehat{M}_v$, and stabilizes $\mathcal{M}'_V$. Since each (F_v, G_v, M_v) is an L-satellite of (F, G, M), we can form the diagonal embedding of $Z(\widehat{M})^\Gamma$ into

$$Z(\widehat{M}_V)^{\Gamma_V} = \prod_{v \in V} Z(\widehat{M}_v)^{\Gamma_v}, \qquad \Gamma_V = \prod_v \Gamma_v.$$

This gives us a set

$$s'_{M_V} Z(\widehat{M})^\Gamma / Z(\widehat{G})^\Gamma = \left\{ s'_V = s'_{M_V} s : \ s \in Z(\widehat{M})^\Gamma / Z(\widehat{G})^\Gamma \right\}$$

of $Z(\widehat{G})^\Gamma$-orbits of semisimple elements in $\widehat{G}_V = \prod_v \widehat{G}_v$. Following the construction of §3, we shall identify this set with a family of $Z(\widehat{G})^\Gamma$-orbits of endoscopic data

$$\mathcal{E} = \mathcal{E}_{M'_V}(G_V, G) = \left\{ (G'_V, \mathcal{G}'_V, s'_V, \xi'_V) \right\}$$

for G_V. We define $\widehat{G}'_V = \prod_v \widehat{G}'_v$ to be the connected centralizer of $s'_V = \prod_v s'_v = \prod_v (s'_{M_v} s)$ in $\widehat{G}_V = \prod_v \widehat{G}_v$, and we take $\mathcal{G}'_V = \prod_v \mathcal{G}'_v$ to be the subgroup $\widehat{G}'_V \mathcal{M}'_V = \prod_v (\widehat{G}'_v \mathcal{M}'_v)$ of ${}^L G_V = \prod_v {}^L G_v$. The third constituent s'_V of a datum equals $s'_{M_V} s$, as above, and the fourth constituent ξ'_V is just the identity embedding of $\mathcal{G}'_V$ into ${}^L G_V$.

As in §3, we shall have to treat the quasisplit case on its own. Assume for a moment that (F_V, G_V, M_V) is quasisplit. In other words, G_v and M_v are quasisplit over F_v, for each $v \in V$. We shall write $\mathcal{E}^1 = \mathcal{E}^1_{M'_V}(G_V, G)$ for the set of elements $G'_V \in \mathcal{E}$ that are isomorphic to $G^*_V = \prod_v G^*_v$, or equivalently, such that s'_V lies in $Z(\widehat{G}_V)^{\Gamma_V} = \prod_v Z(\widehat{G}_v)^{\Gamma_v}$. We set $\mathcal{E}^0 = \mathcal{E}^0_{M'_V}(G_V, G)$ equal to the complement of $\mathcal{E}^1$ in $\mathcal{E}$. If $\mathcal{E}^1$ is nonempty, the endoscopic datum M'_V for M_V is isomorphic to $M^*_V = \prod_v M^*_v$, but unlike the special case of §3, the converse is not true. If it is nonempty, $\mathcal{E}^1$ supports a simply transitive action of the group $Z(\widehat{M})^\Gamma \cap Z(\widehat{G}_V)^{\Gamma_V} / Z(\widehat{G})^\Gamma$. In particular, $\mathcal{E}^1$ could contain more than one element, again unlike the special case of §3. We shall sometimes require separate arguments depending on whether this set is empty or not. If $\mathcal{E}^1$ is empty, it will be convenient to write $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}_{M'_V}(G_V, G)$ for the disjoint union of $\mathcal{E}$ with the one element G^*_V. If $\mathcal{E}^1$ is nonempty, we simply set $\widetilde{\mathcal{E}} = \mathcal{E}$. Finally, we write $\widetilde{\mathcal{E}}^1 = \widetilde{\mathcal{E}}^1_{M'_V}(G_V, G)$ for the complement of $\mathcal{E}^0$ in $\widetilde{\mathcal{E}}$.

If (F_V, G_V, M_V) is not quasisplit, we have no need of such distinctions. In this case we set $\mathcal{E}^0 = \mathcal{E} = \widetilde{\mathcal{E}}$, and we take $\mathcal{E}^1$ and $\widetilde{\mathcal{E}}^1$ to be empty. In general, then, we have disjoint unions

$$\widetilde{\mathcal{E}} \;=\; \mathcal{E}^0 \coprod \widetilde{\mathcal{E}}^1 \;=\; \mathcal{E}^0 \coprod \mathcal{E}^1 \coprod (\widetilde{\mathcal{E}} - \mathcal{E})\,.$$

At least one of the sets $\mathcal{E}^1$ and $\widetilde{\mathcal{E}} - \mathcal{E}$ is empty; they are both empty if and only if (F_V, G_V, M_V) is not quasisplit.

We recall that $\mathcal{M}'_V$ need not be an L-group. However, we can fix an L-homomorphism $\widetilde{\xi}'_V = \prod_v \widetilde{\xi}'_V$ of $\mathcal{M}'_V$ into a group ${}^L\widetilde{M}'_V = \prod_v {}^L\widetilde{M}'_v$ that is the L-group of a central extension $\widetilde{M}'_V = \prod_v \widetilde{M}'_v$ of M'_V by a product $\widetilde{Z}'_V = \prod_v \widetilde{Z}'_v$ of induced tori. The extension comes with a character $\widetilde{\zeta}'_V = \prod_v \widetilde{\zeta}'_v$ on $\widetilde{Z}'_V(F_V)$. The choice of these objects for M'_V determines similar choices for any G'_V in $\mathcal{E}$. We obtain a central extension $\widetilde{G}'_V$ of G'_V by $\widetilde{Z}'_V$, and an extension of $\widetilde{\xi}'_V$ to an L-homomorphism of $\mathcal{G}'_V$ to the L-group ${}^L\widetilde{G}'_V = \widehat{\widetilde{G}}{}'_V \cdot {}^L\widetilde{M}'_V$. We are going to assume that the Langlands-Shelstad transfer conjecture holds for each $v \in V$ and for each endoscopic datum $G'_v \in \mathcal{E}(G_v)$. This gives a mapping $f_V \to f'_V$ from $\mathcal{C}(G_V, \zeta_V)$ to $S\mathcal{I}(\widetilde{G}'_V, \widetilde{\zeta}'_V \zeta_V)$, for any $G'_V \in \mathcal{E}_{M'_V}(G_V, G)$.

The other ingredients we need for the construction are the transfer factors for M_V. We simply take the product of transfer factors

$$\Delta_M(\delta'_V, \gamma_V) \;=\; \prod_{v \in V} \Delta_{M_v}(\delta'_v, \gamma_v)$$

at elements $\gamma_V = \prod_v \gamma_v$ in $\Gamma_{G_V}(M_V)$ and elements $\delta'_V = \prod_v \delta'_v$ in the set $\widetilde{\Gamma}^{\mathcal{E}}_{G_V}(M_V) = \prod_v \widetilde{\Gamma}^{\mathcal{E}}_{G_v}(M_v)$. It follows from Lemma 2.3 that $\Delta_M(\delta'_V, \gamma_V)$ and its adjoint transfer factor

$$\Delta_{M_V}(\gamma_V, \delta'_V) \;=\; \prod_{v \in V} \Delta_{M_v}(\gamma_v, \delta'_v)$$

satisfy relations

$$(5.1) \qquad \sum_{\delta'_V \in \Gamma^{\mathcal{E}}_{G_V}(M_V)} \Delta_{M_V}(\gamma_V, \delta'_V) \Delta_{M_V}(\delta'_V, \gamma_{1,V}) \;=\; \delta(\gamma_V, \gamma_{1,V})$$

and

$$\sum_{\gamma_V \in \Gamma_{G_V}(M_V)} \Delta_{M_V}(\delta'_V, \gamma_V)\Delta_{M_V}(\gamma_V, \delta'_{1,V}) \ = \ \widetilde{\delta}(\delta'_V, \delta'_{1,V})\,. \tag{5.2}$$

In particular, we can define invariant distributions $I_M(\delta'_V)$ on $\mathcal{C}(G_V, \zeta_V)$, parametrized by elements $\delta'_V \in \widetilde{\Gamma}^{\mathcal{E}}_{G_V}(M_V)$, by either of the equivalent formulas

$$I_M(\delta'_V, f_V) \ = \sum_{\gamma_V \in \Gamma_{G_V}(M_V)} \Delta_{M_V}(\gamma'_V, \gamma_V) I_M(\gamma_V, f_V)$$

or

$$I_M(\gamma_V, f_V) \ = \sum_{\delta'_V \in \Gamma^{\mathcal{E}}_{G_V}(M_V)} \Delta_{M_V}(\gamma_V, \delta'_V) I_M(\delta'_V, f_V)\,.$$

We can now give the construction. We are going to define invariant distributions $I^{\mathcal{E}}_M(\delta'_V, f_V)$ and $S^G_M(M'_V, \delta'_V, f_V)$ on $\mathcal{C}(G_V, \zeta_V)$ by an inductive process similar to that of §3. As above, M'_V stands for an elliptic endoscopic datum $(M'_V, \mathcal{M}'_V, s'_V, \xi'_V)$ for M_V, while δ'_V is an element in $\Sigma_{G_V}(\widetilde{M}'_V)$ that we shall assume maps into $\widetilde{\Gamma}^{\mathcal{E}}_{G_V}(M_V)$. The second distribution exists only when (F_V, G_V, M_V) is quasisplit. We shall emphasize the special case that $\mathcal{E}^1$ is nonempty (and, in particular, that $M'_V \cong M^*_V$) by writing

$$S^G_M(\delta_V, f_V) \ = \ S^G_M(M^*_V, \delta^*_V, f_V)\,, \qquad \delta_V \in \Sigma_{G_V}(M_V).$$

For each M'_V and δ'_V, we define the distributions by the formula

$$I^{\mathcal{E}}_M(\delta'_V, f_V) \ = \sum_{G'_V \in \mathcal{E}^0} \widehat{S}^G_M(\delta'_V, f'_V) + |\widetilde{\mathcal{E}}^1| S^G_M(M'_V, \delta'_V, f_V) \tag{5.3}$$

in general, and by the supplementary formula

$$I^{\mathcal{E}}_M(\delta'_V, f_V) \ = \ I_M(\delta'_V, f_V) \tag{5.4}$$

in the case that (F_V, G_V, M_V) is quasisplit (or equivalently, that $\widetilde{\mathcal{E}}^1$ is nonempty). We then set

$$I^{\mathcal{E}}_M(\gamma_V, f_V) \ = \sum_{\delta'_V \in \Gamma^{\mathcal{E}}_{G_V}(M_V)} \Delta_{M_V}(\gamma_V, \delta'_V) I^{\mathcal{E}}_M(\delta'_V, f_V)\,, \tag{5.5}$$

for any $\gamma_V \in \Gamma_{G_V}(M_V)$. It is clear that $I_M^{\mathcal{E}}(\gamma_V, f_V)$ equals $I_M(\gamma_V, f_V)$ in the case that (F_V, G_V, M_V) is quasisplit.

The definition requires further comment. It is of course inductive. We shall first state a formal assumption, on which the definition will ultimately rely. The assumption is based on a fixed subset V_0 of V with the following properties.

(i) V_0 is empty unless (F_V, G_V, M_V) is quasisplit.

(ii) For each $v \in V_0$, (F_v, G_v, M_v) is an *elliptic* satellite of (F, G, M), in the sense that $\mathfrak{a}_{G_v} = \mathfrak{a}_G$.

(iii) For each $v \in V_0$, the embedded subgroup $Z(\widehat{G})$ of $Z(\widehat{G}_v)$ is actually equal to $Z(\widehat{G}_v)$.

Assumption 5.1. *For each $v \in V$, the distributions associated to the triplet (F_v, G_v, M_v) by the basic construction of §3 are all well defined. Furthermore, these distributions satisfy part (b) of Conjecture 3.3 if $v \neq V_0$.*

The first assertion is that the transfer mappings and stability conditions implicit in the inductive definition of §3 are valid. That is to say, they hold for groups obtained from G by a chain of proper endoscopic groups. The second assertion is a further condition, that of course depends on the size of V_0. It would have simplified the discussion to take V_0 to be empty. Our purpose, however, is to carry Assumption 5.1 as an induction hypothesis into a future paper, where we will attack Conjecture 3.3 at places v in a general set V_0. The three conditions on V_0 have been tailored to this end.

Assumption 5.1 is thus to be regarded as our primary induction hypothesis. It will not be resolved in this paper. Our more modest goal here will be to reduce the definition and study of the compound distributions in (5.3) to the simple ones of §3. To this end, we impose a secondary induction hypothesis, that will be resolved presently in terms of the first one. For any datum G'_V in $\mathcal{E}^0$, the distribution $S_M^G(\delta'_V)$ in the summand (5.3) is supposed to be defined on a quasisplit inner K-form of G'_V. We assume that it is in fact defined, and that it is stable at any v in the complement of the set

$$V'_0 \;=\; \{v \in V_0 :\; G'_v = G^*_v\}\,.$$

If U is the complement of V'_0 in V, and $f_V = f_U \times f_{V'_0}$, the function f'_V in (5.3) is then to be understood as the partial Langlands-Shelstad transfer

$$f'_U \otimes f_{V'_0}\,, \qquad\qquad f'_U \in S\mathcal{I}(\widetilde{G}'_U, \widetilde{\zeta}'_U),\;\; f_{V'_0} \in \mathcal{C}(G_{V'_0}, \zeta_{V'_0}).$$

With this harmless abuse of notation, the summand $\widehat{S}^G_M(\delta'_V, f'_V)$ in (5.3) has an obvious meaning. Given this secondary induction hypothesis, we can define $I^{\mathcal{E}}_M(\delta'_V, f_V)$ by (5.4) or (5.3) (according to whether (F_V, G_V, M_V) is quasisplit or not), and we define $S^G_M(M'_V, \delta'_V, f_V)$ by (5.3) in the case that (F_V, G_V, M_V) is quasisplit. It will sometimes be convenient to write

$${}^0S^G_M(M'_V, \delta'_V, f_V) \;=\; \begin{cases} S^G_M(M'_V, \delta'_V, f_V), & \text{if } G^*_V \in \widetilde{\mathcal{E}} - \mathcal{E}, \\ 0, & \text{otherwise,} \end{cases}$$

for a distribution that we expect will always vanish. Then (5.3) can be recast in the form

$$I^{\mathcal{E}}_M(\delta'_V, f_V) \;=\; \sum_{G'_V \in \mathcal{E}} \widehat{S}^G_M(\delta'_V, f'_V) + {}^0S^G_M(M'_V, \delta'_V, f_V)\,, \tag{5.6}$$

since $\widetilde{\mathcal{E}} - \mathcal{E}$ consists of at most the one element G^*_V.

There are still some points in the definition to clarify. The set of summation $\mathcal{E}^0 = \mathcal{E}^0_{M'_V}(G_V, G)$ in (5.3) is infinite (except in the trivial case that $M = G$). However, we have

Lemma 5.2. *(i) The sum in (5.3) can be taken over a finite subset of $\mathcal{E}^0$.*

(ii) Any of the distributions on $G_V(F_V)$ defined by (5.3), (5.4) or (5.5) vanishes unless the space

$$\mathfrak{a}_{M,G_V} \;=\; \bigcap_{v \in V} (\mathfrak{a}_M \cap \mathfrak{a}_{G_v})$$

equals $\mathfrak{a}_G$.

Proof. Assume inductively that (ii) holds if G_V is replaced by a quasisplit inner K-form of any of the groups $G'_V \in \mathcal{E}^0$. Then the summand $\widehat{S}^G_M(\delta'_V, f'_V)$ in (5.3) vanishes unless $\mathfrak{a}_{M,G'_V}$ equals $\mathfrak{a}_G$.

Although the set $Z(\widehat{M})^\Gamma / Z(\widehat{G})^\Gamma$ which parametrizes $\mathcal{E}$ is infinite, it gives rise to only finitely many subgroups $\widehat{G}'_V$ of $\widehat{G}_V$. Any such $\widehat{G}'_V$ comes with an L-action of the group $\Gamma_V = \prod\limits_v \Gamma_v$, and contains the central subgroup

$$Z(\widehat{G}'_V)^{\Gamma_V} \;=\; \prod_v \big(Z(\widehat{G}'_v)^{\Gamma_v}\big)\,.$$

We have in fact four subgroups

$$\begin{array}{ccc} Z(\widehat{G}'_V)^{\Gamma_V} & \hookrightarrow & Z(\widehat{M}'_V)^{\Gamma_V} \\ \cup\!\!\uparrow & & \cup\!\!\uparrow \\ Z(\widehat{G})^\Gamma & \hookrightarrow & Z(\widehat{M})^\Gamma \end{array}$$

of $\widehat{G}'_V$, with the vertical maps being the diagonal embeddings. The Lie algebra of the intersection $Z(\widehat{G}'_V)^{\Gamma_V} \cap Z(\widehat{M})^\Gamma$ (in $Z(\widehat{M}'_V)^{\Gamma_V}$) is isomorphic to

$$\bigcap_{v\in V} (\mathfrak{a}^*_{G_v,\mathbb{C}} \cap \mathfrak{a}^*_{M,\mathbb{C}}) \;\cong\; (\mathfrak{a}_{M,G'_V})^*_{\mathbb{C}}\,.$$

The Lie algebra of $Z(\widehat{G})^\Gamma$ is of course isomorphic to $\mathfrak{a}^*_{G,\mathbb{C}}$. It follows easily from the induction assumption above that the summand in (5.3) vanishes unless $Z(\widehat{G})^\Gamma$ has finite index in $Z(\widehat{G}'_V)^{\Gamma_V} \cap Z(\widehat{M})^\Gamma$. But any element in $Z(\widehat{M})^\Gamma / Z(\widehat{G})^\Gamma$ that gives rise to $\widehat{G}'_V$ must lie in the subgroup $Z(\widehat{G}'_V)^{\Gamma_V} \cap Z(\widehat{M})^\Gamma / Z(\widehat{G})^\Gamma$ of $Z(\widehat{G}'_V)^{\Gamma_V} / Z(\widehat{G})^\Gamma$. The sum in (5.3) may therefore be taken over a finite set.

To establish the assertion (ii), assume that $\mathfrak{a}_{M,G_V}$ is strictly larger than $\mathfrak{a}_G$. If G'_V lies in $\mathcal{E}^0$, the space $\mathfrak{a}_{M,G'_V}$ contains $\mathfrak{a}_{M,G_V}$, and is also strictly larger than $\mathfrak{a}_G$. The corresponding summand $\widehat{S}^G_M(\delta'_V, f'_V)$ in (5.3) therefore vanishes by the induction assumption. It follows from Lemma 4.4 that the various distributions on $G_V(F_V)$ defined by (5.3), (5.4) and (5.5) also vanish. $\square$

Lemma 5.3. *(a) The distribution $I^{\mathcal{E}}_M(\delta'_V, f_V)$ in (5.3) depends only on the image of δ'_V in $\widetilde{\Gamma}^{\mathcal{E}}_{G_V}(M_V)$, and in particular, is independent of the choice of M'_V.*
(b) Suppose that (F_V, G_V, M_V) is quasisplit and that $S = S^G_M(M'_V, \delta'_V)$ is the distribution on $G_V(F_V)$ defined by (5.3). If G^*_V belongs to $\mathcal{E}$, S is stable at each v in the complement of V_0. If G^*_V belongs to $\widetilde{\mathcal{E}} - \mathcal{E}$ and V_0 is empty, S vanishes.

The assertion (a) is required for the right hand side of (5.5) to be well defined. The first assertion in (b) is needed to complete the inductive definition of the compound distributions. The second assertion in (b) is that the distribution ${}^0S^G_M(M'_V, \delta'_V)$ on $G_V(F_V)$ vanishes if V_0 is empty. We will need to apply it inductively during the proof of the stable splitting and descent formulas of the next two sections. The lemma itself will in fact be an easy consequence of these formulas. We shall prove it in two steps, following the proof of each of the theorems of the next two sections. In the meantime, we shall have to impose a third induction hypothesis. We assume that the lemma holds if V is replaced by a proper subset, or in the case that V contains one element v, if (F, G, M) and (F_v, G_v, M_v) are replaced by triplets (F_1, L_1, R_1) and $F_v, L_v, R_v)$, in which L_v and R_v are Levi subgroups of G_v and M_v, and $\dim(\mathfrak{a}_{L_1}) > \dim(\mathfrak{a}_G)$.

Before going on, we note that the definition (5.3) does not have quite the same form as the original one in §3. The coefficients $\iota_{M'}(G, G')$ are absent from the sum in (5.3), and the stable distributions in this sum have been denoted by S^G_M instead of $S^{\widetilde{G}'}_{\widetilde{M}'}$. In particular, it is not immediately clear that the construction of §3 is a special case of the one here. We shall wait until the end of §7 (Corollary 7.3) to check this point.

6. Stable splitting formulas

In the next two sections we shall establish endoscopic and stable analogues of the splitting and descent formulas of §4. Such formulas are important for studying the transfer properties of terms in the local and global trace formulas. They will also allow us to complete the inductive definitions of the last section. The endoscopic formulas will take exactly the same form as their counterparts in §4. However, the stable formulas require the introduction of some new coefficients.

We continue with the setting of §5. Then G_v is a K-group, and (F_v, G_v, M_v) is an L-satellite of (F, G, M), for each v in the finite set V. We shall treat the splitting formulas in this section. As in Proposition 4.3, we suppose that V is a disjoint union of nonempty sets V_1 and V_2, and that for $i = 1, 2$, the image of F_{V_i} in $\mathbb{R}$ under the absolute value is closed. We fix a function in $\mathcal{C}(G_V, \zeta_V)$ of the form

$$f_V = f_{V_1} \times f_{V_2}, \qquad f_{V_i} \in \mathcal{C}(G_{V_i}, \zeta_{V_i}).$$

The splitting formulas are expressed in terms of pairs of Levi subgroups $L_1, L_2 \in \mathcal{L}(M)$. For any such pair, we define a coefficient

$$\mathrm{e}_M^G(L_1, L_2) = d_M^G(L_1, L_2)|Z(\widehat{L}_1)^\Gamma \cap Z(\widehat{L}_2)^\Gamma / Z(\widehat{G})^\Gamma|^{-1}. \tag{6.1}$$

Observe that if $d_M^G(L_1, L_2) \neq 0$, then $\mathfrak{a}_{L_1}^* \cap \mathfrak{a}_{L_2}^* = \mathfrak{a}_G^*$, and the identity component of $Z(\widehat{L}_1)^\Gamma \cap Z(\widehat{L}_2)^\Gamma$ is the same as that of $Z(\widehat{G})^\Gamma$. Therefore $\mathrm{e}_M^G(L_1, L_2)$ is also nonzero. Extending a convention used in Proposition 4.3, we shall generally write L_i as a superscript when it is really the image L_{i,V_i} that is called for.

Theorem 6.1. *(a) Suppose that $\gamma_V = (\gamma_{V_1}, \gamma_{V_2})$ lies in $\Gamma_{G_V}(M_V)$. Then*

$$I_M^{\mathcal{E}}(\gamma_V, f_V) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \widehat{I}_M^{L_1, \mathcal{E}}(\gamma_{V_1}, f_{V_1, L_1}) \widehat{I}_M^{L_2, \mathcal{E}}(\gamma_{V_2}, f_{V_2, L_2}). \tag{6.2}$$

(b) Suppose that (F_V, G_V, M_V) *is quasisplit, and that* $\delta_V = (\delta_{V_1}, \delta_{V_2})$ *lies in* $\Sigma_{G_V}(M_V)$*. Then*

$$(6.3) \qquad S^G_M(\delta_V, f_V) = \sum_{L_1, L_2 \in \mathcal{L}(M)} \mathrm{e}^G_M(L_1, L_2) \widehat{S}^{L_1}_M(\delta_{V_1}, f^{L_1}_{V_1}) \widehat{S}^{L_2}_M(\delta_{V_2}, f^{L_2}_{V_2}) \,.$$

(b′) Suppose that (F_V, G_V, M_V) *is quasisplit, and that* $\delta'_V = (\delta'_{V_1}, \delta'_{V_2})$ *lies in* $\Sigma_{G_V}(\widetilde{M}'_V)$*, for some* $M'_V \in \mathcal{E}_{\mathrm{ell}}(M_V)$*. Then*

$$(6.3') \quad {}^0S^G_M(M'_V, \delta'_V, f_V) = f^{M'}_{V_1}(\delta'_{V_1})\, {}^0S^G_M(M'_{V_2}, \delta'_{V_2}, f_{V_2}) + {}^0S^G_M(M'_{V_1}, \delta'_{V_1}, f_{V_1}) f^{M'}_{V_2}(\delta'_{V_2}) \,.$$

Proof. As in §5, we fix an elliptic endoscopic datum M'_V for M_V and a point $\delta'_V = (\delta'_{V_1}, \delta'_{V_2})$ in $\Sigma_{G_V}(M'_V)$. If G'_V belongs to $\mathcal{E} = \mathcal{E}_{M'_V}(G_V, G)$, the triplets

$$(F_v, G'_v, M'_v) \,, \qquad v \in V,$$

are also L-satellites of (F, G, M), as are the triplets $(F_v, \widetilde{G}'_v, \widetilde{M}'_v)$. We assume inductively that (6.3) holds if (F_V, G_V, M_V) is replaced by a quasisplit inner K-form of $(F_V, \widetilde{G}'_V, \widetilde{M}'_V)$, for any G'_V in the subset $\mathcal{E}^0 = \mathcal{E}^0_{M'_V}(G_V, G)$ of $\mathcal{E}$.

The required formula (6.2) has an analogue

$$(6.4) \qquad I^{\mathcal{E}}_M(\delta'_V, f_V) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d^G_M(L_1, L_2) \widehat{I}^{L_1, \mathcal{E}}_M(\delta'_{V_1}, f_{V_1, L_1}) \widehat{I}^{L_2, \mathcal{E}}_M(\delta'_{V_2}, f_{V_2, L_2})$$

for δ'_V. According to the definition (5.5), the two formulas are equivalent, so for part (a) it will be enough to establish (6.4).

It follows from the definition (5.3) that

$$(6.5) \qquad I^{\mathcal{E}}_M(\delta'_V, f_V) - |\widetilde{\mathcal{E}}^1| S^G_M(M'_V, \delta'_V, f_V)$$

equals

$$\sum_{G'_V \in \mathcal{E}^0} \widehat{S}^G_M(\delta'_V, f'_V) \,.$$

We shall apply (6.3) inductively to each G'_V. If L_i belongs to $\mathcal{L}(M)$, we can form the Levi subgroup L'_{i,V_i} of G'_{V_i}, since (F_v, G'_v, M'_v) is a satellite of (F, G, M) for each $v \in V_i$. According to our convention above, we can write L'_i as a superscript instead of the image L'_{i,V_i}. In fact, we may as well just set $L'_i = L'_{i,V_i}$ in general, as there is no risk of confusion. Since

$$(f'_{V_i})^{L'_i} = (f_{V_i,L_i})^{L'_i} \;=\; f_{V_i}^{L'_i}\,, \qquad\qquad i = 1, 2,$$

we see that (6.5) equals

$$\sum_{G'_V \in \mathcal{E}^0} \sum_{L_1, L_2 \in \mathcal{L}(M)} \mathrm{e}_M^G(L_1, L_2) \widehat{S}_M^{L_1}(\delta'_{V_1}, f_{V_1}^{L'_1}) \widehat{S}_M^{L_2}(\delta'_{V_2}, f_{V_2}^{L'_2})\,.$$

If $M'_V \neq M^*_V$, $\mathcal{E}^0$ equals $\mathcal{E}$. If $M'_V = M^*_V$, however, $\mathcal{E}^0$ could be a proper subset of $\mathcal{E}$. In this case we would have to add a correction term to the last expression to change the sum over $\mathcal{E}^0$ to one over $\mathcal{E}$. In either case, it is the expression

$$\sum_{G'_V \in \mathcal{E}} \sum_{L_1, L_2 \in \mathcal{L}(M)} \mathrm{e}_M^G(L_1, L_2) \widehat{S}_M^{L_1}(\delta'_{V_1}, f_{V_1}^{L'_1}) \widehat{S}_M^{L_2}(\delta'_{V_2}, f_{V_2}^{L'_2}) \tag{6.6}$$

that will be the focal point of the argument. We are going to interchange the sum over G'_V with the double sum over L_1 and L_2.

Fix groups $L_1, L_2 \in \mathcal{L}(M)$, and set

$$\mathcal{E}_i \;=\; \mathcal{E}_{M'_{V_i}}(L_{i,V_i}, L_i)\,, \qquad\qquad i = 1, 2.$$

We would like to compute the contribution of L_1 and L_2 to (6.6). The key step is to observe that there is a natural map

$$\mathcal{E} \longrightarrow \mathcal{E}_1 \times \mathcal{E}_2\,, \tag{6.7}$$

that sends G'_V to the pair (L'_1, L'_2). If G'_V corresponds to the element s'_V in $s'_{M_V} Z(\widehat{M})^\Gamma / Z(\widehat{G})^\Gamma$, (L'_1, L'_2) corresponds to the element

$$(s'_{L_1}, s'_{L_2})\,, \qquad\qquad s'_{L_i} \in s'_{M_{V_i}} Z(\widehat{M})^\Gamma / Z(\widehat{L}_i)^\Gamma,$$

obtained by projecting s'_V onto each factor $s'_{M_{V_i}} Z(\widehat{M})^\Gamma / Z(\widehat{L}_i)^\Gamma$. We can assume that the coefficient $\mathrm{e}^G_M(L_1, L_2)$ is nonzero, since the summand for (L_1, L_2) in (6.6) would otherwise vanish. Therefore $d^G_M(L_1, L_2)$ is nonzero, which implies in particular that $\mathfrak{a}_M = \mathfrak{a}_{L_1} + \mathfrak{a}_{L_2}$ and $\mathfrak{a}^*_{M,\mathbb{C}} = \mathfrak{a}^*_{L_1,\mathbb{C}} + \mathfrak{a}^*_{L_2,\mathbb{C}}$. It follows that the connected component $\big(Z(\widehat{M})^\Gamma\big)^0$ equals the product of $\big(Z(\widehat{L}_1)^\Gamma\big)^0$ and $\big(Z(\widehat{L}_2)^\Gamma\big)^0$. Since $Z(\widehat{M})^\Gamma$ equals the product of $\big(Z(\widehat{M})^\Gamma\big)^0$ with $Z(\widehat{G})^\Gamma$, by Lemma 1.1, we see that

$$Z(\widehat{M})^\Gamma \;=\; Z(\widehat{L}_1)^\Gamma Z(\widehat{L}_2)^\Gamma \,.$$

It follows that the map (6.7) is surjective. Furthermore, the group

$$Z(\widehat{L}_1)^\Gamma \cap Z(\widehat{L}_2)^\Gamma / Z(\widehat{G})^\Gamma \tag{6.8}$$

is finite, and acts simply transitively on the fibres of the map. But the summand in (6.6) depends only on (L'_1, L'_2), and not on the group G'_V in its preimage. We can therefore replace the sum over G'_V in (6.6) by a sum over (L'_1, L'_2) in $\mathcal{E}_1 \times \mathcal{E}_2$, provided that we multiply the summand by the order of the group (6.8). Since the product of $\mathrm{e}^G_M(L_1, L_2)$ with the order of (6.8) equals $d^G_M(L_1, L_2)$, we conclude that the contribution of (L_1, L_2) to (6.6) equals

$$d^G_M(L_1, L_2) \prod_{i=1}^{2} \Big(\sum_{L'_i \in \mathcal{E}_i} \widehat{S}^{L_i}_M(\delta'_{V_i}, f^{L'_i}_{V_i}) \Big) \,. \tag{6.9}$$

The definition (5.3) can be applied to the terms in (6.9). We shall use the equivalent form (5.6), which provides an identity

$$I^{L_i,\mathcal{E}}_M(\delta'_{V_i}, h_{V_i}) \;=\; \sum_{L'_i \in \mathcal{E}_i} \widehat{S}^{L_i}_M(\delta'_{V_i}, h^{L'_i}_{V_i}) + {}^0S^{L_i}_M(M'_{V_i}, \delta'_{V_i}, h_{V_i}) \,,$$

for any function $h_{V_i} \in \mathcal{C}(L_{i,V_i}, \zeta_{V_i})$. Suppose that $L_i \neq G$. There is nothing to rule out $L_{i,v}$ being equal to G_v, for some $v \in V$. However, since

$$\mathfrak{a}_G \subsetneq \mathfrak{a}_{L_i} \subset \mathfrak{a}_{L_{i,v}} \,,$$

no such v can belong to V_0. The analogue of Assumption 5.1, with V_0 empty, then applies to any pair $(L'_{i,v}, M'_v)$. It follows from the induction hypothesis for Lemma 5.3 that

$$ {}^0S_M^{L_i}(M'_{V_i}, \delta'_{V_i}, h_{V_i}) \ = \ 0\,. $$

If neither L_1 nor L_2 equals G, we find that (6.9) equals

$$ d_M^G(L_1, L_2)\widehat{I}_M^{L_1,\mathcal{E}}(\delta'_{V_1}, f_{V_1,L_1})\widehat{I}_M^{L_2,\mathcal{E}}(\delta'_{V_2}, f_{V_2,L_2})\,. $$

If one of the groups L_i equals G, there will also be a supplementary term. Observe that the coefficient $d_M^G(L_1, L_2)$ vanishes in this case unless the other group equals M. Since $d_M^G(G, M) = 1$, and

$$ \widehat{S}_M^M(\delta'_{V_i}, f_{V_i}^{M'}) \ = \ f_{V_i}^{M'_{V_i}}(\delta'_{V_i}) \ = \ f_{V_i}^{M'}(\delta'_{V_i})\,, $$

the supplementary term is just (-1) times the relevant summand on the right hand side of $(6.3')$. Summing the formula we have obtained for (6.9) over L_1 and L_2, we conclude that (6.6) equals the sum of

$$ (6.10) \qquad -\big(f_{V_1}^{M'}(\delta'_{V_1})\,{}^0S_M^G(M'_{V_2}, \delta'_{V_2}, f_{V_2}) + {}^0S_M^G(M'_{V_1}, \delta'_{V_1}, f_{V_1})f_{V_2}^{M'}(\delta'_{V_2})\big) $$

and

$$ (6.11) \qquad \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2)\widehat{I}_M^{L_1,\mathcal{E}}(\delta'_{V_1}, f_{V_1,L_1})\widehat{I}_M^{L_2,\mathcal{E}}(\delta'_{V_2}, f_{V_2,L_2})\,. $$

Notice that (6.10) is (-1) times the right hand side of $(6.3')$, while (6.11) equals the right hand side of (6.4).

Suppose first that $\mathcal{E}^0$ equals $\mathcal{E}$. Then the original expression (6.5) equals (6.6). Moreover, the definitions imply that

$$ |\widetilde{\mathcal{E}}^1|S_M^G(M'_V, \delta'_V, f_V) \ = \ {}^0S_M^G(M'_V, \delta'_V, f_V)\,. $$

Our conclusion in this case is that $I_M^{\mathcal{E}}(\delta'_V, f_V)$ equals the sum of (6.11) and

$$(6.12)\quad {}^0S_M^G(M'_V, \delta'_V, f_V) - \big(f_{V_1}^{M'}(\delta'_{V_1})\,{}^0S_M^G(M'_{V_2}, \delta'_{V_2}, f_{V_2}) + {}^0S_M^G(M'_{V_1}, \delta'_{V_1}, f_{V_1})f_{V_2}^{M'}(\delta'_{V_2})\big)\,.$$

If (F_V, G_V, M_V) is not quasisplit, $\widetilde{\mathcal{E}} - \mathcal{E}$ is empty, and ${}^0S_M^G(M'_V, \delta'_V, f_V)$ vanishes by definition. Moreover, V_0 is empty by assumption. We then deduce that

$${}^0S_M^G(M'_{V_i}, \delta'_{V_i}, f_{V_i}) \;=\; 0\,, \qquad\qquad i = 1, 2,$$

by definition if $(F_{V_i}, G_{V_i}, M_{V_i})$ is not quasisplit, or by applying the induction hypothesis for Lemma 5.3 to V_i (with the empty set $V_0 \cap V_i$ playing the role of V_0) if $(F_{V_i}, G_{V_i}, M_{V_i})$ is quasisplit. The whole expression (6.12) therefore vanishes. Thus, if (F_V, G_V, M_V) is not quasisplit, $I_M^{\mathcal{E}}(\delta'_V, f_V)$ equals (6.11), and the identity (6.4) holds. As we have already noted, this is equivalent to the identity (6.2) in part (a). If (F_V, G_V, M_V) is quasisplit, $|\widetilde{\mathcal{E}}^1|$ is positive, and $I_M^{\mathcal{E}}(\delta'_V, f_V)$ is given by the definition (5.4). The identities (6.2) and (6.4) follow in this case from (5.5) and the splitting formula (4.6) for $I_M(\gamma_V, f_V)$. This implies that (6.12) vanishes, and the required identity (6.3$'$) holds.

Suppose finally that $\mathcal{E}^0 \neq \mathcal{E}$. Then (F_V, G_V, M_V) is quasisplit, and we can set $\delta_V = \delta'_V$. Since the analogues for V_i of the sets $\mathcal{E}^0$ and $\mathcal{E}$ are also not equal, the terms in (6.10) vanish by definition. Therefore (6.10) makes no contribution in this case. On the other hand, $\mathcal{E}$ is a disjoint union of $\mathcal{E}^0$ with the nonempty set $\mathcal{E}^1$, so the original expression (6.5) does not equal (6.6). They differ by the expression given by the product of $|\mathcal{E}^1|$ with the right hand side of (6.3). Our conclusion in this case is that $I_M^{\mathcal{E}}(\delta'_V, f_V)$ equals the sum of (6.11) and

$$(6.13)\qquad |\mathcal{E}^1|\Big(S_M^G(\delta_V, f_V) - \sum_{L_1, L_2} \mathrm{e}_M^G(L_1, L_2)\widehat{S}_M^{L_1}(\delta_{V_1}, f_{V_1}^{L_1})\widehat{S}_M^{L_2}(\delta_{V_2}, f_{V_2}^{L_2})\Big)\,.$$

As before, (6.2) follows from (4.6) and the definition (5.4). This in turn implies that $I_M^{\mathcal{E}}(\delta'_V, f_V)$ equals (6.11). The required identity (6.3) then follows from the fact that (6.13) vanishes. This completes the proof of the theorem. $\square$

Having established endoscopic and stable splitting formulas, we can now give a simple reduction of the proof of Lemma 5.3. The distribution $I_M^{\mathcal{E}}(\delta'_V, f_V)$ in (6.3) satisfies the splitting formula (6.4). We are assuming that V_i, $i = 1, 2$, is a proper subset of V. It follows from the induction hypothesis for Lemma 5.3 that the distributions

$$\widehat{I}_M^{L_i,\mathcal{E}}(\delta'_{V_i}, f_{V_i,L_i}), \qquad\qquad L_i \in \mathcal{L}(M),$$

on the right hand side of (6.4) depend only on the image of δ'_{V_i} in $\widetilde{\Gamma}^{\mathcal{E}}_{G_{V_i}}(M_{V_i})$. Therefore $I_M^{\mathcal{E}}(\delta'_V, f_V)$, as the left hand side of (6.4), depends only on the image of $\delta'_V = (\delta'_{V_1}, \delta'_{V_2})$ in

$$\widetilde{\Gamma}^{\mathcal{E}}_{G_V}(M_V) \;=\; \widetilde{\Gamma}^{\mathcal{E}}_{G_{V_1}}(M_{V_1}) \times \widetilde{\Gamma}^{\mathcal{E}}_{G_{V_2}}(M_{V_2}).$$

This is part (a) of Lemma 5.3.

For part (b), assume that (F_V, G_V, M_V) is quasisplit. If G_V^* belongs to $\mathcal{E}$, set $\delta_V = \delta'_V$, and consider the stable splitting formula (6.3). The induction hypothesis for Lemma 5.3 implies that for $i = 1, 2$, the distributions

$$f_{V_i} \longrightarrow \widehat{S}_M^{L_i}(\delta_{V_i}, f_{V_i}^{L_i}), \qquad\qquad L_i \in \mathcal{L}(M),$$

on the right hand side of (6.3) are stable at each $v \in V_i - V_0$. Therefore the distribution $S_M^G(\delta_V, f_V)$ on the left hand side of (6.3) is stable at any v in $V - V_0$. This is the first assertion of part (b). For the remaining assertion, assume that G_V^* belongs to $\widetilde{\mathcal{E}} - \mathcal{E}$, and that V_0 is empty. If $i = 1, 2$, $G^*_{V_i}$ belongs to the set $\widetilde{\mathcal{E}}_i = \widetilde{\mathcal{E}}_{M'_{V_i}}(G_{V_i}, G)$. If $G^*_{V_i}$ lies in $\widetilde{\mathcal{E}}_i - \mathcal{E}_i$, the distribution ${}^0S(M'_{V_i}, \delta'_{V_i}, f_{V_i})$ on the right hand side of (6.3$'$) vanishes, by the induction hypothesis for Lemma 5.3. If $G^*_{V_i}$ does not lie in $\widetilde{\mathcal{E}}_i - \mathcal{E}_i$, ${}^0S_M^G(M'_{V_i}, \delta'_{V_i}, f_{V_i}) = 0$ by definition. Both terms on the right hand side of (6.3$'$) therefore vanish. It follows that ${}^0S_M^G(M'_V, \delta'_V, f_V)$, the left hand side of (6.3$'$), vanishes. This is the last assertion of Lemma 5.3.

We have reduced the proof of Lemma 5.3 to the case that V contains only one element. We shall complete it in the next section.

7. Stable descent formulas

We shall prove descent formulas for the special case of §5 that V consists of one element v. Suppose that (F_1, G_1, M_1) is an L-satellite of (F, G, M), that R_1 is a Levi subgroup of M_1, and that (F_v, G_v, R_v) is an L-satellite of (F_1, G_1, R_1). As in Proposition 4.2, we write $L_1 \to L_v = L_{1,v}$ for the map from $\mathcal{L}(R_1)$ to $\mathcal{L}(R_v)$. We take f_v to be a fixed function on $\mathcal{C}(G_v, \zeta_v)$.

Since we are dealing with L-satellites, there is an embedding of $Z(\widehat{M})^\Gamma$ into $Z(\widehat{M}_1)^{\Gamma_1}$ and an embedding of $Z(\widehat{G})^\Gamma$ into $Z(\widehat{G}_1)^{\Gamma_1}$. The former gives us an embedding of $Z(\widehat{M})^\Gamma$ into $Z(\widehat{R}_1)^{\Gamma_1}$, while the other provides an embedding of $Z(\widehat{G})^\Gamma$ into $Z(\widehat{L}_1)^{\Gamma_1}$, for each $L_1 \in \mathcal{L}(R_1)$. There are of course also embeddings $Z(\widehat{G})^\Gamma \subset Z(\widehat{M})^\Gamma$ and $Z(\widehat{L}_1)^{\Gamma_1} \subset Z(\widehat{R}_1)^{\Gamma_1}$. We can therefore define a coefficient

$$\mathrm{e}^G_{R_1}(M, L_1) \ = \ d^G_{R_1}(M, L_1)\big|Z(\widehat{M})^\Gamma \cap Z(\widehat{L}_1)^{\Gamma_1}/Z(\widehat{G})^\Gamma\big|^{-1}, \tag{7.1}$$

for each $L_1 \in \mathcal{L}(R_1)$. This is an obvious generalization of (6.1).

Theorem 7.1. *(a) Suppose that γ_v lies in $\Gamma_{G_v}(R_v)$. Then*

$$I^{\mathcal{E}}_M(\gamma_v, f_v) \ = \ \sum_{L_1 \in \mathcal{L}(R_1)} d^G_{R_1}(M, L_1)\widehat{I}^{L_1,\mathcal{E}}_{R_1}(\gamma_v, f_{v,L_v}) \,. \tag{7.2}$$

(b) Suppose that (F_v, G_v, R_v) is quasisplit, and that δ_v lies in $\Sigma_{G_v}(R_v)$. Then

$$S^G_M(\delta_v, f_v) \ = \ \sum_{L_1 \in \mathcal{L}(R_1)} \mathrm{e}^G_{R_1}(M, L_1)\widehat{S}^{L_1}_{R_1}(\delta_v, f^{L_v}_v) \tag{7.3}$$

(b') Suppose that (F_v, G_v, M_v) is quasisplit, and that δ'_v lies in $\Sigma_{G_v}(R'_v)$, for some R'_v in $\mathcal{E}_{\mathrm{ell}}(R_v)$. Then ${}^0S^G_M(M'_v, \delta'_v, f_v)$ vanishes unless $R_1 = M_1$, $\mathfrak{a}_{M_1} = \mathfrak{a}_M$ and $\mathfrak{a}_{G_1} = \mathfrak{a}_G$, in which case we have

$${}^0S^G_M(M'_v, \delta'_v, f_v) \ = \ {}^0S^{G_1}_{M_1}(M'_v, \delta'_v, f_v) \,. \tag{7.3$'$}$$

Proof. The structure of the proof is parallel to that of Theorem 6.1. We fix an elliptic endoscopic datum R'_v for R_v, and a datum $M'_v \in \mathcal{E}_{R'_v}(M_v)$. Then M'_v is determined by a point s'_{M_v} in $s'_{R_v} Z(\widehat{R}_v)^{\Gamma_v}$. Replacing R'_v by another element in its equivalence class, if necessary, we shall assume that s'_{R_v} and s'_{M_v} are equal. We also fix a point δ'_v in $\Sigma_{G_v}(R'_v)$. If G'_v belongs to $\mathcal{E} = \mathcal{E}_{M'_v}(G_v, G)$, the triplets (F_v, G'_v, R'_v) and $(F_v, \widetilde{G}'_v, \widetilde{R}'_v)$ are also L-satellites of (F_1, G_1, R_1). We assume inductively that (7.3) is valid if (F_v, G_v, R_v) is replaced by a quasisplit inner K-form of $(F_v, \widetilde{G}'_v, \widetilde{R}'_v)$, for any G'_v in $\mathcal{E}^0 = \mathcal{E}^0_{M'_v}(G_v, G)$.

The required formula (7.2) has an analogue

$$I^{\mathcal{E}}_M(\delta'_v, f_v) \;=\; \sum_{L_1 \in \mathcal{L}(R_1)} d^G_{R_1}(M, L_1) \widehat{I}^{L_1, \mathcal{E}}_{R_1}(\delta'_v, f_{v, L_v}) \tag{7.4}$$

for δ'_v. The two formulas are equivalent by (5.5), so for part (a) it will be enough to establish (7.4).

It follows from (5.3) that

$$I^{\mathcal{E}}_M(\delta'_v, f_v) - |\widetilde{\mathcal{E}}^1| S^G_M(M'_v, \delta'_v, f_v) \tag{7.5}$$

equals

$$\sum_{G'_v \in \mathcal{E}^0} \widehat{S}^G_M(\delta'_v, f'_v)\,.$$

Applying (7.3) inductively to each G'_v, we see that the last sum equals

$$\sum_{G'_v \in \mathcal{E}^0} \sum_{L_1 \in \mathcal{L}(R_1)} \mathrm{e}^G_{R_1}(M, L_1) \widehat{S}^{L_1}_{R_1}(\delta'_v, f_v^{L'_v})\,.$$

Again, it is the expression

$$\sum_{G'_v \in \mathcal{E}} \sum_{L_1 \in \mathcal{L}(R_1)} \mathrm{e}^G_{R_1}(M, L_1) \widehat{S}^{L_1}_{R_1}(\delta'_v, f_v^{L'_v})\,, \tag{7.6}$$

obtained by summing G'_v over $\mathcal{E}$ instead of $\mathcal{E}^0$, that is the focal point of the argument.

Fix a group $L_1 \in \mathcal{L}(R_1)$, and set

$$\mathcal{E}_1 \;=\; \mathcal{E}_{R'_v}(L_v, L_1)\,.$$

We can then define a map

$$\mathcal{E} \longrightarrow \mathcal{E}_1 \tag{7.7}$$

by sending G'_v to L'_v. If G'_v corresponds to the element s'_v in $s'_{M_v}Z(\widehat{M})^\Gamma/Z(\widehat{G})^\Gamma$, L'_v corresponds to the point s'_{L_v} obtained by projecting s'_v onto $s'_{R_v}Z(\widehat{R}_1)^{\Gamma_1}/Z(\widehat{L}_1)^{\Gamma_1}$. We can assume that the coefficient $\mathrm{e}^G_{R_1}(M, L_1)$ in (7.6) is nonzero. It follows that $d^G_{R_1}(M, L_1) \neq 0$, so that $\mathfrak{a}_{R_1} = \mathfrak{a}_{L_1} + \mathfrak{a}_M$ and $\mathfrak{a}^*_{R_1,\mathbb{C}} = \mathfrak{a}^*_{L_1,\mathbb{C}} + \mathfrak{a}^*_{M,\mathbb{C}}$. Therefore $\big(Z(\widehat{R}_1)^{\Gamma_1}\big)^0$ equals the product of $\big(Z(\widehat{L}_1)^{\Gamma_1}\big)^0$ with $\big(Z(\widehat{M})^\Gamma\big)^0$. Since $Z(\widehat{R}_1)^{\Gamma_1}$ equals the product of $Z(\widehat{L}_1)^{\Gamma_1}$ with $\big(Z(\widehat{R}_1)^{\Gamma_1}\big)^0$, by Lemma 1.1, we have

$$Z(\widehat{R}_1)^{\Gamma_1} \;=\; Z(\widehat{L}_1)^{\Gamma_1} Z(\widehat{M})^\Gamma \,.$$

Therefore the map (7.7) is surjective. Furthermore, the group

$$Z(\widehat{M})^\Gamma \cap Z(\widehat{L}_1)^{\Gamma_1}/Z(\widehat{G})^\Gamma \tag{7.8}$$

is finite and acts simply transitively on the fibres of the map. The summand in (7.6) depends only on L'_v and not on the group G'_v. We can therefore replace the sum over G'_v in (7.6) by the sum over $L'_v \in \mathcal{E}_1$, provided that we multiply the summand by the order of the group (7.8). Since the product of $\mathrm{e}^G_{R_1}(M, L_1)$ with the order of (7.8) equals $d^G_{R_1}(M, L_1)$, we conclude that the contribution of L_1 to (7.6) equals

$$d^G_{R_1}(M, L_1) \sum_{L'_v \in \mathcal{E}_1} \widehat{S}^{L_1}_{R_1}(\delta'_v, f_v^{L'_v}) \,. \tag{7.9}$$

Continuing to follow the proof of Theorem 6.1, we note that for any function h_v in $\mathcal{C}(L_v, \zeta_v)$, the definition (5.6) provides an identity

$$I^{L_1,\mathcal{E}}_{R_1}(\delta'_v, h_v) \;=\; \sum_{L'_v \in \mathcal{E}_1} \widehat{S}^{L_1}_{R_1}(\delta'_v, h_v^{L'_v}) + {}^0S^{L_1}_{R_1}(R'_v, \delta'_v, h_v) \,.$$

Suppose that L_1 is such that $\mathfrak{a}_{L_1}$ is strictly larger than $\mathfrak{a}_G$. Since this implies that $\mathfrak{a}_{L_v}$ is also strictly larger than $\mathfrak{a}_G$, either $L_v \neq G_v$, or the set V_0 is empty. The analogue of

Assumption 5.1, with V_0 empty, then applies to any pair (L'_v, R'_v). It follows from the induction hypothesis for Lemma 5.3 that

$$^0S^{L_1}_{R_1}(R'_v, \delta'_v, h_v) \; = \; 0\,.$$

The expression (7.9) reduces in this case to

$$d^G_{R_1}(M, L_1)\widehat{I}^{L_1,\mathcal{E}}_{R_1}(\delta'_v, f_{v,L_v})\,.$$

In the remaining case that $L_1 = G_1$ and $\mathfrak{a}_{G_1} = \mathfrak{a}_G$, there will also be a supplementary term

$$-d^G_{R_1}(M_1, G_1)\, {}^0S^{G_1}_{R_1}(R'_v, \delta'_v, f_v)\,.$$

But the map

$$\mathfrak{a}^M_{R_1} \oplus \mathfrak{a}^G_{R_1} \longrightarrow \mathfrak{a}^G_{R_1}$$

fails to be an isomorphism unless $R_1 = M_1$, and $\mathfrak{a}_{M_1} = \mathfrak{a}_M$. The supplementary term can therefore be written as

$$(7.10) \qquad -\varepsilon_{R_1}\, {}^0S^{G_1}_{M_1}(M'_v, \delta'_v, f_v)\,,$$

where

$$\varepsilon_{R_1} \; = \; \varepsilon_{R_1}(M_1, M; G_1, G)$$

equals 1 or 0, according to whether the simultaneous conditions $R_1 = M_1$, $\mathfrak{a}_{M_1} = \mathfrak{a}_M$ and $\mathfrak{a}_{G_1} = \mathfrak{a}_G$ hold or not. Summing the formula we have obtained for (7.9) over L, we conclude that (7.6) equals the sum of

$$(7.11) \qquad \sum_{L_1 \in \mathcal{L}(R_1)} d^G_{R_1}(M, L_1)\widehat{I}^{L_1,\mathcal{E}}_{R_1}(\delta'_v, f_{v,L_v})$$

with (7.10).

Suppose that $\mathcal{E}^0$ equals $\mathcal{E}$. Then the original expression (7.5) equals (7.6). Moreover,

$$|\widetilde{\mathcal{E}}^1| S_M^G(M'_v, \delta'_v, f_v) \;=\; {}^0 S_M^G(M'_v, \delta'_v, f_v)\,.$$

Our conclusion in this case is that $I_M^{\mathcal{E}}(\delta'_v, f_v)$ equals the sum of (7.11) and

$$(7.12) \qquad {}^0 S_M^G(M'_v, \delta'_v, f_v) - \varepsilon_{R_1}\, {}^0 S_{M_1}^{G_1}(M'_v, \delta'_v, f_v)\,.$$

If (F_v, G_v, M_v) is not quasisplit, $\widetilde{\mathcal{E}} - \mathcal{E}$ is empty, and both terms in (7.12) vanish by definition. Therefore $I_M^{\mathcal{E}}(\delta'_v, f_v)$ equals (7.11), and the identities (7.2) and (7.4) hold. If (F_v, G_v, M_v) is quasisplit, $|\widetilde{\mathcal{E}}^1|$ is positive, and $I_M^{\mathcal{E}}(\delta'_v, f_v)$ is defined by (5.4). The identities (7.2) and (7.4) then follow from the descent formula (4.5) for $I_M(\gamma_v, f_v)$. Therefore (7.12) vanishes, from which assertion (b$'$) of the theorem follows.

Suppose finally that $\mathcal{E}^0 \neq \mathcal{E}$. Then (F_v, G_v, M_v) is quasisplit, and we set $\delta_v = \delta'_v$. In this case, the expression (7.10) vanishes by definition. On the other hand, the original expression (7.5) differs from (7.6) by the product of $|\mathcal{E}^1|$ with the right hand side of (7.3). Our conclusion in this case is that $I_M^{\mathcal{E}}(\delta'_v, f_v)$ equals the sum of (7.11) and

$$(7.13) \qquad |\mathcal{E}^1| \Big(S_M^G(\delta_v, f_v) - \sum_{L_1 \in \mathcal{L}(R_1)} \mathrm{e}_{R_1}^G(M, L_1) \widehat{S}_{R_1}^{L_1}(\delta_v, f_v^{L_1}) \Big)\,.$$

As above, (7.2) follows from (4.5) and the definition (5.4). This in turn implies that $I_M^{\mathcal{E}}(\delta'_v, f_v)$ equals (7.11). It follows that (7.13) vanishes, and the required identity (7.3) holds. $\square$

Corollary 7.2. *Consider the special case that $R_1 = M_1$, $\mathfrak{a}_{M_1} = \mathfrak{a}_M$ and $\mathfrak{a}_{G_1} = \mathfrak{a}_G$.*

(a) Suppose that $\gamma_v \in \Gamma_{G_v}(M_v)$. Then

$$(7.14) \qquad I_M^{G,\mathcal{E}}(\gamma_v, f_v) \;=\; I_{M_1}^{G_1,\mathcal{E}}(\gamma_v, f_v)\,.$$

(b) Suppose that (F_v, G_v, M_v) is quasisplit, and that $\delta_v \in \Sigma_{G_v}(M_v)$. Then

$$(7.15) \qquad S_M^G(\delta_v, f_v) \;=\; \iota_{M_1}(G, G_1) S_{M_1}^{G_1}(\delta_v, f_v)\,,$$

where

$$\iota_{M_1}(G, G_1) \;=\; \big|Z(\widehat{M}_1)^{\Gamma_1}/Z(\widehat{M})^{\Gamma}\big|\big|Z(\widehat{G}_1)^{\Gamma_1}/Z(\widehat{G})^{\Gamma}\big|^{-1}.$$

Proof. The conditions on M_1 and G_1 imply that

$$d^G_{M_1}(M, L_1) \;=\; \begin{cases} 1, & \text{if } L_1 = G_1, \\ 0, & \text{otherwise,} \end{cases}$$

for any $L_1 \in \mathcal{L}(M_1)$. The formula (7.14) then follows from (7.2). The coefficient $\mathrm{e}^G_{M_1}(M, L_1)$ also vanishes if $L_1 \neq G_1$. In the case $L_1 = G_1$, we have

$$\begin{aligned} \mathrm{e}^G_{M_1}(M, G_1) \;&=\; \big|Z(\widehat{M})^{\Gamma} \cap Z(\widehat{G}_1)^{\Gamma_1}/Z(\widehat{G})^{\Gamma}\big|^{-1} \\ &=\; \big|Z(\widehat{G}_1)^{\Gamma_1}/Z(\widehat{M})^{\Gamma} \cap Z(\widehat{G}_1)^{\Gamma_1}\big|\big|Z(\widehat{G}_1)^{\Gamma_1}/Z(\widehat{G})^{\Gamma}\big|^{-1} \\ &=\; \big|Z(\widehat{G}_1)^{\Gamma_1}Z(\widehat{M})^{\Gamma}/Z(\widehat{M})^{\Gamma}\big|\big|Z(\widehat{G}_1)^{\Gamma_1}/Z(\widehat{G})^{\Gamma}\big|^{-1}. \end{aligned}$$

Since

$$Z(\widehat{G}_1)^{\Gamma_1}Z(\widehat{M})^{\Gamma} \;=\; Z(\widehat{G}_1)^{\Gamma_1}\big(Z(\widehat{M})^{\Gamma}\big)^0 \;=\; Z(\widehat{G}_1)^{\Gamma_1}\big(Z(\widehat{M}_1)^{\Gamma_1}\big)^0 \;=\; Z(\widehat{M}_1)^{\Gamma_1},$$

by Lemma 1.1, we find that

$$\mathrm{e}^G_{M_1}(M, G_1) \;=\; \iota_{M_1}(G, G_1).$$

The formula (7.15) follows from (7.3). □

Corollary 7.3. *The construction of* §5 *reduces to the original definitions (3.4) and (3.5) in the case of the basic distributions of* §3*.*

Proof. Suppose that (F, G, M) is as in §3. This is the special case of the framework of §4 and §5 in which V contains one element v, and $(F_v, G_v, M_v) = (F, G, M)$. We take $f_v = f$ to be fixed function in $\mathcal{C}(G, \zeta)$, and $\delta'_v = \delta'$ to be an elliptic element in $\Sigma_G(\widetilde{M}')$. Consider the summand on the right hand side of (5.3) corresponding to an element $G'_v = G'$ in $\mathcal{E}^0 = \mathcal{E}^0_{M'}(G)$. Taking $(F_1, G_1, M_1) = (F, G', M')$ in (7.15), we can write this summand as

$$\widehat{S}^G_M(\delta', f') \;=\; \iota_{M'}(G, G')\widehat{S}^{G'}_{M'}(\delta', f').$$

(The objects that would have been denoted (F_v, G_v, M_v) and f_v in (7.15) are the objects (F, G', M') and f' here.) In general, δ' and f' are not attached to M' and G', but rather to fixed central extensions $\widetilde{M}'$ and $\widetilde{G}'$ by the induced torus $\widetilde{Z}'$. However, a simple application of the descent formula (7.3) allows us to write

$$\iota_{M'}(G, G')\widehat{S}^{G'}_{M'}(\delta', f') \;=\; \iota_{M'}(G, G')\widehat{S}^{\widetilde{G}'}_{\widetilde{M}'}(\delta', f')\,.$$

Therefore the summand on the right hand side of (5.3) matches the summand on the right hand side of (3.5). Since $|\widetilde{\mathcal{E}}^1|$ equals $\mathcal{E}(G)$ in this case, the two families are the same, and the inductive definitions of §5 reduce to those of §3. $\square$

As we remarked in §4, it would have been possible to derive more general descent formulas. If V parametrizes several L-satellites (F_v, G_v, M_v) of (F_1, G_1, R_1), the descent formulas (7.2), (7.3), (7.14) and (7.15) remain valid as stated. The proofs are similar to the special case $V = \{v\}$ that we established. The more general formulas could also be established by combining this special case with the splitting formulas.

We shall now complete the proof of Lemma 5.3. By the reduction at the end of §6, we can assume that V contains one element $\{v\}$, as we have throughout this section. The lemma pertains to a class $\delta'_v \in \Sigma_{G_v}(\widetilde{M}'_v)$. This class need not, of course, be elliptic over F_v. However, we can find Levi subgroups R_v, R'_v and $\widetilde{R}'_v$ of M_v, M'_v and $\widetilde{M}'_v$ respectively, such that δ'_v is the image of a class in $\Sigma_{G_v,\mathrm{ell}}(\widetilde{R}'_v)$. This follows easily from [14, Lemma 10.2]. We can certainly identify R'_v with an elliptic endoscopic datum for R_v, and $\widetilde{R}'_v$ with a central extension of R'_v. We have then only to apply Theorem 7.1, with $(F_1, G_1, R_1) = (F_v, G_v, R_v)$. The induction hypothesis for Lemma 5.3 applies to all the terms in the relevant descent formula, and establishes the lemma for (F, G, M), except in the case that $R_v = M_v$, $\mathfrak{a}_{M_v} = \mathfrak{a}_M$ and $\mathfrak{a}_{G_v} = \mathfrak{a}_G$. In this latter case, however, we can apply the simpler descent formulas of Corollary 7.2. The proof of the lemma then reduces to the case that $(F, G, M) = (F_v, G_v, M_v)$, and that $\delta'_v \in \Sigma_{G_v}(\widetilde{M}'_v)$ is elliptic. This is the basic setting of §3.

It remains, then, to establish Lemma 5.3 for the distributions of §3. In this case, part (b) of the lemma follows from the definition. More precisely, part (b) is simply a restatement of the second assertion of Assumption 5.1. Part (a) in this case is part of Lemma 3.1. The proof of Lemma 5.3 is thus complete. □

With Lemma 5.3 proved, there remain no more loose ends from the definitions of §5. We have resolved all the induction hypotheses, or rather, we have reduced them to the basic Assumption 5.1. The main point has been that the splitting and descent theorems provide a decomposition of the compound distributions of §5 into the simple ones of §3.

Proposition 7.4. *Assume that V is arbitrary, and that for each $v \in V$, the distributions associated to the triplets (F_v, G_v, M_v) by the basic construction of §3 satisfy Conjecture 3.3.*

(a) Suppose that $\gamma_V \in \Gamma_{G_V}(M_V)$. Then

$$I_M^{\mathcal{E}}(\gamma_V, f_V) \;=\; I_M(\gamma_V, f_V)\,.$$

(b) Suppose that (F_V, G_V, M_V) is quasisplit, and that δ'_V lies in $\Sigma_{G_V}(M'_V)$. Then the distribution

$$f_V \longrightarrow S_M^G(M'_V, \delta'_V, f_V)\,, \qquad\qquad f_V \in \mathcal{C}(G_V, \zeta_V),$$

vanishes unless G^*_V belongs to $\mathcal{E}$, in which case it is stable.

Proof. The splitting and descent formulas of Theorems 6.1 and 7.1 reduce the assertions of the proposition immediately to the corresponding assertions of Conjecture 3.3. □

As we suggested at the end of §4, the general framework of the last few sections has been modelled on two basic cases. The first, in which F is a local field, is that of the local trace formula. We shall see in §8 and §9 that the associated distributions $I_M^{\mathcal{E}}(\gamma_V, f_V)$ and $S_M^G(M'_V, \delta'_V, f_V)$ arise naturally in the stabilized local trace formula. The second case, in which F is a global field, is that of the global trace formula. We shall investigate the stabilized global trace formula in a future paper. It may be that there are other trace formulas that combine groups $\{G_v\}$ that are genuinely distinct. Observe, however, that even in the two basic cases, we are forced to consider the more general setting. The inductive definitions (5.3)–(5.5) give rise to distributions on groups G'_V composed of distinct endoscopic groups G'_v for G_v.

8. Local vanishing theorems

For the study of inner forms and base change for $GL(n)$ [9], it was necessary to establish certain vanishing properties [3, §10] before attempting to compare trace formulas. As further evidence for our conjectural transfer identity, we shall establish general analogues of some of these results. In particular, we shall establish a local vanishing theorem that we shall use in §9 to stabilize part of the local trace formula. We shall save the corresponding global vanishing theorem for a future paper on the stable global trace formula.

The arguments in [3, §10] for $GL(n)$ break down in general. In fact, the naive generalization of the formula for $GL(n)$ turns out to be false. The correct generalization also appears at first glance to be doomed. A closer study, however, will reveal some unexpected cancellation caused by internal signs in the transfer factors. The phenomenon is one more example of the efficacy of the transfer factors.

For the rest of the paper, F will be a local field, and G will be a K-group over F, equipped with a quasisplit inner twist G^*. For simplicity, we fix a minimal Levi subgroup M_0 of G, according to the remarks following the proof of Lemma 2.1. We also fix a Levi subgroup M_0^* of G^* corresponding to M_0, and a Levi subgroup $\widehat{M}_0$ of $\widehat{G}$ dual to M_0. As explained in §1, we then have a bijection $M \to M^*$ from $\mathcal{L}(M_0)$ to $\mathcal{L}(M_0^*)$, and a bijection $M \to \widehat{M}$ from $\mathcal{L}(M_0)$ to $\mathcal{L}(\widehat{M}_0)$.

We are interested in the special case of the framework of §4 that applies to the local trace formula. Then V contains two elements v_1 and v_2, and

$$(F_v, G_v, M_v) \;=\; (F, G, M)\,, \qquad v \in V,$$

for any $M \in \mathcal{L}(M_0)$. The embedding $\mathfrak{a}_M \hookrightarrow \mathfrak{a}_{M_v} = \mathfrak{a}_M$ is defined to be 1 if $v = v_1$ and to be (-1) if $v = v_2$. The same condition applied to the group $Z(\widehat{M})$ gives each triplet (F_v, G_v, M_v) the structure of an L-satellite of (F, G, M). If G' stands for an endoscopic datum $(G', s', \mathcal{G}', \xi')$ for G, we write $\overline{G}'$ for the adjoint endoscopic datum $\big(G', (s')^{-1}, \mathcal{G}', \xi'\big)$. This provides a natural embedding

$$G' \longrightarrow G'_V = G' \times \overline{G}'$$

of $\mathcal{E}(G)$ into $\mathcal{E}(G_V)$. Following the same notation for M, we fix $M' \in \mathcal{E}_{\mathrm{ell}}(M)$ and set $M'_V = M' \times \overline{M}'$. The family $\mathcal{E}_{M'_V}(G_V, G)$ of §5 is by construction in canonical bijection with the family $\mathcal{E}_{M'}(G)$ of §3. Since the embedding of $Z(\widehat{M})^\Gamma$ into $Z(\widehat{M}_V)^{\Gamma_V} = Z(\widehat{M})^\Gamma \times Z(\widehat{M})^\Gamma$ is defined by $s \to (s, s^{-1})$, this bijection is given by the map $G' \to G'_V = G' \times \overline{G}'$ above. We shall usually just identify G' with G'_V, M' with M'_V and $\mathcal{E}_{M'}(G)$ with $\mathcal{E}_{M'_V}(G_V, G)$.

In this section we shall also be interested in Levi subgroups of G^*. The vanishing results will apply only to the case that G is not quasisplit, so we shall take the various constructions for granted in the quasisplit case. We fix a Levi subgroup R of G^*, together with a dual Levi subgroup $\widehat{R} \subset \widehat{G}$, and we assume that Assumption 5.1 holds for any quasisplit inner K-form of (F, G^*, R), with V_0 being the empty subset of V. That is, we assume that the distributions associated in §3 to a quasisplit inner K-form of (F, G^*, R) are defined, and satisfy part (b) of Conjecture 3.3.

Suppose that R' is an elliptic endoscopic datum for R. Following the conventions above for G, we shall usually identify R' with $R'_V = R' \times \overline{R}'$, and $\mathcal{E}_{R'}(G^*)$ with $\mathcal{E}_{R'_V}(G^*_V, G^*)$. Of course, the elements in $\mathcal{E}_{R'}(G^*)$ can also be regarded as endoscopic data for G. Suppose $\sigma' = (\sigma'_1, \sigma'_2)$ is any point in $\Sigma_G(\widetilde{R}'_V)$. Our aim is to study the distribution

$$I^{\mathcal{E}}_R(\sigma', f) \;=\; \sum_{G' \in \mathcal{E}_{R'}(G^*)} \widehat{S}^{G^*}_R(\sigma', f'), \qquad f \in \mathcal{C}(G_V, \zeta_V), \tag{8.1}$$

on $G_V(F_V)$. This is really a hybrid for G and G^* of the distributions (5.3), since R need not come from G, in the sense of being conjugate to an element in $\mathcal{L}(M^*_0)$. In particular, we cannot follow the earlier convention for choosing the base point $(\overline{\delta}', \overline{\gamma})$ implicit in the transfer f'. We shall instead take $(\overline{\delta}', \overline{\gamma})$ to be the diagonal image in $\Sigma_G(\widetilde{G}'_V) \times \Gamma(G_V)$ of any point $(\overline{\delta}', \overline{\gamma})$ in $\Sigma_G(\widetilde{G}') \times \Gamma(G)$ such that $\overline{\delta}'$ is an image $\overline{\gamma}$. We require that the preassigned value of the transfer factor for $(G, \overline{G}')$ at $(\overline{\delta}', \overline{\gamma})$ be the inverse of the one for (G, G'). We may also assume that these preassigned values each have absolute value 1. If R is conjugate to an element $M^* \in \mathcal{L}(M_0^*)$, we are of course free to choose $\overline{\gamma}$ in $\Gamma_G(M)$. The definition (8.1) then matches (5.3).

The independence of (8.1) on the choice of base points is a consequence of the following observation of Kottwitz.

Lemma 8.1. *If G' is an endoscopic datum for G, the relative transfer factor for $(G, \overline{G}')$ is the inverse of the relative transfer factor for (G, G').*

Proof. If $\overline{\Delta}(\delta', \gamma; \overline{\delta}', \overline{\gamma})$ denotes the relative transfer factor for $(G, \overline{G}')$, we have to show that

$$\overline{\Delta}(\delta', \gamma; \overline{\delta}', \overline{\gamma}) = \Delta(\delta', \gamma; \overline{\delta}', \overline{\gamma})^{-1}.$$

This follows readily from an examination of the four terms in the product for $\Delta(\delta', \gamma; \overline{\delta}', \overline{\gamma})$. The main point is to note that if $\{\chi_\alpha\}$ are the χ-data for G' which occur in the factors Δ_{II} and Δ_2, one obtains inverse factors by choosing $\{\chi_\alpha^{-1}\}$ to be the χ-data for $\overline{G}'$. □

Corollary 8.2. *The absolute transfer factor for (G_V, G'_V) satisfies*

$$\Delta(\delta', \gamma) = \Delta(\delta'_1, \gamma_1; \delta'_2, \gamma_2),$$

for any points $\delta' = (\delta'_1, \delta'_2)$ and $\gamma = (\gamma_1, \gamma_2)$ in $\Sigma_G(\widetilde{G}'_V)$ and $\Gamma_G(G_V)$ respectively. In particular, $\Delta(\delta', \gamma)$ is independent of the choice of base point $(\overline{\delta}', \overline{\gamma})$.

Proof. The transfer factor for (G_V, G'_V) is a product of the transfer factors for (G, G') and $(G, \overline{G}')$. Since the preassigned values at $(\overline{\delta}', \overline{\gamma})$ cancel, the lemma gives us

$$\Delta(\delta', \gamma) = \Delta(\delta'_1, \gamma_1; \overline{\delta}', \overline{\gamma})\Delta(\delta'_2, \gamma_2; \overline{\delta}', \overline{\gamma})^{-1}.$$

The corollary follows from [15, Lemma 4.1.A]. □

We now state the main vanishing theorem. We fix $R, \widehat{R}, R'$ and $\sigma' \in \Sigma_G(\widetilde{R}'_V)$ as above, and we fix $f \in \mathcal{C}(G_V, \zeta_V)$.

Theorem 8.3. *(i) Suppose that no $G^*(F)$-conjugate of R belongs to $\mathcal{L}(M_0^*)$. Then*

$$I_R^{\mathcal{E}}(\sigma', f) \;=\; 0\,.$$

(ii) Suppose that R is $G^*(F)$-conjugate to an element M^* in $\mathcal{L}(M_0^*)$. If

$$(M', \delta')\,, \qquad\qquad M' \in \mathcal{E}_{\mathrm{ell}}(M),\ \delta' \in \Sigma_G(\widetilde{M}'_V),$$

is the corresponding image of (R', σ'), then

$$I_R^{\mathcal{E}}(\sigma', f) \;=\; I_M^{\mathcal{E}}(\delta', f)\,.$$

Proof. The identity in (ii) is more or less formal, so we shall concentrate on the vanishing assertion (i). We are certainly free to assume that

$$f \;=\; f_1 \times f_2\,, \qquad\qquad f_i \in \mathcal{C}(G, \zeta_{v_i}).$$

According to the splitting formula (6.3), our distribution

$$I_R^{\mathcal{E}}(\sigma', f) \;=\; \sum_{G' \in \mathcal{E}_{R'}(G^*)} \widehat{S}_R^{G^*}(\sigma', f')$$

can be written in the form

$$\sum_{G'} \sum_{L} \mathrm{e}_R^{G^*}(L) \widehat{S}_R^{L}\big(\sigma', (f')^{L'}\big)\,, \tag{8.2}$$

where L is summed over pairs (L_1, L_2) of elements in $\mathcal{L}(R)$, $\mathrm{e}_R^{G^*}(L)$ equals the constant $\mathrm{e}_R^{G^*}(L_1, L_2)$, $L' = (L_1', \overline{L}_2')$ is the pair of endoscopic data for L defined by G', and

$$\widehat{S}_R^{L}\big(\sigma', (f')^{L'}\big) \;=\; \widehat{S}_R^{L_1}\big(\sigma_1', (f_1^{G'})^{L_1'}\big) \widehat{S}_R^{L_2}\big(\sigma_2', (f_2^{\overline{G}'})^{\overline{L}_2'}\big)\,.$$

As in the proof of Theorem 6.1, we shall study (8.2) by interchanging the sums over G' and L.

Fix a pair $L = (L_1, L_2)$ of groups in $\mathcal{L}(R)$, and a corresponding pair $L' = (L_1', \overline{L}_2')$ of endoscopic data. We shall consider the contribution to (8.2) of those groups in $\mathcal{E}_{R'}(G^*)$ that map to L'. We can assume that $\mathrm{e}_R^{G^*}(L) \neq 0$. Then as in the proof of Theorem 6.1, the group

$$Z(\widehat{L}_1)^\Gamma \cap Z(\widehat{L}_2)^\Gamma / Z(\widehat{G})^\Gamma$$

is finite, and has a simply transitive action $G' \to G_s'$ on the set of G' which map to L'. It will be convenient here to consider the orbits under a smaller group. Recall that $\widehat{L}_{i,\mathrm{sc}}$ denotes the preimage of $\widehat{L}_i$ in $\widehat{G}_{\mathrm{sc}}$. The group

$$Z(\widehat{L}_{1,\mathrm{sc}})^\Gamma \cap Z(\widehat{L}_{2,\mathrm{sc}})^\Gamma / \widehat{Z}_{\mathrm{sc}}^\Gamma \tag{8.3}$$

injects into $Z(\widehat{L}_1)^\Gamma \cap Z(\widehat{L}_2)^\Gamma / Z(\widehat{G})^\Gamma$, and therefore also acts on the set of G' that map to L'. We shall consider the orbit under (8.3) of a fixed G'. The contribution of this orbit to (8.2) equals the product of $\mathrm{e}_{M^*}^{G^*}(L)$ with

$$\sum_s \widehat{S}_R^L\big(\sigma', (f_s')^{L'}\big)\,, \tag{8.4}$$

where s is summed over the group (8.3), and where

$$(f_s')^{L'} \;=\; (f_1^{G_s'})^{L_1'} \times (f_2^{\overline{G}_s'})^{\overline{L}_s'}\,.$$

Since the fixed elements L, L', and G' were arbitrary, Theorem 8.3 will be proved if we can show that (8.4) vanishes.

We can certainly assume that there is an s such that the function $(f_s')^{L'}$ in (8.4) does not vanish. This implies that the endoscopic groups L_1' and $\overline{L}_2'$ both contain rational elements that are images of elements in G. Consequently, the groups L_1 and L_2 in $\mathcal{L}(R)$ are both conjugate to groups in $\mathcal{L}(M_0^*)$. We fix elements ω_1 and ω_2 in $G_{\mathrm{sc}}^*(F)$ such that

$$\mathrm{Int}(\omega_i)L_i \;=\; M_i^*\,, \qquad M_i \in \mathcal{L}(M_0),\ i = 1, 2.$$

As groups in $\mathcal{L}(R)$, L_1 and L_2 have dual Levi subgroups $\widehat{L}_1$ and $\widehat{L}_2$ in $\mathcal{L}(\widehat{R})$, while M_1 and M_2 have dual Levi subgroups $\widehat{M}_1$ and $\widehat{M}_2$ in $\mathcal{L}(\widehat{M}_0)$. We can choose Γ-invariant elements $\widehat{\omega}_1$ and $\widehat{\omega}_2$ in $\widehat{G}_{\mathrm{sc}}$ such that $\mathrm{Int}(\widehat{\omega}_i)\widehat{L}_i = \widehat{M}_i$, and such that the isomorphism $\mathrm{Int}(\widehat{\omega}_i)\colon\ \widehat{L}_i \to \widehat{M}_i$ is dual to $\mathrm{Int}(\omega_i)\colon\ L_i \to M_i$, for $i = 1, 2$. In particular

$$(8.5) \qquad \widehat{\omega}_i Z(\widehat{L}_{i,\mathrm{sc}})^{\Gamma}\widehat{\omega}_i^{-1} \ = \ Z(\widehat{M}_{i,\mathrm{sc}})^{\Gamma} \ \subset \ Z(\widehat{M}_{0,\mathrm{sc}})^{\Gamma}, \qquad i = 1, 2.$$

The group L' is of course independent of the element s in (8.3). It might appear initially that the same is true of the function $(f'_s)^{L'}$. If L_1 and L_2 both actually contained M_0^*, they could be identified with Levi subgroups of G, and the function $f_L = f_{1,L_1} \times f_{2,L_2}$ would make sense. Then $(f'_s)^{L'}$ would equal $(f_L)^{L'}$, and would indeed be independent of s. We used this property at the relevant stage of the proof of Theorem 6.1. As matters stand here, however, L_1 and L_2 contain only conjugates of M_0. To see how this causes $(f'_s)^{L'}$ to vary with s, we have to look at the transfer factors. Let

$$\Delta_s(\delta', \gamma), \qquad \delta' \in \Sigma_G(\widetilde{L}'_V),\ \gamma \in \Gamma_G(G_V),$$

be the restriction to L'_V of the transfer factor from G_V to $G'_{s,V}$. We would like to compare $\Delta_s(\delta', \gamma)$ with the corresponding transfer factor $\Delta(\delta', \gamma)$ from G_V to G'_V.

It follows from (8.5) that

$$(8.6) \qquad s \longrightarrow s_L \ = \ (\widehat{\omega}_1 s \widehat{\omega}_1^{-1})^{-1}(\widehat{\omega}_2 s \widehat{\omega}_2^{-1}), \qquad s \in Z(\widehat{L}_{1,\mathrm{sc}})^{\Gamma} \cap Z(\widehat{L}_{2,\mathrm{sc}})^{\Gamma}/\widehat{Z}_{\mathrm{sc}}^{\Gamma},$$

is a homomorphism from the group (8.3) into $Z(\widehat{M}_{0,\mathrm{sc}})^{\Gamma}$. In §2 we defined a character ζ_G^0 on $Z(\widehat{M}_{0,\mathrm{sc}})^{\Gamma}$ that factors through the group of connected components. This gives us a character

$$s \longrightarrow \zeta_G^0(s_L)$$

on the group (8.3).

Lemma 8.4. *The transfer factors satisfy*

$$\Delta_s(\delta', \gamma) \;=\; \zeta_G^0(s_L)\Delta(\delta', \gamma)\,,$$

for points $\delta' = (\delta_1', \delta_2') \in \Sigma_G(\widetilde{L}_V')$ and $\gamma = (\gamma_1, \gamma_2) \in \Gamma(G_V)$, and for any s in the group (8.3).

Proof. We can arrange that for each G_s', the extension $\widetilde{G}_s'$ belongs to $\mathcal{E}_{\widetilde{M}'}(\widetilde{G})$, for a suitable z-extension $\widetilde{G}$ of G. According to the definitions in [15, (4.4)] and in §2, the transfer factors for (G, G_s') and $(\widetilde{G}, \widetilde{G}_s')$ are essentially the same. Replacing G by $\widetilde{G}$ if necessary, we can assume that $\widetilde{G}_s' = G_s'$ and that the embedding ξ_s' identifies $\mathcal{G}_s'$ with an L-group ${}^L G_s'$.

We recall from Corollary 8.2 that

$$\Delta_s(\delta', \gamma) \;=\; \Delta_s(\delta_1', \gamma_1; \delta_2', \gamma_2)\,.$$

We shall assume that δ_i' is an image of γ_i (in the sense of [15, (1.3)]), since the transfer factor would otherwise vanish. The relative transfer factor on the right is then defined as in (2.3) as a product of four terms.

Fix representatives $\delta_1' \in L_1'(F)$ and $\delta_2' \in L_2'(F)$ within the given stable conjugacy classes, and let $T_1' \subset L_1'$ and $T_2' \subset L_2'$ be their respective centralizers. The individual terms in the product depend on admissible embeddings $T_1' \to T_1$ and $T_2' \to T_2$, for maximal tori $T_1 \subset L_1$ and $T_2 \subset T_2$, defined over F. We are trying to see how $\Delta_s(\delta', \gamma)$ varies with s. The first three terms in the product (2.3) are quotients of absolute factors for (δ_1', γ_1) and (δ_2', γ_2) respectively. An inspection of the definitions in [15, §3] of these terms quickly reveals that the factors in the quotients depend only on the endoscopic data L_i', and not on the datum G_s' that maps to L_i'. They are thus each independent of s. Therefore

$$\begin{aligned}
&\Delta_s(\delta', \gamma)\Delta(\delta', \gamma)^{-1} \\
&= \Delta_s(\delta_1', \gamma_1; \delta_2', \gamma_2)\Delta(\delta_1', \gamma_1; \delta_2', \gamma_2)^{-1} \\
&= \Delta_{1,s}(\delta_1', \gamma_1; \delta_2', \gamma_2)\Delta_1(\delta_1', \gamma_1; \delta_2', \gamma_2)^{-1}\,,
\end{aligned}$$

where we have written $\Delta_{1,s}$ to denote the dependence of the fourth term in the product (2.3) on s.

Keeping in mind that G'_s stands for a full endoscopic datum $(G'_s, s'_s, \mathcal{G}'_s, \xi'_s)$, we recall from the definition in §2 and [15, (3.4)] that

$$\Delta_{1,s}(\delta'_1, \gamma_1; \delta'_2, \gamma_2) \;=\; \Big\langle \mathrm{inv}\Big(\frac{\delta'_1, \gamma_1}{\delta'_2, \gamma_2}\Big), (s'_s)_U \Big\rangle .$$

The point

$$(s'_s)_U \;=\; \big((s'_s)_{T_1}, (s'_s)_{T_2}\big)$$

lies in $\pi_0(\widehat{U}^\Gamma)$, for the dual torus

$$\widehat{U} \;=\; \widehat{T}_{1,\mathrm{sc}} \times \widehat{T}_{2,\mathrm{sc}}/\big\{(z,z):\; z \in \widehat{Z}_{\mathrm{sc}}\big\} .$$

By definition, $(s'_s)_{T_i}$ is a preimage in $\widehat{T}_{i,\mathrm{sc}}$ of the projection onto $\widehat{G}_{\mathrm{ad}}$ of a certain point in $\widehat{T}_i$; the latter is obtained from s'_s and the admissible embedding as in [15, p. 241]. Now $s'_s = \overline{s}s'$, where s' is the semisimple point attached to the fixed endoscopic datum G', and $\overline{s}$ the projection onto $\widehat{G}$ of our variable point s. The torus $\widehat{T}_{i,\mathrm{sc}}$ is contained in $\widehat{L}_{i,\mathrm{sc}}$, and s belongs to the center of $\widehat{L}_{i,\mathrm{sc}}$. It follows that

$$(s'_s)_U \;=\; \big((\overline{s}s')_{T_1}, (\overline{s}s')_{T_2}\big) \;=\; (ss'_{T_1}, ss'_{T_2}) \;=\; s_U s'_U ,$$

where $s_U = (s,s)$. We conclude that

$$\begin{aligned} &\Delta_s(\delta', \gamma)\Delta(\delta', \gamma)^{-1} \\ &= \Big\langle \mathrm{inv}\Big(\frac{\delta'_1, \gamma_1}{\delta'_2, \gamma_2}\Big),\, s_U s'_U \Big\rangle \Big\langle \mathrm{inv}\Big(\frac{\delta'_1, \gamma_1}{\delta'_2, \gamma_2}\Big),\, s'_U \Big\rangle^{-1} \\ &= \Big\langle \mathrm{inv}\Big(\frac{\delta'_1, \gamma_1}{\delta'_2, \gamma_2}\Big),\, s_U \Big\rangle . \end{aligned}$$

It remains to compute this last pairing.

Recall that $\mathrm{inv}\left(\frac{\delta'_1, \gamma_1}{\delta'_2, \gamma_2}\right)$ is the image in $H^1(F, U)$ of a 1-cocycle

$$\tau \longrightarrow \big(v_1(\tau)^{-1}, v_2(\tau)\big) , \qquad \tau \in \Gamma,$$

where

$$v_i(\tau) \;=\; h_i u_{\alpha_i}(\tau)\tau(h_i)^{-1}\,, \qquad\qquad i = 1, 2.$$

We are assuming here that γ_i belongs to the component G_{α_i} in G. Then h_i is an element in G^*_{sc} such that $h_i\psi_{\alpha_i}(\gamma_i)h_i^{-1}$ equals the image of δ'_i in T_i. We can choose the frame (ψ, u) such that for each i, $\psi_{\alpha_i}(M_i) = M_i^*$ and $u_{\alpha_i}(\tau) \in M^*_{i,\mathrm{sc}}$. We are also now regarding γ_1 and γ_2 as fixed points within the given conjugacy classes. We choose them so that $\gamma_i \in M_i(F)$, and so that

$$h_i \;=\; \ell_i\omega_i^{-1}\,, \qquad\qquad \ell_i \in L_{i,\mathrm{sc}},\ i = 1, 2.$$

Then

$$v_i(\tau) \;=\; \ell_i\omega_i^{-1}u_{\alpha_i}(\tau)\omega_i\tau(\ell_i)^{-1}\,.$$

In particular, $v_i(\tau)$ lies in $L_{i,\mathrm{sc}}$.

The pairing is by definition the value at $s_U \in \pi_0(\widehat{U}^\Gamma)$ of a certain character, determined by the K_U-image of the class of the cocycle $\big(v_1(\tau)^{-1}, v_2(\tau)\big)$. But $s_U = (s, s)$ comes from an element s in $Z(\widehat{L}_{1,\mathrm{sc}})^\Gamma \cap Z(\widehat{L}_{2,\mathrm{sc}})^\Gamma/\widehat{Z}^\Gamma_{\mathrm{sc}}$. It follows easily from [14, Theorem 1.2] that the pairing equals

$$\big\langle K_{L_{\mathrm{ad}}}(\overline{u}_{L_{\mathrm{ad}}}), s_U\big\rangle\,,$$

where

$$L_{\mathrm{ad}} \;=\; L_{1,\mathrm{sc}}/Z(G^*_{\mathrm{sc}}) \times L_{2,\mathrm{sc}}/Z(G^*_{\mathrm{sc}})\,,$$

and where

$$\overline{u}_{L_{\mathrm{ad}}} \;=\; (\omega_1^{-1}\overline{u}^{-1}_{\alpha_1,\mathrm{ad}}\omega_1,\, \omega_2^{-1}\overline{u}_{\alpha_2,\mathrm{ad}}\omega_2)$$

is the image of the original cocycle in $H^1(F, L_{\mathrm{ad}})$. In particular, the pairing is independent of ℓ_1 and ℓ_2. Now

$$K_{M^*_{i,\mathrm{ad}}}(\overline{u}_{\alpha_i,\mathrm{ad}}) \;=\; \zeta_G^{M_i}\,, \qquad\qquad i = 1, 2,$$

in the notation of §2. It follows that

$$\begin{aligned}
&\big\langle K_{L_{\mathrm{ad}}}(\overline{u}_{L_{\mathrm{ad}}}), s_U\big\rangle \\
&= \big\langle (\omega_1^{-1}(\zeta_G^{M_1})^{-1}\omega_1, \omega_2^{-1}\zeta_G^{M_2}\omega_2), (s,s)\big\rangle \\
&= \zeta_G^{M_1}(\widehat{\omega}_1 s\widehat{\omega}_1^{-1})^{-1}\zeta_G^{M_2}(\widehat{\omega}_2 s\widehat{\omega}_2^{-1}) \\
&= \zeta_G^{0}(\widehat{\omega}_1 s\widehat{\omega}_1^{-1})^{-1}\zeta_G^{0}(\widehat{\omega}_2 s\widehat{\omega}_2^{-1}) \\
&= \zeta_G^{0}(s_L)\,.
\end{aligned}$$

We have shown that

$$\Delta_s(\delta',\gamma)\Delta(\delta',\gamma)^{-1} \;=\; \zeta_G^0(s_L)\,,$$

as required. □

We can now return to the study of the function $(f'_s)^{L'}$ in (8.4). If δ' belongs to $\Sigma_G(\widetilde{L}'_V)$, we have

$$\begin{aligned}
(f'_s)^{L'}(\delta') \;&=\; \sum_{\gamma\in\Gamma(G_V)} \Delta_s(\delta',\gamma) f_G(\gamma) \\
&=\; \sum_{\gamma\in\Gamma(G_V)} \zeta_G^0(s_L)\Delta(\delta',\gamma) f_G(\gamma) \\
&=\; \zeta_G^0(s_L)(f')^{L'}(\delta')\,,
\end{aligned}$$

by the lemma we have just established. Therefore

$$(f'_s)^{L'} \;=\; \zeta_G^0(s_L)(f')^{L'}\,.$$

The expression (8.4) is the product of $\widehat{S}_M^L\big(\sigma',(f')^{L'}\big)$ with the sum

$$\sum_s \zeta_G^0(s_L)\,, \tag{8.7}$$

over s in the group (8.3).

Lemma 8.5. *The character*

$$s \longrightarrow \zeta_G^0(s_L)\,, \qquad\qquad s \in Z(\widehat{L}_{1,\mathrm{sc}})^\Gamma \cap Z(\widehat{L}_{2,\mathrm{sc}})^\Gamma / \widehat{Z}_{\mathrm{sc}}^\Gamma,$$

is nontrivial.

Proof. We are still working under the conditions of the assertion (i) of Theorem 8.3. The lemma will be a consequence of the fact that no conjugate of R lies in $\mathcal{L}(M_0^*)$.

Set $A = \big(Z(\widehat{R}_{\mathrm{sc}})^\Gamma\big)^0$, where as usual, $\widehat{R}_{\mathrm{sc}}$ is the preimage of $\widehat{R}$ in $\widehat{G}_{\mathrm{sc}}$. We shall also write $A_i = \big(Z(\widehat{L}_{i,\mathrm{sc}})^\Gamma\big)^0$, for $i = 1, 2$. Since $\mathfrak{a}_R$ is the sum of $\mathfrak{a}_{L_1}$ and $\mathfrak{a}_{L_2}$, by our condition that $\mathrm{e}_R^{G^*}(L_1, L_2) \neq 0$, we have $A = A_1 A_2$. Define

$$Z \;=\; \widehat{Z}_{\mathrm{sc}} \cap A \;=\; \widehat{Z}_{\mathrm{sc}}^\Gamma \cap A\,.$$

Since ZA_i is a subgroup of $Z(\widehat{L}_{i,\mathrm{sc}})^\Gamma$, we have an injection

$$ZA_1 \cap ZA_2 / Z \longrightarrow Z(\widehat{L}_{1,\mathrm{sc}})^\Gamma \cap Z(\widehat{L}_{2,\mathrm{sc}})^\Gamma / \widehat{Z}_{\mathrm{sc}}^\Gamma\,.$$

It is enough to show that the character

$$z \longrightarrow \zeta_G^0(z_L) \;=\; \zeta_G^0(\widehat{\omega}_1 z \widehat{\omega}_1^{-1})^{-1} \zeta_G^0(\widehat{\omega}_2 z \widehat{\omega}_2^{-1})\,, \qquad\qquad z \in ZA_1 \cap ZA_2,$$

is nontrivial.

Let $z \to (z_1, z_2)$ be the composition of the two maps

$$ZA_1 \cap ZA_2 \longrightarrow \prod_{i=1}^{2} (ZA_i / A_i) \longrightarrow \prod_{i=1}^{2} (Z / Z \cap A_i)\,.$$

Suppose that (z_1, z_2) is any point in $Z \times Z$. For $i = 1, 2$, let $A^i = A_{\widehat{i}}$, where $\widehat{i}$ is the element in the complement of i in $\{1, 2\}$. We can write

$$z_i \;=\; a_i a^i\,, \qquad\qquad a_i \in A_i,\ a^i \in A^i.$$

Then $a^i = z_i a_i^{-1}$ belongs to ZA_i. Since a^i also belongs to $ZA_{\widehat{i}}$, it lies in the domain $ZA_i \cap ZA_2$ of the map. The image of a^i in ZA_j/A_j equals the image of z_i if $j = i$, and equals 1 if $j \neq i$. Therefore $z = a^1 a^2$ is an element in $ZA_1 \cap ZA_2$ whose image in $\prod_i (Z/Z \cap A_i)$ equals the image of (z_1, z_2). In particular, the map is surjective. Moreover

$$\zeta_G^0(\widehat{\omega}_i z \widehat{\omega}_i^{-1}) \;=\; \prod_{j=1}^{2} \zeta_G^0(\widehat{\omega}_i a^j \widehat{\omega}_i) \;=\; \zeta_G^0(\widehat{\omega}_i a^i \widehat{\omega}_i^{-1}), \qquad i = 1, 2,$$

since for $j \neq i$, $\widehat{\omega}_i A^j \widehat{\omega}_i^{-1} = \widehat{\omega}_i A_i \widehat{\omega}_i^{-1}$ is contained in the subgroup $\big(Z(\widehat{M}_{0,\mathrm{sc}})^\Gamma\big)^0$ of the kernel of ζ_G^0. Furthermore,

$$\zeta_G^0(\widehat{w}_i a^i \widehat{\omega}_i^{-1}) \;=\; \zeta_G(z_i),$$

since $\widehat{\omega}_i a^i \widehat{\omega}_i^{-1}$ and z_i have the same image in $\pi_0\big(Z(\widehat{M}_{0,\mathrm{sc}})^\Gamma\big)$. We conclude that

$$\zeta_G^0(z_L) \;=\; \zeta_G(z_1)^{-1}\zeta_G(z_2) \;=\; \zeta_G(z_1^{-1}z_2), \tag{8.8}$$

for any point (z_1, z_2) in $Z \times Z$.

It follows from Corollary 2.2, and the condition on R, that ζ_G is nontrivial on the subgroup Z of $\widehat{Z}_{\mathrm{sc}}^\Gamma$. In particular, (8.8) is a nontrivial character in (z_1, z_2). It is therefore a nontrivial character in z. □

Having established Lemma 8.5, we can at last complete the proof of the vanishing assertion of Theorem 8.3. The lemma tells us that the sum (8.7) vanishes. Therefore the contribution of (8.4) to the original expansion (8.2) of $I_R^{\mathcal{E}}(\sigma', f)$ vanishes. Since it is a sum of such contributions, (8.2) itself vanishes. We have established that $I_R^{\mathcal{E}}(\sigma', f) = 0$.

The remaining identity (ii) of the theorem is essentially a consequence of the definitions. The summand $\widehat{S}_R^{G^*}(\sigma', f')$ on the right hand side of the definition (8.1) is constructed in terms of embeddings $\widehat{R} \subset \widehat{G}$ and $R' \subset G'$. A variant of Lemma 3.1(i), which takes into account the choice of base points $(\overline{\delta}', \overline{\gamma})$ for (8.1), shows that $\widehat{S}_R^{G^*}(\sigma', f')$ is independent of these embeddings. If R is $G^*(F)$-conjugate to M^* as in (ii), we can take $\widehat{R} = \widehat{M}$ and $R' = M'$. The required identity follows. □

The special case of Theorem 8.3 of inner forms of $GL(n)$ was not discussed in [3], since the local trace formula came later. The main vanishing result of [3] is Proposition 10.2. Its analogue for general inner twists asserts that the function $I_M^{\mathcal{E}}(\cdot, f)$, defined by the special case of (5.3) in which $V = \{v\}$ and $(F_v, G_v, M_v) = (F, G, M)$, is supported on elements that are images from M.

Theorem 8.6. *Suppose that M is a Levi subgroup of G, that M' is an elliptic endoscopic datum for M, and that $\sigma' \in \Sigma_G(\widetilde{M}')$ is not the image of any element in $\Gamma_G(M)$. Then*

$$I_M^{\mathcal{E}}(\sigma', f) \;=\; 0\,, \qquad\qquad f \in \mathcal{C}(G, \zeta).$$

Proof. It is a consequence of [14, Lemma 10.2] that any elliptic element in $\Sigma_G(\widetilde{M}')$ is an image of some element in $\Gamma_G(M)$. Since σ' is not such an image, it must come from a Levi subgroup R' of M' which is not relevant to G. In other words, R' is an elliptic endoscopic datum for a Levi subgroup R of G^* that is not conjugate to an element in $\mathcal{L}(M_0^*)$. Our point σ' is the image in $\Sigma_G(\widetilde{M}')$ of a class in $\Sigma_G(\widetilde{R}')$, that we can also denote by σ'. We can then apply the descent formula (7.3) to the terms in the sum

$$I_M^{\mathcal{E}}(\sigma', f) \;=\; \sum_{G' \in \mathcal{E}_{M'}(G)} \widehat{S}_M^G(\sigma', f')\,.$$

We find that $I_M^{\mathcal{E}}(\sigma', f)$ equals

$$\sum_{G' \in \mathcal{E}_{M'}(G)} \; \sum_{L \in \mathcal{L}(R)} \mathrm{e}_R^G(M, L) \widehat{S}_R^L\big(\sigma', (f')^{L'}\big)\,. \tag{8.9}$$

The expression (8.9) is a direct analogue of the expansion (8.2) for the distribution of Theorem 8.3(i). The proof of the present theorem follows exactly the same lines. We need only summarize the main steps.

There is an action $G' \to G'_s$ of the finite group

$$Z(\widehat{L}_{\mathrm{sc}})^\Gamma \cap Z(\widehat{M}_{\mathrm{sc}})^\Gamma / \widehat{Z}_{\mathrm{sc}}^\Gamma \tag{8.10}$$

on the set of G' that map to a given L'. We are assuming at this point that $\mathrm{e}_R^G(M,L) \neq 0$. We can also assume that L is conjugate to a group in $\mathcal{L}(M_0)$. We can therefore choose elements $\omega \in G^*_{\mathrm{sc}}(F)$ and $\widehat{\omega} \in \widehat{G}^{\Gamma}_{\mathrm{sc}}$ such that $\mathrm{Int}(\omega)L \in \mathcal{L}(M^*_0)$ and $\mathrm{Int}(\widehat{\omega})\widehat{L} \in \mathcal{L}(\widehat{M_0})$, and such that the restriction of $\mathrm{Int}(\widehat{\omega})$ to $\widehat{L}$ is dual to the restriction of $\mathrm{Int}(\omega)$ to L. In particular, $\widehat{\omega}Z(\widehat{L}_{\mathrm{sc}})^{\Gamma}\widehat{\omega}^{-1}$ is contained in $Z(\widehat{M}_{0,\mathrm{sc}})^{\Gamma}$. We therefore obtain a character

$$s \longrightarrow \zeta^0_G(s_{L,M}) \;=\; \zeta^0_G\big((\widehat{\omega}s\widehat{\omega}^{-1})^{-1}s\big) \tag{8.11}$$

on the group (8.10).

In the present situation, we must choose the base point $(\overline{\delta}',\overline{\gamma})$ to be in $\Sigma_G(\widetilde{M}') \times \Gamma_G(M)$. Lemma 8.4 then asserts that the relevant transfer factors satisfy

$$\Delta_s(\delta',\gamma;\overline{\delta}',\overline{\gamma}) \;=\; \zeta^0_G(s_{L,M})\Delta(\delta',\gamma;\overline{\delta}',\overline{\gamma})\,,$$

for points $\delta' \in \Sigma_G(\widetilde{L}')$ and $\gamma \in \Gamma(G)$. As in the proof of the earlier theorem, this immediately leads to an identity

$$(f'_s)^{L'} \;=\; \zeta^0_G(s_{L,M})(f')^{L'}$$

for the functions that occur in (8.9). Finally, Lemma 8.5 asserts that the character (8.11) on the group (8.10) is nontrivial. As in Theorem 8.3, this implies that the contribution to (8.9) of the given orbit $\{G'_s\}$ vanishes. It follows that (8.9) itself vanishes, and that $I^{\mathcal{E}}_M(\sigma',f) = 0$. $\square$

9. Towards a stable local trace formula

We shall now test our constructions on the local trace formula. We shall stabilize the geometric side of this formula in terms of the distributions defined in §5. The result will be important for the further study of the distributions. The process itself can be regarded as a rehearsal for the more difficult stabilization of the global trace formula.

As in the last section, G is a K-group over a local field F, equipped with a quasisplit inner twist G^*, while $M_0 \subset G$ is some fixed minimal Levi subgroup, equipped with a corresponding Levi subgroup $M_0^* \subset G^*$ and a dual Levi subgroup $\widehat{M}_0 \subset \widehat{G}$. It will be convenient also to fix a minimal Levi subgroup R_0 of G^*, with $R_0 \subset M_0^*$. Following the usual trace formula notation, we shall generally write $\mathcal{L} = \mathcal{L}^G$ for the set $\mathcal{L}(M_0) = \mathcal{L}^G(M_0)$, and $W_0 = W_0^G$ for the Weyl group $W(M_0) = W^G(M_0)$. Then W_0 acts on $\mathcal{L}$. Similarly, the Weyl group $W_0^* = W_0^{G^*} = W^{G^*}(R_0)$ of (G^*, R_0) acts on the set $\mathcal{L}^* = \mathcal{L}^{G^*} = \mathcal{L}^{G^*}(R_0)$. We note that the image of $\mathcal{L}$ under the map $M \to M^*$ is not generally $\mathcal{L}^*$, but rather the subset $\mathcal{L}(M_0^*)$ of $\mathcal{L}^*$. If G' is an endoscopic group for G, we have also the set $\mathcal{L}' = \mathcal{L}^{G'}$ and the Weyl group $W_0' = W_0^{G'}$, both taken with respect to some fixed minimal Levi subgroup of G'.

As usual, we fix a central induced torus Z in G over F, and a character ζ on $Z(F)$. The elliptic part of the local trace formula can be regarded as an inner product of two functions. To motivate our later discussion, let us recall how to stabilize this inner product. Suppose that a_G and b_G are two ζ_G^{-1}-equivariant functions on $\Gamma_{\mathrm{ell}}(G)$ that are square-integrable modulo $Z(F)$. Then we have an identity

$$(a_G, b_G) = \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G, G')(a', b'), \tag{9.1}$$

where

$$(a_G, b_G) = \int_{\Gamma_{\mathrm{ell}}(G/Z)} a_G(\gamma)\overline{b_G(\gamma)}\, d\gamma,$$

$$(a', b') = \int_{\Sigma_{G,\mathrm{ell}}(G'/Z)} n(\delta')^{-1} a'(\delta')\overline{b'(\delta')}\, d\delta',$$

$$a'(\delta') \;=\; \sum_{\gamma\in\Gamma_{\mathrm{ell}}(G)} \Delta_G(\delta',\gamma)a_G(\gamma)\,, \qquad\qquad \delta\in\Sigma_{G,\mathrm{ell}}(\widetilde{G}'),$$

and

$$\iota(G,G') \;=\; |\mathrm{Out}_G(G')|^{-1}|Z(\widehat{G}')^\Gamma/Z(\widehat{G})^\Gamma|^{-1}\,,$$

and where $n(\delta')$ is the integer $|\mathcal{K}_{\delta'}|$ defined in [7, p. 509]; the measures $d\gamma$ and $d\delta'$ are defined as in [7, (1.3)]. This is an easy extension of [7, Proposition 3.5], with the role of the p-adic adjoint relations [7, Lemma 2.3] taken by their generalizations (2.6) and (2.7) for K-groups. The proof from [7] was actually for the special case of $Z=\{1\}$ and a_G and b_G in the subspace $\mathcal{I}_{\mathrm{cusp}}(G)$ of $L^2\big(\Gamma_{\mathrm{ell}}(G)\big)$, but the proof extends immediately to the setting here.

We want to relate (9.1) with our earlier construction. We continue with the special case of the general framework of §4-§7 that applies to the local trace formula. Recall from the discussion at the beginning of §8 that if G' is an endoscopic datum for G, G'_V stands for the endoscopic datum $G'\times\overline{G}'$ for G_V. We take ζ_V to be the character $\zeta\times\zeta^{-1}$ on $Z_V(F_V)=Z(F)\times Z(F)$. In order for the notation to match that of [4] and [5], we fix a function in $\mathcal{C}(G_V,\zeta_V)$ of the form

$$f \;=\; f_1\times\overline{f}_2\,, \qquad\qquad f_i\in\mathcal{C}(G,\zeta).$$

Now the absolute transfer factor $\Delta(\cdot,\cdot)$ for (G,G') takes complex values on the unit circle. It follows from Lemma 8.1 that

$$\overline{f}_2^{\,\overline{G}'}(\delta'_2) \;=\; \sum_{\gamma_2\in\Gamma(G)} \overline{\Delta(\delta'_2,\gamma_2)}\,\overline{f_{2,G}(\gamma_2)} \;=\; \overline{f_2^{G'}(\delta'_2)}\,.$$

Therefore

$$f' \;=\; f^{G'_V} \;=\; f_1^{G'}\times\overline{f}_2^{\,\overline{G}'} \;=\; f'_1\times\overline{f'_2}\,.$$

In particular, the function f' is equivariant under the character $(\widetilde{\zeta}'\zeta)_V^{-1}=(\widetilde{\zeta}'\zeta)^{-1}\times\widetilde{\zeta}'\zeta$ on $(\widetilde{Z}'Z)_V(F_V)$. As in §8, we shall identify G' with G'_V when there is no risk of confusion.

We shall also identify any points $\delta' \in \Sigma_G(G')$ and $\gamma \in \Gamma(G)$ with their respective diagonal images $(\delta', \delta') \in \Sigma_{G_V}(G'_V)$ and $(\gamma, \gamma) \in \Gamma(G_V)$.

We can apply the inner product formula (9.1) to the function

$$a_G(\gamma_1) \cdot \overline{b_G(\gamma_2)} \;=\; f_{1,G}(\gamma_1) \cdot \overline{f_{2,G}(\gamma_2)} \;=\; f_G(\gamma_1, \gamma_2)\,, \qquad \gamma_1, \gamma_2 \in \Gamma(G).$$

It takes the form

$$I_{\mathrm{ell}}(f) \;=\; \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G, G') S'_{\mathrm{ell}}(f')\,,$$

where

$$I_{\mathrm{ell}}(f) \;=\; \int_{\Gamma_{\mathrm{ell}}(G/Z)} f_G(\gamma) d\gamma$$

and

$$S'_{\mathrm{ell}}(f') \;=\; \int_{\Sigma_{G,\mathrm{ell}}(G'/Z)} n(\delta')^{-1} f'(\delta') d\delta'\,.$$

Now $I_{\mathrm{ell}}(f)$ is the leading term of the geometric side of the local trace formula. The entire geometric side can be regarded as an expansion

$$I(f) \;=\; \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Gamma_{G,\mathrm{ell}}(M/Z)} I_M(\gamma, f) d\gamma \tag{9.2}$$

of a certain invariant distribution I in terms of the distributions of §4. It is composed of similar expansions [5, (4.10)] for the connected components G_α of G. The inner product formula above and the construction in §5 both suggest what to do next. We set

$$I^{\mathcal{E}}(f) \;=\; \sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G)} \iota(G, G') \widehat{S}'(f') + \varepsilon(G) S^G(f), \tag{9.3}$$

where $\widehat{S}' = \widehat{S}^{\widetilde{G}'}$ is a linear form on $S\mathcal{I}\big(\widetilde{G}'_V, (\widetilde{\zeta}'\zeta)_V\big)$ for each $G' \in \mathcal{E}^0_{\mathrm{ell}}(G)$, that is defined inductively by the supplementary requirement that

$$I^{\mathcal{E}}(f) \;=\; I(f) \tag{9.4}$$

in the case that G is quasisplit.

The definition is obviously in the same spirit as those of §5. We shall assume that for each $M \in \mathcal{L}$, Assumption 5.1 (with $V_0 = V$) holds for (F, G, M). This includes an assertion that for various quasisplit triplets (F, G_1, M_1) (in which G_1 is obtained from G by a succession of proper endoscopic groups), the distributions $\{S_{M_1}^{G_1}(\delta_1) : \delta_1 \in \Sigma_{G,\mathrm{ell}}(M_1)\}$ on $G_1(F)$ are stable. We shall strengthen the assumption slightly by requiring that the entire statement of Conjecture 3.3(b) (rather than just the stability assertion) applies to each (F, G_1, M_1). In other words, the distribution $S_{M_1}^{G_1}(\delta_1')$ vanishes if δ_1' lies in the complement of $\Sigma_{G,\mathrm{ell}}(M_1)$ in $\Gamma_{G,\mathrm{ell}}^{\mathcal{E}}(M_1)$. This is to be the primary induction hypothesis for what follows, and will not be resolved in this paper. As in §5, we impose a secondary induction hypothesis that is to be resolved in terms of the first one. We assume that if G is replaced by a quasisplit inner K-form of $\widetilde{G}'$, for any group $G' \in \mathcal{E}_{\mathrm{ell}}^0(G)$, the corresponding analogue of the distribution S^G is defined and stable. The summands $\widehat{S}'(f')$ in (9.3) then make sense.

Our goal is to establish expansions for $I^{\mathcal{E}}(f)$ and $S^G(f)$ in terms of the distributions in §5. Suppose that $M \in \mathcal{L}$ and that M' is an elliptic endoscopic datum for M. We are identifying M' with the datum $M_V' = M' \times \overline{M}'$ and $\mathcal{E}_{M'}(G)$ with the set $\mathcal{E}_{M_V'}(G_V, G)$. For any $G' \in \mathcal{E}_{M'}(G)$, the corresponding summand in (5.3) satisfies

$$S_M^G(\delta', f') \;=\; \iota_{M'}(G, G') S_{M'}^{G'}(\delta', f') \;=\; \iota_{M'}(G, G') S_{\widetilde{M}'}^{\widetilde{G}'}(\delta', f')\,,$$

by a simple variant of Corollary 7.2. (See the remark following Corollary 7.3.) The definition (5.3) can then be written in a form

$$I_M^{\mathcal{E}}(\delta', f) \;=\; \sum_{G' \in \mathcal{E}_{M'}^0(G)} \iota_{M'}(G, G') \widehat{S}_{\widetilde{M}'}^{\widetilde{G}'}(\delta', f') + \varepsilon(G) S_M^G(M', \delta', f) \tag{9.5}$$

that matches the definition (3.5) of §3.

Theorem 9.1. *(a) In general, we have*

$$I^{\mathcal{E}}(f) \;=\; \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Gamma_{G,\mathrm{ell}}(M/Z)} I_M^{\mathcal{E}}(\gamma,f)d\gamma\,. \tag{9.6}$$

(b) If G is quasisplit,

$$S^G(f) \;=\; \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)}\,\cdot \sum_{M'\in\mathcal{E}_{\mathrm{ell}}(M)} \iota(M,M') \int_{\Sigma_{G,\mathrm{ell}}(M'/Z)} n(\delta')^{-1}S_M^G(M',\delta',f)d\delta'\,. \tag{9.7}$$

Remarks. 1. Suppose that G is quasisplit, and that Conjecture 3.3(b) actually holds for G. Then by Proposition 7.4(b), $S_M^G(M',\delta',f)$ vanishes unless $M'=M^*$, in which case it is stable. The formula (9.7) becomes

$$S^G(f) \;=\; \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Sigma_{G,\mathrm{ell}}(M/Z)} n(\delta)^{-1}S_M^G(\delta,f)d\delta \tag{9.8}$$

under this condition. In particular, S^G is stable. The theorem thus allows us to resolve the secondary induction hypothesis in terms of the first one.

2. The integrals in (9.6) and (9.7) (as well as (9.2)) are absolutely convergent. For by the splitting formulas of Theorem 6.1, it is enough to control the behaviour in γ and δ' of the basic distributions of §3. The definitions (4.4), (5.3), (5.4) and (5.5) reduce the problem to weighted orbital integrals, for which we have a standard estimate

$$J_M(\gamma,h) \;\le\; \nu_n(h)\big(1+\log|D(\gamma)|\big)^p\big(1+\|H_M(\gamma)\|\big)^{-n}\,, \qquad \gamma\in\Gamma_G(M),$$

that holds for any n, and any Schwartz function h on $G(F)$. (See [6, (5.7)].)

Proof. According to the definition (9.3), the difference

$$I^{\mathcal{E}}(f) - \varepsilon(G)S^G(f) \tag{9.9}$$

equals

$$\sum_{G'\in\mathcal{E}^0_{\mathrm{ell}}(G)} \iota(G,G')\widehat{S}'(f')\,.$$

We can assume inductively that part (b) of theorem is valid for quasisplit inner K-forms of $\widetilde{G}'$, for each $G' \in \mathcal{E}^0_{\text{ell}}(G)$. The analogue of (9.8) for $\widetilde{G}'$ then holds, and can be written

$$\widehat{S}'(f') \;=\; \sum_{R' \in \mathcal{L}'} |W_0^{R'}||W_0^{G'}|^{-1} S_{R'}(G')$$

where

$$S_{R'}(G') \;=\; (-1)^{\dim(A_{R'}/A_{G'})} \int_{\Sigma_{G',\text{ell}}(R'/Z)} n(\sigma')^{-1} \widehat{S}^{\widetilde{G}'}_{\widetilde{R}'}(\sigma', f') d\sigma' . \tag{9.10}$$

The difference (9.9) therefore equals

$$\sum_{G' \in \mathcal{E}^0_{\text{ell}}(G)} \iota(G, G') \sum_{R' \in \mathcal{L}'} |W_0^{R'}||W_0'|^{-1} S_{R'}(G') . \tag{9.11}$$

The main step is the following rearrangement formula, that is a property of the connected, quasisplit group G^*.

Lemma 9.2. *Suppose that*

$$S_{R'}(G') , \qquad\qquad G' \in \mathcal{E}(G^*),\ R' \in \mathcal{L}',$$

is a family of complex numbers that depend only on the $\mathrm{Aut}_G(G')$*-orbit of* R'. *Then*

$$\sum_{G' \in \mathcal{E}_{\text{ell}}(G^*)} \iota(G^*, G') \sum_{R' \in \mathcal{L}'} |W_0^{R'}||W_0'|^{-1} S_{R'}(G') \tag{9.12}$$

equals

$$\sum_{R \in \mathcal{L}^*} |W_0^R||W_0^*|^{-1} I_R(G^*) , \tag{9.13}$$

where

$$I_R(G^*) \;=\; \sum_{R' \in \mathcal{E}_{\text{ell}}(R)} \iota(R, R') \sum_{G' \in \mathcal{E}_{R'}(G^*)} \iota_{R'}(G^*, G') S_{R'}(G') .$$

Proof. In general, we shall write S/H for the set of orbits of a group H, acting on a set S. Then (9.12) equals

$$\sum_{G'\in\mathcal{E}(G^*)} \iota(G^*,G') \sum_{R'\in\mathcal{L}'/W_0'} |W'(R')|^{-1} S_{R'}(G')\,,$$

since $W_0^{R'}W'(R')$ is the stabilizer of R' in $W_0'=W_0^{G'}$, and since $\iota(G^*,G')$ vanishes for G' in the complement of $\mathcal{E}_{\mathrm{ell}}(G^*)$ in $\mathcal{E}(G^*)$. The main point is to interchange the sums over G' and R'. This requires several counting arguments.

We can identify $\mathcal{E}(G^*)$ with a set of orbits $\{G'\}/\widehat{G}$ of the dual group $\widehat{G}=\widehat{G}^*$. The underlying set $\{G'\}$ consists of $Z(\widehat{G})$-orbits of endoscopic data $(G',\mathcal{G}',s',\xi')$ for G^*, in which $Z(\widehat{G})$ acts by translation on s', $\mathcal{G}'$ is contained in ${}^LG$, and ξ' is the identity embedding. Given any such G', a choice of Levi subgroup $R'\in\mathcal{L}'/W_0'$, determines Levi subgroups $\widehat{R}'\subset\widehat{G}'$ and $\mathcal{R}'\subset\mathcal{G}'$, up to conjugation by $\widehat{G}'$. The group $\widehat{R}'$ can be identified with a dual group of R', while $\mathcal{R}'$ is a split extension of W_F by $\widehat{R}'$, and $\mathcal{G}'$ equals $\widehat{G}'\mathcal{R}'$. Following the usual convention for endoscopic data, we shall find it convenient to denote a pair $(\widehat{R}',\mathcal{R}')$ by R'. Then $\mathcal{L}'/W_0'$ is bijective with $\{R'\}/\widehat{G}'$. With these changes of notation, (9.12) becomes

$$\sum_{\{G'\}/\widehat{G}}\ \sum_{\{R'\}/\widehat{G}'} \iota(G^*,G')|W'(R')|^{-1}S_{R'}(G')\,.$$

The next step is to combine the iterated sum over G' and R'. Given G' and R', let s_R' be any point in $s'Z(\widehat{G})^\Gamma\big(Z(\widehat{R}')^\Gamma\big)^0$ whose connected centralizer in $\widehat{G}$ equals $\widehat{R}'$. In this way, we can identify R' with an endoscopic datum for G. As G' varies, R' ranges over the endoscopic data $(R',s_R',\mathcal{R}',\xi_R')$ for G^*, taken up to translation of s_R' by $Z(\widehat{G})\big(Z(\widehat{R}')^\Gamma\big)^0$, in which $\mathcal{R}'$ is contained in ${}^LG$ and ξ_R' is the identity embedding. The group $\widehat{G}$ acts on this set $\{R'\}$ in the obvious way by conjugation. We can replace the iterated sum by a sum over the $\widehat{G}$-orbits in the set

$$\big\{(R',G')\big\} \;=\; \big\{(R',G'):\ s'Z(\widehat{G})\subset s_R'Z(\widehat{G})\big(Z(\widehat{R}')^\Gamma\big)^0,\ \mathcal{G}'=\widehat{G}'\mathcal{R}'\big\}\,, \tag{9.14}$$

provided that we scale the summand by the orders of the appropriate stabilizers. The stabilizer of G' in $\widehat{G}$ is

$$\mathrm{Aut}_G(G') \;=\; \bigl\{g \in \widehat{G}:\; gs'g^{-1} \in s'Z(\widehat{G}),\; g\mathcal{G}'g^{-1} = \mathcal{G}'\bigr\}\,.$$

This group is generally larger than $\widehat{G}'$, but its quotient by $\widehat{G}'$ is the finite group $\mathrm{Out}_G(G')$. The stabilizer of R' in $\widehat{G}$, taken modulo the stabilizer of R' in $\widehat{G}'$, is the subgroup $\mathrm{Out}_G(G')^{R'}$ of cosets in $\mathrm{Out}_G(G')$ that have a representative in $\mathrm{Aut}_G(G')$ that stabilizes R'. The scaling factor for the change of summation is therefore the index of $\mathrm{Out}_G(G')^{R'}$ in $\mathrm{Out}_G(G')$. The expression (9.12) becomes

$$\sum_{\{(R',G')\}/\widehat{G}} |W'(R')|^{-1}|\mathrm{Out}_G(G')^{R'}|^{-1}|\mathrm{Out}_G(G')|\iota(G^*,G')S_{R'}(G')\,. \tag{9.15}$$

We now change the sum in (9.15) back to an iterated sum, but with R' taken before G'. Any R' determines a Levi subgroup $\widehat{R}$ of $\widehat{G}$, with the property that $Z(\widehat{R})^\Gamma$ is of finite index in $Z(\widehat{R}')^\Gamma$. Writing

$$Z(\widehat{G})\bigl(Z(\widehat{R}')^\Gamma\bigr)^0 \;=\; Z(\widehat{G})\bigl(Z(\widehat{R})^\Gamma\bigr)^0 \;=\; Z(\widehat{G})Z(\widehat{G})^\Gamma\bigl(Z(\widehat{R})^\Gamma\bigr)^0 \;=\; Z(\widehat{G})Z(\widehat{R})^\Gamma\,,$$

by Lemma 1.1, we see that

$$Z(\widehat{G})\bigl(Z(\widehat{R}')^\Gamma\bigr)^0/Z(\widehat{G}) \;=\; Z(\widehat{R})^\Gamma/Z(\widehat{G})^\Gamma\,.$$

It follows that for any R', the set of G' for which (R',G') belongs to the set (9.14) coincides with the set $\mathcal{E}_{R'}(G)$. Observe that $\widehat{R}'$ stabilizes both R' and G'. The full stabilizer of R' is the larger group

$$\mathrm{Aut}_G(R') \;=\; \bigl\{g \in \widehat{G}:\; gs'_Rg^{-1} \in s'_RZ(\widehat{G})\bigl(Z(\widehat{R}')^\Gamma\bigr)^0,\; g\mathcal{R}'g^{-1} = \mathcal{R}'\bigr\}\,.$$

We shall write $\mathrm{Out}_G(R')^{G'}$ for the stabilizer of any $G' \in \mathcal{E}_{R'}(G)$ in the quotient

$$\mathrm{Out}_G(R') \;=\; \mathrm{Aut}_G(R')/\widehat{R}'\,.$$

We can then change the sum in (9.15) to an iterated sum over $\{R'\}/\widehat{G}$ and $\mathcal{E}_{R'}(G^*)$, provided that we divide the summand by the index of $\mathrm{Out}_G(R')^{G'}$ in $\mathrm{Out}_G(R')$, or in other words, that we multiply the summand by

$$|\mathrm{Out}_G(R')|^{-1}|\mathrm{Out}_G(R')^{G'}|\,.$$

The last step is to change the sum over $\{R'\}/\widehat{G}$ to an iterated sum over Levi subgroups R of G^*, and over those R' that give rise to R as above. The set of such R', taken modulo conjugation by $\widehat{R}$, is just $\mathcal{E}_{\mathrm{ell}}(R)$. Moreover, $\{R\}/\widehat{G}$ can be identified with $\mathcal{L}^*/W_0^*$. The stabilizer of R in $\widehat{G}$ contains $\widehat{R}$, and the quotient of this stabilizer by $\widehat{R}$ may be identified with the Weyl group $W(R) = W^{G^*}(R)$. Let $W(R)^{R'}$ be the subgroup of cosets in $W(R)$ that have a representative in $\widehat{G}$ that stabilizes R'. The sum over $\{R'\}/\widehat{G}$ can then be replaced by an iterated sum over $R \in \mathcal{L}^*/W_0^*$ and $R' \in \mathcal{E}_{\mathrm{ell}}(R)$, provided that the summand is divided by the index of $W(R)^{R'}$ in $W(R)$. It is preferable to sum R over the whole set $\mathcal{L}^*$. We are of course free to do this, as long as we divide the summand further by the index of $W_0^R W(R)$ in W_0^*. We can therefore replace the sum over $\{R'\}/\widehat{G}$ with an iterated sum over $R \in \mathcal{L}^*$ and $R' \in \mathcal{E}_{\mathrm{ell}}(R)$, provided that we multiply the summand by

$$|W_0^R||W_0^*|^{-1}|W(R)^{R'}|\,.$$

We have established that (9.15) equals

$$\sum_{R\in\mathcal{L}^*}|W_0^R||W_0^*|^{-1}\sum_{R'\in\mathcal{E}_{\mathrm{ell}}(R)}\ \sum_{G'\in\mathcal{E}_{R'}(G^*)}\alpha_{R'}(G^*,G')S_{R'}(G')\,,$$

where $\alpha_{R'}(G^*,G')$ equals the product of

$$|W'(R')|^{-1}|\mathrm{Out}_G(G')^{R'}|^{-1}|\mathrm{Out}_G(G')|\iota(G^*,G')$$

with

$$|W(R)^{R'}||\mathrm{Out}_G(R')|^{-1}|\mathrm{Out}_G(R')^{G'}|\,.$$

We have only to simplify $\alpha_{R'}(G^*, G')$.

It follows from the definitions that

$$\begin{aligned} |\mathrm{Out}_G(G')|\iota(G^*, G') &= |Z(\widehat{G}')^\Gamma / Z(\widehat{G})^\Gamma|^{-1} \\ &= |Z(\widehat{R}')^\Gamma / Z(\widehat{R})^\Gamma|^{-1} \iota_{R'}(G^*, G') \\ &= |\mathrm{Out}_R(R')|\iota(R, R')\iota_{R'}(G^*, G') \,. \end{aligned}$$

We may as well choose a representative $(R', \mathcal{R}', s'_R, \xi'_R)$ within the isomorphism class in $\mathcal{E}_{\mathrm{ell}}(R)$ so that $\mathcal{R}' \subset {}^LG$ and $\xi'_R = 1$, and such that s'_R actually centralizes $\mathcal{R}'$ (rather than stabilizing it). It is then easy to see that $\mathrm{Out}_R(R')$ embeds into $\mathrm{Out}_G(R')$. The image of $\mathrm{Out}_R(R')$ is in fact a normal subgroup, whose quotient acts on $\big(Z(\widehat{R}')^\Gamma\big)^0$, and is isomorphic to the subgroup $W(R)^{R'}$ of $W(R)$. Therefore

$$|\mathrm{Out}_R(R')| \;=\; |W(R)^{R'}|^{-1} |\mathrm{Out}_G(R')| \,.$$

Finally, consider the subgroup of elements in $\mathrm{Out}_G(R')^{G'}$ that are induced by conjugation by elements in $\widehat{G}'$ (rather than the larger group $\widehat{G}$). This subgroup is isomorphic to the subgroup of Γ-invariant elements in $W^{\widehat{G}'}(\widehat{R}')$, which is in turn isomorphic to $W'(R') = W^{G'}(R')$. But the subgroup of $\mathrm{Out}_G(R')^{G'}$ in question is normal, and the quotient is easily seen to be isomorphic to $\mathrm{Out}_G(G')^{R'}$. It follows that

$$|\mathrm{Out}_G(R')^{G'}| \;=\; |W'(R')||\mathrm{Out}_G(G')^{R'}| \,.$$

Combining these three observations, we conclude that

$$\alpha_{R'}(G^*, G') \;=\; \iota(R, R')\iota_{R'}(G^*, G') \,.$$

The expression (9.15), which was obtained from the original expression (9.12), becomes

$$\sum_{R \in \mathcal{L}^*} |W_0^R||W_0^*|^{-1} \sum_{R' \in \mathcal{E}_{\mathrm{ell}}(R)} \iota(R, R') \sum_{G' \in \mathcal{E}_{R'}(G^*)} \iota_{R'}(G^*, G') S_{R'}(G') \,.$$

This is just

$$\sum_R |W_0^R||W_0^*|^{-1} I_R(G^*)\,,$$

the required expression (9.13). $\square$

We return to the proof of the theorem. We shall apply the lemma, with $S_{R'}(G')$ defined by (9.10) if G' belongs to $\mathcal{E}^0_{\mathrm{ell}}(G)$, and $S_{R'}(G') = 0$ for any G' in the complement of $\mathcal{E}^0_{\mathrm{ell}}(G)$ in $\mathcal{E}(G^*)$. A simple variant of Lemma 3.1(i) implies that $S^{\widetilde{G}'}_{\widetilde{R}'}(\sigma', f')$ depends only on the orbit of (R', σ') under $\mathrm{Aut}_G(G')$. Therefore $S_{R'}(G')$ satisfies the requirement that it depend only on the $\mathrm{Aut}_G(G')$-orbit of R'. Since $\iota(G, G') = \iota(G^*, G')$ and $\iota_{R'}(G, G') = \iota_{R'}(G^*, G')$, the lemma tells us that the original expression (9.11) equals

$$\sum_{R\in\mathcal{L}^*} |W_0^R||W_0^*|^{-1} \sum_{R'\in\mathcal{E}_{\mathrm{ell}}(R)} \iota(R, R') \sum_{G'\in\mathcal{E}^0_{R'}(G)} \iota_{R'}(G, G') S_{R'}(G')\,.$$

Substituting for $S_{R'}(G)$, we note that $(-1)^{\dim(A_{R'}/A_{G'})}$ equals $(-1)^{\dim(A_R/A_G)}$, and that the integral over $\Sigma_{G',\mathrm{ell}}(R'/Z)$ can be taken over the open dense subset $\Sigma_{G,\mathrm{ell}}(R'/Z)$. In particular, the sign and integral can both be taken outside the sum over G'. We can conclude that (9.11) equals

(9.16)

$$\sum_{R\in\mathcal{L}^*} |W_0^R||W_0^*|^{-1}(-1)^{\dim(A_R/A_G)} \sum_{R'\in\mathcal{E}_{\mathrm{ell}}(R)} \iota(R, R') \int_{\Sigma_{G,\mathrm{ell}}(R'/Z)} n(\sigma')^{-1} B_{R'}(\sigma', f)d\sigma'\,,$$

where

$$B_{R'}(\sigma', f) \;=\; \sum_{G'\in\mathcal{E}^0_{R'}(G)} \iota_{R'}(G, G') S^{\widetilde{G}'}_{\widetilde{R}'}(\sigma', f')\,.$$

We claim that

$$B_{R'}(\sigma', f) \;=\; I^{\mathcal{E}}_R(\sigma', f) - \varepsilon(G) S^G_R(R', \sigma', f)\,.$$

If $\varepsilon(G) = 1$, R belongs to $\mathcal{L}$, and the formula is just (9.5). If $\varepsilon(G) = 0$, $\mathcal{L}$ is a proper subset of $\mathcal{L}^*$ that need not contain R. However, in this case $\mathcal{E}^0_{R'}(G) = \mathcal{E}_{R'}(G)$, and the formula follows from the definition (8.1) and the appropriate variant of Corollary 7.2. Consider, then, the contribution to (9.16) of the two parts of $B_{R'}(\sigma', f)$. The contribution of $-\varepsilon(G)S_R^G(R', \sigma', f)$ is just the product of $-\varepsilon(G)$ with the right hand side of (9.7). For the contribution of $I_R^{\mathcal{E}}(\sigma', f)$, we appeal to Theorem 8.3. According to the vanishing assertion (i) of the theorem, $I_R^{\mathcal{E}}(\sigma', f) = 0$ if R is not $G^*(F)$-conjugate to an element in $\mathcal{L}(M_0^*)$. If R is $G^*(F)$-conjugate to an element in $\mathcal{L}(M_0^*)$, (R, R', σ') lies in the W_0^*-orbit of a triplet

$$(M^*, M', \delta')\,, \qquad\qquad M \in \mathcal{L},\ M' \in \mathcal{E}_{\mathrm{ell}}(M),\ \delta' \in \Sigma_{G,\mathrm{ell}}(\widetilde{M}').$$

In this case, assertion (ii) of the theorem yields

$$I_R^{\mathcal{E}}(\sigma', f) \;=\; I_M^{\mathcal{E}}(\delta', f)\,.$$

The corresponding contribution to (9.16) will then be given by a sum over $M \in \mathcal{L}$, provided that we replace $|W_0^R||W_0^*|^{-1}$ by $|W_0^M||W_0^G|^{-1}$. We have shown that (9.16) equals the sum of

(9.17)
$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \sum_{M'\in\mathcal{E}_{\mathrm{ell}}(M)} \iota(M, M') \int_{\Sigma_{G,\mathrm{ell}}(M'/Z)} n(\delta')^{-1} I_M^{\mathcal{E}}(\delta', f)d\delta'$$

and the product of $-\varepsilon(G)$ with the right hand side of (9.7).

We claim that (9.17) equals

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Gamma_{G,\mathrm{ell}}(M/Z)} I_M^{\mathcal{E}}(\gamma, f)d\gamma\,, \tag{9.18}$$

the right hand side of (9.6). To see this, we begin by writing

$$I_M^{\mathcal{E}}(\delta', f) \;=\; \sum_{L_1, L_2\in\mathcal{L}(M)} d_M^G(L_1, L_2)\widehat{I}_M^{L_1,\mathcal{E}}(\delta', f_{1,L_1})\widehat{I}_M^{L_2,\mathcal{E}}(\delta', \overline{f_{2,L_2}})\,,$$

by the splitting formula (6.2), or rather, its equivalent formulation (6.4). The point δ' is attached to given $M' \in \mathcal{E}_{\mathrm{ell}}(M)$, that we have identified with the datum $(M', \overline{M}')$ in $\mathcal{E}_{\mathrm{ell}}(M_V)$. It follows from Lemma 8.1 and the original definitions of §3 that $\widehat{I}_M^{L_2,\mathcal{E}}(\delta', \overline{f_{2,L_2}})$, with δ' attached to $\overline{M'}$, equals the complex conjugate of $I_M^{L_2,\mathcal{E}}(\delta', f_{2,L_2})$, with δ' attached to M'. We are dealing with functions of $\delta' \in \Sigma_{G,\mathrm{ell}}(\widetilde{M}')$ that are square integrable modulo $\widetilde{Z}'Z$. We can therefore apply the inner product formula (9.1) (with G replaced by M). We find that

$$\sum_{M' \in \mathcal{E}_{\mathrm{ell}}(M)} \iota(M,M') \int_{\Sigma_{G,\mathrm{ell}}(M'/Z)} n(\delta') I_M^{\mathcal{E}}(\delta', f) d\delta'$$
$$= \sum_{L_1, L_2 \in \mathcal{L}(M)} \int_{\Gamma_{G,\mathrm{ell}}(M/Z)} \widehat{I}_M^{L_1,\mathcal{E}}(\gamma, f_{1,L_1}) \widehat{I}_M^{L_2,\mathcal{E}}(\gamma, \overline{f_{2,L_2}}) d\gamma$$
$$= \int_{\Gamma_{G,\mathrm{ell}}(M/Z)} I_M^{\mathcal{E}}(\gamma, f) d\gamma\,,$$

thanks to (3,6), (6.2) and the fact that the complex conjugate of $I_M^{L_2,\mathcal{E}}(\gamma, f_{2,L_2})$ equals $I_M^{L_2,\mathcal{E}}(\gamma, \overline{f_{2,L_2}})$. The claim follows.

We have shown that (9.16) equals the sum of (9.18) and the product of $-\varepsilon(G)$ with the right hand side of (9.7). But (9.16) equals the original expression (9.9), that is the difference of $I^{\mathcal{E}}(f)$ and $\varepsilon(G)S^G(f)$. We conclude that $I^{\mathcal{E}}(f)$ equals the sum of (9.18) and

$$\varepsilon(G)\Bigg(S^G(f) - \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \cdot \sum_{M' \in \mathcal{E}_{\mathrm{ell}}(M)} \iota(M,M') \int_{\Sigma_{G,\mathrm{ell}}(M'/Z)} n(\delta')^{-1} S_M^G(M', \delta', f) d\delta'\Bigg)\,. \tag{9.19}$$

We can now finish the proof of the theorem in the same way we drew the final conclusions of Theorems 6.1 and 7.1. If G is not quasisplit, $\varepsilon(G) = 0$, and the expression (9.19) vanishes. The required expansion (9.6) is given by the resulting equality of $I^{\mathcal{E}}(f)$ with (9.18). If G is quasisplit, $I^{\mathcal{E}}(f) = I(f)$ and $I_M^{\mathcal{E}}(\gamma, f) = I_M(\gamma, f)$, according to the definitions (9.4) and (5.4). In this case, (9.6) reduces to the expansion (9.2). The required expansion (9.7) then follows from the vanishing of (9.19). $\square$

10. A simple application

We shall conclude the paper by looking at the special case of Theorem 9.1 in which one of the components of $f = f_1 \times \overline{f_2}$ is cuspidal. Using the main theorems of [7], we shall establish Conjecture 3.3 for cuspidal functions. This partial result seems to be a necessary local ingredient for the stabilization of the global trace formula. Its general role ought to be analogous to that of [9, Lemma 2.7.2] in the comparison of trace formulas for $GL(n)$ and related groups.

The results of [7] apply to p-adic groups, and are conditional on the fundamental lemma (and its analogue on a Lie algebra). The discussion of this section, as it applies to p-adic groups, is therefore conditional on the same hypothesis. The fundamental lemma has been established for the groups $SL(n)$ [17], $Sp[4]$ [11] and $GSp(4)$ [11], and could presumably also be extended to the corresponding Lie algebras. This being so, our discussion applies at least to inner forms of these p-adic groups. For real groups, the results we require have been known for some time. They are due to Shelstad, and are implicit in the paper [16].

We continue with the setting of Theorem 9.1. Then G is a K-group over a local field F, and

$$f \;=\; f_1 \times \overline{f_2}\,, \qquad\qquad f_i \in \mathcal{C}(G,\zeta).$$

Recall that $\mathcal{C}_{\mathrm{cusp}}(G,\zeta)$ stands for the space of functions f_1 in $\mathcal{C}(G,\zeta)$ that are cuspidal, in the sense that the orbital integral

$$\gamma \longrightarrow f_{1,G}(\gamma) \;=\; J_G(\gamma, f)\,, \qquad\qquad \gamma \in \Gamma(G),$$

is supported on the subset $\Gamma_{\mathrm{ell}}(G)$ of elliptic classes in $\Gamma(G)$. We assume that f_1 is cuspidal. With this condition, we set

$$(10.1) \qquad\qquad I_{\mathrm{disc}}(f) \;=\; \int_{T_{\mathrm{ell}}(G,\zeta)} n(\tau)^{-1} f_{1,G}(\tau)\overline{f_{2,G}(\tau)}\, d\tau\,,$$

where

$$n(\tau) \;=\; |R_{\pi,r}||\det(1-r)_{\mathfrak{a}_L/\mathfrak{a}_G}|\,, \qquad\qquad \tau=(L,\pi,r),$$

in the notation of [7, p. 533]. (If G is not connected, we adopt definitions

$$T(G,\zeta) \;=\; \coprod_{\alpha\in\pi_0(G)} T(G_\alpha,\zeta_\alpha) \;=\; \coprod_\alpha T_{\mathrm{temp}}\big(G_\alpha(F),\zeta_\alpha\big)$$

and

$$T_{\mathrm{ell}}(G,\zeta_\alpha) \;=\; \coprod_{\alpha\in\pi_0(G)} T_{\mathrm{ell}}(G_\alpha,\zeta_\alpha) \;=\; \coprod_\alpha T_{\mathrm{temp,ell}}\big(G_\alpha(F),\zeta_\alpha\big)$$

that are parallel to those in §2 for conjugacy classes.) For the given f, $I_{\mathrm{disc}}(f)$ equals the spectral side of the local trace formula. Equating it with the geometric side studied in §9, we obtain

$$I(f) \;=\; I_{\mathrm{disc}}(f)\,. \tag{10.2}$$

(See [5, Corollary 3.2]. Since f_1 is cuspidal, it is easy to see that the invariant local trace formula does not depend on how we normalize the weighted characters.)

We can regard I_{disc} as a linear form on the subspace

$$\mathcal{C}_{1\text{-cusp}}(G_V,\zeta_V) \;=\; \mathcal{C}_{\mathrm{cusp}}(G,\zeta)\otimes\mathcal{C}(G,\zeta^{-1})$$

of $\mathcal{C}(G_V,\zeta_V)$. Motivated by the definitions (9.3) and (9.4), we set

$$I^{\mathcal{E}}_{\mathrm{disc}}(f) \;=\; \sum_{G'\in\mathcal{E}^0_{\mathrm{ell}}(G)} \iota(G,G')\widehat{S}'_{\mathrm{disc}}(f') + \varepsilon(G)S^G_{\mathrm{disc}}(f)\,, \tag{10.3}$$

for linear forms $\widehat{S}'_{\mathrm{disc}}=\widehat{S}^{\widetilde{G}'}_{\mathrm{disc}}$ on the spaces

$$S\mathcal{I}_{1\text{-cusp}}\big(\widetilde{G}'_V,(\widetilde{\zeta}'\zeta)_V\big) \;=\; S\mathcal{I}_{\mathrm{cusp}}(\widetilde{G}',\widetilde{\zeta}'\zeta)\otimes S\mathcal{I}\big(\widetilde{G}',(\widetilde{\zeta}'\zeta)^{-1}\big)\,,$$

that are defined inductively by setting

$$I^{\mathcal{E}}_{\mathrm{disc}}(f) \;=\; I_{\mathrm{disc}}(f) \tag{10.4}$$

in the case that G is quasisplit.

Proposition 10.1. *(a) For any G, we have*

$$I^{\mathcal{E}}_{\mathrm{disc}}(f) \;=\; I_{\mathrm{disc}}(f)\,, \qquad\qquad f \in \mathcal{C}_{1\text{-cusp}}(G_V,\zeta_V).$$

(b) If G is quasisplit, S^G_{disc} is a stable linear form on $\mathcal{C}_{1\text{-cusp}}(G_V,\zeta_V)$.

Proof. For p-adic groups, the proposition is a recasting of the main results of [7]. We shall take our references from that paper, with the understanding that the corresponding results for real groups are implicit in [16]. We shall also freely adopt notation from [7]. In particular, if G is quasisplit, we have the sets $\Phi(G,\zeta)$ and $\Phi_2(G,\zeta)$ introduced in the p-adic case of [7, §5] to parametrize stable class functions on $G(F)$. In the real case, where G might be disconnected, we can take $\Phi(G,\zeta)$ and $\Phi_2(G,\zeta)$ to be Langlands parameters ϕ: $W_F \to {}^LG$, with Z-central character equal to ζ. These objects parametrize stable (linear combinations of) tempered characters on $G(F)$. There is a bijection $\phi \to \phi^*$ from $\Phi(G,\zeta)$ onto $\Phi(G^*,\zeta^*)$ for which the corresponding map of stable class functions is compatible with the bijection $\delta \to \delta^*$ of $\Sigma(G)$ with $\Sigma(G^*)$.

We can assume that $f = f_1 \times \overline{f_2}$, as above. Recall that $\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$ is a space of functions that can be defined on either $\Gamma(G)$ or $T(G,\zeta)$, by taking either the orbital integrals or the characters of funtions in $\mathcal{C}_{\mathrm{cusp}}(G,\zeta)$. We define two functions $a_G, b_G \in \mathcal{I}_{\mathrm{cusp}}(G,\zeta)$ by setting $a_G(\tau) = f_{1,G}(\tau)$ and $b_G(\tau) = f_{2,G}(\tau)$ if $\tau \in T_{\mathrm{ell}}(G,\zeta)$, and $a_G(\tau) = b_G(\tau) = 0$ if τ lies in the complement of $T_{\mathrm{ell}}(G,\zeta)$ in $T(G,\zeta)$. Recall the inner product formula

$$(a_G, b_G) \;=\; \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G,G')(a',b')$$

given by (9.1) (and [7, Proposition 3.5]). We shall relate this formula to the expansion (10.3).

Since (a_G,b_G) is the geometric side of the local trace formula in the special case of two cuspidal functions, it follows from (10.1) and (10.2) that

$$(a_G,b_G) \;=\; \widehat{I}_{\mathrm{disc}}(a_G \times \overline{b}_G) \;=\; I_{\mathrm{disc}}(f)\,.$$

(See [7, p. 533].) For any G', the functions a' and b' above lie in the space $S\mathcal{I}_{\mathrm{cusp}}(\widetilde{G}', \widetilde{\zeta}'\zeta)$ of stable orbital integrals of cuspidal functions [7, §5]. It follows that

$$(a', b') \ = \ \int_{\Phi_2(\widetilde{G}', \widetilde{\zeta}'\zeta)} n(\phi')^{-1} a'(\phi')\overline{b'(\phi')} d\phi' \, .$$

(See [7, p. 542].) Since f_1 is cuspidal, the definitions in [7, §5] imply that $a'(\phi') = f'_1(\phi')$. For the noncuspidal function f_2, however, the definitions [7, p. 543 and 549] yield only

$$\begin{aligned} b'(\phi') \ &= \sum_{\tau \in T_{\mathrm{ell}}(G,\zeta)} \Delta(\phi', \tau) b_G(\tau) \\ &= \sum_{\tau \in T_{\mathrm{ell}}(G,\zeta)} \Delta(\phi', \tau) f_{2,G}(\tau) \\ &= f'_{2,\mathrm{gr}}(\phi') \, , \end{aligned}$$

for any $\phi' \in \Phi_2(\widetilde{G}', \widetilde{\zeta}'\zeta)$. The purpose of the two main theorems of [7] was to relate this with $f'_2(\phi')$.

Suppose first that G is quasisplit. According to [7, Theorem 6.1], the linear form $f_2 \to f^G_{2,\mathrm{gr}}(\phi)$ on $\mathcal{C}(G, \zeta)$ is stable, for any $\phi \in \Phi_2(G, \zeta)$. It therefore depends only on the function f^G_2 on $\Sigma(G)$ (which we have identified with the image f^*_2 of f_2 in $S\mathcal{I}(G^*, \zeta^*)$). This justifies the definition

$$f^G_2(\phi) \ = \ f^G_{2,\mathrm{gr}}(\phi)$$

of [7, p. 549]. In particular, the linear form

$$^*S^G_{\mathrm{disc}}(f) \ = \ \int_{\Phi_2(G,\zeta)} n(\phi)^{-1} f^G_1(\phi)\overline{f^G_2(\phi)} \, d\phi$$

on $\mathcal{C}_{1\text{-cusp}}(G_V, \zeta_V)$ is stable. Next suppose that G is arbitrary. According to [7, Theorem 6.2], the linear form $f'_2(\phi)$, defined by the composition of the Langlands-Shelstad transfer mapping with the linear form on $S\mathcal{I}(\widetilde{G}', \widetilde{\zeta}'\zeta)$ attached to $\phi' \in \Phi_2(\widetilde{G}', \widetilde{\zeta}'\zeta)$ as above, equals $f'_{2,\mathrm{gr}}(\phi')$. It follows that

$$(a', b') \ = \ \int_{\Phi_2(\widetilde{G}', \widetilde{\zeta}'\zeta)} n(\phi')^{-1} f'_1(\phi')\overline{f'_2(\phi')} \, d\phi' \ = \ ^*\widehat{S}'_{\mathrm{disc}}(f') \, .$$

The inner product formula becomes

$$I_{\mathrm{disc}}(f) \;=\; \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G,G')\, {}^*\widehat{S}'_{\mathrm{disc}}(f') \,. \tag{10.5}$$

If G is quasisplit, it follows inductively from (10.3), (10.4), and (10.5) that ${}^*S^G_{\mathrm{disc}}(f) = S^G_{\mathrm{disc}}(f)$. Since ${}^*S^G_{\mathrm{disc}}(f)$ is stable, we obtain assertion (b) of the proposition. Moreover, for any G, we obtain

$$\begin{aligned} I^{\mathcal{E}}_{\mathrm{disc}}(f) \;&=\; \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G,G')\widehat{S}'_{\mathrm{disc}}(f') \\ &=\; \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G,G')\, {}^*\widehat{S}'_{\mathrm{disc}}(f') \\ &=\; I_{\mathrm{disc}}(f)\,. \end{aligned}$$

This is assertion (a). □

Corollary 10.2. *(a) For any G, we have*

$$I^{\mathcal{E}}(f) \;=\; I(f)\,,$$

if f belongs to the subspace $\mathcal{C}_{1\text{-cusp}}(G_V,\zeta_V)$ of $\mathcal{C}(G_V,\zeta_V)$.

(b) If G is quasisplit, the restriction of S^G to the subspace $\mathcal{C}_{1\text{-cusp}}(G_V,\zeta_V)$ is stable.

Proof. The corollary follows from the proposition, the special case (10.2) of the local trace formula, and the definitions (10.3) and (10.4). □

We can now establish our special case of Conjecture 3.3. Fix a cuspidal function $f_1 \in \mathcal{C}_{\mathrm{cusp}}(G,\zeta)$ and a Levi subgroup $M_1 \in \mathcal{L}$.

Proposition 10.3. *(a) Suppose that $\gamma_1 \in \Gamma_{G,\mathrm{ell}}(M_1)$. Then*

$$I^{\mathcal{E}}_{M_1}(\gamma_1, f_1) \;=\; I_{M_1}(\gamma_1, f_1)\,.$$

(b) *Suppose that G is quasisplit and that $\delta'_1 \in \Sigma_{G,\mathrm{ell}}(M'_1)$ for some $M'_1 \in \mathcal{E}_{\mathrm{ell}}(M_1)$. Then $S^G_{M_1}(M'_1, \delta'_1, f_1)$ vanishes unless M'_1 equals M^*, in which case it is stable as a linear form in $f_1 \in \mathcal{C}_{\mathrm{cusp}}(G, \zeta)$.*

Proof. Set $f = f_1 \times \overline{f_2}$, where $f_2 \in \mathcal{C}(G, \zeta)$ is an arbitrary function. Then f belongs to $\mathcal{C}_{1\text{-cusp}}(G_V, \zeta_V)$, and $I^{\mathcal{E}}(f) = I(f)$ by Corollary 10.2. Applying the two expansions (9.6) and (9.2), we see that the expression

$$\sum_{M \in \mathcal{L}} |W^M_0||W^G_0|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Gamma_{G,\mathrm{ell}}(M/Z)} \big(I^{\mathcal{E}}_M(\gamma, f) - I_M(\gamma, f)\big) d\gamma \tag{10.6}$$

vanishes. Since f_1 is cuspidal, the splitting formulas (6.2) and (4.6) simplify. We obtain a decomposition

$$I^{\mathcal{E}}_M(\gamma, f) - I_M(\gamma, f) \; = \; \big(I^{\mathcal{E}}_M(\gamma, f_1) - I_M(\gamma, f_1)\big)\overline{f_{2,M}(\gamma)}\,,$$

which we can substitute into (10.6). We choose $f_2 \in \mathcal{C}(G, \zeta)$ so that $f_{2,G}$ has compact support modulo $Z(F)$ on $\Gamma(G) = \Gamma_{\mathrm{reg}}\big(G(F)\big)$, and so that $f_{2,G}$ approaches the ζ^{-1}-equivariant Dirac measure at the image of γ_1 in $\Gamma(G)$. The expression then approaches a nonzero multiple of

$$I^{\mathcal{E}}_{M_1}(\gamma_1, f_1) - I_{M_1}(\gamma_1, f_1)\,.$$

The assertion (a) follows.

The proof of (b) is similar. Assuming that G is quasisplit, Corollary 10.2 and Theorem 9.1 tell us that the expression

$$\sum_{M \in \mathcal{L}} |W^M_0||W^G_0|^{-1}(-1)^{\dim(A_M/A_G)} \cdot \sum_{M' \in \mathcal{E}_{\mathrm{ell}}(M)} \iota(M, M') \int_{\Sigma_{G,\mathrm{ell}}(M'/Z)} n(\delta')^{-1} S^G_M(M', \delta', f) d\delta' \tag{10.7}$$

is a stable linear form on $\mathcal{C}_{1\text{-cusp}}(G_V, \zeta_V)$. Since f_1 is cuspidal, the splitting formulas (6.3) and (6.3′) simplify. We obtain a decomposition

$$S^G_M(M', \delta', f) \; = \; S^G_M(M', \delta', f_1)\overline{f^{\mathcal{E}}_{2,M}(\delta')}\,,$$

which we substitute into (10.7). We have written

$$f_{2,M}^{\mathcal{E}}(\delta') \;=\; f_2^{M'}(\delta')\,, \qquad\qquad \delta' \in \Sigma_G(M'),$$

here, as in [7]. We choose f_2 so that $f_{2,G}^{\mathcal{E}}$ has compact support modulo $Z(F)$ on $\Gamma^{\mathcal{E}}(G) = \Gamma_{\mathrm{reg}}^{\mathcal{E}}\big(G(F)\big)$, and so that $f_{2,G}^{\mathcal{E}}$ approaches the ζ^{-1}-equivariant Dirac measure at the image of δ_1' in $\Gamma^{\mathcal{E}}(G)$. The expression (10.7) then approaches a nonzero multiple of $S_{M_1}^G(M_1', \delta_1', f_1)$. If $M_1' \neq M_1^*$, we can also assume that $f_2^* = f_2^{G^*} = 0$. Since it is stable in f_2, (10.7) then vanishes, and we conclude that $S_{M_1}^G(M_1', \delta_1', f_1) = 0$, as required. If $M_1' = M_1^*$, so that $\delta_1' = \delta_1^*$ lies in $\Sigma_{G,\mathrm{ell}}(M_1^*)$, we use the stability of (10.7) in the first function f_1 to conclude that the linear form

$$S_{M_1}(\delta_1, f_1) \;=\; S_{M_1}(M_1^*, \delta_1^*, f_1)$$

is stable in $f_1 \in \mathcal{C}_{\mathrm{cusp}}(G, \zeta)$. This completes the proof of the proposition. □

Remarks. 1. Proposition 10.3 could probably be deduced directly from the results of [7], without recourse to Theorem 9.1. The argument we have given here is quite natural, and has the advantage of being a good illustration of the comparison of trace formulas. It is of course a very simple case. The general case is much more elaborate, and seems to demand a simultaneous comparison of both local and global trace formulas. However, the general case ought to be similar in spirit to the proof here of Proposition 10.3.

2. Suppose that $F = \mathbb{R}$. This is the case of Proposition 10.3 which is not conditional on the fundamental lemma, relying instead on results of [16]. By studying character formulas, Kottwitz has established a result that is equivalent to Proposition 10.3, at least in the case that f_1 is a linear combination of matrix coefficients of discrete series.

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