

Harmonic Analysis in Mathematics

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Harmonic analysis in music is the study of chords, and of how they are used in combination to create musical effects. Harmonic analysis in mathematics takes on a somewhat different meaning. It too has roots in music, or at least in the mathematical analysis of sound. However, the term can also mean a kind of universal duality that runs throughout mathematics, as well as other sciences.

Different musical instruments make different kinds of sound, even when they play the same note. The analysis of this phenomenon can be very complicated, but to a first approximation, it is the *shape* of the instrument that determines the sound it creates. Shape is an obvious *geometric* property. Sound is an example of what is often called a *spectral* phenomenon. The term *spectral* has a precise meaning in mathematics, but we can also think of it in the everyday sense of the word—ghostlike, hard to pin down—the very opposite of geometric. We thus have an example of something *geometric*—the shape of a musical instrument—which corresponds to something *spectral*—the sound produced by the instrument. Harmonic analysis in the broadest sense refers to a general principle in mathematics that links geometric objects with spectral objects. The two kinds of phenomena are sometimes related by explicit formulae, sometimes by parallel mathematical theories and sometimes by laws of physics. The principle seems to be a profound aspect of the inner structure of mathematics, of which sound is but one special case.

Light is another kind of spectral phenomenon. It comes in different colours, or wave lengths, which we can observe by passing sunlight through a prism. We speak of the *spectrum* of light, or more generally of the electromagnetic spectrum, to mean the range of possible wave lengths. This is the scientific origin of the term spectral. One can devise physical experiments with radiation in which the physical or geometric properties of the equipment govern the wave lengths of the radiation that is produced.

The most far-reaching examples from physics are perhaps the ones in quantum mechanics. Quantum mechanics tells us that there is a wave-particle duality for the basic constituents of matter. An electron behaves in some ways like a classical particle, a geometric notion if we think of the path it traces in space. It also behaves like a wave, which is a quintessential spectral concept. The wave in this case does not consist of sound or light; rather it is a wave of probabilities, which quantifies the irrevocable uncertainty of being able to determine the exact location of the electron at any given instant.

I have concentrated on examples from physics since they can be described independently of mathematics. These examples can also be formulated in precise mathematical terms. In this way, they become part of harmonic analysis in mathematics, interpreted in the broad sense above. The general duality between geometric objects and spectral objects permeates mathematics. It has a pervasive influence in such diverse areas as group theory, topology, differential geometry, number theory and algebraic geometry, as well as the areas of analysis and differential equations with which it is more traditionally associated.

I conclude with a few words about some particularly mysterious spectral objects. They were discovered by Alexander Grothendieck, one of the greatest mathematicians of the 20th century. He called them *motifs*. He seems to have chosen the word with an appreciation for the use of motifs in music, literature and the visual arts. Motifs in art are recurring themes that unify a work through its deeper levels of meaning. Motifs in mathematics play a similar role: they represent a magnificent internal structure, hidden and highly abstract, that unifies the venerable subject of algebraic geometry. We do not really understand them. But it does appear that mathematical motifs are a kind of analogue in science of what has long been an essential aspect of art.

Mathematical motifs are now a basic part of what is known as the Langlands programme. The Langlands programme postulates profound relationships between motifs and completely different spectral objects, called *automorphic forms*. Automorphic forms have spectra that behave much like the wave lengths of electromagnetic radiation. For this reason, they are considerably less abstract than motifs. It has been possible in some cases to establish the proposed relationships between motifs and automorphic forms by comparing corresponding geometric data obtained from harmonic analysis.

Although much of it remains conjectural, the Langlands programme might well be the ultimate mathematical expression of symmetry. It provides an extraordinary unification of fundamental mathematical laws that would seem to be completely unrelated. In this sense, the Langlands programme has goals that are comparable with those of present-day theoretical physics. It bears a remarkable resemblance to the quest by physicists to unify the fundamental forces of nature.

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