

Harmonic Analysis of Tempered Distributions on Semisimple Lie Groups of Real Rank One

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SUMMARY. Let G be a real semisimple Lie group. Harish-Chandra has defined the Schwartz space, $\mathcal{S}(G)$, on G . A tempered distribution on G is a continuous linear functional on $\mathcal{S}(G)$.

If the real rank of G equals one, Harish-Chandra has published a version of the Plancherel formula for $L^2(G)$ [**3**(k), §24]. We restrict the Fourier transform map to $\mathcal{S}(G)$, and we compute the image of the space $\mathcal{S}(G)$ [Theorem 3]. This permits us to develop the theory of harmonic analysis for tempered distributions on G [Theorem 5].

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Bibliography

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§1. Introduction. Let G be a real semisimple Lie group. The Fourier transform map, \mathcal{F} , can be regarded as an isometry from $L^2(G)$ onto $L^2(\hat{G})$. $L^2(\hat{G})$ is a Hilbert space defined with the help of the discrete series, \mathcal{E}_d , and the various continuous series, \mathcal{E}_c , of irreducible unitary representations of G . $L^2(\hat{G})$ consists of certain functions whose domain is $\mathcal{E}_d \cup \mathcal{E}_c$, and whose range is the space of Hilbert-Schmidt operators on the Hilbert spaces on which the representations in $\mathcal{E}_d \cup \mathcal{E}_c$ act.

In [3(1)] Harish-Chandra introduces the Schwartz space, $\mathcal{E}(G)$, of functions on G . It is analogous to the space, $\mathcal{S}(\mathbf{R})$, of rapidly decreasing functions on the real line. $\mathcal{E}(G)$ is a Fréchet space. It is dense in $L^2(G)$, and its injection into $L^2(G)$ is continuous. It is of interest to ask about the image of $\mathcal{E}(G)$ in $L^2(\hat{G})$ under \mathcal{F} . There is a candidate, $\mathcal{E}(\hat{G})$, for this image space. $\mathcal{E}(\hat{G})$ is a Fréchet space defined by a natural family of seminorms on $L^2(\hat{G})$.

A tempered distribution on G is a continuous linear functional on $\mathcal{E}(G)$. If we can prove that the Fourier transform gives a topological isomorphism from $\mathcal{E}(G)$ onto $\mathcal{E}(\hat{G})$, we could define the Fourier transform of a tempered distribution as a continuous linear functional on $\mathcal{E}(\hat{G})$. This would include as a special case the theory of Fourier transforms on $L^2(G)$.

We confine ourselves to the case in which the real rank of G equals one. In this case Harish-Chandra has published a version of the Plancherel formula for $L^2(G)$ [3(k), §24]. Our main result is Theorem 3, which asserts the bijectivity between $\mathcal{E}(G)$ and $\mathcal{E}(\hat{G})$ of the Fourier transform, \mathcal{F} .

The most difficult part of this theorem is to prove surjectivity. We have to show that the inverse Fourier transform of an element in $\mathcal{E}(\hat{G})$, which is *a priori* in $L^2(G)$, is actually in $\mathcal{E}(G)$. We use some estimates which Harish-Chandra develops from the study of a differential equation on G [3(1), §27]. In §9 we review his work and show that his estimates are actually uniform, in a sense which will become clear. In §10 we use these estimates to prove that $\mathcal{F}(\mathcal{E}(G))$ contains $\mathcal{E}_0(\hat{G})$, a subspace of $\mathcal{E}(\hat{G})$ associated with the discrete series.

To prove that $\mathcal{F}(\mathcal{E}(G))$ contains $\mathcal{E}_1(\hat{G})$, the subspace of $\mathcal{E}(\hat{G})$ associated with the continuous series, requires more work. It is necessary to derive a formula (Lemma 41) for the norms of certain linear transformations, $c^+(\Lambda)$ and $c^-(\Lambda)$, which arise in §12. This we do in §13 by studying a second-order symmetric differential operator on \mathfrak{a}_p , a one-dimensional subspace of the Lie algebra of G . As a byproduct of this formula we obtain in §14 a condition for irreducibility of certain representations in the continuous series.

For convenience we work with generalized spherical functions. We develop the pertinent information in §5 and then use it in §6 to prove the injectivity of the Fourier transform.

In §16 we define the Fourier transform of a tempered distribution on G . Theorem 6 proves that any continuous linear functional on $\mathcal{E}(\hat{G})$ is a certain sum of tempered distributions on the real line.

It seems likely that some of our methods can be used for proving the analogue of Theorem 3 for arbitrary G . The injectivity of the Fourier transform should

carry over quite easily. Harish-Chandra's estimates are proved in [3(1), §27] for arbitrary G . That these estimates are uniform can also be shown, although the proof of this is somewhat more complicated than in the real rank 1 case. Our proof of Lemma 27 does not carry over in general. However, it gives a good start toward a general proof.

The general Plancherel formula will be complicated by the existence of more than one continuous series of representations. However, in each continuous series linear transformations $c(\lambda)$ can be defined. The formulae in Lemma 41 can probably be proved, although perhaps not by our methods. In general, Lemma 44 would be proved by induction on the real rank of G . Harish-Chandra does this for ordinary spherical functions in [3(h), Theorem 3].

2. Preliminaries. Let G be a connected real semisimple Lie group with Lie algebra \mathfrak{g} . Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be a fixed Cartan decomposition with Cartan involution θ . Let $\mathfrak{a}_{\mathfrak{p}}$ be a fixed maximal abelian subspace of \mathfrak{p} . The dimension of $\mathfrak{a}_{\mathfrak{p}}$ is called the real rank of G . We shall assume that $\dim \mathfrak{a}_{\mathfrak{p}} = 1$.

Let $\mathfrak{a}_{\mathfrak{k}}$ be a subspace of \mathfrak{k} such that

$$\mathfrak{a} = \mathfrak{a}_{\mathfrak{k}} + \mathfrak{a}_{\mathfrak{p}}$$

is a Cartan subalgebra of \mathfrak{g} . Let K be the analytic subgroup of G corresponding to \mathfrak{k} . We assume that G has finite center. This implies that K is compact.

We can make further technical assumptions on G without losing generality. In order to do this we state some definitions of Harish-Chandra.

If L is a connected reductive Lie group over the reals, \mathbf{R} , with Lie algebra \mathfrak{l} , let

$$j: \mathfrak{l} \rightarrow \mathfrak{l}_{\mathbf{C}}$$

be inclusion into the complexification of \mathfrak{l} . (From now on, if \mathfrak{h} is any real Lie algebra we write $\mathfrak{h}_{\mathbf{C}}$ for its complexification.) Then if $L_{\mathbf{C}}$ is a complex analytic group with Lie algebra $\mathfrak{l}_{\mathbf{C}}$, $L_{\mathbf{C}}$ is called a complexification of L if j extends to a homomorphism of L into $L_{\mathbf{C}}$. Let $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{c}$, where \mathfrak{l}_1 is semisimple and \mathfrak{c} is abelian. Let $\mathfrak{l}_{1\mathbf{C}}$ and $\mathfrak{c}_{\mathbf{C}}$ be the respective complexifications of \mathfrak{l}_1 and \mathfrak{c} . Let $L_1, C(L_{1\mathbf{C}}, C_{\mathbf{C}})$ be the analytic subgroups of $L(L_{\mathbf{C}})$ corresponding to $\mathfrak{l}_1, \mathfrak{c}(\mathfrak{l}_{1\mathbf{C}}, \mathfrak{c}_{\mathbf{C}})$ respectively. We call $L_{\mathbf{C}}$ quasi-simply connected (q.s.c.) if $L_{1\mathbf{C}} \cap C_{\mathbf{C}} = \{1\}$ and if $L_{1\mathbf{C}}$ is simply connected. We say that L is q.s.c. if it has a q.s.c. complexification.

Fix a complexification $j: L \rightarrow L_{\mathbf{C}}$ and let \mathfrak{h} be a Cartan subalgebra of \mathfrak{l} . Let A and $A_{\mathbf{C}}$ be the Cartan subgroups of L and $L_{\mathbf{C}}$ corresponding to \mathfrak{h} and $\mathfrak{h}_{\mathbf{C}}$ (that is, the centralizers of \mathfrak{h} and $\mathfrak{h}_{\mathbf{C}}$ in G and $G_{\mathbf{C}}$ respectively). Clearly $j(A) \subseteq A_{\mathbf{C}}$. It is known that $A_{\mathbf{C}}$ is connected [3(j), corollary to Lemma 27]. If λ is a linear functional on $\mathfrak{h}_{\mathbf{C}}$, there exists at most one complex analytic homomorphism

$$\xi_{\lambda}: A_{\mathbf{C}} \rightarrow \mathbf{C}$$

such that for every H in $\mathfrak{h}_{\mathbf{c}}$

$$\xi_{\lambda}(\exp H) = e^{\lambda(H)}.$$

We also write ξ_{λ} for the homomorphism

$$\xi_{\lambda} \circ j: A \rightarrow \mathbf{C}.$$

ξ_{λ} can be seen to be independent of the complexification $L_{\mathbf{c}}$ used, provided that ξ_{λ} is defined on that complexification.

Clearly ξ_{α} exists for any root α of $(\mathfrak{l}_{\mathbf{c}}, \mathfrak{h}_{\mathbf{c}})$. If $P_{\mathfrak{h}}$ is the set of positive roots relative to some ordering, let

$$\rho = \frac{1}{2} \sum_{\alpha \in P_{\mathfrak{h}}} \alpha.$$

It is easy to see that the question of the existence of ξ_{ρ} is independent of the ordering of the roots of $(\mathfrak{l}_{\mathbf{c}}, \mathfrak{h}_{\mathbf{c}})$ and of the choice of Cartan subalgebra \mathfrak{h} . If ξ_{ρ} exists we call $L_{\mathbf{c}}$ acceptable. We say that L is acceptable if it has an acceptable complexification.

If $L_{\mathbf{c}}$ is q.s.c., it is known that it is acceptable [3(j), Lemma 29]. If $L_1 \cap C$ is finite, it is clear that L has a finite, and hence acceptable, cover.

Suppose L is a compact, connected acceptable Lie group with Lie algebra \mathfrak{l} . Let \mathfrak{h} , $P_{\mathfrak{h}}$, A , and ρ be defined as above. For each α define an element H_{α} in $\mathfrak{h}_{\mathbf{c}}$ by

$$B(H_{\alpha}, H) = \alpha(H)$$

for all H in $\mathfrak{h}_{\mathbf{c}}$, where B is the Killing form of $\mathfrak{h}_{\mathbf{c}}$ restricted to $\mathfrak{h}_{\mathbf{c}}$. Put

$$\tilde{\omega} = \prod_{\alpha \in P} H_{\alpha}.$$

$\tilde{\omega}$ is in $S(\mathfrak{h}_{\mathbf{c}})$, the symmetric algebra on $\mathfrak{h}_{\mathbf{c}}$. Let Π be the lattice of linear functionals

$$\lambda: (-1)^{1/2}\mathfrak{h} \rightarrow \mathbf{R}$$

for which ξ_{λ} exists. Let $\Pi' = \{\lambda \in \Pi: \tilde{\omega}(\lambda) \neq 0\}$. If W is the Weyl group of $(\mathfrak{l}_{\mathbf{c}}/\mathfrak{h}_{\mathbf{c}})$, W acts on $(-1)^{1/2}\mathfrak{h}$. Then W acts on Π as follows:

$$s\mu(H) = \mu(s^{-1}H)$$

for μ in Π , s in W , and H in $(-1)^{1/2}\mathfrak{h}$. For s in W , put $\varepsilon(s) = (-1)^{n(s)}$, where $n(s)$ is the number of positive roots that are mapped by s into negative roots. For h a regular element of A , put

$$\Delta(h) = \xi_{\rho}(h) \prod_{\alpha \in P_{\mathfrak{h}}} (1 - \xi_{\alpha}(h^{-1})).$$

LEMMA 1. *There is a map $\mu \rightarrow \sigma(\mu)$ from Π' onto the set of unitary equivalence classes of irreducible representations of L . $\sigma(\mu_1) = \sigma(\mu_2)$ if and only if $\mu_1 = s\mu_2$ for some s in W . Furthermore, if h is a regular element of A ,*

$$\text{tr } \sigma(\mu)(h) = (\text{sign } \tilde{\omega}(\mu)) \cdot \Delta(h)^{-1} \cdot \left(\sum_{s \in W} \varepsilon(s) \xi_{s\mu}(h) \right).$$

Also there exists a constant c_L , independent of μ , such that

$$\dim \sigma(\mu) = c_L |\tilde{\omega}(\mu)|.$$

Finally, if μ is in Π' , and $B(\mu, \alpha) > 0$ for each α in $P_{\mathfrak{h}}$, then $\mu - \rho$ is the highest weight of the representation of the Lie algebra \mathfrak{g} corresponding to $\sigma(\lambda)$.

PROOF. Since L has a finite q.s.c. cover, we will assume without loss of generality that L is q.s.c. We can assume further that L is semisimple. Then L is simply connected, so Π is precisely the lattice of weights of \mathfrak{h} [3(j), Lemma 29]. If μ' is a dominant integral function (in the terminology of [5, p. 215]), and if $\mu = \mu' + \rho$, then $B(\mu, \alpha) > 0$ for any α in $P_{\mathfrak{h}}$ so μ is in Π' . Conversely, if μ is in Π' , there exists a unique s such that $B(s\mu, \alpha) > 0$ for each α in $P_{\mathfrak{h}}$. Then $\mu' = \mu - \rho$ is a dominant integral function on \mathfrak{h} . This demonstrates the relation between μ and the highest weight of $\sigma(\mu)$. The correspondence between representations and dominant integral functions is well known (see [5, Chapter VII]).

The other two statements of the lemma follow from the Weyl character formula [5, p. 255] and the Weyl dimension formula [5, p. 257]. \square

Now let us return to our group G . By going to a finite cover we can assume that G is q.s.c. and hence acceptable. Thus, if $j: \mathfrak{g} \rightarrow \mathfrak{g}_c$ and G_c is a simply connected analytic group with Lie algebra \mathfrak{g}_c then j extends to a homomorphism

$$j: G \rightarrow G_c.$$

Now K is reductive. Therefore, by going to a further finite cover of G , we may also assume that K is acceptable.

If we understand the harmonic analysis of a finite cover, \tilde{G} , of G then we understand the theory for G . We merely throw out those unitary representations of \tilde{G} which are nontrivial on the kernel of the covering projection. Therefore, the above two assumptions can be made with no loss of generality.

There are two possibilities for G .

Case I. There exists a Cartan subalgebra \mathfrak{b} of \mathfrak{g} such that \mathfrak{b} is contained in \mathfrak{k} . We can assume that \mathfrak{b} has been chosen so that it contains \mathfrak{a}_+ . Then it is known that $\{\mathfrak{b}, \mathfrak{a}\}$ is a set of representatives of conjugacy classes of Cartan subalgebras of \mathfrak{g} .

Case II. Such a \mathfrak{b} does not exist. Then there is only one conjugacy class of Cartan subalgebras and it is represented by \mathfrak{a} .

We shall try as far as possible to deal with these two cases together. Whenever we speak of \mathfrak{b} , we shall be implicitly referring to Case I. However, any mention of \mathfrak{a} , unless otherwise stated, will refer to either case.

Let B be the Cartan subgroup of G corresponding to \mathfrak{b} . Since it is a maximal abelian subgroup in the compact connected Lie group K , it is connected [4, Corollary 2.7, p. 247].

Let A be the Cartan subgroup of G corresponding to \mathfrak{a} . Then

$$A = A_I A_p$$

where $A_p = \exp \mathfrak{a}_p$, and A_I is contained in K . In Case II, A_I is a Cartan subgroup of K and is connected. Otherwise, A_I may not be connected. In any case, let \mathfrak{m} and M be the centralizers of \mathfrak{a}_p in \mathfrak{k} and K , respectively. Then M is compact with a finite number of connected components.

Fix compatible orders on the real dual spaces of \mathfrak{a}_p and $\mathfrak{a}_p + (-1)^{1/2}\mathfrak{a}_t$. Let P be the set of positive roots of $(\mathfrak{g}_c, \mathfrak{a}_c)$ relative to this order. Let P_+ be the set of roots in P which do not vanish on \mathfrak{a}_p and let P_M equal $P - P_+$. \mathfrak{a}_t is a Cartan subalgebra of the reductive Lie algebra \mathfrak{m} and we can regard P_M as the set of positive roots of $(\mathfrak{m}, \mathfrak{a}_t)$.

Let M^0 and A_I^0 be the connected components of M and A_I . Let W and W_1 be the Weyl groups of $(\mathfrak{g}/\mathfrak{a})$ and $(\mathfrak{m}/\mathfrak{a}_t)$, respectively. Now in any connected component of M , it is possible to choose an element γ_1 such that

$$\text{Ad } \gamma_1 \cdot \mathfrak{a}_t = \mathfrak{a}_t.$$

But $\text{Ad } \gamma_1$ leaves \mathfrak{a}_p pointwise fixed. Therefore, the action of γ_1 on \mathfrak{a}_t can be regarded as coming from an element of the subgroup of W generated by those roots in P which vanish on \mathfrak{a}_p . That is, the action of $\text{Ad } \gamma_1$ on \mathfrak{a}_t is the same as for some element in W_1 . Therefore, we can choose a new element γ , in the same component of M , that leaves \mathfrak{a}_t pointwise fixed. This means that γ is in A_I . Therefore, A_I has the same number of connected components as M .

As usual, let

$$\rho = \frac{1}{2} \sum_{\alpha \in P} \alpha, \quad \rho_M = \frac{1}{2} \sum_{\alpha \in P_M} \alpha.$$

Then since G is acceptable, it is known that M^0 is also acceptable and that for any a_1 in A_I^0

$$(2.1) \quad \xi_\rho(a_1) = \xi_{\rho_M}(a_1)$$

[see **3(j)**, Lemma 30].

Let \mathcal{E}_M be the set of equivalence classes of irreducible unitary representations of M . Let C be the set of irreducible characters of the group A_I (the set of characters coming from irreducible representations of A_I). For ζ in C and a in A_I write $\langle \zeta, a \rangle$ for the value of ζ at a . It is clear that W_1 operates on C .

Put $\tilde{\omega}^m = \prod_{\alpha \in P_M} H_\alpha$ and let L_1 be the lattice of real linear functionals, μ , on $(-1)^{1/2}\mathfrak{a}_t$ such that ξ_μ exists. Let $L'_1 = \{\mu \in L_1 : \tilde{\omega}^m(\mu) \neq 0\}$. Let $Z(A) = \{\gamma \in A_I : j(\gamma) \in \exp(-1)^{1/2}\mathfrak{a}_R\}$. Then $Z(A)$ is a finite subgroup of A_I . It is known that if γ is in $Z(A)$ and m is in M^0 , then γ and m commute [**3(j)**, Lemma 51]. Also $Z(A)A_I^0 = A_I$, by [**3(k)**, Lemma 20], so $Z(A)M^0 = M$. Let $Z(A)^0 = Z(A) \cap A_I^0$. $Z(A)^0$ is a central subgroup of both $Z(A)$ and M^0 . Then M is the central product of M^0 and $Z(A)$ with respect to $Z(A)^0$ (see [**2**, p. 29]). Thus if $\overline{M} = M^0 \times Z(A)$ and $\overline{Z(A)}^0 = \{(\gamma, \gamma^{-1}) : \gamma \in Z(A)^0\}$ then $\overline{Z(A)}^0$ is a discrete normal subgroup of \overline{M} . M is isomorphic to $\overline{M}/\overline{Z(A)}^0$. Similarly, if $\overline{A_I} = A_I^0 \times Z(A)$, then A_I is isomorphic to $\overline{A_I}/\overline{Z(A)}^0$. Therefore, irreducible representations of M (or A_I) are in one-to-one correspondence with representations of \overline{M} (or

A_I) which are trivial on $\overline{Z(A)}^0$. An irreducible representation of $\overline{A_I}$ is of the form $\xi_\mu \otimes \delta$, where μ is in L_1 and δ is an irreducible representation of $Z(A)$. Let C' be the set of irreducible characters ζ in C that come from representations $\xi_\mu \otimes \delta$ of \overline{M} for which μ is actually in L'_1 . If μ and ζ are so related, we shall write $\mu = \mu_\zeta$. We would like to prove a lemma which will relate the representations in \mathcal{E}_M with characters in C' .

Let σ be an arbitrary representation of M . Then

$$(2.2) \quad \sigma = \sigma_0 \times \varepsilon$$

where σ_0 and ε are irreducible representations of M^0 and $Z(A)$, respectively, such that for any γ_0 in $Z(A)^0$, $\sigma_0(\gamma_0) \otimes \varepsilon(\gamma_0^{-1})$ is the identity. $Z(A)^0$ is in the center of both M^0 and $Z(A)$ so $\sigma_0(\gamma_0)$ and $\varepsilon(\gamma_0)$ are both scalars. Therefore

$$\sigma_0(\gamma_0) = \varepsilon(\gamma_0).$$

Suppose that $\sigma_0 = \sigma_0(\mu)$ in the notation of Lemma 1. μ is a linear functional in L'_1 . Then there exists an s in W_1 such that $s\mu - \rho$ is the highest weight for σ_0 . Let γ_0 be an element in $Z(A)^0$. By looking at the action of $\xi_{s\mu} - \rho(\gamma_0)$ on a highest weight vector for σ_0 we see that the scalar $\sigma_0(\gamma_0)$ is equal to $\xi_{s\mu} - \rho(\gamma_0)$. Therefore

$$\varepsilon(\gamma_0) = \sigma_0(\gamma_0) = \xi_{s\mu}(\gamma_0)\xi_\rho(\gamma_0^{-1}) = \xi_\mu(s^{-1}\gamma_0)\xi_0(\gamma_0^{-1}).$$

However, γ_0 is in the center of M so $s^{-1}\gamma_0 = \gamma_0$. Therefore

$$(2.3) \quad \varepsilon(\gamma_0)\xi_\rho(\gamma_0) = \xi_\mu(\gamma_0)$$

for any γ_0 in $Z(A)^0$.

For any γ in $Z(A)$, define

$$(2.4) \quad \delta(\gamma) = \varepsilon(\gamma)\xi_\rho(\gamma).$$

This is an irreducible representation of $Z(A)$ and by (2.3), $\xi_\mu \otimes \delta$ can be regarded as an irreducible representation of A_I . Let

$$(2.5) \quad \langle \zeta, \gamma a_0 \rangle = \zeta_\mu(a_0) \cdot \text{tr } \delta(\gamma)$$

for a_0 in A_I^0 , γ in $Z(A)$. ζ is an element in C' and $\mu = \mu_\zeta$. Therefore, given a σ in \mathcal{E}_M , we have constructed an element ζ in C' . We write $\sigma = \sigma(\zeta)$.

Conversely, let us start with an element in C' . By working backward we can show that there is a unique element σ in \mathcal{E}_M such that $\sigma = \sigma(\zeta)$.

Suppose that a_0 is a regular element of A_I^0 and γ is in $Z(A)$. We wish to compute the trace of $\sigma(a_0\gamma)$. Define

$$\Delta_M(a_0) = \xi_\rho(a_0) \cdot \prod_{\alpha \in P_M} (1 - \xi_\alpha(a_0^{-1})).$$

In the above notation

$$\text{tr } \sigma(a_0\gamma) = \text{tr } \sigma_0(a_0) \cdot \text{tr } \varepsilon(\gamma).$$

But from Lemma 1,

$$\mathrm{tr} \sigma(a_0) = (\mathrm{sign} \tilde{\omega}^m(\mu)) \cdot \Delta_M(a_0)^{-1} \cdot \left(\sum_{s \in W_1} \varepsilon(s) \xi_{s\mu}(a_0) \right).$$

Therefore, the trace of $\sigma(a_0\gamma)$ is equal to

$$(\mathrm{sign} \tilde{\omega}^m(\mu)) \cdot \Delta_M(a_0)^{-1} \cdot \left(\sum_{s \in W_1} \varepsilon(s) \langle s\zeta, a_0\gamma \rangle \right) \xi_\rho(\gamma^{-1}).$$

Now it is easy to show that if γ_c is in $j(Z(A))$ then $(\gamma_c)^2 = 1$. Therefore if γ is in $Z(A)$, $\xi_\rho(\gamma) = \xi_\rho(\gamma)^{-1}$. For future convenience, we rewrite the trace of $\sigma(a_0\gamma)$ as

$$(2.6) \quad (\mathrm{sign} \tilde{\omega}^m(\mu)) \cdot \Delta_M(a_0)^{-1} \cdot \left(\sum_{s \in W_1} \varepsilon(s) \langle s\zeta, a_0\gamma \rangle \right) \xi_\rho(\gamma).$$

LEMMA 2. *There is a map $\zeta \rightarrow \sigma(\zeta)$ from C' onto \mathcal{E}_M . $\sigma(\zeta_1) = \sigma(\zeta_2)$ if and only if $s\zeta_1 = \zeta_2$ for some s in W_1 . If a_0 is a regular element in A_I^0 and γ is in $Z(A)$ then the trace of $\sigma(\zeta)(a_0\gamma)$ equals*

$$(\mathrm{sign} \tilde{\omega}^m(\mu)) \cdot \Delta_M(a_0)^{-1} \cdot \left(\sum_{s \in W_1} \varepsilon(s) \langle s\zeta, a_0\gamma \rangle \right) \cdot \xi_\rho(\gamma).$$

Also, there exists a constant C_M , independent of ζ , such that

$$\dim \sigma(\zeta) = C_M \cdot |\tilde{\omega}^m(\mu_\zeta)| \cdot \dim \zeta.$$

($\dim \zeta$ means the dimension of the representation of A_I of which ζ is the character.)

PROOF. The dimension formula follows from Lemma 1. All other statements in the lemma follow from the above discussion. \square

Let us say that the linear functional μ_ζ is associated with σ if $\sigma = \sigma(\zeta)$, in the above notation. For any σ in \mathcal{E}_M there are exactly $[W_1]$ associated real linear functionals on \mathfrak{a}_+ .

Now with B there is associated a discrete series of unitary representations of G . With A there is associated a continuous series. We shall describe these.

For the discrete series there is a formal analogy with Lemma 1. Let Σ be the set of positive roots of $(\mathfrak{g}_c, \mathfrak{b}_c)$ relative to some order. For any α in Σ define H_α in $(-1)^{1/2}\mathfrak{b}$ by the formula

$$B(H_\alpha, H) = \alpha(H)$$

for any H in \mathfrak{b}_c . Put $\tilde{\omega}^b = \prod_{\alpha \in \Sigma} H_\alpha$. Let L be the lattice of real linear functionals, λ , on $(-1)^{1/2}\mathfrak{b}$ such that ξ_λ exists. Let $L' = \{\lambda \in L: \tilde{\omega}^b(\lambda) \neq 0\}$. Let $N(B)$ be the normalizer of B in G . Define

$$W_G = W_{G,b} = N(B)/B.$$

This is a finite group. It acts on B and therefore on L .

An irreducible representation π of G on a Hilbert space \mathcal{H} is said to be square-integrable if there exist nonzero vectors Φ_1, Φ_2 in \mathcal{H} such that $(\Phi_1, \pi(x)\Phi_2)$ is a square-integrable function of x . If π and π' are square-integrable representations on \mathcal{H} and \mathcal{H}' and if π and π' are not unitarily equivalent, then for Φ_1, Φ_2 in \mathcal{H} and Φ'_1, Φ'_2 in \mathcal{H}' ,

$$(2.7) \quad \int_G (\Phi_1, \pi(x)\Phi_2)(\pi'(x)\Phi'_2, \Phi'_1) dx = 0.$$

On the other hand, there is a number $\beta(\pi)$, the formal degree of π , such that for every $\Phi_1, \Phi_2, \Psi_1, \Psi_2$ and \mathcal{H} ,

$$(2.8) \quad \int_G (\Phi_1, \pi(x)\Phi_2)(\pi(x)\Psi_2, \Psi_1) dx = \beta(\pi)^{-1}(\Phi_1, \Psi_1)(\Psi_2, \Phi_2).$$

These are the Schur orthogonality relations on G . They are proved in [3(d), Theorem 1].

Let \mathcal{E}_d be the set of unitary equivalence classes of square-integrable representations of G . Harish-Chandra gives a map $\lambda \rightarrow \omega(\lambda)$ from L' onto \mathcal{E}_d [see 3(1), Theorem 16]. $\omega(\lambda_1) = \omega(\lambda_2)$ if and only if there is an s in W_G such that $s\lambda_1 = \lambda_2$. Finally, there is a constant C_G , independent of λ , such that

$$\beta(\omega(\lambda)) = C_G |\tilde{\omega}^b(\lambda)|.$$

LEMMA 3. $\{\beta(\omega) : \omega \in \mathcal{E}_d\}$ is bounded away from zero.

PROOF. It is clearly enough to show that for any α in Σ , $\{\lambda(H_\alpha)\}_{\lambda \in L'}$ is bounded away from zero. Let \tilde{L} be the lattice of real linear functionals on $(-1)^{1/2}\mathfrak{b}$ generated by the roots. Then it is known that L/\tilde{L} is isomorphic to the center of G , which is finite. It is also known that $\{\lambda(H_\alpha)\}_{\lambda \in \tilde{L}}$ is a lattice in \mathbf{R} . Therefore $\{\lambda(H_\alpha)\}_{\lambda \in L}$ is also a lattice in \mathbf{R} . But if λ is in L' , $\lambda(H_\alpha) \neq 0$, so the lemma follows. \square

Now we shall describe the continuous series. There is a linear functional μ_0 from \mathfrak{a}_p to \mathbf{R} such that the restriction of any root in P_+ to \mathfrak{a}_p is either μ_0 or $2\mu_0$. Fix H_0 in \mathfrak{a}_p so that $\mu_0(H_0) = 1$. Extend the definition of μ_0 to \mathfrak{a} by letting it equal zero on \mathfrak{a}_t .

Let $\mathfrak{n}_c = \sum_{\alpha \in P_+} \mathbf{C}X_\alpha$, where for any α in P , X_α is a fixed root vector. Let $\mathfrak{n} = \mathfrak{n}_c \cap \mathfrak{g}$. Let N be the analytic subgroup of G corresponding to \mathfrak{n} . It is well known (see [4, p. 373]) that the map

$$(k, a, n) \rightarrow kan, \quad k \in K, a \in A_p, n \in N,$$

is a diffeomorphism of $K \times A_p \times N$ with G . For f in $C_0^\infty(G)$,

$$(2.9) \quad \int_G f(x) dx = \int_{K \times A_p \times N} f(kan) e^{2\rho(\log a)} dk da dn$$

for a suitable normalization of the Haar measure dx . If $x = kan$, write $K(x) = k$ and $H(x) = \log a$.

It is clear that $P = MA_pN$ is a subgroup of G . If σ in \mathcal{E}_M acts on a finite dimensional Hilbert space V_σ , and if Λ is in \mathbf{R} , then the map σ_Λ from P into $\text{End}(V_\sigma)$ given by

$$\sigma_\Lambda(m \cdot \exp tH_0 \cdot n) = \sigma(m)e^{-i\Lambda t}, \quad m \in M, n \in N, t \in \mathbf{R}$$

is an irreducible unitary representation of P . (We shall sometimes write i instead of $(-1)^{1/2}$.) Let $\pi_{\sigma,\Lambda}$ be the unitary representation of G on the Hilbert space $\mathcal{H}_{\sigma,\Lambda}$ obtained by inducing σ_Λ from P to G .

Then $\mathcal{H}_{\sigma,\Lambda}$ is the set of functions Φ from G into V_σ such that

$$(2.10) \quad \Phi(x\xi^{-1}) = \sigma_\Lambda(\xi)\Phi(x), \quad x \in G, \xi \in P,$$

$$(2.11) \quad \Phi(k) \text{ is a Borel function on } K,$$

$$(2.12) \quad \int_K \|\Phi(k)\|^2 dk < \infty.$$

The inner product on $\mathcal{H}_{\sigma,\Lambda}$ is given by

$$(\Phi, \Psi) = \int_K (\Phi(k), \Psi(k))_{V_\sigma} dk, \quad \Phi, \Psi \in \mathcal{H}_{\sigma,\Lambda},$$

where $(\cdot, \cdot)_{V_\sigma}$ is the inner product in V_σ . If Φ is in $\mathcal{H}_{\sigma,\Lambda}$, $\pi_{\sigma,\Lambda}(y)\Phi$ is given by

$$(2.13) \quad (\pi_{\sigma,\Lambda}(y)\Phi)(x) = \Phi(y^{-1}x)e^{-\rho(H(y^{-1}x)) + \rho(H(x))}, \quad x, y \in G.$$

For any real Λ , and any Φ in $\mathcal{H}_{\sigma,\Lambda}$ we can define a function $\tilde{\Phi}$ from K to V_σ by restricting Φ to K . This identifies $\mathcal{H}_{\sigma,\Lambda}$ with a Hilbert space, \mathcal{H}_σ , of square-integrable functions from K into V_σ . \mathcal{H}_σ is independent of Λ . In fact, if π_σ is the representation of K obtained by inducing σ to K , \mathcal{H}_σ is the Hilbert space on which π_σ acts. The above equivalence between \mathcal{H}_σ and $\mathcal{H}_{\sigma,\Lambda}$ gives an intertwining operator between π_σ and $\pi_{\sigma,\Lambda}|_K$, the restriction of $\pi_{\sigma,\Lambda}$ to K .

Let M' be the normalizer of \mathfrak{a}_p in K . M is a normal subgroup of M' . M'/M is a group consisting of two elements, $\{1, \delta\}$ say. δ acts on \mathfrak{a}_p by reflection. δ also induces an automorphism of M , modulo the group of inner automorphisms. Therefore δ defines a bijection.

$$\delta: \sigma \rightarrow \sigma'$$

of \mathcal{E}_M onto itself. If we let δ act on P , we can transform the representation σ_Λ into the representation $(\sigma')_{-\Lambda}$. Now, if Λ is real and σ is in \mathcal{E}_M , it is known that the representation $\pi_{\sigma,\Lambda}$ is equivalent to $\pi_{\sigma',-\Lambda}$. Furthermore, the representations $\{\pi_{\sigma,\Lambda}\}$, $\sigma \in \mathcal{E}_M$, $\Lambda > 0$ are all irreducible and inequivalent [see 1, Theorem 7; 2].

For each σ in \mathcal{E}_M and $\Lambda \neq 0$, let $N_\sigma(\Lambda)$ be a fixed unitary intertwining operator between the representations $\pi_{\sigma,\Lambda}$ and $\pi_{\sigma',-\Lambda}$. Then

$$N_\sigma(\Lambda)\pi_{\sigma,\Lambda}(x)N_\sigma(\Lambda)^{-1} = \pi_{\sigma',-\Lambda}(x), \quad x \in G.$$

Notice that since $\pi_{\sigma,\Lambda}$ is irreducible,

$$(2.14) \quad N_{\sigma'}(-\Lambda) = N_\sigma(\Lambda)^{-1}.$$

It will be convenient to assign a positive real number to any equivalence class of representations in either \mathcal{E}_d or \mathcal{E}_M . If ω is in \mathcal{E}_d , choose λ in L' such that $\omega = \omega(\lambda)$. The Killing form, B , of $\mathfrak{g}_\mathbb{C}$ can be regarded as a positive definite form on either $(-1)^{1/2}\mathfrak{b}$ or its real dual space. Then put

$$|\omega|^2 = B(\lambda, \lambda).$$

Since W_G acts on $(-1)^{1/2}\mathfrak{b}$ as a group of isometries under the Killing form, $|\omega|$ is well defined. Similarly, for σ in \mathcal{E}_M , we define

$$|\sigma|^2 = B(\mu_\sigma, \mu_\sigma)$$

where μ_σ is any real linear functional on $(-1)^{1/2}\mathfrak{a}_\mathfrak{k}$ associated with σ . $|\sigma|$ is well defined by the above argument.

Let \mathcal{E}_K be the set of unitary equivalence classes of irreducible representations of K . Let \mathfrak{h} be the subspace of \mathfrak{k} which is equal to either \mathfrak{b} or $\mathfrak{a}_\mathfrak{k}$, depending on whether we are in Case I or Case II. \mathfrak{h} is a Cartan subalgebra of \mathfrak{k} . In either case, we have already ordered the dual space of \mathfrak{h} . K is acceptable by assumption, so the representations in \mathcal{E}_K can be indexed by certain real linear functionals on $(-1)^{1/2}\mathfrak{h}$ as in Lemma 1. If τ is in \mathcal{E}_K and $\tau = \tau(\mu)$ for some real linear functional μ on $(-1)^{1/2}\mathfrak{h}$, then we write

$$|\tau|^2 = B(\mu, \mu).$$

$|\tau|$ is well defined.

3. Plancherel formula for $L^2(G)$. In order to put the Plancherel formula for G into the form we want, we must discuss characters of unitary representations of G . To do this we must introduce some more notation of Harish-Chandra.

For t in \mathbf{R} , put $h_t = \exp tH_0$. For g in $C_0^\infty(M^0A_p)$, write

$$F_g^M(a_0h_t) = \Delta_M(a_0) \cdot \int_{M^0/A_I^0} g(m^{*-1}a_0h_tm^*) dm^*$$

for a_0 in A_I^0 and a_0h_t a regular element in A . Here dm^* is the invariant measure on the homogeneous space M^0/A_I^0 . It is known that there exists a constant $c_1 > 0$ such that for any g in $C_0^\infty(M^0A_p)$

$$(3.1) \quad \int_{M^0 \times \mathbf{R}} g(m_0h_t) dm_0 dt = c_1 \int_{A_I^0 \times \mathbf{R}} \overline{\Delta_M(a_0)} \cdot F_g^M(a_0h_t) da_0 dt$$

(see [3(j), Lemma 41]).

For a in A_I and ah_t a regular element in A write

$$\begin{aligned} \Delta(ah_t) &= \xi_\rho(ah_t) \cdot \prod_{\alpha \in P} (1 - \xi_\alpha(ah_t)^{-1}), \\ \varepsilon_{\mathbf{R}}(ah_t) &= 1 \quad \text{if } t > 0, \quad = -1 \quad \text{if } t < 0. \end{aligned}$$

If f is in $C_0^\infty(G)$ write

$$F_f(ah_t) = \varepsilon_{\mathbf{R}}(ah_t) \cdot \Delta(ah_t) \cdot \int_{G^*} f(x^{*-1}ah_tx^*) dx^*.$$

(G^* is the homogeneous space G/A_0 where A_0 is the center of A . Let dx^* be the G invariant measure on G^*). It is clear that

$$(3.2) \quad F_f(sah_t) = \varepsilon(s)F_f(ah_t), \quad s \in W_1.$$

It is known that if f is in $C_0^\infty(G)$, then F_f extends to an infinitely differentiable function on A (see [3(f), Lemma 40]). Furthermore, F_f has compact support in A [3(f), Theorem 2].

Let $\tilde{\omega} = \tilde{\omega}^a = \prod_{\alpha \in P} H_\alpha$. Let

$$q = \frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{k} - \text{rank } \mathfrak{g} + \text{rank } \mathfrak{k}).$$

It is known that q is an integer. If we are in Case II, the Cartan subgroup A is fundamental, in the terminology of [3(f), p. 759]. Then Harish-Chandra's limit formula applies to F_f [3(f), Theorem 4]. Namely, there is a positive constant c such that for any f in $C_0^\infty(G)$

$$(3.3) \quad cf(1) = (-1)^q F_f(1; \tilde{\omega}).$$

Here $\tilde{\omega}$ is to be regarded as a differential operator on A .

For f in $C_0^\infty(G)$ define a function g_f in $C_0^\infty(MA_p)$ by

$$g_f(mh_t) = e^{\rho(tH_0)} \int_N \int_K f(kmh_tnk^{-1}) dk dn, \quad m \in M, t \in \mathbf{R}.$$

For γ in $Z(A)$ and m_0 in M^0 put

$$g_{f,\gamma}(m_0h_t) = g_f(\gamma m_0h_t).$$

Then in [3(j), Lemma 52] it is shown that there is a constant $c_2 > 0$ such that

$$(3.4) \quad F_f(\gamma a_0h_t) = c_2 \cdot \xi_\rho(\gamma) F_{g_{f,\gamma}}^M(a_0h_t).$$

While we are at it, we shall state another Jacobian formula which we shall need later in the paper (see [4, p. 381, Proposition 1.17]). The map from $K \times \mathfrak{a}_p^+ \times K$ into G given by

$$(k_1, tH_0, k_2) \rightarrow k_1 \cdot \exp tH_0 \cdot k_2$$

is a diffeomorphism onto G . (We write $\mathfrak{a}_p^+ = \{tH_0 : t > 0\}$.) Furthermore, there is a constant $c > 0$ such that for any f in $C_0^\infty(G)$

$$(3.5) \quad \begin{aligned} \int_G f(x) dx &= c \int_0^\infty \int_{K \times K} f(k_1 \cdot \exp tH_0 \cdot k_2) |D(t)| dk_1 dk_2 dt \\ &= \frac{c}{2} \int_{-\infty}^\infty \int_{K \times K} f(k_1 \cdot \exp tH_0 \cdot k_2) |D(t)| dk_1 dk_2 dt. \end{aligned}$$

Here $D(t) = (e^t - e^{-t})^{r_1} \cdot (e^{2t} - e^{-2t})^{r_2}$, where r_1 and r_2 are the number of roots in P_+ which, when restricted to \mathfrak{a}_p , are respectively equal to μ_0 and $2\mu_0$.

Let π be an irreducible unitary representation of G on the Hilbert space \mathcal{H} . Let f be a function in $C_0^\infty(G)$. It is known that the operator

$$\pi(f) = \int_G f(x) \pi(x) dx$$

is of trace class. The map

$$f \rightarrow \text{tr } \pi(f)$$

is a distribution on $C_0^\infty(G)$ (see [3(c), §5]). This distribution is called the character of π .

If σ is in \mathcal{E}_M , and Λ is in \mathbf{R} , let $\Theta_{\sigma,\Lambda}$ be the character of the representation $\pi_{\sigma,\Lambda}$. Let $m = \frac{1}{2}(\dim \mathfrak{g} - \text{rank } \mathfrak{g})$. Choose ζ in C' and the associated μ_ζ in L_1 such that $\sigma = \sigma(\zeta)$ as in Lemma 2.

THEOREM 1. *There exists a constant $c_0 > 0$ such that for every f in $C_0^\infty(G)$, $\Theta_{\sigma,\Lambda}(f)$ is equal to*

$$c_0(-1)^{m+\iota}(\text{sign } \tilde{\omega}^m(\mu_\zeta)) \cdot \int_{A_I \times \mathbf{R}} F_f(ah_t)\langle \zeta, a \rangle e^{-i\Lambda t} da dt,$$

where ι equals 1 or 0 depending on whether we are in Case I or Case II.

PROOF. Let A be the operator

$$\int_G f(x)\pi_{\sigma,\Lambda}(x) dx$$

on $\mathcal{H}_{\sigma,\Lambda}$. We want to compute the trace of A . $\text{tr } A$ is equal to $\overline{\text{tr } A^*}$ where A^* is the adjoint operator of A and the bar denotes complex conjugation. If Φ is in $\mathcal{H}_{\sigma,\Lambda}$ and k_1 is in K

$$\begin{aligned} (A^*\Phi)(k_1) &= \left(\int_G \overline{f(x)}\pi(x^{-1}) dx \cdot \Phi \right) (k_1) \\ &= \int_G \overline{f(x)}\Phi(xk_1)e^{-\rho(H(xk_1))} dx \\ &= \int_G \overline{f(xk_1^{-1})}\Phi(x)e^{-\rho(H(x))} dx. \end{aligned}$$

Assume that the Haar measure on A_p has been normalized so that $dh_t = dt$. Then by (2.9) the above integral equals

$$\int_{K \times \mathbf{R} \times N} \overline{f(kh_tnk_1^{-1})}e^{i\Lambda t}e^{\rho(tH_0)}\Phi(k) dk dt dn.$$

In this integral, substitute km for k and integrate with respect to M . Then $(A^*\Phi)(k_1)$ equals

$$\int_{K \times M \times \mathbf{R} \times N} \overline{f(kmh_tnk_1^{-1})}\sigma(m^{-1})e^{i\Lambda t}e^{\rho(tH_0)}\Phi(k) dm dt dn dk.$$

Now to deal further with this expression we consider the principal fiber bundle

$$M \rightarrow K \rightarrow K/M.$$

The map $m \rightarrow \sigma(m^{-1})$ defines a complex vector bundle E_σ over K/M with fiber V_σ , the space on which σ acts. Let $F(k_1, k)$ be the function

$$\int_{M \times \mathbf{R} \times N} \overline{f(kmh_tnk_1^{-1})}\sigma(m^{-1})e^{i\Lambda t}e^{\rho(tH_0)} dm dt dn.$$

Now it is easy to check that M normalizes N and that for fixed m in M the measures dn and $d(mnm^{-1})$ on N are equal. Then for \bar{m}_1, \bar{m} in M ,

$$F(k_1\bar{m}_1, k\bar{m}) = \sigma(\bar{m}_1^{-1})F(k_1, k)\sigma(\bar{m}).$$

Therefore $F(k_1, k)$ can be regarded as a section of $E_\sigma \boxtimes E_\sigma^*$, where E_σ^* is the adjoint bundle of E_σ and $E_\sigma \boxtimes E_\sigma^*$ is the exterior tensor product of E_σ and E_σ^* , a bundle with base space $K/M \times K/M$ and fiber $V_\sigma \otimes V_\sigma^*$.

Now there is a natural equivalence between $\mathcal{H}_{\sigma, \Lambda}$ and the space \mathcal{H}_σ defined earlier. However, \mathcal{H}_σ is the space of square-integrable sections of E_σ with respect to a K -invariant measure on K/M . $F(k_1, k)$ can be regarded as the kernel of the linear operator A^* on this space. Then for any Φ on \mathcal{H}_σ

$$(A^*\Phi)(k_1) = \int_K F(k_1, k)\Phi(k) dk.$$

To evaluate the trace of A^* we appeal to the following lemma.

LEMMA 4. *Let X be a compact infinitely differentiable manifold of dimension n . Let dx be a positive nowhere-vanishing differentiable n -form on X . If $E \rightarrow X$ is a differentiable Hilbert bundle of fiber dimension s , let $L^2(E)$ be the Hilbert space of square-integrable sections of E . If $F(x_1, x)$ is a continuous section of $E \boxtimes E^*$, $F(x_1, x)$ defines a bounded linear operator F on $L^2(E)$ in the obvious manner. Then if $F(x_1, x)$ is differentiable in both variables, F is of trace class. Furthermore*

$$\text{tr } F = \int_X (\text{tr } F(x, x)) dx.$$

PROOF. Let T be the closed unit n -cube with opposite sides identified. T is an n -torus and there is a canonical n -form dt on T . Let

$$S = \{t \in \mathbf{R}^n : |t| < 1\}.$$

S is an open subset of T .

Choose a finite differentiable partition of unity $\{\Psi_\alpha\}_{\alpha \in I}$ and a collection $\{U_\alpha\}_{\alpha \in I}$ of open subsets of X such that the support of Ψ_α is contained in U_α . We assume that for every (α, β) in $I \times I$ there is a diffeomorphism $\lambda_{\alpha\beta}$ from $U_\alpha \cup U_\beta$ onto $S_{\alpha\beta}$, an open subset of S . It can be seen that with no loss of generality we may also assume that

- (i) $\lambda_{\alpha\beta}^*(dt) = dx$.
- (ii) If $E_{\alpha\beta}$ is the restriction of E to $U_\alpha \cup U_\beta$, then $E_{\alpha\beta}$ is trivial.
- (iii) The map $\lambda_{\alpha\beta}$ lifts to a bundle map

$$\Lambda_{\alpha\beta}: E_{\alpha\beta} \rightarrow S_{\alpha\beta} \times \mathbf{R}^s$$

which is an isomorphism between Hilbert bundles preserving the inner product on each fiber. (We assume that \mathbf{R}^s is equipped with the natural scalar product.)

Let $F_{\alpha\beta}$ be the integral operator on $L^2(E)$ with kernel

$$F_{\alpha\beta}(x_1, x) = \Psi_\alpha(x_1)F(x_1, x)\Psi_\beta(x).$$

It is clear that if each $F_{\alpha\beta}$ is of trace class then so is F . In that case $\text{tr } F = \sum_{\alpha\beta} \text{tr } F_{\alpha\beta}$. Furthermore

$$\begin{aligned} \int_X (\text{tr } F(x, x)) dx &= \int_X \left(\sum_{\alpha} \Psi_{\alpha}(x) \right) (\text{tr } F(x, x)) \left(\sum_{\beta} \Psi_{\beta}(x) \right) dx \\ &= \sum_{\alpha\beta} \int_X (\text{tr } F_{\alpha\beta}(x, x)) dx. \end{aligned}$$

Therefore it is enough to prove our lemma for the operators $F_{\alpha\beta}$.

$L^2(E_{\alpha\beta})$ is a closed subspace of $L^2(E)$. It is an invariant subspace for the operator $F_{\alpha\beta}$. $F_{\alpha\beta}$ equals zero on the complement of $L^2(E_{\alpha\beta})$ in $L^2(E)$, so the trace of $F_{\alpha\beta}$ is equal to the trace of the restriction of $F_{\alpha\beta}$ to $L^2(E_{\alpha\beta})$. Let $\mathcal{E}(\mathbf{R}^s)$ be the space of linear transformations of \mathbf{R}^s . Use the map $\Lambda_{\alpha\beta}$ to transform $F_{\alpha\beta}(x_1, x)$ into a section $R(t_1, t)$ of $(S_{\alpha\beta} \times S_{\alpha\beta}) \times \mathcal{E}(\mathbf{R}^s)$. Then we can regard $R(t_1, t)$ as an element in $C^\infty(T \times T) \times \mathcal{E}(\mathbf{R}^s)$. We have reduced our lemma to the case where $X = T$, $dx = dt$, $E = T \times \mathbf{R}^s$ and $F = R$.

Let $\{\phi_1(t), \phi_2(t), \dots\}$ be an orthonormal basis $L^2(T) \otimes \mathbf{R}^s$, consisting of functions of the form

$$e^{2\pi i(\nu, t)} \otimes v.$$

Here ν will be an n -tuple of integers and v will be a unit vector in \mathbf{R}^s . Let

$$(3.6) \quad r_{ij} = \int_{T \times T} (R(t_1, t) \phi_i(t), \phi_j(t_1)) dt dt_1.$$

The above inner product is of course in \mathbf{R}^s . Since $R(t_1, t)$ is differentiable, we can show from the harmonic analysis of the group $T \times T$ that if m_1, m_2 are any positive integers,

$$(3.7) \quad \sup_{ij} |r_{ij}| (1+i)^{m_1} (1+j)^{m_2} < \infty.$$

This shows that R is of trace class.

If v is in \mathbf{R}^s , then from (3.6) we can show that for any t_1, t in T

$$R(t_1, t)v = \sum_{ij} r_{ij} (\phi_i(t), v) \phi_j(t_1).$$

Therefore

$$(3.8) \quad \text{tr } R(t_1, t) = \sum_{ij} r_{ij} (\phi_i(t), \phi_j(t_1)).$$

We now compute the trace of the operator R .

$$\begin{aligned} \text{tr } R &= \sum_i r_{ii} = \sum_i \int_T r_{ii}(\phi_i(t), \phi_i(t)) dt \\ &= \sum_{ij} \int_T r_{ij}(\phi_i(t), \phi_j(t)) dt. \end{aligned}$$

This last expression is absolutely convergent by (3.7). Therefore

$$\operatorname{tr} R = \int_T \sum_{ij} r_{ij}(\phi_i(t), \phi_j(t)) dt.$$

By (3.8) this expression is equal to

$$\int_T (\operatorname{tr} R(t, t)) dt.$$

This completes the proof of Lemma 4. \square

Let us return to the proof of the theorem. By the lemma, $\operatorname{tr} A^*$ equals

$$\int_{K \times M \times \mathbf{R} \times N} \overline{f(kmh_tnk_1^{-1})} \cdot \operatorname{tr} \sigma(m^{-1}) e^{i\Lambda t} e^{\rho(tH_0)} dk dm dt dn.$$

Therefore

$$\begin{aligned} \operatorname{tr} A &= \overline{\operatorname{tr} A^*} \\ &= \int_{K \times M \times \mathbf{R} \times N} f(kmh_tnk^{-1}) \overline{\operatorname{tr} \sigma(m^{-1})} e^{-i\Lambda t} e^{\rho(tH_0)} dk dm dt dn \\ &= \int_{M \times \mathbf{R}} g_f(mh_t) \cdot \operatorname{tr} \sigma(m) \cdot e^{-i\Lambda t} dm dt, \end{aligned}$$

since

$$\overline{\operatorname{tr} \sigma(m^{-1})} = \overline{\operatorname{tr} \sigma(m)^*} = \operatorname{tr} \sigma(m).$$

Let Z_A be a set of representatives of cosets of $Z(A)/Z(A)^0$. Then M is diffeomorphic with $Z_A \times M^0$. Therefore the trace of A equals

$$\sum_{\gamma \in Z_A} \int_{M^0 \times \mathbf{R}} g_f(\gamma m_0 h_t) \cdot \operatorname{tr} \sigma(\gamma m_0) e^{-i\Lambda t} dm_0 dt.$$

For any finite set S let $[S]$ denote the number of elements in S . Then $[P] = m$. Recall that

$$q = \frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{k} - \operatorname{rank} \mathfrak{g} + \operatorname{rank} \mathfrak{k}).$$

Then

$$\begin{aligned} q &= \frac{1}{2}([P_+] + 1) \quad \text{in Case I,} \\ q &= \frac{1}{2}[P_+] \quad \text{in Case II.} \end{aligned}$$

Therefore

$$[P_M] = [P] - [P_+] = m - 2q + \iota.$$

If a_0 is in A_1^0 then

$$\overline{\Delta_M(a_0)} = \Delta_M(a_0)(-1)^{[P_M]} = \Delta_M(a_0)(-1)^{m+\iota}.$$

Now for any m_0 in M

$$\operatorname{tr} \sigma(m_0^{-1} \gamma a_0 m_0) = \operatorname{tr} \sigma(\gamma a_0).$$

Therefore, from (3.1) we see that $\operatorname{tr} A$ equals

$$c_1 \sum_{\gamma \in Z_A} \int_{A_1^0 \times \mathbf{R}} F_{g_f, \gamma}^M(a_0 h_t) \cdot \overline{\Delta_M(a_0)} \cdot \operatorname{tr} \sigma(\gamma a_0) \cdot e^{-i\Lambda t} da_0 dt.$$

By Lemma 2, this equals

$$\begin{aligned}
 & (\text{sign } \tilde{\omega}^m(\mu_\zeta))(-1)^{m+\iota} c_1 \sum_{\gamma \in Z_A} \int_{A_I^0 \times \mathbf{R}} \xi_\rho(\gamma) F_{g_f, \gamma}^M(a_0 h_t) \\
 & \quad \times e^{-i\Lambda t} \left(\sum_{s \in W_1} \varepsilon(s) \langle s_\zeta, \gamma a_0 \rangle \right) da_0 dt.
 \end{aligned}$$

By formula (3.4) this expression then equals

$$\begin{aligned}
 & (\text{sign } \tilde{\omega}^m(\mu_\zeta))(-1)^{m+\iota} \left(\frac{c_1}{c_2} \right) \sum_{\gamma \in Z_A} \int_{A_I^0 \times \mathbf{R}} F_f(\gamma a_0 h_t) e^{-i\Lambda t} \\
 & \quad \times \left(\sum_{s \in W_1} \varepsilon(s) \langle s_\zeta, \gamma a_0 \rangle \right) da_0 dt \\
 & = (\text{sign } \tilde{\omega}^m(\mu_\zeta))(-1)^{m+\iota} \left(\frac{c_1}{c_2} \right) \int_{A_I \times \mathbf{R}} F_f(a h_t) e^{-i\Lambda t} \\
 & \quad \times \left(\sum_{s \in W_1} \varepsilon(s) \langle s_\zeta, a \rangle \right) da dt.
 \end{aligned}$$

Now if s is in W_1 , substitute sa for a in the above expression. From (3.2) we obtain the formula

$$\text{tr } A = (\text{sign } \tilde{\omega}^m(\mu_\zeta)) \cdot (-1)^{m+\iota} \cdot \left(\frac{c_1}{c_2} \right) \cdot [W_1] \cdot \int_{A_I \times \mathbf{R}} F_f(a h_t) e^{-i\Lambda t} \langle s, a \rangle da dt.$$

This proves the theorem if we let $c_0 = (c_1/c_2)[W_1]$. \square

For every ζ in C' there is associated a unique μ_ζ in L'_1 . For any real Λ we write

$$\tilde{\omega}(\zeta, \Lambda) = \tilde{\omega}(\mu_\zeta + i\Lambda\mu_0).$$

For our discussion of the Plancherel formula it is necessary to examine this expression separately for Cases I and II. We have the formula

$$(3.9) \quad \tilde{\omega}(\zeta, \Lambda) = \tilde{\omega}^m(\mu_\zeta) \cdot \prod_{\alpha \in P_+} \langle \mu_\zeta + i\Lambda\mu_0, H_\alpha \rangle.$$

Now P_+ is the union of the positive real roots, $P_{\mathbf{R}}$, and the positive complex roots $P_{\mathbf{C}}$. Let η be the conjugation of $\mathfrak{g}_{\mathbf{C}}$ with respect to the real form \mathfrak{g} . η acts as a permutation of period 2 on P_+ . A root in P_+ is fixed by η if and only if it is a real root, so the positive complex roots occur in pairs. Since $\dim \mathfrak{a}_{\mathfrak{p}} = 1$, there can be at most one positive real root. Now it is known that \mathfrak{a} has a real root if and only if \mathfrak{a} is not fundamental. Therefore there exists a real root if and only if we are in Case I.

If α is a complex root and α^η is its conjugate root, then

$$\langle \mu_\zeta + i\Lambda\mu_0, H_\alpha \rangle \cdot \langle \mu_\zeta + i\Lambda\mu_0, H_{\alpha^\eta} \rangle = -(\mu_\zeta(H_\alpha))^2 + \Lambda^2 \mu_0(H_\alpha)^2.$$

Therefore the sign of the real number

$$\prod_{\alpha \in P_{\mathbf{C}}} \langle \mu_\zeta + i\Lambda\mu_0, H_\alpha \rangle$$

is equal to $(-1)^{|P_c|/2}$, which equals $(-1)^{q+\iota}$. Therefore

$$(3.10) \quad \begin{aligned} \tilde{\omega}(\zeta, \Lambda) \cdot |\tilde{\omega}(\zeta, \Lambda)|^{-1} &= i(-1)^{q+\iota} \cdot \text{sign } \Lambda \cdot \text{sign } \tilde{\omega}^m(\mu_\zeta) \quad \text{in Case I,} \\ \tilde{\omega}(\zeta, \Lambda) \cdot |\tilde{\omega}(\zeta, \Lambda)|^{-1} &= (-1)^q \cdot \text{sign } \tilde{\omega}^m(\mu_\zeta) \quad \text{in Case II.} \end{aligned}$$

It is also clear that

$$(3.11) \quad \tilde{\omega}(\zeta, -\Lambda) = (-1)^\iota \tilde{\omega}(\zeta, \Lambda).$$

If ζ is in C' , choose σ in \mathcal{E}_M such that $\sigma = \sigma(\zeta)$. In §2 we defined the representation σ' . Choose ζ' in C' such that $\sigma' = \sigma'(\zeta')$. Given ζ , ζ' is not uniquely defined. However the expression

$$\text{sign } \tilde{\omega}^m(\mu_{\zeta'}) \cdot \tilde{\omega}(\zeta', \Lambda)$$

is well defined for any real Λ . Furthermore

$$(3.12) \quad \text{sign } \tilde{\omega}^m(\mu_{\zeta'}) \cdot \tilde{\omega}(\zeta', \Lambda) = \text{sign } \tilde{\omega}^m(\mu_\zeta) \cdot \tilde{\omega}(\zeta, \Lambda).$$

For any ω in \mathcal{E}_d , let Θ_ω and $\beta(\omega)$ be the character and formal degree of ω . A formula for $\beta(\omega)$ was quoted in §2. It is clear that there is a polynomial p such that

$$(3.13) \quad \beta(\omega) \leq p(|\omega|), \quad \omega \in \mathcal{E}_d.$$

LEMMA 5. *There exists a nonnegative function $\beta(\sigma, \Lambda)$ on $\mathcal{E}_M \times \mathbf{R}$ such that for any f in $C_0^\infty(G)$,*

$$f(1) = \sum_{\omega \in \mathcal{E}_d} \beta(\omega) \Theta_\omega(f) + \sum_{\sigma \in \mathcal{E}_M} \int_0^\infty \beta(\sigma, \Lambda) \Theta_{\sigma, \Lambda}(f) d\Lambda.$$

In addition $\beta(\sigma, \Lambda)$ has the following properties.

- (i) $\beta(\sigma, \Lambda) = \beta(\sigma, -\Lambda) = \beta(\sigma', \Lambda)$.
- (ii) *For any σ in \mathcal{E}_M , $\beta(\sigma, \Lambda)$ is the restriction to \mathbf{R} of a meromorphic function on \mathbf{C} with no real poles.*
- (iii) *If σ is in \mathcal{E}_M and $\Lambda \neq 0$, then $\beta(\sigma, \Lambda) \neq 0$.*
- (iv) *For every $r > 0$, there are polynomials p_1, p_2 such that for σ in \mathcal{E}_M , Λ in \mathbf{R} ,*

$$\left| \left(\frac{d}{d\Lambda} \right)^r \beta(\sigma, \Lambda) \right| \leq p_1(|\sigma|) \cdot p_2(|\Lambda|).$$

PROOF. We deal with Case I first. Although the lemma is true in general for Case I, Harish-Chandra in [3(k), §24] proves it only in case j is one-to-one; that is, $G \overline{C} G_c$. We shall content ourselves with dealing with this situation.

Let $C^\pm = \{\zeta \in C: \langle \zeta, \exp(-1)^{1/2} \pi H_0 \rangle = \pm 1\}$. Since $(\exp(-1)^{1/2} \pi H_0)^2 = 1$, any element in C is in either C^+ or C^- . Then in [3(k), Lemma 56], Harish-Chandra shows that there is a constant, which he writes as c_A/cc_B , such that

for every f in $C_0^\infty(G)$

$$f(1) = \sum_{\omega \in \mathcal{E}_d} \beta(\omega) \Theta_\omega(f) - (c_A/c c_B) \cdot i \cdot (-1)^{m+q} \left\{ \sum_{\zeta \in C^+} \int_0^\infty \coth \frac{\pi\Lambda}{2} \cdot \tilde{\omega}(\zeta, \Lambda) \left(\int_{A_I \times \mathbf{R}} F_f(ah_t) \langle \zeta, a \rangle \cdot e^{i\Lambda t} da dt \right) d\Lambda + \sum_{\zeta \in C^-} \int_0^\infty \tanh \frac{\pi\Lambda}{2} \cdot \tilde{\omega}(\zeta, \Lambda) \left(\int_{A_I \times \mathbf{R}} F_f(ah_t) \langle \zeta, a \rangle e^{i\Lambda t} da dt \right) d\Lambda \right\}.$$

This equals

$$\sum_{\omega \in \mathcal{E}_d} \beta(\omega) \Theta_\omega(f) - (c_A/c_0 c c_B) \cdot i \cdot (-1)^{q-1} \left\{ \sum_{\zeta \in C^+} \int_0^\infty \coth \frac{\pi\Lambda}{2} \cdot \tilde{\omega}(\zeta, \Lambda) \cdot \text{sign } \tilde{\omega}^m(\mu_\zeta) \cdot \Theta_{\sigma(\zeta), -\Lambda}(f) + \sum_{\zeta \in C^-} \int_0^\infty \tanh \frac{\pi\Lambda}{2} \cdot \tilde{\omega}(\zeta, \Lambda) \cdot \text{sign } \tilde{\omega}^m(\mu_\zeta) \cdot \Theta_{\sigma(\zeta), -\Lambda}(f) \right\}.$$

We then define

$$(3.14) \quad \beta(\zeta, \Lambda) = (c_A/c_0 c c_B) (-i) (-1)^{q-1} \begin{cases} \coth \frac{\pi\Lambda}{2} \\ \tanh \frac{\pi\Lambda}{2} \end{cases} \cdot \text{sign } \tilde{\omega}^m(\mu_\zeta),$$

where $\coth(\pi\Lambda/2)$ or $\tanh(\pi\Lambda/2)$ is used depending on whether ζ is in C^+ or C^- .

Now let us deal with Case II. Then \mathcal{E}_d is empty. We can use the limit formula (3.3). Therefore

$$f(1) = (1/c) (-1)^q F_f(1; \tilde{\omega}).$$

By the Fourier inversion formula on the connected abelian group $A_I \times \mathbf{R}$,

$$f(1) = \left(\frac{1}{c}\right) (-1)^q \sum_{\zeta \in C} \int_{-\infty}^\infty \left[\int_{A_I \times \mathbf{R}} F_f(ah_t; \tilde{\omega}) \langle \zeta, a \rangle e^{i\Lambda t} da dt \right] d\Lambda.$$

Since $m = [P]$ and since F_f is in $C_0^\infty(A_I \times \mathbf{R})$, we see by integration by parts that $f(1)$ is equal to

$$\left(\frac{1}{c}\right) (-1)^{q+m} \sum_{\zeta \in C} \int_{-\infty}^\infty \tilde{\omega}(\zeta, \Lambda) \left[\int_{A_I \times \mathbf{R}} F_f(ah_t) \langle \zeta, a \rangle e^{i\Lambda t} da dt \right] d\Lambda.$$

By Theorem 1 this expression equals

$$\left(\frac{1}{c_0 c}\right) (-1)^q \sum_{\zeta \in C} \int_{-\infty}^\infty \tilde{\omega}(\zeta, \Lambda) \cdot \text{sign } \tilde{\omega}^m(\mu_\zeta) \cdot \Theta_{\sigma(\zeta), -\Lambda}(f) d\Lambda.$$

We then define

$$(3.15) \quad \beta(\zeta, \Lambda) = (2/c_0 c) (-1)^q \cdot \tilde{\omega}(\zeta, \Lambda) \cdot \text{sign } \tilde{\omega}^m(\mu_\zeta).$$

In either case, we see from (3.10) that $\beta(\zeta, \Lambda)$ is nonnegative. Also from (3.11) we see that

$$\beta(\zeta, -\Lambda) = \beta(\zeta, \Lambda).$$

It is clear that the expression $\beta(\zeta', \Lambda)$ is well defined. (3.12) implies the formula

$$\beta(\zeta', \Lambda) = \beta(\zeta, \Lambda).$$

Since for any $\Lambda \neq 0$ the representations $\pi_{\sigma', \Lambda}$ and $\pi_{\sigma, -\Lambda}$ are equivalent,

$$\Theta_{\sigma', \Lambda} = \Theta_{\sigma, -\Lambda}.$$

Therefore in either Case I or Case II we obtain the formula

$$f(1) = \sum_{\omega \in \mathcal{E}_d} \beta(\omega) \Theta_\omega + \sum_{\zeta \in C} \int_0^\infty \beta(\zeta, \Lambda) \Theta_{\sigma(\zeta), \Lambda}(f) d\Lambda.$$

It is clear in either case that

$$\beta(s\zeta, \Lambda) = \beta(\zeta, \Lambda), \quad s \in W_1.$$

Then if σ is in \mathcal{E}_M , choose any ζ in C' such that $\sigma = \sigma(\zeta)$. Define

$$\beta(\sigma, \gamma) = [W_1] \beta(\zeta, \Lambda).$$

Then $\beta(\sigma, \Lambda)$ is well defined and $\beta(\sigma, \Lambda)$ satisfies the formula of the lemma.

Property (i) of the lemma follows from the above discussion. Properties (ii), (iii), and (iv) follow easily from formulae (3.14) and (3.15). \square

For ω in \mathcal{E}_d , let π_ω be a representation in the class of ω , acting on the Hilbert space \mathcal{H}_ω . Let $\mathcal{H}_2(\omega)$ be the space of Hilbert-Schmidt operators on \mathcal{H}_ω with the Hilbert-Schmidt norm $\|\cdot\|_2$. Similarly, for σ in \mathcal{E}_M , write $\mathcal{H}_2(\sigma)$ as the space of Hilbert-Schmidt operators on \mathcal{H}_σ .

Let $L_0^2(\hat{G})$ be the set of functions

$$a_0: \mathcal{E}_d \rightarrow \bigoplus_{\omega \in \mathcal{E}_d} \mathcal{H}_2(\omega)$$

such that

(i) $a_0(\omega)$ is in $\mathcal{H}_2(\omega)$ for each ω in \mathcal{E}_d .

(ii) $\|a_0\|^2 = \sum_{\omega \in \mathcal{E}_d} \|a_0(\omega)\|_2^2 \beta(\omega) < \infty$.

Notice that if we are in Case II, \mathcal{E}_d is empty so that $L_0^2(\hat{G})$ is empty.

Let $L_1^2(\hat{G})$ be the set of functions

$$a_1: \mathcal{E}_M \times \mathbf{R} \rightarrow \bigoplus_{\sigma \in \mathcal{E}_M} \mathcal{H}_2(\sigma)$$

such that

(i) $a_1(\sigma, \Lambda)$ is in $\mathcal{H}_2(\sigma)$ for each σ in \mathcal{E}_M and Λ in \mathbf{R} .

(ii) $a_1(\sigma', -\Lambda) = N_\sigma(\Lambda) a_1(\sigma, \Lambda) N_\sigma(\Lambda)^{-1}$, $\sigma \in \mathcal{E}_M$, $\Lambda \neq 0$.

(iii) For any σ in \mathcal{E}_M , $a_1(\sigma, \Lambda)$ is a Borel function of Λ .

(iv)

$$\|a_1\|^2 = \frac{1}{2} \sum_{\sigma \in \mathcal{E}_M} \int_{-\infty}^{\infty} \|a_1(\sigma, \Lambda)\|_2^2 \beta(\sigma, \Lambda) d\Lambda < \infty.$$

(In (ii) we can regard the operators $N_\sigma(\Lambda)$ as maps from \mathcal{H}_σ to $\mathcal{H}_{\sigma'}$ if we recall the canonical isomorphisms $\mathcal{H}_{\sigma, \Lambda} \leftrightarrow \mathcal{H}_\sigma$, $\mathcal{H}_{\sigma', -\Lambda} \leftrightarrow \mathcal{H}_{\sigma'}$.)

Notice that since $N_\sigma(\Lambda)$ is unitary, condition (ii) implies that

$$\|a_1(\sigma', -\Lambda)\|_2^2 = \|a_1(\sigma, \Lambda)\|_2^2.$$

Therefore

$$(3.16) \quad \|a_1\|^2 = \sum_{\sigma \in \mathcal{E}_M} \int_0^\infty \|a_1(\sigma, \Lambda)\|_2^2 \beta(\sigma, \Lambda) d\Lambda.$$

$L_0^2(\hat{G})$ and $L_1^2(\hat{G})$ are Hilbert spaces. Let $L^2(\hat{G}) = L_0^2(\hat{G}) \oplus L_1^2(\hat{G})$. If f is in $C_0^\infty(G)$, define \hat{f} in $L^2(\hat{G})$ by

$$\begin{aligned} \hat{f} &= (\hat{f}_0(\omega), \hat{f}_1(\sigma, \Lambda)), \\ \hat{f}_0(\omega) &= \int_G f(x) \pi_\omega(x) dx, \quad \omega \in \mathcal{E}_d, \\ \hat{f}_1(\sigma, \Lambda) &= \int_G f(x) \pi_{\sigma, \Lambda}(x) dx, \quad \sigma \in \mathcal{E}_M, \Lambda \in \mathbf{R}. \end{aligned}$$

(We can regard $\hat{f}_1(\sigma, \Lambda)$ as an operator on \mathcal{H}_σ .)

THEOREM 2 (PLANCHEREL FORMULA). *The map*

$$f \rightarrow \hat{f}, \quad f \in C_0^\infty(G),$$

extends uniquely to an isometry from $L^2(G)$ onto $L^2(\hat{G})$.

PROOF. Fix f in $C_0^\infty(G)$. Define

$$g(x) = \int_G f(y) \overline{f(x^{-1}y)} dy, \quad x \in G.$$

Clearly g is in $C_0^\infty(G)$ and $g(1)$ equals $\|f\|_2^2$. If π is an irreducible unitary representation of G ,

$$\begin{aligned} \pi(g) &= \int_{G \times G} f(y) \overline{f(x^{-1}y)} dy \cdot \pi(x) dx \\ &= \int_{G \times G} f(y) \overline{f(x^{-1})} \cdot \pi(yx) dy dx \\ &= \left(\int_G f(y) \cdot \pi(y) dy \right) \left(\int_G f(x) \cdot \pi(x) dx \right)^* \\ &= \pi(f) \cdot \pi(f)^* \end{aligned}$$

where $\pi(f)^*$ is the adjoint of $\pi(f)$. Therefore

$$\text{tr } \pi(g) = \|\pi(f)\|_2^2 = \|\hat{f}(\pi)\|_2^2.$$

Therefore, applying Lemma 5 to $g(x)$ we see that

$$\|f\|_2^2 = \|\hat{f}\|^2.$$

Thus, the map $f \rightarrow \hat{f}$ is an isometry. We need only show that it is surjective.

By the Schur orthogonality relations (2.7) and (2.8), the map is onto $L_0^2(\hat{G})$. We must show that it is onto $L_1^2(\hat{G})$.

Let $\hat{\rho}_1$ be the representation of $G \times G$ on $L_1^2(\hat{G})$ given by

$$\hat{\rho}_1(x, y)a_1(\sigma, \Lambda) = \pi_{\sigma, \Lambda}(x)a_1(\sigma, \Lambda)\pi_{\sigma, \Lambda}(y^{-1})$$

for $\sigma \in \mathcal{E}_M$, $\Lambda \in \mathbf{R}$, and $(x, y) \in G \times G$. $G \times G$, being semisimple, is of type I [3(a), p. 30], so $\hat{\rho}_1$ is of type I. Let $\mathbf{R}^+ = \{\Lambda \in \mathbf{R} : \Lambda > 0\}$, $S = \mathcal{E}_M \times \mathbf{R}^+$, and let C be the measure class on S defined by the discrete measure on \mathcal{E}_M and Lebesgue measure on \mathbf{R}^+ . $\beta(\sigma, \Lambda)$ does not vanish for any (σ, Λ) in S , and the representations $\{\pi_{\sigma, \Lambda} \times \pi_{\sigma, \Lambda} : (\sigma, \Lambda) \in S\}$ of $G \times G$ are all irreducible and inequivalent. $\hat{\rho}_1$ is clearly the direct integral of these representations of $G \times G$ with respect to the measure class C . Therefore $\hat{\rho}_1$ is multiplicity-free by [6(b), Theorem 5]. This means that the algebra $R(\hat{\rho}_1, \hat{\rho}_1)$ of intertwining operators of $\hat{\rho}_1$ is commutative.

Let ρ be the two-sided regular representation of $G \times G$ on $L^2(G)$. Then the map

$$f \rightarrow \hat{f}_1, \quad f \in L^2(G),$$

is an intertwining operator between ρ and $\hat{\rho}_1$. Thus if L is the closure of the set $\{\hat{f}_1 : f \in L^2(G)\}$, and P is the orthogonal projection of $L_1^2(\hat{G})$ onto L , then P is in $R(\hat{\rho}_1, \hat{\rho}_1)$. But since $R(\hat{\rho}_1, \hat{\rho}_1)$ is commutative, it is well known that P is of the form P_E , where E is a Borel subset of S and

$$P_E = \{a_1 \in L_1^2(\hat{G}) : a_1 \text{ vanishes outside } E\}.$$

To complete the proof of the surjectivity of the map $f \rightarrow \hat{f}_1$, we need only show that the complement of E in S is a null set with respect to C .

Let us assume the contrary. Then there is a σ in \mathcal{E}_M and a subset R_1 of \mathbf{R}^+ of positive Lebesgue measure such that for any f in $C_0^\infty(G)$,

$$\hat{f}_1(\sigma, \Lambda) = 0 \quad \text{for almost all } \Lambda \text{ in } R_1.$$

Choose a τ in \mathcal{E}_K for which there is a nonzero intertwining operator T between the restriction of τ to M and σ . Choose a vector ξ in the space on which τ acts such that $T\xi \neq 0$. Define

$$\Phi(k) = T(\tau(k^{-1})\xi), \quad k \in K.$$

Then Φ is in \mathcal{H}_σ . For any f in $C_0^\infty(G)$,

$$\begin{aligned} (\hat{f}(\sigma, \Lambda)\Phi)(1) &= \left(\int_G f(x)\pi_{\sigma, \Lambda}(x) dx \cdot \Phi \right) (1) \\ &= \int_G f(x^{-1})\Phi(x)e^{-\rho(H(x))} dx. \end{aligned}$$

Then by (2.9), $(\hat{f}(\sigma, \Lambda)\Phi)(1)$ is equal to

$$\int_{K \times \mathbf{R} \times N} f(n^{-1} \cdot \exp(-tH_0) \cdot k^{-1})e^{(i\Lambda + \rho(H_0))t}\Phi(k) dk dt dn.$$

Let $f(n^{-1} \cdot \exp(-tH_0) \cdot k^{-1})$ equal

$$\chi(k) \cdot \alpha(t) \cdot \nu(n)$$

where $\chi(k) = (\tau(k)\xi, \xi)$ and ν is any function in $C_0^\infty(N)$ such that $\int_N \nu(n) dn = 1$. α is some function in $C_0^\infty(\mathbf{R})$ such that $\int_{-\infty}^\infty \alpha(t)e^{i\Lambda + \rho(H_0)t} dt$ is not equal to zero for any Λ belonging to a subset R_2 of R_1 of positive measure. Clearly such an α exists.

For a fixed Λ_0 in R_2 ,

$$(\hat{f}(\sigma, \Lambda_0)\Phi)(1) = T(\xi) \cdot \int_{-\infty}^\infty \alpha(t)e^{i\Lambda_0 + \rho(H_0)t} dt.$$

This is a nonzero vector in the space on which σ acts. However, $(\hat{f}(\sigma, \Lambda_0)\Phi)(k)$ is a continuous function of k , so $(\hat{f}(\sigma, \Lambda_0)\Phi)(k)$ is nonzero for a subset of K of positive measure. Therefore $\hat{f}(\sigma, \Lambda_0)\Phi$ is a nonzero vector in \mathcal{H}_σ . This means that the operators $\hat{f}_1(\sigma, \Lambda)$ do not vanish for any Λ in R_2 . We have a contradiction. The proof of Theorem 2 is now complete. \square

4. Statement of Theorem 3. For x in G , define

$$\Xi(x) = \int_K e^{-\rho(H(xk))} dk.$$

Define a norm on \mathfrak{g} by putting

$$\|X\|^2 = -B(X, \theta X), \quad X \in \mathfrak{g},$$

where B is the Killing form on \mathfrak{g} . Since $G = KA_pK$ there exist a unique function σ on G such that

- (i) $\sigma(k_1 x k_2) = \sigma(x)$, $k_1, k_2 \in K$, $x \in G$;
- (ii) $\sigma(\exp H) = \|H\|$, $H \in \mathfrak{a}_p$.

It is known that there exist numbers c , d such that for any a in A_p^+ ($= \{\exp tH_0 : t \geq 0\}$),

$$(4.1) \quad 1 \leq \Xi(a)e^{\rho(\log a)} \leq c(1 + \sigma(a))^d$$

(see [3(g), Theorem 3 and Lemma 36]). Also there is an $r_0 > 0$ such that

$$(4.2) \quad \int_G \Xi(x)^2(1 + \sigma(x))^{-r_0} dx = N(r_0) < \infty$$

(see [3(1), Lemma 11]).

Choose δ in K such that $\delta^{-1}a\delta = a^{-1}$ for any a in A_p . We obtain the formulae

$$(4.3) \quad \begin{aligned} \Xi(a^{-1}) &= \Xi(\delta^{-1}a\delta) = \Xi(a), \\ \sigma(a^{-1}) &= \sigma(\delta^{-1}a\delta) = \sigma(a). \end{aligned}$$

Let \mathcal{B} be the universal enveloping algebra of \mathfrak{g}_c . We can identify \mathcal{B} with the algebra of left invariant differential operators on G . Let ρ be the canonical anti-isomorphism with \mathcal{B} and the algebra of right invariant differential operators on G . If g_1 and g_2 are in \mathcal{B} and f is a differentiable function on G , then the actions

of $\rho(g_1)$ and g_2 on f commute. We denote the resultant of this action at any x in G by $f(g_1; x; g_2)$.

Now for every g_1, g_2 , in \mathcal{B} and s in \mathbf{R} , we define a seminorm on $C^\infty(G)$ by

$$\|f\|_{g_1, g_2, s} = \sup_{x \in G} |f(g_1; x; g_2)| \Xi(x)^{-1} (1 + \sigma(x))^s, \quad f \in C^\infty(G).$$

Let $\mathcal{E}(G) = \{f \in C^\infty(G) : \|f\|_{g_1, g_2, s} < \infty, \text{ for any } g_1, g_2 \text{ in } \mathcal{B} \text{ and } s \text{ in } \mathbf{R}\}$. These seminorms make $\mathcal{E}(G)$ into a Fréchet space.

Clearly

$$C_0^\infty(G) \bar{\subset} \mathcal{E}(G)$$

is a continuous inclusion, and it is known that $C_0^\infty(G)$ is dense in $\mathcal{E}(G)$ [3(1), Theorem 2]. Also from (4.2) we see that there is a continuous inclusion of $\mathcal{E}(G)$ into $L^2(G)$. $\mathcal{E}(G)$ is called the Schwartz space of G .

We wish to define a subspace of $L^2(\hat{G})$ which will ultimately turn out to be the image of $\mathcal{E}(G)$ under the Fourier transform map, $f \rightarrow \hat{f}$. We shall need to fix appropriate bases for the Hilbert spaces \mathcal{H}_ω and $\mathcal{H}_{\sigma, \Lambda}$.

For each ω in \mathcal{E}_d let π_ω be a representation in the class of ω acting on the Hilbert space \mathcal{H}_ω . We can choose an orthonormal basis

$$(4.4) \quad \{\Phi_{\tau, i} = \Phi_{\tau, i}(\omega)\}_{\tau \in \mathcal{E}_K}$$

of \mathcal{H}_ω such that $\Phi_{\tau, i}$ transforms under $\pi_\omega|_K$, the restriction of π_ω to K , according to the irreducible representation τ of K . The second subscript, i , ranges from 1 to $[\omega : \tau] \cdot \dim \tau$, where $[\omega : \tau]$ is the multiplicity of τ in $\pi_\omega|_K$. It is known that $[\omega : \tau] \leq \dim \tau$ (see [3(b), Theorem 4]).

We shall construct explicit bases for the Hilbert spaces $\mathcal{H}_{\sigma, \Lambda}$. As we remarked earlier, there is a canonical intertwining operator between the representations $\pi_{\sigma, \Lambda}|_K$ and π_σ of K . Therefore we shall choose a fixed orthonormal basis for the Hilbert space \mathcal{H}_σ .

The multiplicity of τ in $\pi_{\sigma, \Lambda}|_K$ equals the multiplicity of τ in π_σ . But π_σ is just the representation σ induced to K . Therefore by the Frobenius reciprocity theorem for compact groups [6(a), Theorem 8.2], these multiplicities are just equal to $[\tau : \sigma]$, the multiplicity of σ in $\tau|_M$ ($\tau|_M$ is the restriction of τ to M).

Fix τ in \mathcal{E}_K and σ in \mathcal{E}_M acting on the Hilbert spaces V_τ and V_σ of dimension t and s respectively. Let $R(\tau, \sigma)$ be the set of intertwining operators from V_τ to V_σ for $\tau|_M$ and σ . The Hilbert-Schmidt norm makes $R(\tau, \sigma)$ into a Hilbert space of dimension $[\tau : \sigma]$.

Now suppose T is in $R(\tau, \sigma)$. Since σ is irreducible, we can assume that there are orthonormal bases $\{\xi_1, \dots, \xi_t\}$ and $\{\eta_1, \dots, \eta_s\}$ of V_τ and V_σ respectively such that there is a constant c for which

$$\begin{aligned} T\xi_i &= c\eta_i, & i \leq s, \\ T\xi_i &= 0, & i > s. \end{aligned}$$

Suppose T has been normalized such that $c = (t/s)^{1/2}$. Then

$$(4.5) \quad \begin{aligned} T\xi_i &= (t/s)^{1/2} \eta_i, & i \leq s, \\ \|T\| &= t^{1/2}. \end{aligned}$$

Fix an element ξ of norm 1 in V_τ . Write $\tau^*(k)$ for $\tau(k^{-1})$ if k is in K . Define

$$\Phi(k) = T(\tau^*(k)\xi), \quad k \in K.$$

Then

(i)

$$\begin{aligned} \Phi(km^{-1}) &= T(\tau(m)\tau^*(k)\xi) = \sigma(m)T(\tau^*(k)\xi) \\ &= \sigma(m)\Phi(k), \quad m \in M, k \in K. \end{aligned}$$

Therefore Φ is an element in \mathcal{H}_σ .

(ii) $\|\Phi\| = 1$, because

$$\begin{aligned} (\Phi, \Phi) &= \int_K (T(\tau^*(k)\xi), T(\tau^*(k)\xi)) dk \\ &= \int_K \sum_{ij} (T[(\tau^*(k)\xi, \xi_i)\xi_i], T[(\tau^*(k)\xi, \xi_j)\xi_j]) dk \\ &= \sum_{i=1}^s \int_K ((\tau^*(k)\xi, \xi_i)\eta_i, (\tau^*(k)\xi, \xi_i)\eta_i) dk \cdot \left(\frac{t}{s}\right) \\ &= \sum_{i=1}^s \int_K (\tau^*(k)\xi, \xi_i)\overline{(\tau^*(k)\xi, \xi_i)} dk \cdot \left(\frac{t}{s}\right) \\ &= \left(\frac{t}{s}\right) \cdot \left(\frac{\dim \sigma}{\dim \tau}\right) \text{ (by the Schur orthogonality relations on } K) \\ &= 1. \end{aligned}$$

Conversely, let Φ be any unit vector in \mathcal{H}_σ such that Φ transforms under π_σ according to τ . Then there exists a unit vector ξ in V_τ and a T in $R(\tau, \sigma)$ with $\|T\| = (\dim \tau)^{1/2}$ such that

$$\Phi(k) = T(\tau^*(k)\xi), \quad k \in K.$$

For Φ defined as above, the vector $N_\sigma(\Lambda)\Phi$ is in $\mathcal{H}_{\sigma'}$. Clearly $N_\sigma(\Lambda)\Phi$ transforms under $\pi_{\sigma'}$ according to τ . Then there exists a unique T' in $R(\tau, \sigma')$ with $\|T'\| = (\dim \tau)^{1/2}$ such that

$$(N_\sigma(\Lambda)\Phi)(k) = T'(\tau^*(k)\xi), \quad k \in K.$$

The map $T \rightarrow T'$ from $R(\tau, \sigma)$ into $R(\tau, \sigma')$ will be denoted $n_\sigma(\Lambda)$, so $T' = n_\sigma(\Lambda)T$. $n_\sigma(\Lambda)$ is norm-preserving and hence unitary.

Fix an orthonormal base $\{T_1, \dots, T_r\}$ of $R(\tau, \sigma)$ of elements of norm equal to $(\dim \tau)^{1/2}$. For $1 \leq l \leq r$, $1 \leq j \leq t$, and k in K , define

$$(4.6) \quad \Phi_{\tau, (l-1)t+j}(k) = T_l(\tau^*(k)\xi_j).$$

Then $\{\Phi_{\tau, i}; \tau \in \mathcal{E}_K, 1 \leq i \leq [\tau: \sigma] \dim \tau\}$ is an orthonormal base for \mathcal{H}_σ .

The bases (4.4) and (4.6) can be used to define a collection of seminorms on $L_0^2(\hat{G})$ and $L_1^2(\hat{G})$ respectively. For each triplet (p, q_1, q_2) of polynomials we define a seminorm on $L_0^2(\hat{G})$ by letting $\|a_0\|_{p, q_1, q_2}$ be the supremum over ω , (τ_1, i_1) , (τ_2, i_2) of the expressions

$$(4.7) \quad |(\Phi_{\tau_1, i_1}, a_0(\omega)\Phi_{\tau_2, i_2})| p(|\omega|) q_1(|\tau_1|) q_2(|\tau_2|), \quad a_0 \in L_0^2(\hat{G}).$$

Let $\mathcal{E}_0(\hat{G})$ be the set of all a_0 in $L_0^2(\hat{G})$ for which $\|a_0\|_{p,q_1,q_2} < \infty$ for every triplet (p, q_1, q_2) .

For each set of polynomials (p_1, p_2, q_1, q_2) and each integer n define a seminorm on $L_1^2(\hat{G})$ as follows: put $\|a_1\|_{(p_1, p_2, q_1, q_2; n)} = \infty$ if for some σ in \mathcal{E}_M and some Φ_{τ_1, i_1} and Φ_{τ_2, i_2} the function $(\Phi_{\tau_1, i_1}, a_1(\sigma, \Lambda)\Phi_{\tau_2, i_2})$ is not n times continuously differentiable in Λ . Otherwise, let $\|a_1\|_{(p_1, p_2, q_1, q_2; n)}$ equal the supremum over $(\sigma, \Lambda), (\tau_1, i_1), (\tau_2, i_2)$ of the expressions

$$(4.8) \quad \left| \left(\frac{d}{d\Lambda} \right)^n (\Phi_{\tau_1, i_1}, a_1(\sigma, \Lambda)\Phi_{\tau_2, i_2}) \right| p_1(|\sigma|) p_2(|\Lambda|) q_1(|\tau_1|) q_2(|\tau_2|).$$

Let $\mathcal{E}_1(\hat{G})$ be the set of all a_1 in $L_1^2(\hat{G})$ for which $\|a_1\|_{(p_1, p_2, q_1, q_2; n)} < \infty$ for every set $(p_1, p_2, q_1, q_2; n)$.

The above seminorms define topologies on $\mathcal{E}_0(\hat{G})$ and $\mathcal{E}_1(\hat{G})$. Define

$$\mathcal{E}(\hat{G}) = \mathcal{E}_0(\hat{G}) \oplus \mathcal{E}_1(\hat{G}).$$

$\mathcal{E}(\hat{G})$ is a Fréchet space.

THEOREM 3. *The map $f \rightarrow \hat{f}$ gives a topological isomorphism of $\mathcal{E}(G)$ onto $\mathcal{E}(\hat{G})$.*

We shall spend most of the rest of this paper proving this theorem.

5. Spherical functions. In this section we shall define τ -spherical functions on G and develop some of their elementary properties.

A unitary double representation τ of the compact group K is a Hilbert space on which there is both a left and a right unitary K action. In addition, these actions are required to commute with each other. We denote both the left and the right action of K by τ . If τ is a unitary double representation of K on the vector space V_τ , define a representation τ' of $K \times K$ on V_τ by

$$\tau'(k_1, k_2)v = \tau(k_1)v\tau(k_2^{-1}), \quad v \in V_\tau, \quad k_1, k_2 \in K.$$

There is a one-to-one correspondence between double representations of K and representations of $K \times K$.

Suppose τ is a unitary double representation of K on the vector space V_τ . A function ϕ from G to V_τ is said to be τ -spherical if for every k_1, k_2 in K and x in G ,

$$\phi(k_1 x k_2) = \tau_1(k_1)\phi(x)\tau_2(k_2).$$

We shall write $|\phi(x)|$ to indicate the norm of $\phi(x)$ in V_τ .

Suppose $f(x)$ is a continuous complex-valued function on G such that the left and right translates of f by elements in K span a finite-dimensional space of functions on G . We shall use f to define a spherical function.

Let ϕ be the function from G into $L^2(K \times K)$ defined by

$$\phi(x)(k_1, k_2) = f(k_1^{-1} x k_2^{-1}), \quad x \in G, \quad k_1, k_2 \in K.$$

Define a double K representation μ on $L^2(K \times K)$ by

$$\begin{aligned} [\mu(\bar{k}_1)u](k_1, k_2) &= u(\bar{k}_1^{-1}k_1, k_2), \\ [u\mu(\bar{k}_2)](k_1, k_2) &= u(k_1, k_2\bar{k}_2^{-1}), \end{aligned}$$

for u in $L^2(K \times K)$ and $k_1, k_2, \bar{k}_1, \bar{k}_2$ in K . Let V_μ equal $sp_{x \in G}\{\phi(x)\}$, the finite-dimensional subspace of $L^2(K \times K)$ spanned by $\{\phi(x) : x \in G\}$. Then for any x in G , and $\bar{k}_1, \bar{k}_2, k_1, k_2$ in K ,

$$\phi(\bar{k}_1 x \bar{k}_2)(k_1, k_2) = f(k_1^{-1} \bar{k}_1 x \bar{k}_2 k_2^{-1}) = f((\bar{k}_1^{-1} k_1)^{-1} x (k_2 \bar{k}_2^{-1})^{-1}).$$

This expression equals

$$(\mu(\bar{k}_1)\phi(x)\mu(\bar{k}_2))(k_1, k_2).$$

Therefore ϕ is a μ -spherical function, which we shall call the μ -spherical function associated with f .

Notice that if τ is an irreducible unitary double representation of K on the finite-dimensional Hilbert space V_τ , then τ can be regarded as an irreducible representation $\tau_1 \otimes \tau_2^*$ of $K \times K$ on $V_1 \otimes V_2^*$. Here τ_1 and τ_2 are irreducible representations of K on the spaces V_1 and V_2 , and τ_2^* is the dual representation of τ_2 acting on V_2^* , the dual space of V_2 . We write τ as (τ_1, τ_2) and $|\tau|$ as $|\tau_1| + |\tau_2|$. Let \mathcal{E}_K^2 be the set of equivalence classes of irreducible unitary double representations of K .

Suppose that $f(x)$ is the function $(\Phi_1, \pi(x)\Phi_2)$ where π is a unitary representation of G on a Hilbert space \mathcal{H} . We assume that for $\alpha = 1$ or 2 , Φ_α is a unit vector in \mathcal{H} that transforms under $\pi|_K$ according to the irreducible unitary representation τ_α of K , acting on the Hilbert space V_α . Let $\tau = (\tau_1, \tau_2) \cdot \tau$ is in \mathcal{E}_K^2 and acts on the Hilbert space $V_\tau = V_1 \otimes V_2^*$. We shall find a formula for the spherical function ϕ associated with f . Then we shall specialize to the case where π is one of the induced representations $\pi_{\sigma, \Lambda}$ defined in §2.

Let τ_α have dimension t_α and let $\{\xi_{\alpha 1}, \dots, \xi_{\alpha t_\alpha}\}$ be an orthonormal base for V_α , for $\alpha = 1$ or 2 . Let V'_α be the subspace of \mathcal{H} spanned by $\{\pi(k)\Phi_\alpha : k \in K\}$. Choose an orthonormal base $\{\Phi_{\alpha 1}, \dots, \Phi_{\alpha t_\alpha}\}$ of V'_α such that the correspondence

$$\xi_{\alpha i} \leftrightarrow \Phi_{\alpha i}, \quad i = 1, 2, \dots, t_\alpha,$$

gives an intertwining operator between τ_α and $\pi|_K$ acting on the space V'_α . Define functions $e_{1i}(k_1)$ and $e_{2j}(k_2)$ as follows:

$$\begin{aligned} e_{1i}(k_1) &= (\pi(k_1)\Phi_1, \Phi_{1i}), \quad k_1 \in K, \\ e_{2j}(k_2) &= \overline{(\pi(k_2^{-1})\Phi_2, \Phi_{2j})} = (\pi(k_2)\Phi_{2j}, \Phi_2), \quad k_2 \in K. \end{aligned}$$

Then

$$\phi(x)(k_1, k_2) = f(k_1^{-1} x k_2^{-1}) = (\pi(k_1)\Phi_1, \pi(x)\pi(k_2^{-1})\Phi_2).$$

This is equal to the expression

$$\sum_{ij} e_{1i}(k_1) e_{2j}(k_2) (\Phi_{1i}, \pi(x)\Phi_{2j}).$$

