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Suppose that G is a reductive Lie group, with Lie algebra \mathfrak{g} . To be safe, let us assume that G is the set of real points of a reductive algebraic group defined over \mathbb{R} . Harish-Chandra has defined the Schwartz space, $C(G)$, on G . It is a certain space of functions on G such that

$$C_c^\infty(G) \subset C(G) \subset L^2(G).$$

In each case the inclusion is a continuous map from one space onto a dense subspace of the second. A distribution, F , on G is called tempered if it extends to a continuous linear functional on $C(G)$. F is said to be invariant if for fixed $f \in C_c^\infty(G)$, $\langle F, f^Y \rangle$ is independent of $y \in G$. Here

$$f^Y(x) = f(y^{-1}xy).$$

EXAMPLE: Suppose that T is a Cartan subgroup of G with Lie algebra \mathfrak{A} . If $\gamma \in T_{\text{reg}}$, the set of points in T whose centralizer is T , define

$$D(\gamma) = \det(1 - \text{Ad}(\gamma))_{\mathfrak{g}/\mathfrak{A}}.$$

Harish-Chandra has shown that the map which sends $f \in C(G)$ to

$$F(\gamma, f) = F^{G,T}(\gamma, f) = |D(\gamma)|^{1/2} \int_{T/G} f(x^{-1}\gamma x) dx$$

is well defined, and is a tempered invariant distribution on G . The distributions so obtained, known as the orbital integrals of f , play a central role in the harmonic analysis

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on G . They also arise naturally in the study of $L^2(\Gamma \backslash G)$, where Γ is a discrete co-compact subgroup of G .

PROBLEM: Define the notion of Fourier transform for an invariant tempered distribution.

In this lecture we shall describe a solution to this problem. We shall then use our solution to define a wide class of invariant tempered distributions, of which the orbital integrals are special cases.

Let K be a fixed maximal compact subgroup of G . It is the set of fixed points of a Cartan involution θ of G . Suppose that P is a parabolic subgroup of G . Then $P = NM$, and $M = M^1 \cdot A$, where N is the unipotent radical of P , and M^1 and A are uniquely determined θ -stable reductive groups such that M^1 has compact center and A is a vector group. Any $m \in M$ can be written $m^1 \exp H$, where $m^1 \in M^1$ and H belongs to \mathfrak{a} , the Lie algebra of A . We denote H by $H_M(m)$. Since $G = PK$, any x in G can be written

$$x = nmk, \quad n \in N, \quad m \in M, \quad k \in K.$$

It is customary to denote the vector $H_M(m)$ by $H_P(x)$. Suppose that σ is an irreducible representation of M . We can pull σ back from $M \cong N \backslash P$ to P and then induce to G . The result is a representation $\tilde{I}_P(\sigma)$ of G on a Hilbert space $\tilde{H}_P(\sigma)$ of functions on G , which is unitary if σ is unitary. Any function in $\tilde{H}_P(\sigma)$ is uniquely determined by its restriction to K . Let $H_P(\sigma)$ be the space of functions on K obtained in this way, and let $I_P(\sigma)$ be the correspond-

ing representation of G on $H_P(\sigma)$. The reason for introducing this latter space is that it does not change if σ is twisted by a character of A . In other words, if

$$\sigma_A(m) = \sigma(m) e^{\Lambda(H_M(m))}, \quad m \in M,$$

for any quasi-character Λ of \mathcal{M} , the representations

$$\{I_P(\sigma_A) : \Lambda \in \text{Hom}(\mathcal{M}, \mathbb{C})\}$$

all act on the same space.

A representation π of G is said to be tempered if its character is a tempered distribution. This implies that for $f \in \mathcal{C}(G)$,

$$\text{ch}_\pi(f) = \text{tr} \left(\int_G f(x) \pi(x) dx \right)$$

is well defined. We obtain a map from $\mathcal{C}(G)$ to the space of complex valued functions on $E_T(G)$, the set of classes of irreducible tempered representations of G . The problem is to characterize the image of this map.

It is known that π is tempered if and only if it is a subrepresentation of some $I_P(\sigma)$, where σ is an irreducible unitary representation of M which is square integrable modulo A . Let Δ_M be any second order left and right invariant differential operator on M such that for all such σ the operator $\sigma(\Delta_M)$, a priori a scalar, is actually positive. Then if π is any subrepresentation of $I_P(\sigma)$, define

$$\|\pi\| = \sigma(\Delta_M).$$

It is easy to show that this number depends only on the tempered representation π and not on P and σ . For any P , and any $\sigma \in E_T(M)$, we can define $\|\sigma\|$ the same way. Suppose that s is a complex valued function on $E_T(G)$. If P is a parabolic subgroup, and $\sigma \in E_T(M)$, then $I_P(\sigma)$ is a finite sum of tempered representations of G . We define $s(I_P(\sigma))$ additively. If $D = D_\Lambda$ is a differential operator of constant coefficients on the real vector space $\text{Hom}(\mathfrak{a}, i\mathbb{R})$, and n is a positive integer, define

$$\|s\|_{D,n} = \sup_{\sigma \in E_T(M)} (\|\sigma\|^n |D_\Lambda s(I_P(\sigma_\Lambda))_{\Lambda=0}|)$$

if all the derivatives in this formula exist. Otherwise, set $\|s\|_{D,n} = \infty$. Let $I(\hat{G})$ be the set of all functions

$$s: E_T(G) \rightarrow \mathbb{C}$$

such that

$$\|s\|_{D,n} < \infty$$

for all P , D and n . With the topology defined by the semi-norms $\| \cdot \|_{D,n}$, $I(\hat{G})$ becomes a Frechet space.

THEOREM 1: The map

$$\text{ch}: (f, \pi) \rightarrow \text{ch}_\pi(f), \quad f \in C(G), \quad \pi \in E_T(G),$$

is a continuous surjection from $C(G)$ onto $I(\hat{G})$.

COROLLARY: The transpose

$$\text{ch}' : I'(\hat{G}) \rightarrow C'(G)$$

is a continuous injection.

THEOREM 2: The image of ch' is exactly the space of invariant tempered distributions on G .

COROLLARY: For $f \in C(G)$ suppose that either

- (a) $\text{ch}_\pi(f) = 0$ for every $\pi \in E_T(G)$, or
- (b) $F^{G,T}(\gamma, f) = 0$ for every Cartan subgroup T and all $\gamma \in T_{\text{reg}}$

Then $I(f) = 0$ for every invariant tempered distribution I .

The corollary is easy to prove. Recall that if

$$\pi \in E_T(G),$$

$$\text{ch}_\pi(f) = \sum_{\{T\}} \int_{T_{\text{reg}}} \phi_\pi^T(\gamma) F^{G,T}(\gamma, f) d\gamma,$$

where $\{T\}$ is a set of representatives for conjugacy classes of Cartan subgroups of G , and $\phi_\pi^T(\gamma)$ is a bounded function on T . Therefore, (b) implies (a). But

$$I(f) = (\text{ch}'(S))(f) = S(\text{ch } f),$$

for some $S \in I'(G)$. Therefore (a) implies that $I(f) = 0$.

The proofs of Theorems 1 and 2 are more difficult and will appear elsewhere. However, they give a solution to our problem. If I is a tempered invariant distribution on G , we define its Fourier transform to be $(\text{ch}')^{-1}(I)$. It is an element in the dual space of $I(\hat{G})$.

Suppose that A is a vector subgroup of G such that $\theta A = A$. Let M be the centralizer of A in G and let \mathfrak{a} be the Lie algebra of A . A is called a special subgroup of G if it is the split component of a parabolic subgroup P . Suppose that this is the case and let $\mathcal{P}(A)$ be the set of all such P . If $P \in \mathcal{P}(A)$, let Φ_P be the set of simple roots of (P, A) . Then

$$\mathfrak{a}_P^+ = \{H \in \mathfrak{a} : \alpha(H) > 0, \alpha \in \Phi_P\}$$

is called the positive chamber of P . As P varies throughout $\mathcal{P}(A)$, the associated positive chambers are disjoint and dense in \mathfrak{a} . Groups P and P' in $\mathcal{P}(A)$ are said to be adjacent if their chambers share a common wall.

Let us identify \mathfrak{a} with its dual space by means of a suitable positive definite bilinear form \langle , \rangle on \mathfrak{a} . Suppose that

$$\{c_P(\lambda) : P \in \mathcal{P}(A), \lambda \in \mathfrak{a}_P^+\}$$

is a family of analytic functions. We shall call it a compatible family if whenever P and P' are adjacent and λ is in their common wall,

$$c_P(\lambda) = c_{P'}(\lambda).$$

LEMMA 1: If the family is compatible,

$$\sum_{P \in \mathcal{P}(A)} c_P(\lambda) \prod_{\alpha \in \Phi_P} \langle \alpha, \lambda \rangle^{-1}$$

is an analytic function of Λ .

PROOF: The only possible singularities of the function are $\langle \beta, \Lambda \rangle = 0$ for roots β of (G, A) . Given β the only summands for which $\langle \beta, \Lambda \rangle = 0$ is a singularity occur in adjacent pairs, P and P' . Moreover, we can assume $\Phi_P \cup (-\Phi_{P'}) = \{\beta\}$. The contribution in the above sum from (P, P') is

$$\langle \beta, \Lambda \rangle^{-1} \{ c_P(\Lambda) \left(\prod_{\alpha \in \Phi_P \setminus \{\beta\}} \langle \alpha, \Lambda \rangle \right)^{-1} - c_{P'}(\Lambda) \left(\prod_{\alpha \in \Phi_{P'} \setminus \{-\beta\}} \langle \alpha, \Lambda \rangle \right)^{-1} \}.$$

If Λ is in the common wall of P and P' , the expression in the curly brackets is 0. It follows that $\langle \beta, \Lambda \rangle = 0$ is not a singularity. []

EXAMPLE: Fix $x \in G$. It is easy to check that

$$\{ c_P(\Lambda) = e^{\langle \Lambda, H_P(x) \rangle} : P \in \mathcal{P}(A) \}$$

is a compatible family. Therefore

$$v(x) = \lim_{\Lambda \rightarrow 0} \sum_{P \in \mathcal{P}(A)} e^{\langle \Lambda, H_P(x) \rangle} \left(\prod_{\alpha \in \Phi_P} \langle \beta, \Lambda \rangle \right)^{-1}$$

is well defined. If $m \in M$,

$$H_P(mx) = H_P(x) + H_M(m),$$

so that

$$v(mx) = e^{\langle 0, H_M(m) \rangle} v(x) = v(x).$$

Suppose that T is a Cartan subgroup in M , and $\gamma \in T_{\text{reg}}$. It can be shown that

$$|D(\gamma)|^{1/2} \int_{T \setminus G} f(x^{-1} \gamma x) v(x) dx, \quad f \in C(G),$$

is a tempered distribution. However, it is not invariant. We would like to modify it to obtain an invariant distribution.

LEMMA 2: Suppose that σ is an irreducible unitary representation of M . Then there exist unitary operators

$$R_{P' | P}(\sigma): H_P(\sigma) \rightarrow H_{P'}(\sigma), \quad P, P' \in P(A),$$

such that

$$(i) \quad R_{P' | P}(\sigma) I_P(\sigma, x) = I_{P'}(\sigma, x) R_{P' | P}(\sigma), \quad x \in G,$$

$$(ii) \quad R_{P'' | P}(\sigma) = R_{P'' | P'}(\sigma) R_{P' | P}(\sigma), \quad P, P', P'' \in P(A),$$

(iii) If P and P' are adjacent and Λ lies in their common wall, $R_{P' | P}(\sigma_\Lambda) = R_{P' | P}(\sigma)$.

Suppose that $P_0 \in P(A)$, $f \in C(G)$, and $\sigma \in E_T(M)$.

Then

$$c_P(\Lambda) = \text{tr}(R_{P | P_0}(\sigma)^{-1} R_{P | P_0}(\sigma_\Lambda) I_{P_0}(\sigma, f))$$

is a compatible family. Therefore

$$\phi_A(F, \sigma) = \lim_{\Lambda \rightarrow 0} \sum_P c_P(\Lambda) (\prod_{\alpha \in \Phi_P} \langle \alpha, \Lambda \rangle)^{-1}$$

is well defined. It is independent of P_0 .

LEMMA 3: The map

$$\sigma \rightarrow \phi_A(f, \sigma), \quad \sigma \in E_T(M),$$

is in $I(M)$.

It follows from Theorem 1 that there is a function $\phi_A(f)$ in $C(M)$ such that

$$\text{ch}_\sigma(\phi_A(f)) = \phi_A(f, \sigma)$$

for every $\sigma \in E_T(M)$. $\phi_A(f)$ is not uniquely defined, but it follows from Theorem 2 that if I is any invariant tempered distribution on M , $I(\phi_A(f))$ is uniquely defined. Suppose that $\gamma \in T_{\text{reg}}$. We define a number $F^{G,T,A}(\gamma, f)$ inductively by

$$\sum_{\{A^*: A^* \subset A\}} F^{M^*, T, A}(\gamma, \phi_{A^*}(f)) = |D(\gamma)|^{1/2} \int_{T \setminus G} f(x^{-1}\gamma x) v(x) dx.$$

The sum on the left is over the finite set of special subgroups which are contained in A . We are assuming inductively that we have defined all the summands except the one for which A^* is the split component of the center of G . In this latter case, of course,

$$F^{M^*, T, A}(\gamma, \phi_{A^*}(f)) = F^{G, T, A}(\gamma, f).$$

Notice that if A is the split component of the center of G there is only one term in the sum. In this case $F^{G, T, A}(\gamma, f)$ is just the orbital integral $F^{G, T}(\gamma, f)$ defined earlier.

THEOREM 3: The map

$$F^{G, T, A}(\gamma) : f \rightarrow F^{G, T, A}(\gamma, f)$$

is an invariant tempered distribution on G .

Notice that this theorem is necessary for our inductive definition to work. The theorem, along with Lemmas 2 and 3, is proved in [1]. The distributions $F^{G,T,A}(\gamma)$ seem to be the natural replacements of the orbital integrals when one tries to study $L^2(\Gamma \backslash G)$ and $\Gamma \backslash G$ is assumed only to have finite volume.

Reference

- [1] J. Arthur, On the invariant distribution associated to weighted orbital integrals, preprint.

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